Fuzzy Quasi-Metric Spaces: Bicompletion, Contractions on Product Spaces, and Applications to Access Predictions

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Certifico: que la presente memoria "Fuzzy Quasi-Metric Spaces: Bicompletion, Contractions on Product Spaces, and Applications to Access Predictions", ha sido realizada bajo mi dirección por D. Francisco Castro Company, en el Departamento de Matemática Aplicada de la Universidad Politécnica de Valencia, y constituye su tesis para optar al grado de Doctor por la Universidad Politécnica de Valencia.

Y para que así conste, presento la referida tesis, firmando el presente certificado.

Valencia, Marzo de 2010

Fdo. Salvador Romaguera Bonilla

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Abstract

Since L.A. Zadeh introduced the theory of fuzzy sets in 1965, it has been used in a range of areas of mathematics and applied to a great variety of real life scenarios. These scenarios cover complex processes with no simple mathematical model such as industrial control devices, planning and scheduling, pattern recognition, etc. or systems managing inaccurate or highly unpredictable information.

Fuzzy topology is one important example of use of L.A. Zadeh's theory. Through the years, authors of this field have pursued the definition of a fuzzy metric space in order to measure the distance between elements of a set according to degrees of closeness.

This work deals with the bicompletion of fuzzy quasi-metric spaces in the sense of Kramosil and Michalek. Sherwood proved that every fuzzy metric space has a completion which is unique up to isometry based on properties of Lévy's metric. Here we prove that each fuzzy quasi-metric space has a bicompletion. Our construction is performed using directly the suprema of subsets of $[0, 1]$ and lower limits of sequences in $[0, 1]$ instead of using Lévy's metric.

We take advantage of the bicompleteness and bicompletion of fuzzy quasi-metric spaces as well as of the properties of fuzzy and intuitionistic fuzzy metric spaces in order to introduce several applications to computer science problems.

Thus, the existence and uniqueness of solution for the recurrence equations associated to certain algorithms with two recursive procedures is studied. To carry out a complexity analysis of algorithms we apply the Banach contraction principle both in a certain product of (non-Archimedean) fuzzy quasi-metrics on the domain of words and in the product quasi-metric of two

Schellekens' complexity quasi-metric spaces.

Finally, we study an application of fuzzy metric spaces to information systems based on accesses locality. For that means we use equivalence classes in order to compare elements and we take advantage of the suitability of fuzzy constructions related to problems that evolve during time. This approach allows to define a dynamical framework to decide on an object classification into different classes. As a natural extension of the model we use the notion of an intuitionistic fuzzy metric space to measure both the degree of closeness and remoteness between two elements of a fuzzy set.

Resumen

Desde que L.A. Zadeh presentó la teoría de conjuntos difusos en 1965, se ha usado en una serie de áreas de las matemáticas y se ha aplicado en una gran variedad de escenarios de la vida real. Estos escenarios cubren procesos complejos sin modelo matemático sencillo tales como dispositivos de control industrial, planificación y programación, reconocimiento de patrones, etc. o sistemas que gestionen información imprecisa o altamente impredecible.

La topología difusa es un importante ejemplo de uso de la teoría de L.A. Zadeh. Durante años, los autores de este campo han buscado obtener la definición de un espacio métrico difuso para medir la distancia entre elementos según grados de proximidad.

El presente trabajo trata acerca de la bicompletación de espacios casi-métricos difusos en el sentido de Kramosil y Michalek. Sherwood probó que todo espacio métrico difuso tiene una completación que es única excepto por isometría basándose en propiedades de la métrica de Lévy. Probamos aquí que todo espacio casi-métrico difuso tiene bicompletación. Nuestra construcción se realiza usando directamente el supremo de conjuntos en $[0, 1]$ y límites inferiores de secuencias en $[0, 1]$ en lugar de usar la métrica de Lévy.

Aprovechamos tanto la bicompletitud y bicompletación de espacios casi-métricos difusos como las propiedades de los espacios métricos difusos y espacios métricos difusos intuicionistas para presentar varias aplicaciones a problemas del campo de la informática.

De esta manera, se estudia la existencia y unicidad de una solución para las ecuaciones de recurrencia asociadas a ciertos algoritmos formados por dos procedimientos recursivos. Para realizar el análisis de complejidad de algoritmos aplicamos el principio de contracción de Banach tanto en un producto de casi-métricas (no-Arquimedianas) en el dominio de las palabras como en

la casi-métrica producto de dos espacios de complejidad casi-métricos de Schellekens.

Finalmente, estudiamos una aplicación de espacios métricos difusos a sistemas de información basados en localidad de accesos. Para ello usamos clases de equivalencia para comparar elementos y aprovechamos la idoneidad de las construcciones difusas para modelar problemas que evolucionan con el tiempo. Esta aproximación permite definir un marco dinámico para decidir acerca de la clasificación de un elemento en clases. Como extensión natural del modelo usaremos la noción de un espacio métrico intuicionista para modelar tanto el grado de proximidad como el de lejanía de dos elementos de un conjunto difuso.

Resum

Des de que L.A. Zadeh va presentar la teoria de conjunts difusos en 1965, s'ha gastat en una serie d'árees de les matemàtiques i s'ha aplicat en una gran varietat d'escenaris de la vida real. Estos escenaris cobrixen procesos complexes sense model matemàtic senzill com dispositius de control industrial, planificació o programació, reconeiximent de patrons, etc. o també sistemes que gestionen informació imprecisa o altament impredictible.

La topologia difusa es un important exemple d'us de la teoria de L.A. Zadeh. Durant anys, els autors d'este camp han buscat obtindre la definició d'un espai metric difus per a medir la distancia entre elements d'un conjunt segons graus de proximitat.

El present treball tracta de la bicompletació d'espais casi-mètrics difusos en el sentit de Kramosil i Michalek. Sherwood va provar que tot espai metric difus té una completació que es única excepte per isometria basant-se en propietats de la mètrica de Lévy. Ací provem que tot espai casi-mètric difus té bicompletació. La nostra construcció s'obté gastant directament el suprem de conjunts en $[0,1]$ i límits inferiors de seqüencies en $[0,1]$ en lloc de gastar la mètrica de Lévy.

Aprofitem tant la bicompletitud i bicompletació d'espais casi-mètrics difusos com les propietats d'espais mètrics difusos i espais mètrics difusos intuicionistes per a presentar distintes aplicacions a problemes del camp de la informàtica.

D'esta manera, s'estudia l'existència i unicitat d'una solució per a les ecuacions de recurrència associades a certs algorismes formats per dos procediments recursius. Per a fer l'anàlisis de complexitat d'algorismes apliquem el principi de contracció de Banach tant en un producte de casi-mètriques (no-Arquimedianes) en el domini de les paraules com en la casi-mètrica pro-

ducte de dos espais de complexitat casi-mètrics de Schellekens.

Finalment, estudiem una aplicació d'espais mètrics difusos a sistemes d'informació basats en localitat d'accesos. Per a ixe motiu gastem classes d'equivalència per a comparar elements i aprofitem la idoneïtat de les construccions difuses per a modelar problemes que evolucionen durant el temps. Esta aproximació permitix definir un marc dinàmic per a decidir al voltant de la classificació d'un element en classes distintes. Com a extensió natural del model gastem la noció d'un espai mètric intuicionista per a modelar tant el grau de proximitat com el de lluntania de dos elements d'un conjunt difus.

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Chapter 1

Introduction

Since the theory of fuzzy sets, introduced by L.A. Zadeh [68] appeared in 1965 it has been used in a range of areas of mathematics. One of these areas, fuzzy logic, has allowed to apply fuzzy behaviour to implement industrial control devices and to use multivalued logic concepts in real scenarios.

These techniques are profitable for complex processes with no simple mathematical model. To name a few, tunning and maintenance of home appliances (domotics) or industrial devices (elevators, air conditioners, lightning) are systems whose linearity shall be approached using fuzzy constructions; we can take advantage of fuzziness to monitor public transportation schedules, define routes according to space coverage, etc.; some existing works show the application of fuzzy constructions to pattern recognition or defects detection in colour images [41]; Geographic Information Systems (GIS) are also suited for fuzziness as boundaries of geographic objects and areas are usually inaccurate.

A fuzzy set is a set whose elements may be divided into the ones that belong to the set, the ones that do not belong set and the ones for which it is not possible to decide without a certain degree of uncertainty whether they belong to the set or not. Following Zadeh's idea, K. Atanassov [3] introduced the concept of intuitionistic fuzzy set to allow grouping elements according to degrees of truth.

Fuzzy topology is another example of use of Zadeh's theory. Authors of this field have pursued the definition of a fuzzy metric space from different points of view (see for instance [34, 17, 32, 2]) so that the distance between different elements can be established according to degrees of closeness and remoteness.

Our work deals with the bicompletion of fuzzy quasi-metric spaces, a compelling topic from fuzzy topology, and with the application of fuzzy constructions to several areas of computer science. In order to emphasize these objectives the text is structured in chapters according to three main subjects: bicompletion of fuzzy quasi-metric spaces (Chapter 3), application of fuzzy metric constructions to the complexity of algorithms (Chapter 4) and to information systems based on access locality (Chapter 5). In each chapter, conclusions extracted from our work are given.

After this brief introduction, Chapter 2 recalls for several general definitions that will be used in the remaining chapters. Notice that not all the definitions have been placed in Chapter 2, to ease the readability other definitions will be found where they are meant to be used. This initial chapter is meant for basic concepts and for definitions that are relevant for several chapters.

The core of this study is based on the Kramosil and Michalek notion of a fuzzy metric space (see [34], Definition 2.2.3 and the ones based on it) and on the definition of an intuitionistic fuzzy metric space (see [47] and [1]).

Sherwood proved in the framework of probabilistic metric spaces [65] that every fuzzy metric space has a completion which is unique up to isometry, with the help of the completeness properties of Lévy's metric. In Chapter 3 we will prove that the bicompletion of fuzzy quasi-metric spaces can be achieved avoiding the use of Lévy's metric [19, 38] directly using the suprema of subsets of $[0,1]$ and lower limits of sequences in $[0,1]$.

In fact, completeness and completion are very useful properties in the context of (fuzzy)metric spaces. We will take advantage of this desirable characteristic in order to introduce several applications of fuzzy constructions to computer science problems.

Fuzzy sets can be classified to apply in several information-driven types of tasks: classification and data analysis, decision-making problems and approximate reasoning. We have chosen two different classification and data analysis problems as applications. One of them, see Chapter 4, is borrowed from algorithms theory. The other problem shown in Chapter 5 is a quite general one of computer science scenarios; systems based on access locality are widely spread. Even though we can find uses of access locality in basic foundations of computer science we have focused on an applied scenario to try to offer a glimpse of the potential applications of the of fuzzy metric spaces.

In Chapter 4 we show the existence and uniqueness of solution for the recurrence equations associated to certain algorithms with two recursive procedures by applying the Banach contraction principle both in a certain product of fuzzy quasi-metrics on the domain of words and in the product quasi-metric of two complexity spaces. In particular we use Schellekens' complexity quasi-metric space $(\mathcal{C}, d_{\mathcal{C}})$ [59] as a model for the complexity analysis of algorithms.

Several approaches can be found in algorithms theory in order to analyze their complexity (see [5] or [10]). Take into account that complexity analysis is essential as it classifies the cost of execution of computer programs. These programs are composed of data structures, algorithms, functions from the main system and related subsystems, thus a classification of algorithms efficiency in terms of estimated execution time and data and resources usage is vital in order to decide on the suitability of computer solutions.

For the general case, asymptotic cost analysis shall be used. This technique compares the order of magnitude of the cost (time consumption, generally) of an algorithm with a known cost (polynomic, logarithmic, etc.). Comparisons for the worst, best or average case can be performed in order to bound theoretically the cost – $\mathcal{O}$, Ω, Θ functions – as it is shown in Section 4.1.

Recursive algorithms are those defined in terms of calls to the algorithm itself. In fact, recursiveness is an unifying theory for algorithmic problems based on recurrence equations. Unfortunately there is no general solution for recurrence equations. One of the existing methods to solve them is to use the celebrated Banach fixed point theorem. The contraction principle and the completeness of the metric space used ensures the uniqueness of the recurrence equation solution.

Recently, Park introduced and studied in [47] a notion of intuitionistic fuzzy metric space that generalizes the concept of fuzzy metric space due to George and Veeramani [17]. These spaces were initially motivated from a high energy physics point of view (see [42, 43, 44, 45, 46], etc. from M.S. El Naschie).

Almost simultaneously C. Alaca, D. Turkoglu and C. Yildiz proved in [1] intuitionistic fuzzy versions of the Banach fixed point theorem by means of a notion of intuitionistic fuzzy metric space which is based on the concept of fuzzy metric space due to Kramosil and Michalek [34].

In [56] the authors generalize the notions of intuitionistic fuzzy metric space by Alaca et al to the quasi-metric setting and obtain, among other results, an intuitionistic fuzzy quasi-metric version of the Banach contraction principle which is applied to deduce the existence of solution for the recurrence equation which is typically associated to the complexity analysis of Quicksort algorithm [30].

In order to complement previous works we have chosen a kind of recurrence algorithm composed by two recurrences, extracted from [4]:

```
function P(n)
    if n > 0 then Q(n-1); C; P(n-1); C; Q(n-1)


function Q(n)
    if n > 0 then P(n-1); C; Q(n-1); C; P(n-1); C; Q(n-1)
```

In Chapter 5 we study an application of fuzzy metric spaces to information systems (with a special focus on distributed systems) based on accesses locality. In this chapter we make an extensive use of equivalence classes in order to compare elements. Fuzzy metric space constructions are used to achieve accesses optimization in general information systems based on the classification of element classes. Previous approaches are based on the statistical nature of data accesses [31].

Among the variety of information systems, we choose those based on access locality in the sense of proximity among the elements of the set. This characteristic appears quite often in basic information systems (compilation, physical memory accesses, transaction isolation, etc.) and also suits finely the way human organizations are structured (headquarters and geographically scattered delegations for instance).

After a first approach using a quasi-metric lattice in Section 5.2, we tackled the problem using a fuzzy metric space construction (see Section 5.3). The first approach served to obtain a function v such that $v(x)$ allows us to compare element x access histories.

The fuzzy metric space approach from Section 5.3 allows us to move from a late classification to a more dynamical classification. Here we take advantage of the suitability of fuzzy constructions related to problems that evolve under known patterns during time. Due to this fact we need to change

v definition to an appropriate function $v(x,t)$ for which we associate t directly to the concept of "time" and that will lead to the definition of $M(x,y,t)$.

This adaptability is a strong attribute of fuzzy structures as lightweight calculations allow progressive tunning.

For our objective we show that the Kramosil and Michalek definition of fuzzy metric [34] is the one that is better suited for our purposes. Comparison is now performed using the metric space $(X, M, *)$ where X is the set of elements and M is defined by:

$$M(x,y,t) = v(x,t) * v(y,t)$$

where $*$ is any t-norm on the elements of the set which allows to use fuzzy metrics to improve our previous results.

Later in Section 5.4, we use the notion of an intuitionistic fuzzy metric space as a natural generalization of a fuzzy metric space which provides mechanisms to measure the degree of closeness, with a metric based on a t-norm, and also remoteness, with another metric based on a t-conorm, between two elements of a fuzzy set according to a parameter t.

As the tackled problem has a great variability we have taken an approach similar to the one used when analysing the complexity of algorithms: worst, best and average scenarios have been tested to perform an empirical analysis of all approaches (see Sections 5.2.2, 5.3.2 and 5.4.2).

Due to this variability of scenarios and due to the fact that we also pursue fast computation of element comparisons, traditional t-norms and t-conorms are the ones that are best suited as the intuitionistic fuzzy metric tuple components.

Nevertheless, for this study, t-norms, t-conorms and families of t-norms and t-conorms selected from [13] are evaluated in order to provide a large pool of choices for any real scenario.

Chapter 2

Preliminary concepts and definitions

Most of our work relies on the set of concepts we are about to introduce in this chapter. Here we show the ones that are used throughout all the chapters or the ones whose relevance for our objectives is evident.

2.1 Basic definitions

We introduce firstly the notions of quasi-metric and order which are used in Section 5.2 for our initial approach to model information systems based on access locality.

Our basic references for quasi-uniform and quasi-metric spaces are [15, 36] and for order theory they are [11, 18].

Definition 2.1.1. *A quasi-metric on a set X is a nonnegative real valued function defined on $X \times X$ such that for all $x, y, z \in X$:*

(i) $d(x, y) = d(y, x) = 0 \Longleftrightarrow x = y$;

(ii) $d(x, y) \leq d(x, z) + d(y, z)$.

Definition 2.1.2. *A quasi-metric space is a pair (X, d) such that X is a set and d is a quasi-metric on X.*

Each quasi-metric d on X induces a T_0 topology τ_d on X which has as a base the family of open balls $\{B_d(x,r) : x \in X, \varepsilon > 0\}$, where

$$B_d(x,\varepsilon) = \{y \in X : d(x,y) < \varepsilon\}$$

for all $x \in X$ and $\varepsilon > 0$.

Given a quasi-metric d on X, then the function d^{-1} defined by $d^{-1}(x,y) = d(y,x)$, is also a quasi-metric on X, called the conjugate of d, and the function d^s defined by $d^s(x,y) = \max\{d(x,y), d^{-1}(x,y)\}$ is a metric on X.

A quasi-metric space (X, d) is said to be bicomplete if (X, d^s) is a complete metric space. In this case we say that d is a bicomplete quasi-metric on X.

A topological space (X, τ) is called quasi-metrizable if there is a quasi-metric d on X such that $\tau = \tau_d$.

Definition 2.1.3. *An order on a nonempty set X is a binary relation $\leq$ on X such that for all $x, y, z \in X$:*

(i) *$x \leq x$ (reflexivity);*

(ii) *$x \leq y$ and $y \leq x \Rightarrow x = y$ (antisymmetry);*

(iii) *$x \leq y$ and $y \leq z \Rightarrow x \leq z$ (transitivity).*

Definition 2.1.4. *An ordered set is a pair $(X, \leq)$ such that $\leq$ is an order on the (nonempty) set X.*

A totally ordered set is an ordered set $(X, \leq)$ such that for each $x, y \in X$, $x \leq y$ or $y \leq x$.

Definition 2.1.5. *If d is a quasi-metric on a (nonempty) set X, the order $\leq_d$ induced by d is defined by:*

$$x \leq_d y \Longleftrightarrow d(x,y) = 0.$$

and it is called the specialization order.

It is clear that indeed $(X, \leq_d)$ is an ordered set.

Definition 2.1.6. *An ordered set $(X, \leq)$ is called a lattice if each $x, y \in X$ have a supremum $x \vee y$ and an infimum $x \wedge y$.*

It is clear that each totally ordered set $(X, \leq)$ is a lattice because if for $x, y \in X$ we have, for instance, $x \leq y$, then we can define $x \vee y = y$ and $x \wedge y = x$.

Definition 2.1.7. *A quasi-metric lattice is a triple $(X, d, \leq)$ such that (X, d) is a quasi-metric space and $(X, \leq)$ is a lattice such that:*

$$d(x \vee z, y \vee z) \leq d(x, y) \quad \text{and} \quad d(x \wedge z, y \wedge z) \leq d(x, y)$$

The very foundation of our study of closeness or remoteness among elements of a set is based on the following definitions of t-norm and t-conorm.

Definition 2.1.8. [61]. *A continuous t-norm is a binary operation $* : [0, 1] \times [0, 1] \to [0, 1]$ such that: (i) $*$ is commutative and associative, (ii) $*$ is continuous, (iii) $a * 1 = a$ for all $a \in [0, 1]$, and (iv) $a * b \leq c * d$ when $a \leq c$ and $b \leq d$ $(a, b, c, d \in [0, 1])$.*

Three paradigmatic examples of continuous t-norms are $\wedge$, Prod and $*_L$ (the Lukasiewicz t-norm), which are defined by $a \wedge b = \min\{a, b\}$, $a\text{Prod}b = a \cdot b$ and $a *_L b = \max\{a + b - 1, 0\}$, respectively.

Remark. *Note that by conditions (iii), (iv), above, $* \leq \wedge$ for every continuous t-norm $*$.*

Definition 2.1.9. [40]. *A continuous t-conorm is a binary operation $\diamond : [0, 1] \times [0, 1] \to [0, 1]$ such that: (i) $\diamond$ is commutative and associative, (ii) $\diamond$ is continuous, (iii) $a \diamond 0 = a$ for all $a \in [0, 1]$ and (iv) $a \diamond b \leq c \diamond d$ when $a \leq c$ and $b \leq d$ $(a, b, c, d \in [0, 1])$.*

Remark. *It is known that* $x \vee y \leq x \diamond y$ *for any continuous t-conorm* $\diamond$*, if* $x \vee y$ *is the continuous t-conorm* $\max(x, y)$.

The minimum and maximum t-norms where proposed by L.A. Zadeh [68] to perform several logical operations on fuzzy sets.

Definition 2.1.10. [13]. *For a given t-norm* $*$ *and* $x, y \in [0, 1]$*, the t-conorm* $*'$ *defined as:*

$$x *' y = 1 - ((1 - x) * (1 - y))$$

is called the dual t-conorm of $*$.

Similarly, for any given t-conorm $\diamond$*, the t-norm* $\diamond'$ *defined as:*

$$x \diamond' y = 1 - ((1 - x) \diamond (1 - y))$$

is called the dual t-norm of $\diamond$.

Definition 2.1.11. [13]. *If* $*$ *is a t-norm (respectively* $\diamond$ *is a t-conorm) such that:*

$$\lim_{n \to \infty} (x * x)^{(n)} = 0$$

where $(x * x)^{(n)}$ *denotes the nth power of the t-norm (respectively t-conorm) defined as:* $x * x * x ... * x$*, then we say that* $*$ *(respectively* $\diamond$*) is Archimedean.*

2.2 Fuzzy structures definitions

This section shows the definitions we have used for quasi-metric space structures.

Definition 2.2.1. [22]. *A KM-fuzzy quasi-pseudo-metric on a set* X *is a pair* $(M, *)$ *such that* $*$ *is a continuous t-norm and* M *is a fuzzy set in* $X \times X \times [0, \infty)$ *such that for all* $x, y, z \in X$*:*

(KM1) $M(x, y, 0) = 0$;

(KM2) *$M(x,x,t) = 1$ for all $t > 0$;*

(KM3) *$M(x,z,t+s) \geq M(x,y,t) * M(y,z,s)$ for all $t,s \geq 0$;*

(KM4) *$M(x,y,\_) : [0,\infty) \to [0,1]$ is left continuous.*

Definition 2.2.2. [8, 22]. *A KM-fuzzy quasi-metric on a set X is a KM-fuzzy quasi-pseudo-metric $(M,*)$ on X that satisfies the following condition:*

(KM2') *$x = y$ if and only if $M(x,y,t) = M(y,x,t) = 1$ for all $t > 0$.*

Remark. *Note that, in their notion of fuzzy metric space, Kramosil and Michalek require condition $\lim_{t\to\infty} M(x,y,t) = 1$. However this condition is not necessary in our context.*

Definition 2.2.3. [34]. *A KM-fuzzy (pseudo-)metric on a set X is a KM-fuzzy quasi-(pseudo-)metric $(M,*)$ on X such that for each $x,y \in X$:*

(KM5) *$M(x,y,t) = M(y,x,t)$ for all $t > 0$.*

Definition 2.2.4. [8, 22]. *A KM-fuzzy (quasi-)(pseudo-)metric space is a triple $(X,M,*)$ such that X is a nonempty set and $(M,*)$ is a KM-fuzzy (quasi-)(pseudo-)metric on X.*

Remark. *The following useful fact is well-known [8, 22]: Let $(X,M,*)$ be a KM-fuzzy quasi-metric space. Then, for each $x,y \in X$, the function $M(x,y,\_)$ is nondecreasing.*

Definition 2.2.5. [22]. *A KM-fuzzy quasi-metric space $(X,M,*)$ such that $M(x,y,t) \geq \min\{M(x,z,t), M(z,y,t)\}$ for all $x,y,z,\in X$, $t > 0$, is called a non-Archimedean KM-fuzzy quasi-metric space, and $(M,*)$ is called a non-Archimedean KM-fuzzy quasi-metric on X.*

If $(M, *)$ is a KM-fuzzy quasi-metric on a set X, it is obvious that $(M^{-1}, *)$ is also a KM-fuzzy quasi-metric on X, where M^{-1} is the fuzzy set in $X \times X \times [0, \infty)$ defined by

$$M^{-1}(x, y, t) = M(y, x, t).$$

Moreover, if we denote by M^i the fuzzy set in $X \times X \times [0, \infty)$ given by

$$M^i(x, y, t) = \min\{M(x, y, t), M^{-1}(x, y, t)\},$$

then $(M^i, *)$ is, clearly, a KM-fuzzy metric on X.

Similarly to the fuzzy metric case, each KM-fuzzy quasi-metric $(M, *)$ on a set X induces a T_0 topology τ_M on X which has as a base the family of open balls

$$\{B_M(x, \varepsilon, t) : x \in X, 0 < \varepsilon < 1, t > 0\},$$

where $B_M(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon\}$ (see [22, 24]).

Example 2.2.1. [22]. *Let (X, d) be a quasi-metric space and let M_d be the function defined on $X \times X \times [0, \infty)$ by $M_d(x, y, 0) = 0$ and*

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}$$

for all $t > 0$. Then, for each continuous t-norm $*$, $(M_d, *)$ is a KM-fuzzy quasi-metric on X called the KM-fuzzy quasi-metric induced by d, and $(X, M_d, *)$ is called the standard KM-fuzzy quasi-metric space of (X, d). Furthermore, it is easy to check that $(M_d)^{-1} = M_{d^{-1}}$ and $(M_d)^i = M_{d^s}$. Finally, the topology τ_d coincides with the topology τ_{M_d}.

We say that a topological space (X, τ) admits a compatible KM-fuzzy quasi-metric if there is a KM-fuzzy quasi-metric $(M, *)$ on X such that $\tau = \tau_M$.

Then, it follows from Example 2.2.1 above that every quasi-metrizable topological space admits a compatible KM-fuzzy quasi-metric.

Conversely, it was shown in [22] that for each KM-fuzzy quasi-metric space $(X, M, *)$, the countable family $\{U_n : n = 2, 3, ...\}$ is a base for a quasi-uniformity $\mathcal{U}_M$ on X compatible with τ_M,where

$$U_n = \{(x, y) \in X \times X : M(x, y, 1/n) > 1 - 1/n\}.$$

Consequently, for every KM-fuzzy quasi-metric space $(X, M, *)$, the topological space (X, τ_M) is quasi-metrizable.

Moreover, the conjugate quasi-uniformity $(\mathcal{U}_M)^{-1}$ coincides with $\mathcal{U}_{M^{-1}}$ and it is compatible with $\tau_{M^{-1}}$.

A KM-fuzzy metric space $(X, M, *)$ is complete [17, 67] provided that each Cauchy sequence in X is convergent with respect to τ_M, where a sequence $(x_n)_n$ in X is said to be a Cauchy sequence if for each $\varepsilon \in (0, 1)$ and each $t > 0$ there is $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for all $n, m \geq n_0$.

According to [22, 24], a KM-fuzzy quasi-metric space $(X, M, *)$ is called bicomplete if $(X, M^i, *)$ is a complete fuzzy metric space. In this case, we say that $(M, *)$ is a bicomplete fuzzy quasi-metric on X.

Due to its relevance, a more recent definition of a fuzzy quasi-metric [17] is given:

Definition 2.2.6. [17]. *A GV-fuzzy quasi-metric on a set X is a pair $(M, *)$ such that $*$ is a continuous t-norm and M is a fuzzy set in $X \times X \times (0, \infty)$ such that for $x, y, z \in X$ and $t, s > 0$:*

(GV1) $M(x, y, t) > 0$.

(GV2) *$x = y$ if and only if $M(x, y, t) = M(y, x, t) = 1$.*

(GV3) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$.

(GV4) *$M(x, y, \_) : (0, \infty) \to [0, 1]$ is continuous.*

Definition 2.2.7. [17]. *A GV-fuzzy (pseudo-)metric on a set X is a GV-fuzzy quasi-(pseudo-)metric $(M, *)$ on X such that for each $x, y \in X$:*

(GV5) $M(x, y, t) = M(y, x, t)$.

Definition 2.2.8. [17]. *A GV-fuzzy (quasi-)(pseudo-)metric space is a triple $(X, M, *)$ such that X is nonempty set and $(M, *)$ is a GV-fuzzy (quasi-)(pseudo-)metric $(M, *)$ on X.*

Obviously, each GV-fuzzy (quasi-)metric space $(X, M, *)$ can be considered as a fuzzy (quasi-)metric space, in the sense of Definition 2.2.2, by defining $M(x, y, 0) = 0$ for all $x, y \in X$. Therefore, each GV-fuzzy quasi-metric space induces a topology τ_M defined as in the KM-case. Moreover, the properties of KM-fuzzy quasi-metrics given above remain valid for GV-fuzzy quasi-metrics.

Finally, observe that the standard KM-fuzzy quasi-metric space $(X, M_d, *)$ of Example 2.2.1 is actually a GV-fuzzy quasi-metric space.

Definition 2.2.9. [56]. *An intuitionistic fuzzy quasi-metric on a set X is a 4-tuple $(M, N, *, \diamond)$ such that $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and M, N are fuzzy sets in $X \times X \times [0, \infty)$ such that for all $x, y, z \in X$:*

(1) $M(x, y, t) + N(x, y, t) \leq 1$ *for all* $t \geq 0$;

(2) $M(x, y, 0) = 0$;

(3) $x = y$ *if and only if* $M(x, y, t) = M(y, x, t) = 1$ *for all* $t > 0$;

(4) $M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$ *for all* $t, s \geq 0$;

(5) $M(x, y, \_) : [0, \infty) \to [0, 1]$ *is left continuous;*

(6) $N(x, y, 0) = 1$;

(7) *$x = y$ if and only if $N(x,y,t) = N(y,x,t) = 0$ for all $t > 0$;*

(8) *$N(x,y,t) \diamond N(y,z,s) \geq N(x,z,t+s)$ for all $t,s \geq 0$;*

(9) *$N(x,y,\_) : [0,\infty) \to [0,1]$ is left continuous.*

Definition 2.2.10. [1]. *An intuitionistic fuzzy metric on a set X is an intuitionistic fuzzy quasi-metric $(M,N,*,\diamond)$ on X such that for each $x,y \in X$:*

(10) *$M(x,y,t) = M(y,x,t)$*

(11) *$N(x,y,t) = N(y,x,t)$ for all $t > 0$.*

Remark. *Note that the authors of [1] require in their notion of intuitionistic fuzzy metric space $\lim_{t\to\infty} M(x,y,t) = 1$ and $\lim_{t\to\infty} N(x,y,t) = 0$ conditions. However these conditions are not necessary in our context.*

Definition 2.2.11. [56]. *An intuitionistic fuzzy (quasi-)metric space is a 5-tuple $(X,M,N,*,\diamond)$ such that $(M,N,*,\diamond)$ is an intuitionistic fuzzy (quasi-) metric on a set X.*

Remark. *It is clear that if $(X,M,N,*,\diamond)$ is an intuitionistic fuzzy (quasi-) metric space, then $(X,M,*)$ is a KM-fuzzy (quasi-)metric space.*

If $(M,N,*,\diamond)$ is an intuitionistic fuzzy quasi-metric on X, then one has that $(M^{-1},N^{-1},*,\diamond)$ is also an intuitionistic fuzzy quasi-metric on X, where M^{-1} is the fuzzy set in $X \times X \times [0,\infty)$ defined by $M^{-1}(x,y,t) = M(y,x,t)$ and N^{-1} is the fuzzy set in $X \times X \times [0,\infty)$ defined by $N^{-1}(x,y,t) = N(y,x,t)$. Moreover, if we define M^i as above and denote by N^s the fuzzy set in $X \times X \times [0,\infty)$ given by $N^s(x,y,t) = \max\{N(x,y,t), N^{-1}(x,y,t)\}$ then $(M^i,N^s,*,\diamond)$ is an intuitionistic fuzzy metric on X.

In order to construct a suitable topology on an intuitionistic fuzzy quasi-metric space $(X, M, N, *, \diamond)$, Romaguera and Tirado considered in [56] the natural "balls" $B(x, \varepsilon, t)$ defined, similarly to [47] and [1], by:

$$B(x, \varepsilon, t) = \{y \in X : M(x, y, t) > 1 - \varepsilon, N(x, y, t) < \varepsilon\}$$

for all $x \in X$, $0 < \varepsilon < 1$, and $t > 0$.

Then, they proved that $B(x, \varepsilon, t) = B_M(x, \varepsilon, t)$ (compare [25] for the metric case), and thus the topology induced by $(M, N, *, \diamond)$ coincides with the topology τ_M induced by $(M, *)$.

In [47] Park introduced the notion of a complete intuitionistic fuzzy metric space. It is proved in [25] that an intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is complete if and only if $(X, M, *)$ is complete.

For the quasi-metric case we have the following.

Definition 2.2.12. [56]. *(a) A sequence $(x_n)_n$ in an intuitionistic fuzzy quasi-metric space $(X, M, N, *, \diamond)$ is called a Cauchy sequence if for each $\varepsilon \in (0, 1)$, $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$, and $N(x_n, x_m, t) < \varepsilon$, for all $n, m \geq n_0$.*

(b) An intuitionistic fuzzy quasi-metric space $(X, M, N, *, \diamond)$ is called bicomplete if $(X, M^i, N^s, *, \diamond)$ is a complete intuitionistic fuzzy metric space.

It is shown in [56] that a sequence in an intuitionistic fuzzy quasi-metric space $(X, M, N, *, \diamond)$ is a Cauchy sequence if and only if it is a Cauchy sequence in the fuzzy metric space $(X, M^i, *)$, and that an intuitionistic $(X, M, N, *, \diamond)$ is bicomplete if and only if the fuzzy quasi-metric space $(X, M, *)$ is bicomplete.

Chapter 3

The bicompletion of fuzzy quasi-metric spaces

3.1 Introduction

The problem of the completion of fuzzy metric spaces and fuzzy quasi-metric spaces has received a certain attention in the last years. In this chapter we will discuss this problem for KM-fuzzy quasi-metric spaces.

Kramosil and Michalek introduced in [34] their celebrated notion of a fuzzy metric space. This notion has an evident appeal due to its close relationship with probabilistic metric spaces. In particular, they observed that the class of fuzzy metric spaces in their sense, is "equivalent" to the class of Menger spaces having a continuous t-norm. Sherwood proved in [65] that every Menger space belonging to this class has a completion which is unique up to isometry, and thus one can easily deduce that every KM-fuzzy metric space has a completion which is unique up to isometry.

However, the problem of the completion of fuzzy metric spaces in the sense of George and Veeramani is quite different. In fact, an example of a GV-fuzzy metric space which does not admit a GV-fuzzy completion was presented in [21], whereas in [23] it was obtained a characterization of those GV-fuzzy metric spaces that are GV-completable.

Recently, J. Gutiérrez García, M.A. de Prada Vicente and S. Romaguera [27, 28] have established connections between these kinds of fuzzy metric spaces and uniform structures in the sense of Hutton. While J. Gutiérrez García, S. Romaguera and M. Sanchis have extended the notion of fuzzy metric in the sense of Kramosil and Michalek to a uniform setting [29].

The concept of metric fuzziness was generalized to the quasi-metric setting in [8, 22], where several properties of these structures were discussed (see Chapter 2). Recently, there were given in [52, 56, 57], applications of fixed point theorems, in the realm of fuzzy quasi-metric spaces, to deduce the existence and uniqueness of solution for the recurrence equations associated to some types of algorithms, whereas in [48] it was presented a study of a notion of Hausdorff fuzzy quasi-pseudo-metric on the collection of nonempty subsets of a given fuzzy quasi-metric space.

In this context, the completion of fuzzy quasi-metric spaces appears as a natural and attractive question, which was discussed for GV-fuzzy quasi-metric spaces in [24]. Here we shall show that every KM-fuzzy quasi-metric space has a (KM-fuzzy quasi-metric) bicompletion which is unique up to isometry. Then, the completion of a KM-fuzzy metric space is restated as a particular case. Finally, we shall apply our constructions to study the bicompletion of non-Archimedean fuzzy quasi-metric spaces and intuitionistic fuzzy quasi-metric spaces respectively.

We emphasize at this point that while Sherwood's construction is strongly based on the properties of Lévy's metric, our construction avoids the use of Lévy's metric and directly uses suprema of subsets of [0,1] and lower limits of sequences in [0,1].

3.2 The completion of a fuzzy metric space

In this section we recall some known and crucial results on the completion of KM-fuzzy metric spaces mentioned in Section 3.1 which should be useful to a better understanding of our motivation and also of the constructions made in the rest of the chapter.

A distribution function [62] is a function $F : \mathbb{R} \to [0, 1]$ such that:

(i) F is nondecreasing (i.e., $F(s) \leq F(t)$ whenever $s \leq t$);

(ii) F is left continuous;

(iii) $\lim_{t \to -\infty} F(t) = 0$ and $\lim_{t \to +\infty} F(t) = 1$.

We denote by Δ the set of distribution functions, and by Δ^+ the subset of Δ consisting of those distribution functions F such that $F(0) = 0$.

Remark. *Since in our context condition $\lim_{t \to +\infty} F(t) = 1$ is not necessary, the family of functions $F : \mathbb{R} \to [0, 1]$ satisfying conditions (i), (ii), above, and $\lim_{t \to -\infty} F(t) = 0$, will be denoted by Γ in the following, and the family of elements F of Γ such that $F(0) = 0$ will be denoted by Γ^+.*

A (generalized) probabilistic metric space [62] is a pair $(X, \mathcal{F})$ such that X is a set and $\mathcal{F}$ is a mapping from $X \times X$ into Δ^+ (resp. into Γ^+) such that for all $x, y, z \in X$:

(i) $\mathcal{F}(x, y)(t) = 1$ for all $t > 0$ if and only if $x = y$;

(ii) $\mathcal{F}(x, y) = \mathcal{F}(y, x)$;

(iii) If $\mathcal{F}(x, y)(t) = 1$ and $\mathcal{F}(y, z)(s) = 1$, then $\mathcal{F}(x, z)(t + s) = 1$.

As usual, we shall write F_{xy} instead of $\mathcal{F}(x, y)$ if no confusion arises.

In the following, generalized probabilistic metric spaces will be simply called g-probabilistic metric spaces.

A sequence $(x_n)_n$ in a g-probabilistic metric space $(X, \mathcal{F})$ is said to be a Cauchy sequence if for each $r \in (0,1)$ and each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $F_{x_n x_m}(t) > 1 - r$ for all $n, m \geq n_0$.

$(X, \mathcal{F})$ is called complete if for each Cauchy sequence $(x_n)_n$ there exists $x \in X$ satisfying the following condition: for each $r \in (0,1)$ and each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $F_{xx_m}(t) > 1 - r$ for all $n \geq n_0$; i.e., if for each $t > 0$, $\lim_n F_{xx_n}(t) = 1$.

A (g-)Menger space is a triple $(X, \mathcal{F}, *)$ such that $(X, \mathcal{F})$ is a (g-)probabilistic metric space and $*$ is a t-norm such that for all $x, y, z \in X$ and $t, s \geq 0$:

(iv) $\mathcal{F}(x,y)(t+s) \geq \mathcal{F}(x,z)(t) * \mathcal{F}(z,y)(s)$.

Since every (g-)Menger space can be considered as a (g-)probabilistic metric space, the notions of a Cauchy sequence in (g-)Menger spaces and of a complete (g-)Menger space are defined in the obvious manner.

In fact, each (g-)Menger space $(X, \mathcal{F}, *)$ induces a topology $\tau_{\mathcal{F}}$ on X defined as follows:

$$\tau_{\mathcal{F}} = \{A \subseteq X : \text{for each } x \in A \text{ there exist } r \in (0,1),\ t > 0, \text{ such that } B_{\mathcal{F}}(x,r,t) \subseteq A\},$$

where $B_{\mathcal{F}}(x,r,t) = \{y \in X : F_{xy}(t) > 1 - r\}$ for all $x \in X$, $r \in (0,1)$, $t > 0$.

Furthermore, if the t-norm $*$ is continuous, then $(X, \tau_{\mathcal{F}})$ is a metrizable topological space because the countable collection

$$\{\{(x,y) \in X \times X : F_{xy}(1/n) > 1 - 1/n\} : n \in \mathbb{N}\},$$

is a base for a uniformity on X such that its induced topology coincides with $\tau_{\mathcal{F}}$ (see [62]).

Note that, in particular, a sequence $(x_n)_n$ in $(X, \mathcal{F}, *)$ converges to $x \in X$ with respect to $\tau_{\mathcal{F}}$ if and only if for each $t > 0$, $\lim_n F_{xx_n}(t) = 1$.

A mapping f from a (g-)Menger space $(X, \mathcal{F}, *)$ to a (g-)Menger space $(Y, \mathcal{G}, \star)$ is called an isometry if for each $x, y \in X$:

$$\mathcal{G}(f(x), f(y)) = \mathcal{F}(x, y).$$

It is clear that any isometry is one-to-one.

Two (g-)Menger spaces $(X, \mathcal{F}, *)$ and $(Y, \mathcal{G}, \star)$ are called isometric if there is an isometry from $(X, \mathcal{F}, *)$ onto $(Y, \mathcal{G}, \star)$.

A complete (g-)Menger space is said to be a completion of a given (g-) Menger space $(X, \mathcal{F}, *)$ if it has a dense subspace isometric to $(X, \mathcal{F}, *)$.

Sherwood proved in [65] that every (g-)Menger space $(X, \mathcal{F}, *)$ such that $*$ is continuous has a completion which is unique up to isometry (a different construction of the completion, based on the theory of fixed point, was obtained by Sempi in [64]).

Sherwood's construction is strongly based on the properties of Lévy's metric L on Δ (in fact on Γ), which is defined as follows [19, 38]:

$$L(F, G) = \inf\{h > 0 : F(t-h) - h \leq G(t) \leq F(t+h) + h, \text{ for all } t > 0\},$$

whenever $F, G \in \Gamma$.

It is well known that (Γ, L) is a complete metric space. Since Γ^+ is a closed subset of Γ it follows that (Γ^+, L) is also a complete metric space.

Then, and following Sherwood [65], denote by $\sim$ the binary relation defined on the set S of all Cauchy sequences in $(X, \mathcal{F}, *)$ by

$$(x_n)_n \sim (y_n)_n \Longleftrightarrow \lim_n F_{x_n y_n}(t) = 1 \quad \text{for all } t > 0,$$

whenever $(x_n)_n, (y_n)_n \in S$.

Thus $\sim$ is an equivalence relation on S. Let $\widetilde{X_{\mathcal{F}}}$ be the quotient $S/\sim$. The elements of $\widetilde{X_{\mathcal{F}}}$ will be denoted by $[(x_n)_n]$, where $(x_n)_n \in S$.

For each pair $(x_n)_n, (y_n)_n \in S$, it follows that $(F_{x_n y_n})_n$ is a Cauchy sequence in the complete metric space (Γ^+, L), so this sequence converges to an element of Γ^+ which we denote by $\lim_n^L F_{x_n y_n}$.

Furthermore, for each $(x'_n)_n \in [(x_n)_n]$, and each $(y'_n)_n \in [(y_n)_n]$, one has that

$$\lim_n^L F_{x_n y_n} = \lim_n^L F_{x'_n y'_n}.$$

Consequently, we can define a function $\widetilde{\mathcal{F}} : \widetilde{X_{\mathcal{F}}} \times \widetilde{X_{\mathcal{F}}} \to [0,1]$, by

$$\widetilde{\mathcal{F}}([(x_n)_n], [(y_n)_n])(0) = 0,$$

and

$$\widetilde{\mathcal{F}}([(x_n)_n], [(y_n)_n])(t) = \lim_n^L F_{x_n y_n}(t)$$

whenever $t > 0$.

Then $(\widetilde{X_{\mathcal{F}}}, \widetilde{\mathcal{F}}, *)$ is a complete g-Menger space (a complete Menger space if $(X, \mathcal{F})$ is a Menger space), and the mapping

$$i_{\mathcal{F}} : X \to \widetilde{X_{\mathcal{F}}}$$

given by

$$i_{\mathcal{F}}(x) = [(x, x, ...)]$$

whenever $x \in X$, is an isometry between $(X, \mathcal{F}, *)$ and a dense subspace of $(\widetilde{X_{\mathcal{F}}}, \widetilde{\mathcal{F}}, *)$.

Moreover, if $(Y, \mathcal{G}, \star)$ is a complete g-Menger space having a dense subspace isometric to $(X, \mathcal{F}, *)$, then $(Y, \mathcal{G}, \star)$ is isometric to $(\widetilde{X_{\mathcal{F}}}, \widetilde{\mathcal{F}}, *)$.

The (complete) g-Menger space $(\widetilde{X_{\mathcal{F}}}, \widetilde{\mathcal{F}}, *)$ is called the completion of $(X, \mathcal{F}, *)$.

Next we shall apply Sherwood's construction to directly obtain the completion of any KM-fuzzy metric space.

In the remainder of this thesis, KM-fuzzy (quasi-)metric spaces will be simply called fuzzy (quasi-)metric spaces.

As we indicated above, Kramosil and Michalek observed in [34] that there exists a natural "equivalence" between the class of g-Menger spaces with continuous t-norm and the class of fuzzy metric spaces.

Indeed, if $(X, \mathcal{F}, *)$ is a g-Menger space such that $*$ is a continuous t-norm, then we define $M_{\mathcal{F}} : X \times X \times [0, +\infty) \to [0, 1]$ by

$$M_{\mathcal{F}}(x, y, t) = F_{xy}(t),$$

for all $t \geq 0$, and thus $(X, M_{\mathcal{F}}, *)$ is a fuzzy metric space.

Conversely, if $(X, M, *)$ is a fuzzy metric space, then we define $\mathcal{F}_M : X \times X \to \Gamma^+$ by

$$\mathcal{F}_M(x, y)(t) = M(x, y, t),$$

for all $t > 0$, and $\mathcal{F}_M(x, y)(t) = 0$ for all $t \leq 0$.

Thus $(X, \mathcal{F}_M, *)$ is a g-Menger space, and we shall write M_{xy} instead of $\mathcal{F}_M(x, y)$ if no confusion arises.

A complete fuzzy metric space is said to be a completion of a given fuzzy metric space $(X, M, *)$ if it has a dense subspace isometric to $(X, M, *)$.

A mapping f from a fuzzy metric space $(X, M, *)$ to a fuzzy metric space $(Y, N, \star)$ is called an isometry if for each $x, y \in X$ and each $t > 0$, $N(f(x), f(y), t)) = M(x, y, t)$. It is clear that every isometry is a one-to-one mapping.

Two fuzzy metric spaces $(X, M, *)$ and $(Y, N, \star)$ are called isometric if there is an isometry from X onto Y.

Then, we can immediately adapt Sherwood's construction to the fuzzy metric context as follows (this approach is taken from [50]).

Let $(X, M, *)$ be a fuzzy metric space. Denote by $\sim$ the binary relation defined on the set S of all Cauchy sequences in $(X, M, *)$ by

$$(x_n)_n \sim (y_n)_n \Longleftrightarrow \lim_n M(x_n, y_n, t) = 1 \quad \text{for all } t > 0,$$

whenever $(x_n)_n, (y_n)_n \in S$.

Then $\sim$ is an equivalence relation on S. Let $\widetilde{X_M}$ be the quotient $S/\sim$. The elements of $\widetilde{X_M}$ will be denoted by $[(x_n)_n]$, where $(x_n)_n \in S$.

For each pair $(x_n)_n, (y_n)_n \in S$, we have that $(M_{x_n y_n})_n$ is a Cauchy sequence in the complete metric space (Γ^+, L), so this sequence converges to an element of Γ^+ which is denoted by $\lim_n^L M_{x_n y_n}$.

Furthermore, for each $(x'_n)_n \in [(x_n)_n]$, and each $(y'_n)_n \in [(y_n)_n]$, one has that

$$\lim_n^L M_{x_n y_n} = \lim_n^L M_{x'_n y'_n}.$$

Consequently, we can define a function $\widetilde{M} : \widetilde{X_M} \times \widetilde{X_M} \times [0, \infty) \to [0, 1]$, by

$$\widetilde{M}([(x_n)_n], [(y_n)_n], 0) = 0,$$

and

$$\widetilde{M}([(x_n)_n], [(y_n)_n], t) = \lim_n^L M_{x_n y_n}(t)$$

whenever $t > 0$.

Thus $(\widetilde{X_M}, \widetilde{M}, *)$ is a complete fuzzy metric space, and the mapping

$$i_M : X \to \widetilde{X_M}$$

given by

$$i_M(x) = [(x, x, ...)]$$

whenever $x \in X$, is an isometry between $(X, M, *)$ and a dense subspace of $(\widetilde{X_M}, \widetilde{M}, *)$.

Moreover, if $(Y, M_Y, *_Y)$ is a complete fuzzy metric space having a dense subspace isometric to $(X, M, *)$, then $(Y, M_Y, *_Y)$ is isometric to $(\widetilde{X_M}, \widetilde{M}, *)$.

Therefore we have the following.

Theorem 3.2.1. *Every fuzzy metric space has a completion which is unique up to isometry.*

The (complete) fuzzy metric space $(\widetilde{X_M}, \widetilde{M}, *)$ is called the completion of $(X, M, *)$.

In the rest of this section we will consider the completion of GV-fuzzy metric spaces.

The notions of a Cauchy sequence, completeness and completion for GV-fuzzy metric spaces are defined as for fuzzy metric spaces.

In [21] it was obtained an example of a GV-fuzzy metric space which does not admit a GV-fuzzy metric completion.

Such an example suggests, in a natural way, the problem of characterizing completable GV-fuzzy metric spaces, i.e., those GV-fuzzy metric spaces that admit a fuzzy metric completion which is a GV-fuzzy metric space; such a completion if exists is called a GV-fuzzy metric completion. This problem was solved in [23] as follows.

Theorem 3.2.2. *A GV-fuzzy metric space $(X, M, *)$ is completable if and only if for each pair $(a_n)_n, (b_n)_n$, of Cauchy sequences in X, the assignment*

$$t \mapsto \lim_n M(a_n, b_n, t)$$

is a continuous function on $(0, \infty)$ with values in $(0, 1]$.

Furthermore, if a GV-fuzzy metric space is completable, then its GV-fuzzy metric completion is unique up to isometry.

3.3 The bicompletion of a fuzzy quasi-metric space

Definition 3.3.1. [22, 24]. *A mapping f from a fuzzy quasi-metric space $(X, M, *)$ to a fuzzy quasi-metric space $(Y, N, \star)$ is said to be an isometry if*

$M(x,y,t) = N(f(x), f(y), t)$ for each $x, y \in X$ and each $t > 0$.

The fuzzy quasi-metric spaces $(X, M, *)$ and $(Y, N, \star)$ are called isometric if there is an isometry from X onto Y.

Definition 3.3.2. [22, 24]. *Let $(X, M, *)$ be a fuzzy quasi-metric space. A (fuzzy quasi-metric) bicompletion of $(X, M, *)$ is a bicomplete fuzzy quasi-metric space $(Y, N, \star)$ such that $(X, M, *)$ is isometric to a τ_{N^i}-dense subspace of Y.*

In the sequel we shall construct the bicompletion of a fuzzy quasi-metric space.

Indeed, let $(X, M, *)$ be a fuzzy quasi-metric space.

Denote by S the collection of all Cauchy sequences in $(X, M^i, *)$.

Define a relation $\sim$ on S by

$$(x_n)_n \sim (y_n)_n \Longleftrightarrow \sup_{0<s<t} \underline{\lim}\, M^i(x_n, y_n, s) = 1 \text{ for all } t > 0,$$

where by $\underline{\lim}\, M^i(x_n, y_n, s)$ we denote, as usual, the lower limit of the sequence $(M^i(x_n, y_n, s))_n$, i.e.,

$$\underline{\lim}\, M^i(x_n, y_n, s) = \sup_k \inf_{n \geq k} M^i(x_n, y_n, s).$$

Then:

Lemma 3.3.1. $\sim$ *is an equivalence relation on S.*

Proof:

Let $(x_n)_n, (y_n)_n, (z_n)_n \in S$. Clearly $(x_n)_n \sim (x_n)_n$ because $M^i(x_n, x_n, s) = 1$ for all $n \in \mathbb{N}$ and $s > 0$, so that for each $t > 0$,

$$\sup_{0<s<t} \underline{\lim}\, M^i(x_n, x_n, s) = 1.$$

Moreover, if $(x_n)_n \sim (y_n)_n$, it immediately follows that $(y_n)_n \sim (x_n)_n$ because $M^i(x_n, y_n, s) = M^i(y_n, x_n, s)$ for all $n \in \mathbb{N}$ and $s > 0$, so that for each $t > 0$,

$$\sup_{0<s<t} \underline{\lim}\, M^i(y_n, x_n, s) = \sup_{0<s<t} \underline{\lim}\, M^i(x_n, y_n, s) = 1.$$

Finally, suppose that $(x_n)_n \sim (y_n)_n$ and $(y_n)_n \sim (z_n)_n$. Let $t > 0$. We shall prove that $\sup_{0<s<t} \underline{\lim}\, M^i(x_n, z_n, s) = 1$.

To this end, choose an arbitrary $\varepsilon \in (0, 1)$. Then, there exists $\delta \in (0, 1)$ such that $(1 - \delta) * (1 - \delta) > 1 - \varepsilon$. Hence, there exists $s' \in (0, t)$ such that $\underline{\lim}\, M^i(x_n, y_n, s') > 1 - \delta$, and consequently there exists $k_1 \in \mathbb{N}$ such that

$$M^i(x_n, y_n, s') > 1 - \delta,$$

for all $n \geq k_1$.

Now choose $r > 0$ such that $s' + r < t$. Since $(y_n)_n \sim (z_n)_n$, we have that $\sup_{0<s<r} \underline{\lim}\, M^i(y_n, z_n, s) = 1$. Hence, there exist $s'' \in (0, r)$ and $k_2 \geq k_1$ such that

$$M^i(y_n, z_n, s'') > 1 - \delta,$$

for all $n \geq k_2$.

Therefore

$$M^i(x_n, z_n, s' + s'') \geq M^i(x_n, y_n, s') * M^i(y_n, z_n, s'') \geq (1-\delta) * (1-\delta) > 1-\varepsilon,$$

for all $n \geq k_2$, which implies that

$$\underline{\lim}\, M^i(x_n, z_n, s' + s'') \geq 1 - \varepsilon.$$

Since $0 < s' + s'' < t$, we deduce that

$$\sup_{0<s<t} \underline{\lim}\, M^i(x_n, z_n, t) = 1,$$

and hence $(x_n)_n \sim (z_n)_n$.

□

Now define a function $M_S : S \times S \times [0, \infty) \to [0, 1]$ as follows:

$$M_S((x_n)_n, (y_n)_n, 0) = 0,$$

and

$$M_S((x_n)_n, (y_n)_n, t) = \sup_{0<s<t} \underline{\lim}\, M(x_n, y_n, s),$$

for all $t > 0$.

Then:

Lemma 3.3.2. *M_S is a KM-fuzzy quasi-pseudo-metric on S.*

Proof:

Condition (KM1) is obviously satisfied by the definition of M_S.

Let $(x_n)_n, (y_n)_n, (z_n)_n \in S.$, $t, s > 0$, and put $\alpha = M_S((x_n)_n, (y_n)_n, t)$, $\beta = M_S((y_n)_n, (z_n)_n, s)$ and $\gamma = M_S((x_n)_n, (z_n)_n, t + s)$. We shall show that $\alpha * \beta \leq \gamma$.

If $\alpha = 0$ or $\beta = 0$, the conclusion is obvious. So we assume that $\alpha > 0$ and $\beta > 0$. Choose an arbitrary $\varepsilon \in (0, \min\{\alpha, \beta\}/2)$. Then, there exist $t' \in (0, t)$ and $s' \in (0, s)$ such that

$\alpha - \varepsilon < M_S((x_n)_n, (y_n)_n, t')$ and $\beta - \varepsilon < M_S((y_n)_n, (z_n)_n, s')$.

Furthermore, there exists n_ε such that for each $k \geq n_\varepsilon$,

$$M_S((x_n)_n, (y_n)_n, t') - \varepsilon < M(x_k, y_k, t'), \quad \text{and}$$

$$M_S((y_n)_n, (z_n)_n, s') - \varepsilon < M(y_k, z_k, s').$$

Then

$$\begin{aligned}(\alpha - 2\varepsilon) * (\beta - 2\varepsilon) &\leq (M_S((x_n)_n, (y_n)_n, t') - \varepsilon) * (M_S((y_n)_n, (z_n)_n, s') - \varepsilon)\\ &\leq M(x_k, y_k, t') * M(y_k, z_k, s')\\ &\leq M(x_k, z_k, t' + s').\end{aligned}$$

for all $k \geq n_\varepsilon$.

Therefore

$$\begin{aligned}(\alpha-2\varepsilon)*(\beta-2\varepsilon) &\leq \inf_{k\geq n_\varepsilon} M(x_k,z_k,t'+s') \leq \underline{\lim}\, M(x_n,z_n,t'+s') \\ &\leq \sup_{0<r<t+s} \underline{\lim}\, M(x_n,z_n,t'+s')=\gamma.\end{aligned}$$

By continuity of $*$, it follows that $\alpha*\beta\leq\gamma$. So condition (KM3) is satisfied.

Finally, fix $(x_n)_n,(y_n)_n\in S$ and $t>0$, and let $(t_j)_j$ be a strictly increasing sequence of positive real numbers such that $\lim_j t_j=t$. Since $t_j<t$ for all j, we clearly have that:

$$M_S((x_n)_n,(y_n)_n,t_j)\leq M_S((x_n)_n,(y_n),t),$$

for all j.

Moreover, given $\varepsilon>0$, there is $s_\varepsilon\in(0,t)$ such that

$$M_S((x_n),(y_n),t)<\varepsilon+\sup_n\inf_{k\geq n} M(x_k,y_k,s_\varepsilon).$$

Let j_ε such that $t_j>s_\varepsilon$ for all $j\geq j_\varepsilon$. From the preceding relation, we deduce that

$$M_S((x_n),(y_n),t)\leq\varepsilon+\sup_{0<s<t_j}\underline{\lim}\, M(x_n,y_n,s),$$

for all $j\geq j_\varepsilon$. So

$$M_S((x_n)_n,(y_n)_n,t_j)\leq M_S((x_n),(y_n),t)\leq\varepsilon+M_S((x_n)_n,(y_n)_n,t_j),$$

for all $j\geq j_\varepsilon$. We conclude that $M_S((x_n)_n,(y_n)_n,\_)$ is left continuous. Therefore, condition (KM4) is satisfied.

□

Now denote by $\widetilde{X}$ the quotient $S/\sim$, and by $[(x_n)_n]$ the class of the element $(x_n)_n$ of S.

Lemma 3.3.3. *For each $(x_n)_n, (y_n)_n \in S$ and each $(a_n)_n \in [(x_n)_n]$, $(b_n)_n \in [(y_n)_n]$, one has*

$$M_S((x_n)_n, (y_n)_n, t) = M_S((a_n)_n, (b_n)_n, t),$$

for all $t > 0$.

Proof:
Let $t > 0$. Given $\varepsilon \in (0, t/2)$ we obtain

$$\begin{aligned} & M_S((x_n)_n, (y_n)_n, t) \\ \geq\ & M_S((x_n)_n, (a_n)_n, \varepsilon) * M_S((a_n)_n, (b_n)_n, t - 2\varepsilon) * M_S((b_n)_n, (y_n)_n, \varepsilon) \\ =\ & M_S((a_n)_n, (b_n)_n, t - 2\varepsilon). \end{aligned}$$

Since $M_S((a_n)_n, (b_n)_n, \_)$ is left continuous, we deduce that

$$\lim_{\varepsilon \to 0} M_S((a_n)_n, (b_n)_n, t - 2\varepsilon) = M_S((a_n)_n, (b_n)_n, t).$$

Thus $M_S((x_n)_n, (y_n)_n, t) \geq M_S((a_n)_n, (b_n)_n, t)$.

The same argument shows that $M_S((a_n)_n, (b_n)_n, t) \geq M_S((x_n)_n, (y_n)_n, t)$.

□

Lemma 3.3.4. [22]. *Let $(X, M, *)$ be a fuzzy quasi-metric space and $(Y, N, \star)$ a bicomplete fuzzy quasi-metric space. If there is a τ_{M^i}-dense subset A of X and an isometry $f : (A, M, *) \to (Y, N, \star)$, then there exists a unique isometry $F : (X, M, *) \to (Y, N, \star)$ such that $F |_A = f$.*

Now, for each $[(x_n)_n], [(y_n)_n] \in \widetilde{X}$, define

$$\widetilde{M}([(x_n)_n], [(y_n)_n], 0) = 0,$$

and

$$\widetilde{M}([(x_n)_n], [(y_n)_n], t) = M_S(x_n, y_n, t),$$

for $t > 0$.

Then $\widetilde{M}$ is a function from $\widetilde{X} \times \widetilde{X} \times [0, \infty)$ to $[0, 1]$ (indeed, it is well-defined by Lemma 3.3.3).

We also define $i : X \to \widetilde{X}$ such that, for each $x \in X$, $i(x)$ is the class of the constant sequence $x, x, ...$

From the above constructions we obtain:

Theorem 3.3.1. *Let $(X, M, *)$ be a fuzzy quasi-metric space. Then:*

(a) *$(\widetilde{M}, *)$ is a fuzzy quasi-metric on $\widetilde{X}$.*

(b) *$i(X)$ is dense in $(\widetilde{X}, \widetilde{M^i}, *)$.*

(c) *$(X, M, *)$ is isometric to $(i(X), \widetilde{M}, *)$.*

(d) *$(\widetilde{M}, *)$ is bicomplete.*

(e) *If $(Y, N, *)$ is a bicomplete fuzzy quasi-metric space such that $(X, M, *)$ is isometric to a τ_{N^i}-dense subspace of Y, then $(Y, N, *)$ and $(\widetilde{X}, \widetilde{M}, *)$ are isometric.*

Proof:

(a) $(\widetilde{M}, *)$ satisfies conditions (KM1), (KM3) and (KM4) of Definition 2.2.2 as an immediate consequence of Lemma 3.3.2.

Now let $(x_n)_n, (y_n)_n \in S$ such that $\widetilde{M}([(x_n)_n], [(y_n)_n], t) = 1$ for all $t > 0$. If $(z_n)_n \in [(y_n)_n]$, it follows from Lemma 3.3.3 that $M_S((z_n)_n, (y_n)_n, t) = 1$ for all $t > 0$, i.e., $(z_n)_n \in [(y_n)_n]$. The same argument shows that $(z_n)_n \in [(x_n)_n]$ whenever $(z_n)_n \in [(y_n)_n]$. We conclude that $\widetilde{M}([(x_n)_n], [(y_n)_n], t) = 1$ for all $t > 0$, if and only if $[(x_n)_n] = [(y_n)_n]$. Consequently $(\widetilde{M}, *)$ is a fuzzy quasi-metric on $\widetilde{X}$.

(b) Let $(x_n)_n \in S$, $\varepsilon \in (0, 1)$ and $t > 0$. Choose an $s_\varepsilon \in (0, t)$. Since $(x_n)_n$ is a Cauchy sequence in $(X, M^i, *)$ there is $n_\varepsilon \in \mathbb{N}$ such that $M^i(x_k, x_{n_\varepsilon}, s_\varepsilon) >$

$1-\varepsilon/2$ for all $k \geq n_\varepsilon$. Thus

$$\begin{aligned}
\widetilde{M}([(x_n)_n], i(x_{n_\varepsilon}), t) &= \sup_{0<s<t} \sup_n \inf_{k\geq n} M(x_k, x_{n_\varepsilon}, s) \\
&\geq \sup_n \inf_{k\geq n} M(x_k, x_{n_\varepsilon}, s_\varepsilon) \\
&\geq \inf_{k\geq n_\varepsilon} M(x_k, x_{n_\varepsilon}, s_\varepsilon) \\
&\geq \inf_{k\geq n_\varepsilon} M^i(x_k, x_{n_\varepsilon}, s_\varepsilon) \\
&\geq 1-\varepsilon/2 > 1-\varepsilon.
\end{aligned}$$

Similarly, we deduce that

$$\widetilde{M}^{-1}([(x_n)_n], i(x_{n_\varepsilon}), t) > 1-\varepsilon.$$

We have shown that $i(X)$ is dense in $(\widetilde{X}, \widetilde{M^i}, *)$.

(c) This is almost obvious because for each $x, y \in X$ and $t > 0$ we have

$$\widetilde{M}(i(x), i(y), t) = \sup_{0<s<t} M(x, y, s) = M(x, y, t).$$

(d) Let $(\widetilde{x}_n)_n$ be a Cauchy sequence in $(\widetilde{X}, \widetilde{M^i}, *)$. Then, there is an increasing sequence $(n_k)_k$ in $\mathbb{N}$ such that

$$\widetilde{M^i}(\widetilde{x}_n, \widetilde{x}_m, 2^{-k}) > 1-2^{-k},$$

for all $n, m \geq n_k$.

Since $i(X)$ is dense in $(\widetilde{X}, \widetilde{M^i}, *)$, for each $k \in \mathbb{N}$ there is $y_k \in X$ such that

$$\widetilde{M^i}(\widetilde{x}_{n_k}, i(y_k), 2^{-k}) > 1-2^{-k},$$

for all $k \in \mathbb{N}$.

We show that $(y_k)_k$ is a Cauchy sequence in $(X, M^i, *)$. To this end, choose $\varepsilon \in (0,1)$ and $t > 0$. Take $j \in \mathbb{N}$ such that $2^{-j} < t/3$ and

$$(1-2^{-j}) * (1-2^{-j}) * (1-2^{-j}) > 1-\varepsilon.$$

Then, for each $k, m \geq j$, we have

$$\begin{aligned} M(y_k, y_m, t) &= \widetilde{M}(i(y_k), i(y_m), t) \geq \widetilde{M}(i(y_k), i(y_m), 3 \cdot 2^{-j}) \\ &\geq \widetilde{M}(i(y_k), \widetilde{x}_{n_k}, 2^{-j}) * \widetilde{M}(\widetilde{x}_{n_k}, \widetilde{x}_{n_m}, 2^{-j}) * \widetilde{M}(\widetilde{x}_{n_m}, i(y_m), 2^{-j}) \\ &\geq \widetilde{M}(i(y_k), \widetilde{x}_{n_k}, 2^{-k}) * \widetilde{M}(\widetilde{x}_{n_k}, \widetilde{x}_{n_m}, 2^{-(k \wedge m)}) \\ &\quad * \widetilde{M}(\widetilde{x}_{n_m}, i(y_m), 2^{-m}) \\ &\geq (1 - 2^{-k}) * (1 - 2^{-(\min\{k,m\})}) * (1 - 2^{-m}) \\ &\geq (1 - 2^{-j}) * (1 - 2^{-j}) * (1 - 2^{-j}) > 1 - \varepsilon, \end{aligned}$$

and consequently $(y_k)_k$ is a Cauchy sequence in $(X, M^i, *)$. Therefore $\widetilde{y} \in \widetilde{X}$, where $\widetilde{y} := [(y_k)_k]$.

Finally, we prove that the sequence $(\widetilde{x}_n)_n$ converges to $\widetilde{y}$ in $(\widetilde{X}, \widetilde{M^i}, *)$. Indeed, as in part (c) above, choose $\varepsilon \in (0, 1)$ and $t > 0$. Take $j \in \mathbb{N}$ such that $2^{-j} < t/3$, and

$$(1 - 2^{-j}) * (1 - 2^{-j}) * (1 - 2^{-j}) > 1 - \varepsilon.$$

Since $(y_k)_k$ is a Cauchy sequence in $(\widetilde{X}, \widetilde{M^i}, *)$, the proof of part (b) shows that there is $k \geq j$ such that

$$\widetilde{M^i}(\widetilde{y}, i(y_k), 2^{-j}) > 1 - 2^{-j}.$$

Then, for $n \geq n_k$ we obtain

$$\begin{aligned} \widetilde{M^i}(\widetilde{y}, \widetilde{x}_n, t) &\geq \widetilde{M^i}(\widetilde{y}, i(y_k), 2^{-j}) * \widetilde{M^i}(i(y_k), \widetilde{x}_{n_k}, 2^{-j}) * \widetilde{M^i}(\widetilde{x}_{n_k}, \widetilde{x}_n, 2^{-j}) \\ &\geq (1 - 2^{-j}) * \widetilde{M^i}(i(y_k), \widetilde{x}_{n_k}, 2^{-k}) * \widetilde{M^i}(\widetilde{x}_{n_k}, \widetilde{x}_n, 2^{-k}) \\ &\geq (1 - 2^{-j}) * (1 - 2^{-k}) * (1 - 2^{-k}) \\ &\geq (1 - 2^{-j}) * (1 - 2^{-j}) * (1 - 2^{-j}) > 1 - \varepsilon. \end{aligned}$$

We conclude that $(\widetilde{X}, \widetilde{M}, *)$ is bicomplete.

(e) This follows directly from Lemma 3.3.4 and standard arguments.

□

Remark. *The preceding theorem implies that every fuzzy quasi-metric space* $(X, M, *)$ *has a bicompletion which is unique up to isometry. We refer to* $(\widetilde{X}, \widetilde{M}, *)$ *as the bicompletion of* $(X, M, *)$.

Remark. *Note that if* $(X, M, *)$ *is a fuzzy quasi-metric space, then* $\widetilde{M}^{-1} = \widetilde{M^{-1}}$ *on* $\widetilde{X}$. *On the other hand, if* $(X, M, *)$ *is a fuzzy metric space, then* $(\widetilde{M}, *)$ *is a fuzzy metric on* $\widetilde{X}$, *and thus the complete fuzzy metric space* $(\widetilde{X}, \widetilde{M}, *)$ *is the completion of* $(X, M, *)$.

Next we apply our constructions to study the bicompletion of some paradigmatic examples of fuzzy quasi-metric spaces. In order to help to the reader we recall the construction of the bicompletion of a quasi-metric space (see [12, 58] or p. 163 of [37]).

Let (X, d) be a quasi-metric space. Denote by Y the set of all Cauchy sequences in the metric space (X, d^s). For each $(x_n)_n, (y_n)_n \in Y$ put $(x_n)_n \sim (y_n)_n$ if and only if $\lim_n d^s(x_n, y_n) = 0$. Then $\sim$ is an equivalence relation on Y. Denote by X^B the quotient $Y/\sim$. For each $[(x_n)_n], [(y_n)_n] \in X^B$, let $d^B([(x_n)_n], [(y_n)_n]) = \lim_n d(x_n, y_n)$. Then (X^B, d^B) is a bicomplete quasi-metric space such that (X, d) is isometric to a dense subspace of the metric space $(X^B, (d^B)^s)$. The space (X^B, d^B) is said to be the bicompletion of (X, d). Furthermore, the bicompletion coincides with the standard completion when (X, d) is a metric space.

Example 3.3.1. *Let* (X, d) *be a quasi-metric space and let* $*$ *be a continuous t-norm. Then, the pair* $(M_{d,01}, *)$ *is a fuzzy quasi-metric on* X, *where* M *is the fuzzy set in* $X \times X \times [0, \infty)$ *given by* $M_{d,01}(x, y, t) = 0$ *if* $d(x, y) \geq t$ *and* $M_{d,01}(x, y, t) = 1$ *if* $d(x, y) < t$. *Moreover, the topology* τ_d, *induced by* d, *coincides with the topology* $\tau_{M_{d,01}}$ *induced by* $(M_{d,01}, *)$.

It is almost obvious that a sequence in X *is a Cauchy sequence in* (X, d^s) *if and only if it is a Cauchy sequence in* $(X, (M_{d,01})^i, *)$, *and thus it easily* *follows that* $\widetilde{X} = X^B$. *For each* $[(x_n)_n], [(y_n)_n] \in \widetilde{X}$ *and* $t > 0$ *we have*

$$d^B([(x_n)_n], [(y_n)_n]) < t \Leftrightarrow \widetilde{M_{d,01}}([(x_n)_n], [(y_n)_n]) = 1,$$

and hence $\widetilde{M_{d,01}} = M_{d^B,01}$ *on* $\widetilde{X}$.

Example 3.3.2. *Let* $(X, M, *)$ *be a fuzzy quasi-metric space such that* $*_L \leq *$. *Similarly to the metric case, the function* $d_M : X \times X \to [0, \infty)$ *given by* $d_M(x, y) = \sup\{t \geq 0 : 1 - M(x, y, t) \geq t\}$, *is a quasi-metric on* X *whose induced topology coincides with* τ_M *(compare [8, Remark 7.6.1]).*

It is clear that a sequence in X *is Cauchy in* $(X, (d_M)^s)$ *if and only if it is Cauchy in* $(X, M^i, *)$.

We show that $(d_M)^B = d_{\widetilde{M}}$ *on* X^B.

Let $[(x_n)_n], [(y_n)_n] \in X^B$. *Put* $\alpha = (d_M)^B([(x_n)_n], [(y_n)_n])$ *and* $\beta = d_{\widetilde{M}}([(x_n)_n], [(y_n)_n])$.

Then $\alpha = \lim_n(\sup\{t \geq 0 : t \leq 1 - M(x_n, y_n, t\})$, *and*
$\beta = \sup\{t \geq 0 : t \leq 1 - \widetilde{M}([(x_n)_n], [(y_n)_n], t)\}$.

We first show that $\beta \leq \alpha$:

Let $t > 0$ *such that* $t \leq 1 - \widetilde{M}([(x_n)_n], [(y_n)_n], t)$. *Then*

$$\sup_{0<s<t} \sup_k \inf_{n \geq k} M(x_n, y_n, s) \leq 1 - t.$$

Thus, for each $s \in (0, t)$ *and each* k, *we have*

$$\inf_{n \geq k} M(x_n, y_n, s) \leq 1 - t.$$

Given $\varepsilon \in (0, t/2)$, *by definition of* α *there is* n_0 *such that for each* $n \geq n_0$, $\gamma_n < \alpha + \varepsilon$, *where*

$$\gamma_n = \sup\{r \geq 0 : r \leq 1 - M(x_n, y_n, r)\}.$$

Choose $s \in (0, t)$ *such that* $t < s + \varepsilon$. *Then, for* $k = n_0$, *there is* $n_1 \geq n_0$ *such that*

$$M(x_{n_1}, y_{n_1}, s) < 1 - t + \varepsilon.$$

So $0 < s - \varepsilon < t - \varepsilon < 1 - M(x_{n_1}, y_{n_1}, s) \le 1 - M(x_{n_1}, y_{n_1}, s - \varepsilon)$, *and consequently* $s - \varepsilon \le \gamma_{n_1}$. *Therefore* $t < \gamma_{n_1} + 2\varepsilon$, *and thus* $t < \alpha + 3\varepsilon$, *so that* $t \le \alpha$. *We conclude that* $\beta \le \alpha$.

Next we show that $\alpha \le \beta$:

Claim: If $t \le 1 - M(x_n, y_n, t)$ *eventually, then* $t \le 1 - \widetilde{M}([(x_n)_n], [(y_n)_n], t)$:

Indeed, by hypothesis, there is n_0 *such that* $\sup_{n \ge n_0} M(x_n, y_n, t) \le 1 - t$. *Let* $s \in (0, t)$. *For each* k *choose* $n_k \ge \max\{n_0, k\}$. *Then*

$$\sup_k \inf_{n \ge k} M(x_n, y_n, s) \le \sup_k M(x_{n_k}, y_{n_k}, s) \le$$

$$\sup_k M(x_{n_k}, y_{n_k}, t) \le \sup_{n \ge n_0} M(x_n, y_n, t) \le 1 - t.$$

Hence

$$\widetilde{M}([(x_n)_n], [(y_n)_n], t) = \sup_{0<s<t} \sup_k \inf_{n \ge k} M(x_n, y_n, s) \le 1 - t.$$

Finally, given $\varepsilon > 0$, *there is* n_0 *such that*

$$\alpha < \varepsilon + \sup\{t \ge 0 : t \le 1 - M(x_n, y_n, t)\},$$

for all $n \ge n_0$. *By our claim*

$$\alpha < \varepsilon + \sup\{t \ge 0 : t \le 1 - \widetilde{M}([(x_n)_n], [(y_n)_n], t)\} = \varepsilon + \beta.$$

We conclude that $\alpha \le \beta$.

We conclude this section with some observations on the bicompletion of GV-fuzzy quasi-metric spaces, for which, the situation is quite different to the corresponding one for KM-fuzzy quasi-metric spaces, of course. In fact, it was proved in [24] the following quasi-metric extension of Theorem 3.2.2.

Theorem 3.3.2. *A GV-fuzzy quasi-metric space* $(X, M, *)$ *is bicompletable if and only if for each pair* $(a_n)_n, (b_n)_n$, *of Cauchy sequences in* $(X, M^i, *)$, *the assignment*

$$t \mapsto \lim_n M(a_n, b_n, t)$$

is a continuous function on $(0, \infty)$ *with values in* $(0, 1]$.

Furthermore, if a GV-fuzzy quasi-metric space is bicompletable, then its GV-fuzzy quasi-metric bicompletion is unique up to isometry.

Moreover, in Example 2 of [24] it was shown that if (X, d) is a quasi-metric space, then the bicompletion $(\widetilde{X}, \widetilde{M_d}, *)$ of the standard GV-fuzzy quasi-metric space $(X, M_d, *)$ is (isometric to) the GV-fuzzy quasi-metric space $(X^B, M_{d^B}, *)$, i.e., the bicompletion of the standard GV-fuzzy quasi-metric space $(X, M_d, *)$ is the standard GV-fuzzy quasi-metric space of the bicompletion of (X, d).

A version of the results introduced in this section, has been submitted for possible publication.

3.4 The bicompletion of a non-Archimedean fuzzy quasi-metric space

In this section we prove, with the help of the construction made in the preceding section, the following result.

Theorem 3.4.1. *The bicompletion of a non-Archimedean fuzzy quasi-metric space is a non-Archimedean fuzzy quasi-metric space.*

Proof:
Let $(X, M, *)$ be a non-Archimedean fuzzy quasi-metric space and let $(x_n)_n$, $(y_n)_n$ and $(z_n)_n$ be Cauchy sequences in the (non-Archimedean) fuzzy metric space $(X, M^i, *)$.

For each $t > 0$ put $\widetilde{M}([(x_n)_n], [(y_n)_n], t) = F(t)$, $\widetilde{M}([(y_n)_n], [(z_n)_n], t) = G(t)$ and $\widetilde{M}([(x_n)_n], [(z_n)_n], t) = H(t)$.

We shall prove that $H(t) \geq F(t) \wedge G(t)$.

Indeed, fix $t > 0$ and assume, without loss of generality that $F(t) > 0$ and $G(t) > 0$.

Choose an arbitrary $\varepsilon > 0$. Then there is $s \in (0, t)$ such that

$$F(t) < \varepsilon + \underline{\lim}\, M(x_n, y_n, s) \quad \text{and} \quad G(t) < \varepsilon + \underline{\lim}\, M(y_n, z_n, s).$$

Now there is $k_\varepsilon \in \mathbb{N}$ such that

$$\underline{\lim}\, M(x_n, y_n, s) < \varepsilon + \inf_{n \geq k_\varepsilon} M(x_n, y_n, s) \quad \text{and}$$

$$\underline{\lim}\, M(y_n, z_n, s) < \varepsilon + \inf_{n \geq k_\varepsilon} M(y_n, z_n, s).$$

On the other hand, since $M(x_n, y_n, s) \wedge M(y_n, z_n, s) \leq M(x_n, z_n, s)$ for all n, we deduce that

$$(\inf_{n \geq k_\varepsilon} M(x_n, y_n, s)) \wedge (\inf_{n \geq k_\varepsilon} M(y_n, z_n, s)) \leq \inf_{n \geq k_\varepsilon} M(x_n, z_n, s).$$

Therefore, from the above relations it follows that

$$\begin{aligned} F(t) \wedge G(t) &< (\varepsilon + \underline{\lim}\, M(x_n, y_n, s)) \wedge (\varepsilon + \underline{\lim}\, M(y_n, z_n, s)) \\ &< (2\varepsilon + \inf_{n \geq k_\varepsilon} M(x_n, y_n, s)) \wedge (2\varepsilon + \inf_{n \geq k_\varepsilon} M(y_n, z_n, s)) \\ &\leq 2\varepsilon + \inf_{n \geq k_\varepsilon} M(x_n, z_n, s) \\ &\leq 2\varepsilon + \underline{\lim}\, M(x_n, y_n, s) \\ &\leq 2\varepsilon + H(t). \end{aligned}$$

We conclude that $F(t) \wedge G(t) \leq H(t)$, and hence the bicompletion of $(X, M, *)$ is a non-Archimedean fuzzy quasi-metric space.

□

Corollary. *The completion of a non-Archimedean fuzzy metric space is a non-Archimedean fuzzy metric space.*

3.5 The bicompletion of an intuitionistic fuzzy quasi-metric space

In this section we shall show, with the help of the construction of the bicompletion of a fuzzy quasi-metric space made above, that every intuitionistic fuzzy quasi-metric space in the sense of [56] (see Chapter 2), has a bicompletion which is unique up to isometry. To this end we shall use some properties of these spaces which were explained in Chapter 2.

Definition 3.5.1. *A mapping f from an inutitionistic fuzzy quasi-metric space $(X, M, N, *, \diamond)$ to an intuitionistic fuzzy quasi-metric space $(Y, M_Y, N_Y, *_Y, \diamond_Y)$ is called an isometry if for each $x, y \in X$ and each $t > 0$,*

$$M_Y(f(x), f(y), t)) = M(x, y, t) \quad \textit{and} \quad N_Y(f(x), f(y), t)) = N(x, y, t).$$

It is clear that every isometry is a one-to-one mapping.

Two intuitionistic fuzzy quasi-metric spaces $(X, M, N, *, \diamond)$ and $(Y, M_Y, N_Y, *_Y, \diamond_Y)$ are called isometric if there is an isometry from X onto Y.

Definition 3.5.2. *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy quasi-metric space. A bicompletion of $(X, M, N, *, \diamond)$ is a bicomplete fuzzy quasi-metric space $(Y, M_Y, N_Y, *_Y, \diamond_Y)$ such that $(X, M, N, *, \diamond)$ is isometric to a $\tau_{(M_Y)^i}$-dense subspace of Y.*

Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy quasi-metric space. Consider the fuzzy quasi-metric space $(X, M, *)$, and let $(\widetilde{X_M}, \widetilde{M}, *)$ be its bicompletion as constructed above.

Recall that, in particular, the mapping $i : X \to \widetilde{X_M}$ as defined in the preceding section, is an isometry between $(X, M, *)$ and a dense subspace of $(\widetilde{X_M}, \widetilde{M}, *)$.

In the following we will refer to i as i_M

Note also that $(X, 1 - N, \diamond')$ is a fuzzy quasi-metric space.

Now we construct a fuzzy set $\widetilde{1-N}$ in $\widetilde{X_M} \times \widetilde{X_M} \times [0,\infty) \to [0,1]$ by

$$\widetilde{1-N}([(x_n)_n],[(y_n)_n],0)=0,$$

and

$$\widetilde{1-N}([(x_n)_n],[(y_n)_n],t)=\sup_{0<s<t} \underline{\lim}\,(1-N)(x_n,y_n,s),$$

whenever $t>0$.

From the fact that $M+N\leq 1$, it follows that every Cauchy sequence in the fuzzy metric space $(X,M^i,*)$ is a Cauchy sequence in the fuzzy metric space $(X,(1-N)^i,\diamond')$, so that, similarly to the technique used in the section above, we can prove that $(\widetilde{X_M},\widetilde{1-N},\diamond')$ is a fuzzy quasi-metric space.

Since $M\leq 1-N$, we deduce that

$$\sup_{0<s<t} \underline{\lim}\, M(x_n,y_n,s) \leq \sup_{0<s<t} \underline{\lim}(1-N)(x_n,y_n,s)$$

for each $t>0$ and each pair $(x_n)_n,(y_n)_n$, of Cauchy sequences in $(X,M^i,*)$. Thus

$$\widetilde{M}\leq \widetilde{1-N}.$$

Since $(\widetilde{X_M},\widetilde{M},*)$ is a bicomplete fuzzy quasi-metric space, we conclude that $(\widetilde{X_M},\widetilde{M},1-(\widetilde{1-N}),*,\diamond)$ is a bicomplete intuitionistic fuzzy quasi-metric space.

Furthermore, the mapping i_M satisfies for each $x,y\in X$ and $t>0$,

$$\widetilde{M}(i_M(x),i_M(y),t)=M(x,y,t),$$

and, also,

$$(1-(\widetilde{1-N}))(i_M(x),i_M(y),t)=N(x,y,t).$$

Thus i_M is an isometry between $(X,M,N,*,\diamond)$ and the subspace $i_M(X)$ of $(\widetilde{X_M},\widetilde{M},1-(\widetilde{1-N}),*,\diamond)$, which is dense with respect to τ_{M^i}.

We have proved that $(\widetilde{X_M},\widetilde{M},1-(\widetilde{1-N}),*,\diamond)$ is a bicompletion of $(X,M,N,*,\diamond)$.

Finally, suppose that $(Y, M_Y, N_Y, *_Y, \diamond_Y)$ is any bicompletion of $(X, M, N, *, \diamond)$. Then, there is an isometry j from $(X, M, N, *, \diamond)$ to $(Y, M_Y, N_Y, *_Y, \diamond_Y)$. On the other hand, since the fuzzy quasi-metric space $(Y, M_Y, *_Y)$ is a bicompletion of $(X, M, *)$, there is a unique isometry F from $(\widetilde{X_M}, \widetilde{M}, *)$ onto $(Y, M_Y, *_Y)$ such that $F(i_M) = j$. Taking into account that every Cauchy sequence in $(X, M^i, N^i, *, \diamond)$ is a Cauchy sequence in $(X, (1-N)^i, \diamond')$ we deduce from standard arguments (see for instance the proof of [22, Proposition 4.5]) that

$$N_Y(F(\widetilde{x}), F(\widetilde{y}), t) = (1 - (\widetilde{1 - N}))(\widetilde{x}, \widetilde{y}, t),$$

whenever $\widetilde{x}, \widetilde{y} \in \widetilde{X_M}$ and $t > 0$. Therefore F is an isometry from $(\widetilde{X_M}, \widetilde{M}, 1 - (\widetilde{1 - N}), *, \diamond)$ onto $(Y, M_Y, N_Y, *_Y, \diamond_Y)$.

Thus, we have proved the following.

Theorem 3.5.1. *Let $(X, M, N, *, \diamond)$ be an intuitionistic fuzzy quasi-metric space. Then:*

(a) *$(\widetilde{M}, 1 - (\widetilde{1 - N}), *, \diamond)$ is an intuitionistic fuzzy quasi-metric on $\widetilde{X_M}$.*

(b) *$i_M(X)$ is dense in $(\widetilde{X_M}, \widetilde{M}^i, *)$.*

(c) *$(X, M, N, *, \diamond)$ is isometric to $(i_M(X), \widetilde{M}, 1 - (\widetilde{1 - N}), *, \diamond)$.*

(d) *$(\widetilde{M}, 1 - (\widetilde{1 - N}), *, \diamond)$ is bicomplete.*

(e) *If $(Y, M_Y, N_Y, *_Y, \diamond_Y)$ is a bicomplete intuitionistic fuzzy quasi-metric space such that $(X, M, N, *, \diamond)$ is isometric to a $\tau_{(M_Y)^i}$-dense subspace of Y, then $(Y, M_Y, N_Y, *_Y, \diamond_Y)$ and $(\widetilde{X_M}, \widetilde{M}, 1 - (\widetilde{1 - N}), *, \diamond)$ are isometric.*

Remark. *The above theorem shows that each intuitionistic fuzzy quasi-metric space has a bicompletion which is unique up to isometry.*

3.6 Conclusions

The main result that can be extracted from this chapter is the fact that each fuzzy quasi-metric space in the sense of Kramosil and Michalek admits bicompletion. We have shown how bicompletion of fuzzy quasi-metric spaces can be achieved using the suprema of subsets of $[0,1]$ and lower limits of sequences of $[0,1]$ (Section 3.3).

Thus, in Section 3.4 it has been shown that the bicompletion of a non-Archimedean fuzzy quasi-metric space is a non-Archimedean fuzzy quasi-metric space. As we will see in next chapter, this property is relevant when studying (non-Archimedean) quasi-metrics constructed on the domain of words.

Finally, in Section 3.5, the bicompletion of intuitionistic fuzzy quasi-metric spaces is obtained.

In all cases bicompletion is constructed directly using the suprema of subsets of $[0,1]$ and lower limits of sequences of $[0,1]$. We find that this approach is more natural than what one would find taking advantage of Lévy's metric completeness properties.

Chapter 4

Contraction maps on fuzzy quasi-metric spaces and algorithms with two recurrence procedures

4.1 Introduction

In the last years some authors applied fixed point theorems on the domain of words, equipped with suitable bicomplete fuzzy quasi-metrics, to prove the existence (and uniqueness) of solution for the recurrence equations typically associated to "Divide and Conquer" algorithms and Quicksort algorithms, respectively ([52, 56, 57]). In this chapter we show that this approach is also useful to obtain the existence and uniqueness of solution for the system of two recurrence equations asocciated to certain algorithms with two recurrence procedures as analyzed by Atkinson ([4]). We also emphasize on the importance of the fact, proved in Chapter 3, that the bicompletion of a non-Archimedean fuzzy quasi-metric space is a non-Archimedean fuzzy quasi-metric space in our approach (see the remark at the end of Section 4.2). We will also deduce the existence and uniqueness of the solution by using contraction maps on the product of complexity spaces, as defined by Schellekens in [59].

Denotational semantics theory has proved to be suitable for the complexity analysis of "Divide and Conquer" algorithms. We study the suitability of the same principles for the complexity analysis of algorithms based on recurrence equations in general. For that means and in order to complement previous studies we have chosen a recurrence equation that models the execution times of two recursive procedures dependent the one with the other.

Schellekens introduced [59] the complexity (quasi-metric) space based on the Smyth completion [66] in order to construct a suitable mathematical model for the complexity analysis of algorithms. In fact, he proved in Section 6 of [59] the existence and uniqueness of solution for the recurrence equations associated to "Divide and Conquer" algorithms by applying a quasi-metric version of the Banach fixed point theorem to the complexity space. Recently it was shown in [16] that Schellekens' technique can be successfully systematized to deduce the existence and uniqueness of solution for the recurrence equations associated to "Probabilistic Divide and Conquer" algorithms, and for the recurrence inequations associated to *expoDC* Algorithms, respectively (see [14] and Section 7.7 of [5] for a study of such algorithms).

While *mergesort* (see [10] for a design study) was used there as an example of "Divide and Conquer" algorithm, in [54] authors proved the suitability of denotational semantics for *expoDC* algorithms, see [5] also. This last study proved that the theory was able to deal with an algorithm whose recurrence was an inequation and also had several parameters.

Here we show that the complexity space also provides an efficient framework to prove the existence and uniqueness of solution for the pair of recurrence equations associated to a class of algorithms with two recurrence procedures, as considered by Atkinson [4]. With the help of the notion of an improver we also deduce that if (f_0, g_0) denotes the solution of such recurrences, then $f_0(e^{2n}) \in \mathcal{O}(e^{2n})$ and $g_0(e^{2n}) \in \mathcal{O}(e^{2n})$. In order to prove these results we will need to apply the Banach fixed point theorem to the "product complexity space" instead to the original one because this kind of algorithms involves two equations.

In Section 4.2, the aforementioned algorithm is shown as a pair of recurrence equations expressed in terms of P and Q procedures. Concrete examples of this class of algorithms could be extracted from language theory scenarios; such a system of equations may represent a couple of rules of a grammar dependent the one on the other. Another scenario where many cases can be found is object-oriented design. This algorithm expresses a situation of highly coupled design; a pair of objects from the system with methods that rely non-interactively the one on the other to fulfill a given task.

Recall that asymptotic notation introduces three functions to denote the "order of" an algorithm f in the set of all possible functions. For a lower order threshold, $\mathcal{O}$-notation is used:

$$\begin{aligned}\mathcal{O}(g(n)) &= \{f : \omega \to [0, \infty) : \exists c > 0, n_0 \in \omega, \text{ such that} \\ &\quad f(n) \leq cg(n) \text{ for all } n \geq n_0\}\end{aligned}$$

for a superior order threshold, Ω-notation is used:

$$\begin{aligned}\Omega(g(n)) &= \{f : \omega \to [0, \infty) : \exists c > 0, n_0 \in \omega, \text{ such that} \\ &\quad f(n) \geq cg(n) \text{ for all } n \geq n_0\}\end{aligned}$$

and for the exact order of magnitude of an algorithm, Θ is:

$$\Theta(g(n)) = \mathcal{O}(g(n)) \cap \Omega(g(n))$$

We will perform complexity analysis based on $(\mathcal{C}, d_{\mathcal{C}})$ complexity space to find the algorithm cost with more precision than we would find by using asymptotic analysis.

4.2 Application of the Banach fixed point theorem on fuzzy quasi-metric spaces to algorithms with two recurrence equations

In [20], M. Grabiec introduced the following notions in order to obtain a fuzzy version of the classical Banach fixed point theorem:

A sequence $(x_n)_n$ in a fuzzy metric space $(X, M, *)$ is Cauchy provided that $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$ for each $t > 0$ and $p \in \mathbb{N}$.

A fuzzy metric space $(X, M, *)$ is complete provided that every Cauchy sequence in X is convergent. In this case, $(M, *)$ is called a complete fuzzy metric on X.

In the sequel, and according to [26] and [67], a Cauchy sequence in Grabiec's sense will be called G-Cauchy and a complete fuzzy metric space in Grabiec's sense will be called G-complete.

On the other hand, following [63], a B-contraction on a fuzzy metric space $(X, M, *)$ is a self-map f on X such that there is a constant $k \in (0, 1)$ satisfying

$$M(f(x), f(y), kt) \geq M(x, y, t)$$

for all $x, y \in X$, $t > 0$.

Thus, Grabiec's fixed point theorem can be formulated as follows.

Theorem 4.2.1. [20]. *Let $(X, M, *)$ be a G-complete fuzzy metric space such that $\lim_{t\to\infty} M(x, y, t) = 1$ for all $x, y \in X$. Then every B-contraction on X has a unique fixed point.*

The following quasi-metric generalizations of the notions of B-contraction and G-completeness were introduced in [52].

Definition 4.2.1. *A B-contraction on a fuzzy quasi-metric space* $(X, M, *)$ *is a self-map* f *on* X *such that there is a constant* $k \in]0, 1[$ *satisfying*

$$M(f(x), f(y), kt) \geq M(x, y, t)$$

for all $x, y \in X$, $t > 0$. *The number* k *is then called a contraction constant of* f.

Definition 4.2.2. *A sequence* $(x_n)_n$ *in a fuzzy quasi-metric space* $(X, M, *)$ *is called G-Cauchy if it is a G-Cauchy sequence in the fuzzy metric space* $(X, M^i, *)$.

Definition 4.2.3. *A fuzzy quasi-metric space* $(X, M, *)$ *is called G-bicomplete if the fuzzy metric space* $(X, M^i, *)$ *is G-complete. In this case, we say that* $(M, *)$ *is a G-bicomplete fuzzy quasi-metric on* X.

Then, Grabiec's theorem was generalized to fuzzy quasi-metric spaces in [52] as follows.

Theorem 4.2.2. [52]. *Let* $(X, M, *)$ *be a G-bicomplete fuzzy quasi-metric space such that* $\lim_{t \to \infty} M(x, y, t) = 1$ *for all* $x, y \in X$. *Then every B-contraction on* X *has a unique fixed point.*

Although G-(bi)completeness is a very strong kind of completeness, we have the following nice and useful fact for our approach.

Theorem 4.2.3. [52]. *Each bicomplete non-Archimedean fuzzy quasi-metric space is G-bicomplete.*

The following result implies that Theorem 4.2.2 applies to the standard fuzzy quasi-metric space of any bicomplete non-Archimedean quasi-metric space.

Theorem 4.2.4. *Let (X, d) be a bicomplete non-Archimedean quasi-metric space. Then $(X, M_d, \wedge)$ is a G-bicomplete (non-Archimedean) fuzzy quasi-metric space such that $\lim_{t\to\infty} M_d(x, y, t) = 1$ for all $x, y \in X$.*

Proof:

It is clear and well-known that $\lim_{t\to\infty} M_d(x, y, t) = 1$ for all $x, y \in X$, and that $(X, M_d, \wedge)$ is bicomplete (resp. non-Archimedean) if and only if (X, d) is bicomplete (resp. non-Archimedean). The conclusion follows from Theorem 4.2.3.

□

Next we recall several pertinents facts and results on the domain of words and some non-Archimedean (fuzzy) quasi-metrics that one can construct on it.

The domain of words Σ^∞ ([35, 39, 51, 60, 66, etc]) consists of all finite and infinite sequences ("words") over a nonempty set ("alphabet") Σ, ordered by the so-called information order $\sqsubseteq$ on Σ^∞ , i.e., $x \sqsubseteq y \Leftrightarrow x$ is a prefix of y, where we assume that the empty sequence ϕ is an element of Σ^∞.

For each $x, y \in \Sigma^\infty$ denote by $x \sqcap y$ the longest common prefix of x and y, and for each $x \in \Sigma^\infty$ denote by $\ell(x)$ the length of x. Thus $\ell(x) \in [1, \infty]$ whenever $x \neq \phi$, and $\ell(\phi) = 0$.

Given a nonempty alphabet Σ, Smyth introduced in [66] a non-Archimedean quasi-metric $d_\sqsubseteq$ on Σ^∞ given by $d_\sqsubseteq(x, y) = 0$ if $x \sqsubseteq y$, and $d_\sqsubseteq(x, y) = 2^{-\ell(x\sqcap y)}$ otherwise (see also [35, 49, 52, etc]).

This quasi-metric has the advantage that its specialization order coincides with the order $\sqsubseteq$, and thus the quasi-metric space $(\Sigma^\infty, d_\sqsubseteq)$ preserves the information provided by $\sqsubseteq$. Moreover, the metric $(d_\sqsubseteq)^s$ is given by $(d_\sqsubseteq)^s(x, y) = 0$ if $x = y$, and $(d_\sqsubseteq)^s(x, y) = 2^{-\ell(x\sqcap y)}$ otherwise; so that $(d_\sqsubseteq)^s$ is exactly the celebrated Baire metric on Σ^∞.

Consequently $d_\sqsubseteq$ is a bicomplete non-Archimedean quasi-metric on Σ^∞.

In order to apply techniques of fixed point for obtaining the existence and uniqueness of solution for the two recurrence equations associated to algorithms with two recurrence procedures, we shall combine the above results with some facts on the product of (non-Archimedean) fuzzy quasi-metrics that we present in the sequel.

Similarly to [9] the product (fuzzy quasi-metric) space of two fuzzy quasi-metric spaces $(X_1, M_1, *)$ and $(X_2, M_2, *)$ is the fuzzy quasi-metric space $(X_1 \times X_2, M_1 \times M_2, *)$ such that for each $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$ and each $t \geq 0$,

$$(M_1 \times M_2)((x_1, x_2), (y_1, y_2), t) = M_1(x_1, y_1, t) * M_2(x_2, y_2, t).$$

In particular, if $(X_1, M_1, \wedge)$ and $(X_2, M_2, \wedge)$ are non-Archimedean, then $(X_1 \times X_2, M_1 \times M_2, \wedge)$ is non-Archimedean.

Furthermore, it is clear that if $(X_1, M_1, *)$ and $(X_2, M_2, *)$ are bicomplete, then $(X_1 \times X_2, M_1 \times M_2, *)$ is bicomplete.

By applying the above results and remarks to the standard fuzzy quasi-metric space of $(\Sigma^\infty, d_\sqsubseteq)$ when $* = \wedge$, we immediately deduce the following.

Theorem 4.2.5. $(\Sigma^\infty \times \Sigma^\infty, M_{d_\sqsubseteq} \times M_{d_\sqsubseteq}, \wedge)$ *is a bicomplete non-Archimedean fuzzy quasi-metric space such that* $\lim_{t\to\infty} (M_{d_\sqsubseteq} \times M_{d_\sqsubseteq})((x_1, x_2), (y_1, y_2), t) = 1$ *for all* $(x_1, x_2), (y_1, y_2) \in \Sigma^\infty \times \Sigma^\infty$. *Therefore, every B-contraction on this space has a unique fixed point.*

Following Atkinson [4, p. 16-17], consider the two recursive procedure algorithm defined, for two procedures P and Q, and $n \in \omega$, by:

```
function P(n)
  if n > 0 then Q(n-1); C; P(n-1); C; Q(n-1)


function Q(n)
  if n > 0 then P(n-1); C; Q(n-1); C; P(n-1); C; Q(n-1)
```

where C denotes any statements taking time independent of n.

Then, the execution times $S(n)$ and $T(n)$ of $P(n)$ and $Q(n)$, satisfy, at least approximately, the recurrences

$$S(n) = S(n-1) + 2T(n-1) + K_1,$$

and

$$T(n) = 2S(n-1) + 2T(n-1) + K_2,$$

for $n \in \mathbb{N}$, and with K_1, K_2, nonnegative constants. (We assume that $S(0) > 0$ and $T(0) > 0$).

We shall deduce the existence and uniqueness of solution for the recurrences S and T by means of a version of the Banach fixed point theorem on a suitable (product) fuzzy quasi-metric space constructed on a certain product of domain of words.

To this end, consider the recurrences A and B given by $A(0) > 0$, $B(0) > 0$, and

$$A(n) = pA(n-1) + qB(n-1) + K_1,$$

and

$$B(n) = rA(n-1) + sB(n-1) + K_2,$$

for all $n \in \mathbb{N}$, where p, q, r, s, K_1, K_2, are nonnegative constants with $p, q, r, s > 0$.

Note that recurrences S and T are a particular case of A and B for $p = 1$, $q = r = s = 2$.

In the rest of this section by Σ^∞ we shall denote the domain of words where the alphabet Σ is the set of nonnegative real numbers.

Recurrences A and B suggest the construction of the functional

$$\Phi : \Sigma^\infty \times \Sigma^\infty \to \Sigma^\infty \times \Sigma^\infty,$$

given for each pair $x^1, x^2 \in \Sigma^\infty$, by

$$\Phi(x^1, x^2) = (u^1, u^2),$$

where

$$(u^1)_0 = A(0), \quad (u^2)_0 = B(0),$$

and

$$(u^1)_n = p(x^1)_{n-1} + q(x^2)_{n-1} + K_1, \quad (u^2)_n = r(x^1)_{n-1} + s(x^2)_{n-1} + K_2,$$

for all $n \in \mathbb{N}$ such that $n \leq (\ell(x^1) \wedge \ell(x^2)) + 1$.

Note that then $\ell(u^1) = \ell(u^2) = (\ell(x^1) \wedge \ell(x^2)) + 1$.

Next we prove that for each $(x^1, x^2), (y^1, y^2) \in \Sigma^\infty \times \Sigma^\infty$ and each $t > 0$, one has

$$(M_{d_\sqsubseteq} \times M_{d_\sqsubseteq})(\Phi((x^1, x^2)), \Phi((y^1, y^2)), t/2) \geq M_{d_\sqsubseteq}(x^1, y^1, t) \wedge M_{d_\sqsubseteq}(x^2, y^2, t)$$

Indeed, put $\Phi(x^1, x^2) = (u^1, u^2)$ and $\Phi(y^1, y^2) = (v^1, v^2)$ and let $t > 0$.

First observe that if $u^1 \sqsubseteq u^2$ and $v^1 \sqsubseteq v^2$, we obtain

$$(M_{d_\sqsubseteq} \times M_{d_\sqsubseteq})(\Phi((x^1, x^2)), \Phi((y^1, y^2)), t/2) =$$

$$M_{d_\sqsubseteq}(u^1, v^1, t/2) \wedge M_{d_\sqsubseteq}(u^2, v^2, t/2) = 1.$$

Otherwise, we will take into account that, by the construction of u^1, u^2, v^1 and v^2, we have

$$\ell(u^k \sqcap v^k) \geq (\ell(x^1 \sqcap y^1) \wedge \ell(x^2 \sqcap y^2)) + 1, \text{ for } k = 1, 2.$$

Consequently

$$
\begin{aligned}
&(M_{d_\sqsubseteq} \times M_{d_\sqsubseteq})(\Phi((x^1,x^2)),\Phi((y^1,y^2)),t/2)\\
&\quad= M_{d_\sqsubseteq}(u^1,v^1,t/2) \wedge M_{d_\sqsubseteq}(u^2,v^2,t/2)\\
&\quad= \frac{t/2}{t/2+d_\sqsubseteq(u^1,v^1)} \wedge \frac{t/2}{t/2+d_\sqsubseteq(u^2,v^2)}\\
&\quad= \frac{t}{t+2^{-\ell(u^1\sqcap v^1)+1}} \wedge \frac{t}{t+2^{-\ell(u^2\sqcap v^2)+1}}\\
&\quad\geq \frac{t}{t+2^{-(\ell(x^1\sqcap y^1)\wedge \ell(x^2\sqcap y^2))}}\\
&\quad= \frac{t}{t+2^{-\ell(x^1\sqcap y^1)}} \wedge \frac{t}{t+2^{-\ell(x^2\sqcap y^2)}}\\
&\quad= M_{d_\sqsubseteq}(x^1,y^1,t) \wedge M_{d_\sqsubseteq}(x^2,y^2,t)\\
&\quad= (M_{d_\sqsubseteq} \times M_{d_\sqsubseteq})((x^1,x^2),(y^1,y^2),t).
\end{aligned}
$$

We have shown that Φ is a B-contraction of the G-bicomplete (non-Archimedean) fuzzy quasi-metric space $(\Sigma^\infty \times \Sigma^\infty, M_{d_\sqsubseteq} \times M_{d_\sqsubseteq}, \wedge)$. By Theorem 4.2.5, Φ has a unique fixed point which is obviously the solution of the recurrences A and B.

Remark. *In practice one actually works on the set Σ^F of all finite words (over the alphabet $[0,\infty)$), that endowed with the restriction of the fuzzy quasi-metric $(M_{d_\sqsubseteq}, \wedge)$ provides a non-Archimedean fuzzy quasi-metric space which is not bicomplete. In fact the product space $(\Sigma^F \times \Sigma^F, M_{d_\sqsubseteq} \times M_{d_\sqsubseteq}, \wedge)$ is also a non-bicomplete non-Archimedean fuzzy quasi-metric space. By Theorems 3.3.1, 3.4.1 it is bicompletable and its bicompletion is a non-Archimedean fuzzy quasi-metric space for which we can apply Theorem 4.2.5 (in fact, the bicompletion is isometric to $(\Sigma^\infty \times \Sigma^\infty, M_{d_\sqsubseteq} \times M_{d_\sqsubseteq}, \wedge)$). In particular, for each pair $x^1, x^2 \in \Sigma^F$, the sequence of iterations $(\Phi^k(x^1,x^2))_k$ converges, in $(\Sigma^\infty \times \Sigma^\infty, (M_{d_\sqsubseteq} \times M_{d_\sqsubseteq})^i, \wedge)$, to the element that constitutes the solution for the pair of recurrence equations A and B.*

4.2.1 Application of the Banach fixed point theorem on complexity spaces to algorithms with two recurrence equations

Schellekens introduced in [59] the complexity (quasi-metric) space to construct a suitable mathematical model for the complexity analysis of algorithms. In fact, he proved in Section 6 of [59] the existence and uniqueness of solution for the recurrence equations associated to “Divide and Conquer” algorithms by applying a quasi-metric version of the Banach fixed point theorem to the complexity space. Recently it was shown in [16] and [54] that Schellekens’ technique can be successfully systematized to deduce the existence and uniqueness of solution for the recurrence equations associated to “Probabilistic Divide and Conquer” algorithms, and for the recurrence inequations associated to *expoDC* Algorithms, respectively (see [14] and [5, Section 7.7] for a study of such algorithms).

Here we show that the complexity space also provides an efficient framework to prove the existence and uniqueness of solution for the pair of recurrence equations considered in the section above. With the help of the notion of an improver (see its definition below) we also deduce that if (f_0, g_0) denotes the solution of recurrences S and T , then $f_0(n) \in \mathcal{O}(e^{2n})$ and $g_0(n) \in \mathcal{O}(e^{2n})$. In order to prove these results we will need to apply the Banach fixed point theorem to the “product complexity space” instead to the original one because this kind of algorithms involves two equations.

A quasi-metric space (X, d) is said to be bicomplete if (X, d^s) is a complete metric space.

By a contraction map on a quasi-metric space *(X, d)* we mean a self-map f of X such that $d(fx, fy) \leq kd(x, y)$ for all $x, y \in X$, where k is a constant with $0 < k < 1$. The number k is called a contraction constant for *f*.

It is clear that if f is a contraction map on a quasi-metric space (X, d)

with contraction constant k, then f is a contraction map on the metric space (X, d^s) with contraction constant k.

Therefore, the classical Banach contraction principle can be generalized to the quasi-metric setting as follows (see for instance [39, Lemma 2.4]).

Theorem 4.2.6. *Let f be a contraction map on a bicomplete quasi-metric space (X,d). Then, for each $x \in X$, the sequence of iterations $(f^n x)_{n\in\omega}$ is convergent in (X, d^s) to a point $x_0 \in X$ which is the unique fixed point of f.*

Let us recall that the product quasi-metric space of two quasi-metric spaces (X, d) and (Y, e) is the quasi-metric space $(X \times Y, d \times e)$, where $d \times e$ is defined by

$$(d \times e)((x_1, y_1), (x_2, y_2)) = d(x_1, x_2) \vee e(y_1, y_2),$$

for all $(x_1, y_1), (x_2, y_2) \in X \times Y$.

In this case, $d \times e$ is called the product (or box) quasi-metric of d and e.

The so-called complexity space ([59]) is the quasi-metric space $(\mathcal{C}, d_{\mathcal{C}})$, where

$$\mathcal{C} = \left\{ f : \omega \to (0, \infty] : \sum_{n=0}^{\infty} 2^{-n} \frac{1}{f(n)} < \infty \right\},$$

and d_C is the quasi-metric on $\mathcal{C}$ given by

$$d_{\mathcal{C}}(f, g) = \sum_{n=0}^{\infty} 2^{-n} \left(\left(\frac{1}{f(n)} - \frac{1}{g(n)} \right) \vee 0 \right)$$

for all $f, g \in \mathcal{C}$. (We adopt the convention that $1/\infty = 0$.)

The elements of $\mathcal{C}$ are called complexity functions.

The following useful result is a consequence of [53, Theorem 1, and Remark on p. 317].

Theorem 4.2.7. *The complexity space $(\mathcal{C}, d_{\mathcal{C}})$ is bicomplete.*

In our next result we construct a monotone increasing functional Φ, associated with the two recurrences equations A and B constructed in the preceding section, which is a contraction on $(\mathcal{C}\times\mathcal{C}, d_{\mathcal{C}}\times d_{\mathcal{C}})$. Then, its unique fixed point (f_0, g_0) will be the solution of the recurrence equations A and B.

Theorem 4.2.8. *Let Φ be the functional on $\mathcal{C}\times\mathcal{C}$ defined by*

$$\Phi(f,g)(0) = (A(0), B(0)),$$

and

$$\Phi(f,g)(n) = \left(pf(n-1) + qg(n-1) + K_1,\ rf(n-1) + sg(n-1) + K_2\right),$$

for $n\in\mathbb{N}$ and $f,g\in\mathcal{C}$.

If $\alpha < 1$, where

$$\alpha = \frac{1}{2}\left(\frac{1}{p\wedge r} + \frac{1}{q\wedge s}\right),$$

then:

(1) Φ is a monotone increasing contraction on $(\mathcal{C}\times\mathcal{C}, d_{\mathcal{C}}\times d_{\mathcal{C}})$ with contraction constant α.

(2) Φ has a unique fixed point (f_0, g_0).

Proof:

(1) We first note that if $(f,g)\in\mathcal{C}\times\mathcal{C}$, then $\Phi(f,g)\in\mathcal{C}\times\mathcal{C}$, because

$$\sum_{n=1}^{\infty} 2^{-n}\frac{1}{pf(n-1) + qg(n-1) + K_1} \leq \frac{1}{q}\sum_{n=1}^{\infty} 2^{-n}\frac{1}{g(n-1)} < \infty$$

and

$$\sum_{n=1}^{\infty} 2^{-n}\frac{1}{rf(n-1) + sg(n-1) + K_2} \leq \frac{1}{s}\sum_{n=1}^{\infty} 2^{-n}\frac{1}{g(n-1)} < \infty$$

Now let $(f_1,g_1),(f_2,g_2)\in\mathcal{C}\times\mathcal{C}$ be such that $f_1\le f_2$ and $g_1\le g_2$. It is straightforward to chech that then $\Phi(f_1,g_1)\le\Phi(f_2,g_2)$.

Next we show that

$$(d_{\mathcal{C}}\times d_{\mathcal{C}})(\Phi(f_1,g_1),\Phi(f_2,g_2))\le\frac{\alpha}{2}(d_{\mathcal{C}}\times d_{\mathcal{C}})((f_1,g_1),(f_2,g_2)),$$

for all $(f_1,g_1),(f_2,g_2)\in\mathcal{C}\times\mathcal{C}$.

Indeed, given $(f_1,g_1),(f_2,g_2)\in\mathcal{C}\times\mathcal{C}$, put

$$\Phi(f_1,g_1)=(f_1',g_1')\quad\text{and}\quad\Phi(f_2,g_2)=(f_2',g_2').$$

Thus

$$(d_{\mathcal{C}}\times d_{\mathcal{C}})(\Phi(f_1,g_1),\Phi(f_2,g_2))=d_{\mathcal{C}}(f_1',f_2')\vee d_{\mathcal{C}}(g_1',g_2').$$

We have

$$\begin{aligned}
d_{\mathcal{C}}(f_1',f_2') &= \sum_{n=0}^{\infty}2^{-n}\left(\left(\frac{1}{f_2'(n)}-\frac{1}{f_1'(n)}\right)\vee 0\right)\\
&= \sum_{n=1}^{\infty}2^{-n}\left(\left(\frac{1}{pf_2(n-1)+qg_2(n-1)+K_1}\right.\right.\\
&\quad\left.\left.-\frac{1}{pf_1(n-1)+qg_1(n-1)+K_1}\right)\vee 0\right)\\
&\le \sum_{n=1}^{\infty}2^{-n}\left(\frac{p(f_1(n-1)-f_2(n-1))+q(g_1(n-1)-g_2(n-1))}{p^2f_1(n-1)f_2(n-1)+q^2g_1(n-1)g_2(n-1)}\vee 0\right)\\
&\le \sum_{n=1}^{\infty}2^{-n}\left(\left(\frac{f_1(n-1)-f_2(n-1)}{pf_1(n-1)f_2(n-1)}\vee 0\right)\right.\\
&\quad\left.+\left(\frac{(g_1(n-1)-g_2(n-1))}{qg_1(n-1)g_2(n-1)}\vee 0\right)\right)\\
&= \frac{1}{p}\sum_{n=1}^{\infty}2^{-n}\left(\left(\frac{1}{f_2(n-1)}-\frac{1}{f_1(n-1)}\right)\vee 0\right)\\
&\quad+\frac{1}{q}\sum_{n=1}^{\infty}2^{-n}\left(\left(\frac{1}{g_2(n-1)}-\frac{1}{g_1(n-1)}\right)\vee 0\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2p}\sum_{n=0}^{\infty} 2^{-n}\left(\left(\frac{1}{f_2(n)} - \frac{1}{f_1(n)}) \vee 0\right)\right. \\
&\quad + \frac{1}{2q}\sum_{n=0}^{\infty} 2^{-n}\left(\left(\frac{1}{g_2(n)} - \frac{1}{g_1(n)}\right) \vee 0\right) \\
&= \frac{1}{2p}d_{\mathcal{C}}(f_1, f_2) + \frac{1}{2q}d_{\mathcal{C}}(g_1, g_2) \\
&\le \frac{1}{2}\left(\frac{1}{p} + \frac{1}{q}\right)(d_{\mathcal{C}}(f_1, f_2) \vee d_{\mathcal{C}}(g_1, g_2)).
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
d_{\mathcal{C}}(g_1', g_2') &= \sum_{n=0}^{\infty} 2^{-n}\left(\left(\frac{1}{g_2'(n)} - \frac{1}{g_1'(n)}\right) \vee 0\right) \\
&= \sum_{n=1}^{\infty} 2^{-n}\left(\left(\frac{1}{rf_2(n-1) + sg_2(n-1) + K_2}\right.\right. \\
&\quad \left.\left. - \frac{1}{rf_1(n-1) + sg_1(n-1) + K_2}\right) \vee 0\right) \\
&\le \frac{1}{2r}d_{\mathcal{C}}(f_1, f_2) + \frac{1}{2s}d_{\mathcal{C}}(g_1, g_2) \\
&\le \frac{1}{2}\left(\frac{1}{r} + \frac{1}{s}\right)(d_{\mathcal{C}}(f_1, f_2) \vee d_{\mathcal{C}}(g_1, g_2)).
\end{aligned}$$

Consequently

$$\begin{aligned}
(d_{\mathcal{C}} \times d_{\mathcal{C}})(\Phi(f_1, g_1), \Phi(f_2, g_2)) &= d_{\mathcal{C}}(f_1', f_2') \vee d_{\mathcal{C}}(g_1', g_2') \\
&\le \left(\frac{1}{2p}d_{\mathcal{C}}(f_1, f_2) + \frac{1}{2q}d_{\mathcal{C}}(g_1, g_2)\right) \\
&\le \left(\frac{1}{2p}d_{\mathcal{C}}(f_1, f_2) + \frac{1}{2q}d_{\mathcal{C}}(g_1, g_2)\right) \\
&\quad \vee \left(\left(\frac{1}{2r}d_{\mathcal{C}}(f_1, f_2) + \frac{1}{2s}d_{\mathcal{C}}(g_1, g_2)\right)\right) \\
&\le \frac{1}{2}\left(\frac{1}{p \wedge r} + \frac{1}{q \wedge s}\right)(d_{\mathcal{C}}(f_1, f_2) \vee d_{\mathcal{C}}(g_1, g_2))
\end{aligned}$$

$$= \alpha(d_{\mathcal{C}} \times d_{\mathcal{C}})((f_1, g_1), (f_2, g_2)).$$

(2) Since, by Theorem 4.2.7, $(\mathcal{C}, d_{\mathcal{C}})$ is bicomplete, then $(\mathcal{C} \times \mathcal{C}, d_{\mathcal{C}} \times d_{\mathcal{C}})$ is bicomplete. Hence, we can apply Theorem 4.2.6 and thus there exists a unique $(f_0, g_0) \in \mathcal{C} \times \mathcal{C}$ such that $\Phi((f_0, g_0)) = (f_0, g_0)$.

□

Remark. *Note that by the construction of the functional* Φ, (f_0, g_0) *is the solution for the recurrences* A *and* B.

We conclude this section by showing that in the case that the pair (f_0, g_0) is the solution of the recurrence equations S and T associated with the algorithm discussed by Atkinson [4], then $f_0(n) \in \mathcal{O}(e^{2n})$ and $g_0(n) \in \mathcal{O}(e^{2n})$.

This will be done by constructing an appropriate element of $\mathcal{C} \times \mathcal{C}$ for which Φ is an improver.

The following extension to our context of Definition 6.2 of [59] will be needed.

Definition 4.2.4. *A functional* Φ *from* $(\mathcal{C} \times \mathcal{C}, d_{\mathcal{C}} \times d_{\mathcal{C}})$ *into itself is an improver with respect to an element* $(f, g) \in \mathcal{C} \times \mathcal{C}$ *if for each* $n \in \omega$, $\Phi^{n+1}(f, g) \leq \Phi^n(f, g)$.

Note that if Φ is monotone increasing (i.e., $\Phi(f_1, g_1) \leq \Phi(f_2, g_2)$ whenever $f_1 \leq f_2$ and $g_1 \leq g_2$), to show that Φ is an improver with respect to (f, g) it suffices to verify that $\Phi(f, g) \leq (f, g)$.

Intuitively (compare, for instance, [16, p. 348]), an improver is a functional that corresponds to a transformation on algorithms and satisfies the following condition: the iterative applications of the transformation to a given algorithm yield an improved algorithm at each step of the iteration.

Put $c = (S(0) + T(0) + K_1 + K_2)(e^2 - 4)^{-1}$, and let $u, v : \omega \to (0, \infty)$ given by

$u(0) = S(0)$, $v(0) = T(0)$, and $u(n) = v(n) = ce^{2n}$ for all $n \in \mathbb{N}$.

Clearly $u, v \in \mathcal{C}$. Next we show that $\Phi((u,v)) \leq (u,v)$, and thus Φ is an improver with respect to (u,v).

Indeed, we have

$$\Phi((u,v))(0) = (S(0), T(0)) = (u(0), v(0)),$$

$$\begin{aligned}
\Phi((u,v))(1) &= (u(0) + 2v(0) + K_1, 2u(0) + 2v(0) + K_2) \\
&= (S(0) + 2T(0) + K_1, 2S(0) + 2T(0) + K_2) \\
&\leq ((S(0) + T(0) + K_1 + K_2)\frac{e^2}{e^2 - 4}, \\
&\quad (S(0) + T(0) + K_1 + K_2)\frac{e^2}{e^2 - 4}) \\
&= (ce^2, ce^2) \\
&= (u(1), v(1)) \\
&= (u,v)(1).
\end{aligned}$$

and for $n > 1$,

$$\begin{aligned}
\Phi((u,v))(n) &= (u(n-1) + 2v(n-1) + K_1, 2u(n-1) + 2v(n-1) + K_2) \\
&= (ce^{2(n-1)} + 2ce^{2(n-1)} + K_1, 2ce^{2(n-1)} + 2ce^{2(n-1)} + K_2) \\
&\leq (4ce^{2(n-1)} + K_1 + K_2, 4ce^{2(n-1)} + K_1 + K_2) \\
&\leq (4ce^{2(n-1)} + c(e^2 - 4), 4ce^{2(n-1)} + c(e^2 - 4)) \\
&\leq (ce^{2(n-1)}(4 + (e^2 - 4)), ce^{2n}(4 + (e^2 - 4))) \\
&= (ce^{2n}, ce^{2n}) \\
&= (u,v)(n).
\end{aligned}$$

Since Φ is increasing it follows that $\Phi^{n+1}((u,v)) \leq \Phi^n((u,v))$ for all $n \in \omega$. Therefore, from the fact (see Theorem 1) that $(\Phi^n((u,v)))_{n\in\omega}$ converges to (f_0, g_0) in $(\mathcal{C} \times \mathcal{C}, (d_{\mathcal{C}} \times d_{\mathcal{C}})^s)$, it follows that $(f_0, g_0) \leq (u, v)$. Consequently $f_0(n) \in \mathcal{O}(e^{2n})$ and $g_0(n) \in \mathcal{O}(e^{2n})$.

A version of the results introduced in this section, has been submitted for possible publication.

4.3 Conclusions

Our starting point is the fact that the analysis of complexity or execution times for algorithms can be carried out using the theory of recurrence relations. There is no general method of solution for any arbitrary recurrence equation, however there are broad classes of recurrence relations for which solution techniques are known. Schellekens' model is built upon a complexity quasi-metric space, and uses fixed point techniques to find the solution of recurrence equations.

In previous works, the Banach fixed point theorem has been used to solve recurrences of "Divide and Conquer" algorithms in the form of equations, [59] or [16], or inequations [55]. Here we have extended the model to show the existence and uniqueness of the solution of the equations of an algorithm defined with several recurrence equations. Moreover, the use of an improver guarantees an efficient and precise solution.

Moreover, we have approached the complexity analysis using the product of non-Archimedean fuzzy quasi-metrics on the domain of words. The result obtained in Theorem 3.4.1 allows us to tackle this approach on the domain of finite words, whose bicompletion is the domain of infinite words and it is also a non-Archimedean fuzzy quasi-metric space.

Working on finite words domain is a more natural approach for an application to algorithms complexity analysis. Even though complexity analysis is performed in an asymptotic context, a correct algorithm design must ensure that at least one of the algorithm termination conditions is always met, so that the number of iterations to perform is always finite in any case.

Chapter 5

Application of quasi-metric lattice and intuitionistic fuzzy metric structures to optimize information systems based on access locality

5.1 Introduction

Most information systems one can find in production environments are based on access locality. For production environments we mean a systems whose objective is to be used by a given public. Say, for instance, banking systems for customer access, ticket sales systems for potential event audiences, health care systems for doctors or patients, enrollment web pages for students, etc.

Generally, from basic theory such as modern techniques for language compilers to advanced web software development, data and access patterns locality is an intrinsic quality of all of these systems.

The following paragraph reviews some of these examples in order to give samples of the meaning of locality we will be using here.

Algorithms and data structures theory enforces the use of encapsulation and abstraction as the smallest pieces characteristics in order to build more complex systems. As a natural consequence, programmers are encouraged to develop code with no long instruction jumps between sentences. This is a clear form of locality: code jumps cannot be avoided (as programs are composed of statements, conditions and loops) but the programming approach should aim to minimize the length of these jumps. The benefits of this approach are not to be underestimated as code is simpler, clearer, more maintenable and reusable. In the next step, compilers use this code to generate machine language that can be later executed. Small jumps in between machine instructions produces faster and more optimizable programs. This approach is the very basis of today's software development industry: compiler's optimization and then also encapsulation of code in methods for modularity and reusability.

In the first paragraph of this section, examples are more end-user oriented. This kind of software programs are called *distributed systems*. Say for instance applications for event ticket sales or for students enrollment requests. For these applications, usually accesses precede given events such as scheduled event programmes and every year's opening of enrollment requests at the start of each course. This means that the system is used according to low access load patterns during the rest of the year and that access peaks are localised when the load is high and the importance of the system is at its most.

In a similar way, health care systems patient appointments are usually requested during the morning. Furthermore, due to the way we are organised as a society other forms of locality appear. For the cases of banking and health care applications, organisations are usually structured in a hierarchical way: We do have central bank offices as well as local offices (we shall continue exploiting this structure as ATM machines that depend on close-by bank offices). We also have hospitals as well as clinics or outpatient departments. Users (patients or customers) tend to use the nearest office or hospital.

We can infer from this examples that locality appears very frequently in many scenarios. The differences lay on the degree of locality. For example a bank customer is expected to use different ATM machines to withdraw money more frequently than a patient is to visit different clinics. This degree depends on environmental factors such as the information system adaptability for low locality degrees or on the scenario's nature itself.

This hierarchical structure is typically connected via a communications network. In general terms, each office, hospital or enrolling point is a network *node*. Each bank account, clinical history record or student appointment is called an *object* or an *element*. In enterprise software systems these objects are stored in databases.

How the same user is able to, for instance, withdraw money from several geographically scattered ATM machines or to request appointments in different hospitals depends on how the distributed system nodes access these objects. The objects may be centralised in a single node (one server) or replicated (not necessarily all of them) in each node (many servers). This replication may be "eager" (replicate always) or "lazy" (replicate only when needed). These factors influence the degree of system availability.

Nevertheless the system must guarantee that the user receives an updated version of each object anywhere. If at the request moment the object is outdated then the system needs to reach for an updated version prior to the object delivery. This fact implies the need to carry out certain coordination of actions over a set of elements.

An analysis of the objects' access histories would allow the system to predict the object state prior to its effective access. This prediction would allow the system to perform background look-ahead updates or to save on the total amount of needed network messages.

We have defined a mathematical framework based on fuzzy constructions in order to tackle this problem. The framework empirical tests are applied to best, average and worst case usage scenarios because we intend the study to be applied in a general way.

The starting point of our application is based on a quasi-metric lattice structure, see Section 5.2.

Extending our work on a model for object access prediction using a quasi-metric lattice we take a step further in order to exploit the possibilities of the stored history values. In Sections 5.3 and 5.4 we show how we can tune the fuzzy metric results in order to predict access histories working on variations of the fuzzy constructions.

For these metric constructions we will use a set of continuous t-norms and t-norm families (as well as with t-conorms) to build fuzzy constructions in order to advance the time of prediction and object class classification. T-norms suitability will be evaluated according to their computation time and the sensitiveness of the t-norm for different representative cases.

5.2 Starting point: Quasi-metric lattice

In [7] we tackled the problem of detecting data access patterns with several degrees of locality using a quasi-metric lattice.

For each $x \in X$, denote by $k(x)$ the number of *uses* of x in $[0, T]$, where T is the instant of time when we want to predict x's value reliability and x is an object in an information system such as a replicated database object, for instance (see [33] for "eager" replicated consistency protocols and [31] for a "lazy" approach).

Now, for each $x \in X$ with $k(x) > 0$, we construct a function also denoted by x, from $[0, T]$ into $\mathbb{N} \cup \{0\}$ as follows:

$x(t) = 0$, if $t = 0$;

$x(t) = 1$, if $0 < t \leq t_{1(x)}$;

$x(t) = 2$, if $t_{1(x)} < t \leq t_{2(x)}$;

....

$x(t) = k(x) - 1$, if $t_{(k(x)-2)(x)} < t \leq t_{(k(x)-1)(x)}$;

$x(t) = k(x)$, if $t_{(k(x)-1)(x)} < t \leq T$.

If $k(x) = 0$, we define $x : [0, T] \to [0, 1]$ by

$x(t) = 0$, if $0 \leq t \leq T$.

$x(t)$ represents object x history of accesses during time.

Example 5.2.1. *If $T = 4$ and $x \in X$ such as $k(x) = 3$, with $t_{1(x)} = 0.5$, $t_{2(x)} = 3$, and $t_{3(x)} = t_3 = 3.5$, then:*

$x(0) = 0$;

$x(t) = 1$, *if* $0 < t \leq 0.5$;

$x(t) = 2$, *if* $0.5 < t \leq 3$, *and*

$x(t) = 3$, *if* $3 < t \leq 4$.

Remark. *Each history starts on instant 0 as it is relative to the first access to the object. This fact allows us to compare whichever two different objects regardless whether they are being concurrently accessed or not.*

Notice how $x(t)$ definition relates to the concept of object version (which allows us to decide whether an object is updated or outdated).

We are interested in constructing a function v on X such that $v(x)$ provides a sufficiently satisfactory value of possibility of element usage according to locality access patterns. The condition formulated in Definition 5.2.1 below yields a suitable tool to model this behaviour, as we will see.

Definition 5.2.1. *Let X be a (nonempty) set and let v be a function from X into [0,1]. We say that v satisfies the Proximity and Frequency Condition if for each pair $x, y \in X$ satisfying $0 < k(y) \leq k(x)$ the following holds (we assume $k(x)(x) = k(x)$ for all $x \in X$):*

(PFC) $v(x) \leq v(y)$ whenever $t_{j(y)} \leq t_{(j+k(x)-k(y))(x)}$ for all $j \in \{1, ..., k(y)\}$, and $v(x) < v(y)$ if, in addition, there is $h \in \{1, ..., k(y)\}$ such that $t_{h(y)} < t_{(h+k(x)-k(y))(x)}$.

i.e., v allows us to compare two elements histories in a way that if the second element history adds closer to T accesses in between the first element history then v value for the second element is greater than for the first element.

A relatively easy function which is a suitable candidate to provide an efficient model in this context is the function $v : X \rightarrow [0,1]$ defined as follows: $v(x) = 0$ if $k(x) = 0$, and

$$v(x) = \sum_{j=1}^{k(x)} 2^{-j} \frac{t_{(k(x)-(j-1))(x)}}{T}$$

whenever $k(x) > 0$.

Example 5.2.2. *Let $T = 4$, and x as defined in Example 5.2.1. Moreover, let $y \in X$ such that $k(y) = 3$, with $t_{1(y)} = 1$, $t_{2(y)} = 3$, and $t_{3(y)} = 3.5$. Then:*

$$v(x) = \frac{1}{4}\left[\frac{0.5}{2^3} + \frac{3}{2^2} + \frac{3.5}{2}\right] = \frac{41}{64}, \quad \textit{and} \quad v(y) = \frac{1}{4}\left[\frac{1}{2^3} + \frac{3}{2^2} + \frac{3.5}{2}\right] = \frac{21}{32}.$$

$v(y) > v(x)$, as specified in the Proximity and Frequency Condition (Definition 5.2.1).

Proposition 5.2.1. *The function v as defined above satisfies the Proximity and Frequency Condition (Definition 5.2.1).*

Proof:

Let $x, y \in X$ satisfying $0 < k(y) \leq k(x)$. Suppose that $t_{j(y)} \leq t_{(j+k(x)-k(y))(x)}$ for all $j \in \{1, ..., k(y)\}$. Then $t_{(k(y)-(j-1))(y)} \leq t_{(k(x)-(j-1))(x)}$ for all $j \in \{1, ..., k(y)\}$. Since

$$\begin{aligned} v(x) - v(y) &= \frac{1}{T}\sum_{j=1}^{k(y)} 2^{-j}\left[t_{(k(x)-(j-1))(x)} - t_{(k(y)-(j-1))(y)}\right] \\ &+ \frac{1}{T}\sum_{j=k(y)+1}^{k(x)} 2^{-j} t_{(k(x)-(j-1))(x)}, \end{aligned}$$

it follows that $v(x) - v(y) \geq 0$.

Finally, it is clear that if, in addition, $t_{h(y)} \leq t_{(h+k(x)-k(y))(x)}$ for some $h \in \{1, ..., k(y)\}$, then $v(x) - v(y) > 0$. This concludes the proof.

□

Some of the aspects we took into account when choosing function v are:

(i) v is bounded. In fact $0 \leq v(x) \leq 1$, for all $x \in X$.

(ii) v is expressed in negative powers of 2, which makes its calculation fast in computers as these operations in binary are performed using bit shifting.

(iii) Each term calculation is useful for a new access to the element (an increment of $k(x)$).

(iv) If working with distributed systems, calculations are most likely to be local to a single node. No coordination is needed for v calculation, but only for its results possible consequences.

From the point of view of the application of our techniques, it is important for the calculus of the predictions to be fast and also adaptive to to multiple kinds of system usages.

The following sections explain how the quasi-metric lattice offers an adequate framework to explain the pattern accesses properties by grouping objects in classes $[x] = \{y \in X : v(x) = v(y)\}$ in a way that if we compare two classes $[x] \sqsubseteq [y] \iff v(x) \leq v(y)$, then $\widetilde{X} := \{[x] : x \in X\}$ admits a lattice structure and $(\widetilde{X}, d, \sqsubseteq)$ is a quasi-metric lattice.

5.2.1 Quasi-metric lattice framework

In this section we shall prove that the notion of a quasi-metric lattice provides an appropriate framework to explain the properties of a pattern access.

To this end, we first construct a binary relation R on X as follows:

$$xRy \iff v(x) = v(y).$$

Clearly R is an equivalence relation. Denote by $[x]$ the class of $x \in X$, i.e.

$$[x] = \{y \in X : v(x) = v(y)\},$$

Let $\widetilde{X} := \{[x] : x \in X\}$ be the set of all classes. We shall show that $\widetilde{X}$ can be endowed with the structure of a lattice.

Indeed, given $x, y \in X$, define

$$[x] \sqsubseteq [y] \iff v(x) \leq v(y).$$

As usual, we write $[x] \sqsubset [y]$ if $[x] \sqsubseteq [y]$ but $[x] \neq [y]$, i.e.

$$[x] \sqsubset [y] \iff v(x) < v(y).$$

Next we observe that $(\widetilde{X}, \sqsubseteq)$ is totally ordered:

Reflexivity: for each $x \in X$ we have $[x] \sqsubseteq [x]$ because $v(x) \leq v(x)$.

Antisymmetry: if $\{[x] \sqsubseteq [y]$ and $[y] \sqsubseteq [x]$, then $v(x) = v(y)$, so $[x] = [y]$.

Transitivity: if $[x] \sqsubseteq [y]$ and $[y] \sqsubseteq [z]$, then $v(x) \leq v(y) \leq v(z)$, so $[x] \sqsubseteq [z]$.

Thus $\sqsubseteq$ is an order on $\widetilde{X}$. Moreover, given $x, y \in X$, it is clear that $[x] \sqsubseteq [y]$ or $[y] \sqsubseteq [x]$ because $v(x) \leq v(y)$ or $v(y) \leq v(x)$.

We conclude that $(\widetilde{X}, \sqsubseteq)$ is totally ordered, and hence $(\widetilde{X}, \sqcup, \sqcap)$ is a lattice, where, as usual, if $[x] \sqsubset [y]$, we define $[x] \sqcup [y] = [y]$ and $[x] \sqcap [y] = [x]$.

Now, in a natural way, we define a function $V : \widetilde{X} \to [0, 1]$, representing the prediction value for a class of objects, by

$$V([x]) = v(x),$$

and thus we may define a function $d : \widetilde{X} \times \widetilde{X} \to [0, 1]$, in order to measure the difference between two classes of objects, by

$$d([x], [y]) = \max \{V([x]) - V([y]), 0\}.$$

We check that d is a quasi-metric on $\widetilde{X}$. Indeed:

$$d([x], [y]) = d([y], [x]) = 0 \Longleftrightarrow V([x]) = V([y])$$

$$\Longleftrightarrow v(x) = v(y) \Longleftrightarrow [x] = [y].$$

and

$$d([x], [y]) = \max \{V([x]) - V([y]), 0\}$$

$$= \max \{V([x]) - V([z]) + V([z]) - V([y]), 0\}$$

$$\leq \max \{V([x]) - V([z]), 0\} + \max \{V([z]) - V([y]), 0\}$$

$$= d([x], [z]) + d([z], [y]).$$

Furthermore, for each $x, y, z \in X$, we have:

$d([x] \sqcup [z], [y] \sqcup [z]) = \max\{(V([x]) \sqcup V([z]) - (V([y]) \sqcup V([z]), 0\}$

$\leq \max\{V([x]) - V([y]), 0\} = d(x, y),$

and

$d([x] \sqcap [z], [y] \sqcap [z]) = \max\{(V([x]) \sqcap V([z]) - (V([y]) \sqcap V([z]), 0\}$

$\leq \max\{V([x]) - V([y]), 0\} = d(x, y).$

We have shown that $(\widetilde{X}, d, \sqsubseteq)$ is a quasi-metric lattice.

Note that

$$d([x],[y]) = 0 \Longleftrightarrow [x] \sqsubseteq [y] \Longleftrightarrow v(x) \leq v(y),$$

Hence the order $\sqsubseteq$ coincides with the order $\leq_d$ induced by d on $\widetilde{X}$.

Moreover, for $[x] \neq [y]$, condition $d([x],[y]) = 0$ is equivalent to $v(x) < v(y)$, so this condition indicates the existence of an increase in the possibility of use when x is replaced by y.

Notice that this important assertion would not be obtained if one considered the metric D given by the Euclidean distance: $D([x],[y]) = |V([x]) - V([y])|$, because in this case $D([x],[y]) = 0 \Leftrightarrow xRy$ and this is true only when $[x] = [y]$. Thus the metric would provide less information than the quasi-metric.

We also observe that if $v(y) \to v(x)$, then $V([y]) \to V([x])$, and consequently

$$d([x],[y]) \to 0 \quad \text{and} \quad d([y],[x]) \to 0.$$

In this direction, it is illustrative to compute $d([x],[y])$ in some interesting particular cases.

For instance, put $X_1 = \{x \in X : k(x) = 1\}$, and suppose that there is $x_0 \in X_1$ with $t_{1(x_0)} = T$. Then $v(x_0) = 1/2$. Since for each $y \in X_1$ we have

$v(y) = t_{1(y)}/2T$, it follows that $v(y) \leq v(x_0)$, so

$$d([y],[x_0]) = 0,$$

for all $y \in X_1$. On the other hand, for each $y \in X_1$ such that $t_{1(y)} < T$, we obtain:

$$d([x_0],[y]) = V([x_0]) - V([y]) = v(x_0) - v(y) = \frac{1}{2}\left(1 - \frac{t_{1(y)}}{T}\right).$$

In particular, if $t_{1(y)} \to T$, it follows that

$$d([x_0],[y]) \to 0.$$

Now suppose that there is $x_1 \in X_1$ with $t_{1(x_1)} = T/2$. Then $v(x_1) = 1/4$. So for each $y \in X_1$ such that $t_{1(y)} > T/2$, we obtain $v(y) > v(x_1)$, and thus

$$d([x_1],[y]) = 0, \quad \text{and} \quad d([y],[x_1]) = \frac{1}{2}\left(\frac{t_{1(y)}}{T} - \frac{1}{2}\right).$$

Consequently, if $t_{1(y)} \to T/2$, it follows that

$$d([y],[x_1]) \to 0.$$

5.2.2 Empirical Results

In this section, graphical representation of empirical tests show that our selection of v provides a suitable setting to our study.

Results are organized in *consecutive* and *uniform* accesses to a given object during a period of time ranging from 0 to T. The first cases show the behaviour of v when accesses are performed consecutively one after another during certain periods of time. The case of uniform accesses shows the behaviour of v when an object is accessed with a given periodicity.

In all figures the y-axis shows $v(x)$ values. For the consecutive accesses experiments, the x-axis shows the total amount of accesses (*steps*) performed over the object. Notice that for the uniform accesses experiments, the x-axis shows the separation between uses of the object.

Consecutive accesses far from T

Figure 5.1 shows the behaviour of $v(x)$ when there are consecutive accesses far from the measuring time T. This situation is achieved performing accesses to x as soon as the experiment starts and models a situation where an object was oftenly used but it is not used anymore.

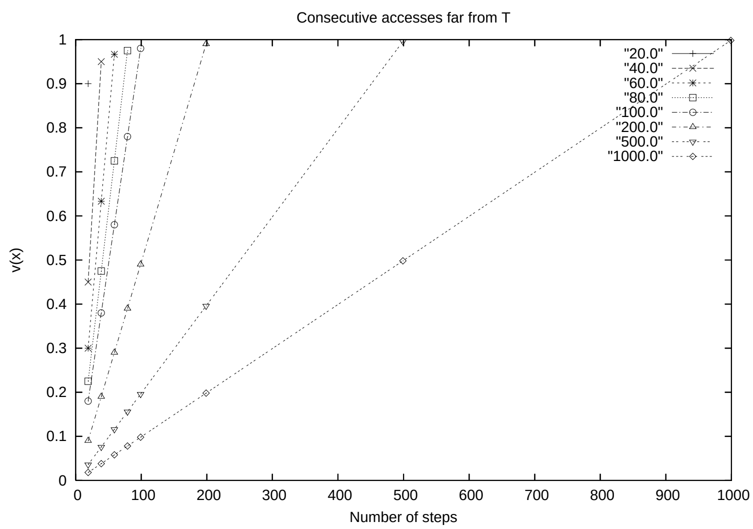

Figure 5.1: Consecutive far-from-T accesses for fixed T values.

Each line in the figure connects several experiments for different number of accesses (20, 40, etc.) to an object for fixed T values (20, 40, 60, etc.). For example, for a T of 500 its line contains the union of points representing $v(x)$ values for 19 accesses (from 1 to 19), 39 accesses (from 1 to 39), 59 accesses (from 1 to 59) and so on.

It can be seen that $v(x)$ grows linearly when more accesses are performed. This happens when the distance between T and the group of accesses gets smaller. The more accesses, the closer the distance between the last one and T is.

These results show that our function discards in a natural way too old history values. The meaning of "too old" is not tuned selecting a window of $x(t)$ values but a choosing a convenient threshold that discriminates from updated to outdated appropriately.

Consecutive accesses centered at $T/2$

Figure 5.2 shows the behaviour of $v(x)$ when there are consecutive accesses centered at the middle of the interval $[0, T]$. This is, for a T of 200, 40 accesses would happen between the moment 80 and 120.

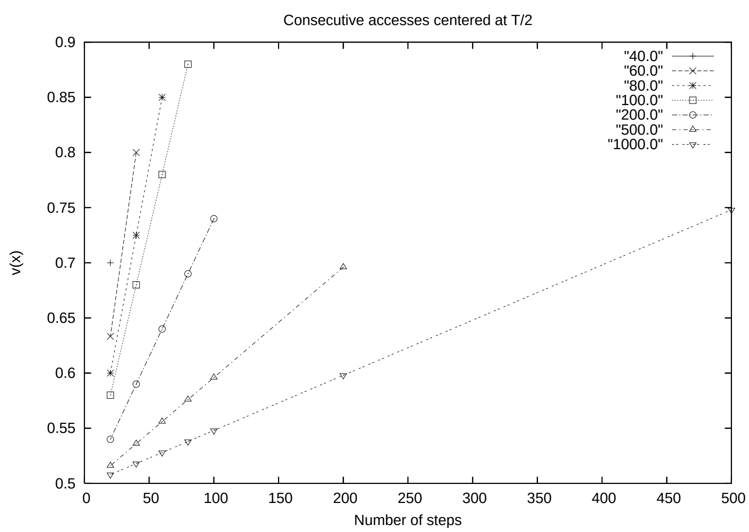

Figure 5.2: Consecutive centered between 0 and T accesses for fixed T values.

Again, each line of the figure holds values for different T tests. If we compare this figure with the previous one (figure 5.1), now $v(x)$ produces bigger values. See, for example, for $T = 1000$ we obtain a value close to 0.75 for 500 accesses while in Figure 5.1 we obtain a value close to 0.5. While the number of accesses is the same in this case, accesses centered at the middle of the studied history are closer to T.

This test shows that the weight of closer to T accesses is important and it shows that this importance decreases the further we measure from T.

Consecutive accesses near from T

Figure 5.3 shows, compared to Figure 5.2 and even more to Figure 5.1, how much close-to-T accesses influence $v(x)$ value. Again, lines group values for a given T. For example, for a T of 100, the line points represent 19 accesses (from 80 to 99), 39 accesses (from 60 to 99), 59 accesses (from 40 to 99) and 80 accesses (from 20 to 99).

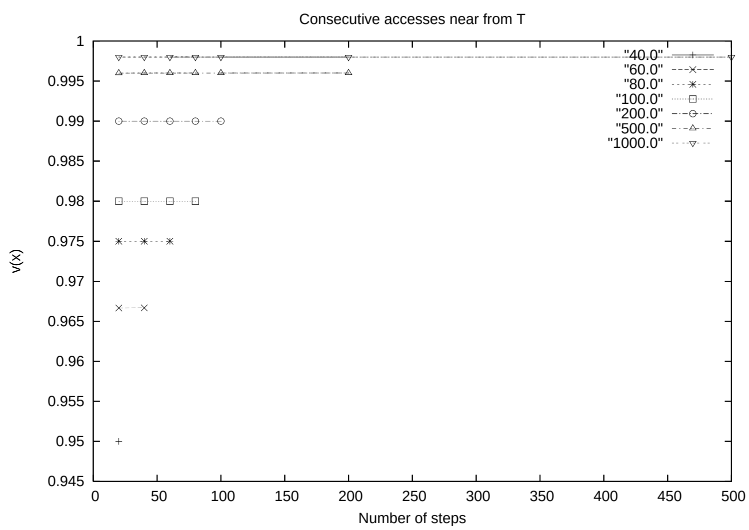

Figure 5.3: Consecutive near-from-T accesses for fixed T values.

Notice that $v(x)$ values are all over 0.9, a value big enough to be over a reliability decision threshold. The figure shows horizontal lines for each T experiment only because of the graphical representation precision but values are slightly bigger (differences appear at the third decimal value) when more accesses are performed.

Uniformly scattered accesses

Figure 5.4 represents v values for an object that has been accessed regularly with a given periodicity. The x-axis represents the separation between accesses: notice that a separation of 50 requires 19 accesses for a T of 1000.

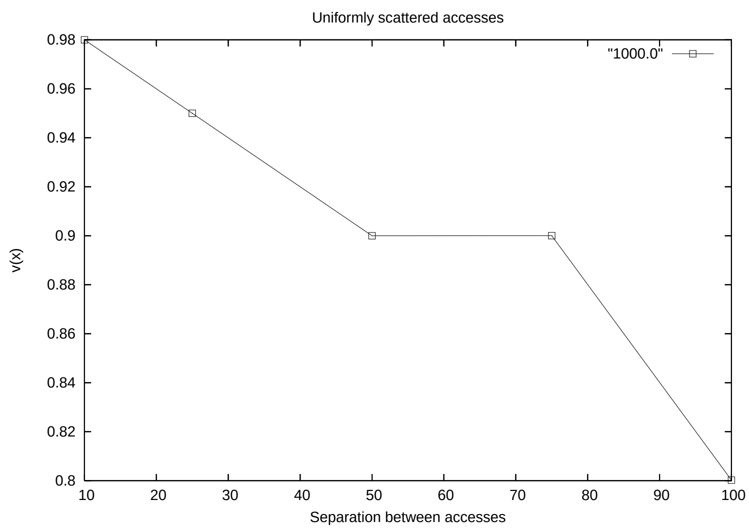

Figure 5.4: Periodical accesses for fixed T values.

We have chosen this figure from our set of experiments because it shows an interesting situation. Initially we would expect a strictly decreasing line but Figure 5.4 displays a different result:

> $v(x)$ values are high but decrease *almost* linearly. For 75-wide steps, $v(x) = 0.9000091552734375$ with the last step 25 units of time away from T, while $v(x) = 0.9000000953674316$ for 50-wide steps with a distance of 50 units.

Due to this fact, for that case, the function grows instead of decreasing: We cannot say that the smaller the step is, the closer the last step will be from T. Due to the way that $v(x)$ weights t values, for close-to-T cases, the distance from the last accesses to T is more determinant than the number of accesses.

Another thing that can be noticed is that $v(x)$ values are all high (in this experiment the lowest result is 0.8001953125). Uniform accesses are good cases in general as they usually represent batch processes such as nightly maintenance operations, monthly reports, etc. that can be easily identified. Furthermore, if an object is regularly accessed it is surely updated.

The effect of T

In the following figures, 5.5 and 5.6, each function represents different values for different T cases. While the previous figures show values for a fixed T, these two figures use a fixed number of accesses and a variable set of T: In Figure 5.5 the number of consecutive accesses is set to 20 and in Figure 5.6 it is set to 80.

For these number of accesses, we test different T values for the three scenarios that have been studied for consecutive accesses: accesses near from T (line labeled as *final*), far from T (line labeled as *initial*) and centered in the middle of the interval $[0, T]$ (line labeled as *middle*).

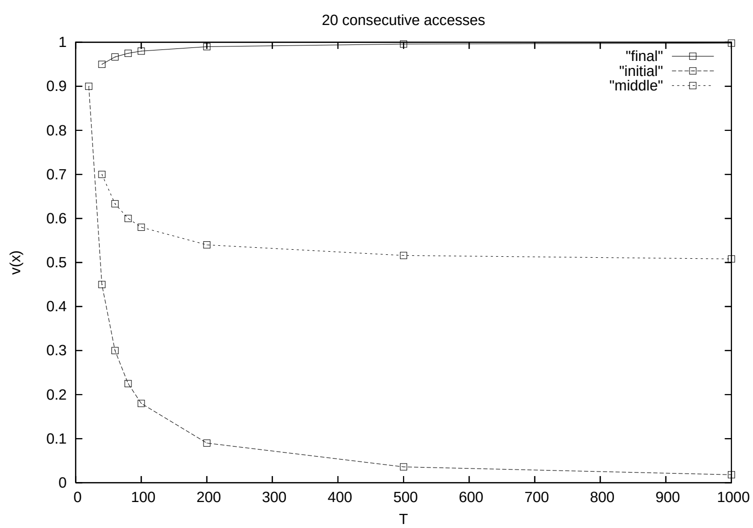

Figure 5.5: 20 Consecutive accesses for several T values.

It can be seen that the further the accesses are from T, the smaller $v(x)$ gets. This results in decreasing functions when accesses are far. When accesses are close, the t/T terms are bigger each time and easily counteract the

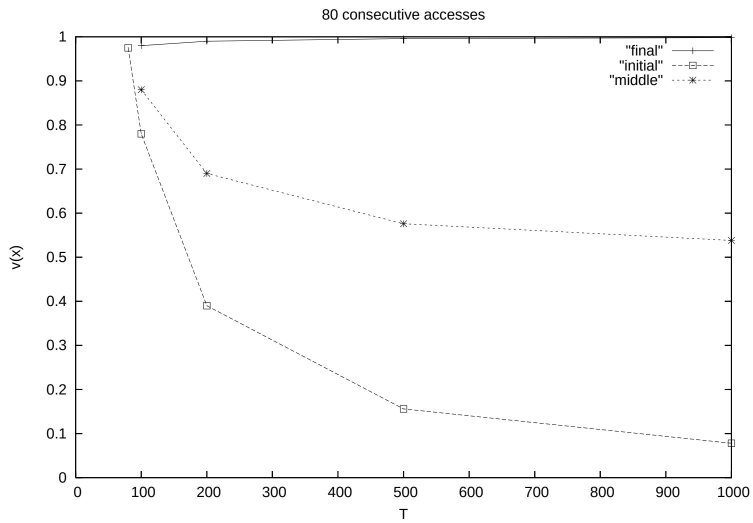

Figure 5.6: 80 Consecutive accesses for several T values.

exponential term. This explains why scenario changes (from further accesses to closer ones) result in bigger values for $v(x)$ gradually.

5.2.3 Conclusions

The main conclusion from this approach is the fact that optimizing access prediction in information systems based on accesses locality can be conceived by our quasi-metric lattice mathematical framework. We consider v values as metadata because this information is not related to the system application itself, but to the mathematical framework. Thus, we can obtain metadata v associated with each object by considering access patterns $[x]$.

This collected metadata v will be able to foretell with a high confidence whether the object is up-to-date and the initial data access is right. If that is the case few operations should be carried previous to the delivery of correct data to the users. On the contrary, if an object is out-dated, some additional operations should be performed to obtain the up-to-date version and the effective access will be slower. This procedure is called a consistency protocol.

A particular case of access locality is the case of consistency protocols for replicated databases. In [31] statistics are used to minimize the transactions rate. Authors achieve metadata collection using timestamps when reading objects ahead and a formula calculation is performed in order to update possibly out-dated objects. We endow the set of access patterns, $\widetilde{X}$, with a lattice framework, and a quasi-metric on this set is defined.
We have noticed too that over a certain value $v(x)$ is big but differences between different tests are insignificant. Moreover, if $v(x)$ is over a certain threshold then can improve the performance of our protocol and reduce both the required computation and storage needs.

Our next approach is to try to construct on the sets of the form $[0, T]$ an appropriate fuzzy valuation in order to provide information systems with more accurate think ahead capabilities.

5.3 Model extension in time: Fuzzy metric space

As a natural continuation of the initial study we pretend to take advantage of the intermediate elements' accesses values. For this means we will base our study on the Kramosil and Michalek definition of a fuzzy metric space, see 2.2.3.

In our model, for two consecutive accesses t_{prev} and t_{next} of an element x, we will have that if $t \in (t_{prev}, t_{next}]$ then $v(x, t) = v(x, t_{next})$. In particular, $v(x, t) = v(x, t_{1(x)})$ if $t \in (0, t_{1(x)}]$. This means that we can ensure left continuity (by taking $v(x, t) = 1$ whenever $t > T$) according to condition (KM5) from 2.2.3.

While in our initial approach we chose $k(x)$ as the number of *uses* between

0 and T of an element x, and the instant of time when the prediction is estimated, now we choose $t \in (t_{(i-1)(x)}, t_{i(x)}]$, and the number of computed accesses $k(x,t)$ are those happened until the instant $t_{i(x)}$ (we assume $t_{0(x)} = 0$), in such a way that if $t_{k(x)} < T$, we define $k(x,t) = k(x)$ for $t \in (t_{k(x)}, T]$.

Consequently we put $k(x,t) = i$ if $t \in (t_{(i-1)(x)}, t_{i(x)}]$ with $i < k(x)$, and $k(x,t) = k(x)$ if $t \in (t_{(k-1)(x)}, T]$, and then we define a function $v : X \times [0,\infty) \to [0,1]$ by $v(x,0) = 0$,

$$v(x,t) = \sum_{j=1}^{k(x,t)} 2^{-j} \frac{t_{(k(x,t)-(j-1))(x)}}{T},$$

if $0 < t \leq T$, and $v(x,t) = 1$ if $t > T$.

Note that, in particular,$v(x,t) = v(x)$ for $t \in (t_{(k-1)(x)}, T]$.

Moreover, it is clear that for each $x \in X$, $v(x\_)$ is left continuous and nondecreasing (i.e., $v(x,t) \leq v(x,s)$ whenever $t \leq s$).

The fact that now v depends on x and on t is the basis to represent fuzzy behaviour.

These early prediction capabilities can be used by the affected information system in order to modify its behaviour based on its experience handling previously accessed elements in a preventive fashion.

We shall suppose that $v(x,T)$ offers a "reasonable" value of certainty of element x state. Then we can compare $v(x,t)$ and $v(y,t)$ and if they show similar values then we can try to advance the prediction of y's class, which will most possibly be the same as x's, i.e. $[y] = [x]$.

This comparison will be represented by a fuzzy metric space $(X, M, *)$ where M_* is defined by $M_*(x,y,t) = v(x,t) * v(y,t)$.

Actually, we can deduce this fact for a more general result.

Proposition 5.3.1. *Let X be a nonempty set, $v : X \times [0,\infty) \to [0,1)$ be a function such that for each $x \in X$ $v(x,\_)$ is left continuous and nondecreasing and $*$ be a t-norm. Then $(X, M_*, *)$ is a fuzzy metric space where M_* is defined by:*

(a) $M_*(x, y, 0) = 0$,

(b) $M_*(x, x, t) = 1$ *for each* $t > 0$ *and each* $x \in X$.

(c) $M_*(x, y, t) = v(x, t) * v(y, t)$ *for each* $t > 0$ *and each* $x, y \in X$ *with* $x \neq y$.

Proof:

Condition (KM1) in Definition 2.2.1 follows from (a).

On the other hand, since for each $x, y \in X$ with $x \neq y$, and each $t > 0$,

$$M_*(x, y, t) = v(x, t) * v(y, t) = v(y, t) * v(x, t) = M_*(y, x, t),$$

we deduce that condition (KM5) in Definition 2.2.3 also holds.

Now suppose that $M_*(x, y, t) = 1$ for each $t > 0$. If $x \neq y$ we obtain

$$M_*(x, y, t) = v(x, t) * v(y, t) \leq v(x, t) \wedge v(y, t) < 1,$$

a contradiction. Therefore condition (KM2') in Definition 2.2.2 holds.

Now let x, y, z three different elements in X and $t, s > 0$. Then

$$\begin{aligned} M_*(x, z, t+s) &= v(x, t+s) * v(z, t+s) \geq v(x, t) * v(z, t) \\ &\geq v(x, t) * v(y, t) * v(z, s) * v(y, s) = M_*(x, y, t) * M_*(z, y, s), \end{aligned}$$

so condition (KM3) in Definition 2.2.1 is satisfied.

Finally, for each $x, y \in X$, $M_*(x, y, \_) : [0, \infty) \to [0, 1]$ is left continuous because $v(x, \_)$ and $v(y, \_)$ are left continuous and $*$ is continuous.

We have shown that $(X, M_*, *)$ is a fuzzy metric space.

□

The preceding result will be used in following sections to deduce empirical results based on experiments using a variety of continuous t-norms and t-norm families.

5.3.1 Evaluated t-norms

We have considered different continuous t-norms (check Dubois-Prade [13]) and compared them according to their results with the fuzzy metric in $v(x,t)$:

- Minimum: $x \wedge y := \min(x,y) := \begin{cases} x & \text{if} \quad x \leq y \\ y & \text{if} \quad y < x \end{cases}$.
- Product: $x \ Prod \ y := \prod(x,y) := xy$.
- Lukasiewicz: $x *_L y := \mathrm{W}(x,y) := \max\{x+y-1,0\}$.
- t-norm families: different parameter values will be compared.
  - Frank family: $log_s\left(1+\dfrac{(s^x-1)(s^y-1)}{s-1}\right)$ where $s>0, s\neq 1$.
  - Hamacher family: $\dfrac{xy}{\alpha+(1-\alpha)(x+y-xy)}$ where $\alpha \geq 0$.
  - Sugeno-Weber family: $\max\left\{\dfrac{x+y-1+\lambda xy}{1+\lambda},0\right\}$ where $\lambda > -1$.
  - Schweizer-Sklar family: $(\max\{x^{-p}+y^{-p}-1,0\})^{-1/p}$.
  - Yager family: $\max\{1-((1-x)^p+(1-y)^p)^{1/p},0\}$ where $p \in (0,\infty)$.
  - Dombi family: $\dfrac{1}{1+((\frac{1-x}{x})^\lambda+(\frac{1-y}{y})^\lambda)^{1/\lambda}}$ where $\lambda \in (0,\infty)$.
  - Dubois-Prade family: $\dfrac{xy}{max(x,y,\gamma)}$ where $\gamma \in [0,1]$.

All of them except the Dubois-Prade family are archimedean t-norms (see Definition 2.1.11).

Comparisons using continuous t-norm families allow us to tune the predictions precision (check related figures to see how the parameter affects the evaluation).

T-norms ordering and properties

In [6] we showed empirical results based on experiments using a variety of continuous t-norms and t-norm families.

After a straightforward evaluation of boundary conditions, we can order the t-norms:

- $\min(x, y) \geq$ Dubois-Prade$(x, y) \geq \prod(x, y) \geq$ W(x, y).
- $\min(x, y) \geq \prod(x, y) \geq$ Sugeno-Weber$_{>0}(x, y) \geq$ W$(x, y) \geq$ Sugeno-Weber$_{<0}(x, y)$.
- Hamacher$_{<1}(x, y) \geq \prod(x, y)$.
- The order of Hamacher$_{<1}(x, y)$ and Dubois-Prade(x, y) depends on the parameters values. For $\alpha, \gamma \in [0, 1)$:
  - If $\alpha + (1 - \alpha)(x + y - xy) < \max(x, y, \gamma)$ then
  $$\text{Dubois-Prade}(x, y) \geq \text{Hamacher}_{<1}(x, y).$$
  - Otherwise:
  $$\text{Hamacher}_{<1}(x, y) \geq \text{Dubois-Prade}(x, y).$$
- As it happens with $\lambda < 0$ for Sugeno-Weber(x, y), for "big" values of α of Hamacher(x, y) and depending on $v(x)$ and $v(y)$ values, the t-norm can be smaller than W(x, y).

Comparisons using continuous t-norm families allow us to tune the predictions precision (check related figures to see how the parameter affects the evaluation).

If observed carefully, figures in Section 5.3.2 show that for a moment t close to T, the values of $M_{\cdot}(x, y, t)$ are "close" to the values of $M_{*_L}(x, y, t)$, where $\cdot$ and $*_L$ denote the t-norm product and the Lukasiewicz t-norm respectively. The analytical explanation of this fact is the following:

$$\begin{aligned} & \lim_{v(x,t),v(y,t)\to 1} (M_{\cdot}(x,y,t) - M_{*_L}(x,y,t)) \\ = & \lim_{v(x,t),v(y,t)\to 1} (v(x,t)\cdot v(y,t) - (v(x,t)+v(y,t)-1)) \\ = & \lim_{v(x,t),v(y,t)\to 1} ((v(x,t)-1)(v(y,t)-1)) = 0. \end{aligned}$$

5.3.2 Empirical results: Comparison

We evaluate v in $[0,T]$ and we arbitrarily set $T = 1000$ to allow the prediction to range from no uses to plenty of them. t instants have been chosen uniformly scattered through the interval.

Our tests are based on comparisons of v values during $[0,T]$ for two different objects x, y (fuzzy set elements) using continuous t-norms. These differences are achieved applying localized variations in the first object x to obtain y. That is how we model element accesses with degrees of closeness as it happens in systems with strong locality components.

Computing the variation of y is a simple implementation of the Proximity and Frequency Condition (Definition 5.2.1). We have tried three kinds of variations as we display in the following subsections.

Random variations – Random

Figures 5.7, 5.8 and 5.9 show n additional accesses to element y which are performed randomly through the interval $[0,T]$.

Figure 5.8 shows $v(x,t)$ and $v(y,t)$ values: The final value for both experiments is high because accesses are performed throughout the end of the study for both elements. Figure 5.9 shows the fuzzy metric results for the minimum, product and Lukasiewicz t-norms. Figures 5.10, 5.11, 5.12 are examples of the results obtained for the t-norm families.

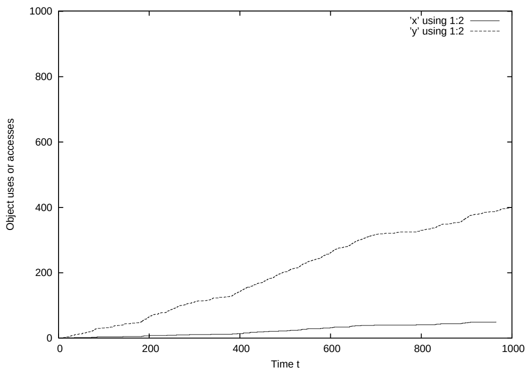

Figure 5.7: Random histories.

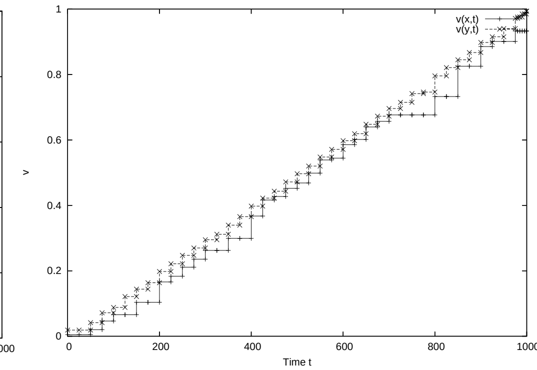

Figure 5.8: Random: $v(x)$ and $v(y)$.

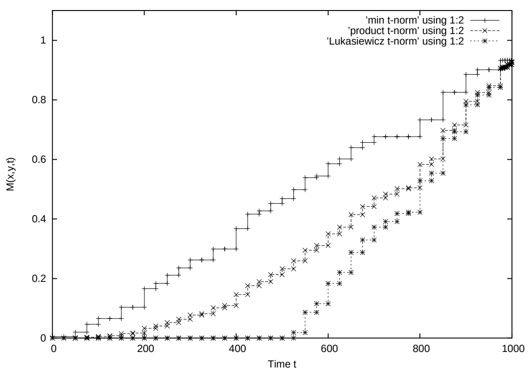

Figure 5.9: Random: M using minimum, product, Lukasiewicz.

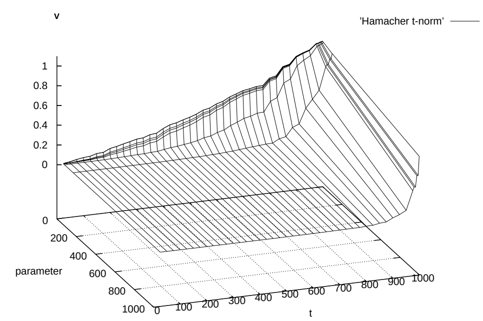

Figure 5.10: Random: M using Hamacher.

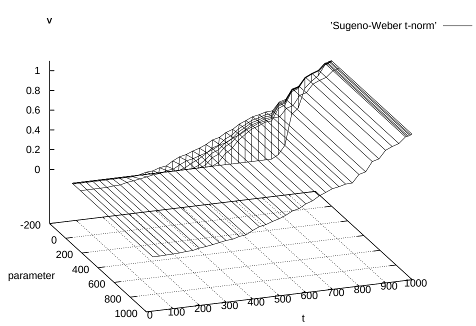

Figure 5.11: Random: M using Sugeno-Weber.

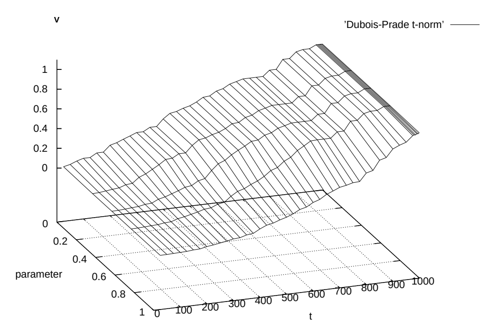

Figure 5.12: Random: M using Dubois-Prade.

For the case of the Hamacher family, there is no abrupt change in the family behaviour when we introduce changes in the parameter α value. For close to 1 values, Hamacher family gets close to the product results because as the parameter is placed in at the denominator, α increments mean that M values will decrease. As there is no upper bound for α this family allows us to get lower values than the ones obtained using the Lukasiewicz t-norm for the metric construction.

For the Sugeno-Weber t-norm with λ values in between -1 and 0 the t-norm results converge to the Lukasiewicz ones and for λ values greater than 0 results converge to the product ones.

For the Dubois-Prade results, γ increments imply a progressive decrement of M.

Left-random variations – Left-Random (L-Random)

In this set of experiments, the n additional accesses appear randomly close to the beginning of the history. It can be seen how the behaviour of the metric is similar to the one found in the previous case with completely random variations. This is due to the fact that early accesses have less impact on the final result and, in the end, the accesses distribution is quite similar to the random case.

See Figures 5.13, 5.14 and 5.15 for the data accesses distribution and v and M evaluations. Figures 5.16, 5.17, 5.18 are examples of the results obtained for M built using three different t-norm families.

Right-random variations – Right-Random (R-Random)

Now the n additional accesses appear randomly close to the end of the history. As soon as the histories start diverging the metrics behaviour is similar to the previous experiments. In these experiments we can see how equal histories

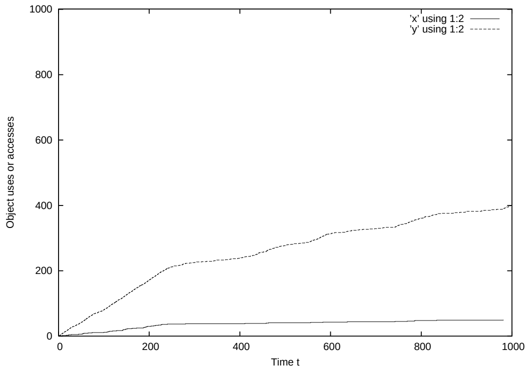

Figure 5.13: Left-Random accesses.

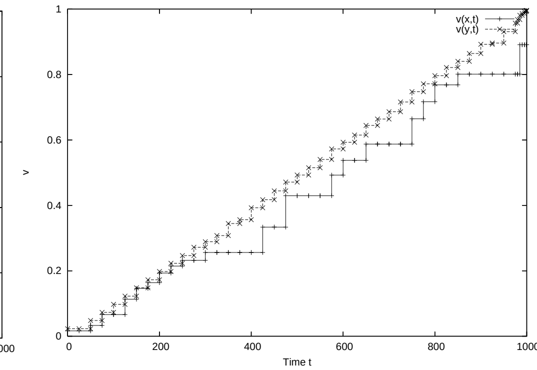

Figure 5.14: L-Random: $v(x)$, $v(y)$.

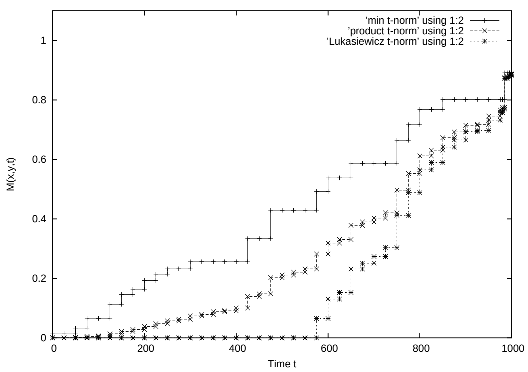

Figure 5.15: Left-Random: M using minimum, product and Lukasiewicz.

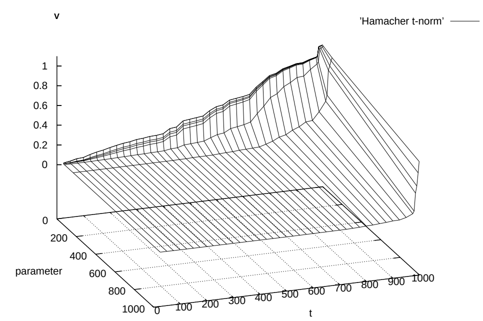

Figure 5.16: Left-Random: M using Hamacher.

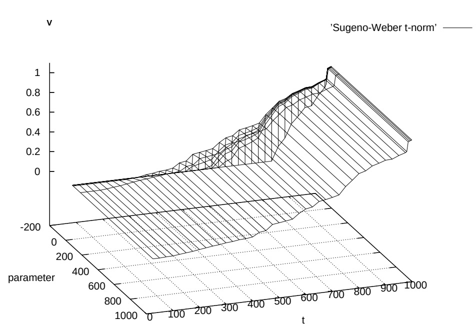

Figure 5.17: Left-Random: M using Sugeno-Weber.

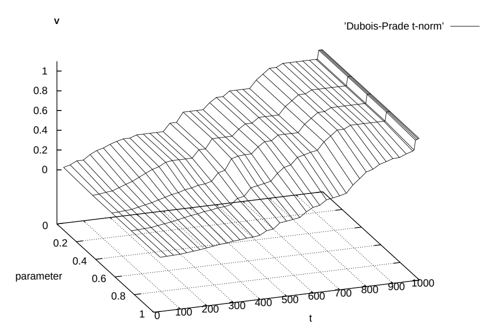

Figure 5.18: Left-Random: M using Dubois-Prade.

are perfectly identified as it happens simply with v evaluations but, as in all the experiments, M evaluation allows a richer identification of differences.

See Figures 5.19, 5.20 and 5.21 for the distribution of accesses and Figures 5.22, 5.23, 5.24 as representative examples of the results obtained for different fuzzy metrics built using different t-norm families.

Two independent histories – Opposite histories (Opposite)

One of them has accesses performed mainly at the beginning of the experiment and the other has them at the end. Experiments can be observed in Figures 5.25, 5.26 and 5.27. Figures 5.28, 5.29, 5.30 are examples of the results obtained for the t-norm families.

In all previous experiment groups, final values for M were big enough to consider that the element was likely to be accessed again soon. The reason for this constant behaviour is the fact that there were always close to T accesses. The experiment with two independent histories shows a different case and M evaluation does not seem to imply a prompt access happening. Even though the prediction is different, it can be noticed that the t-norms comparison is still valid.

Additional t-norms

Next, we show in a separate group of figures families Schweizer-Sklar (Figures 5.43, 5.44, 5.45, 5.46), Dombi (Figures 5.31, 5.32, 5.33, 5.34), Frank (Figures 5.35, 5.36, 5.37, 5.38) or Yager (Figures 5.39, 5.40, 5.41, 5.42).

They require costly calculations which do not easily apply to our fast prediction intentions and neither they introduce any characteristic which could not be found in the faster families or the traditional t-norms. While they may prove to be very useful for other application scenarios (for instance non real-time systems) we have discarded them due to their complexity.

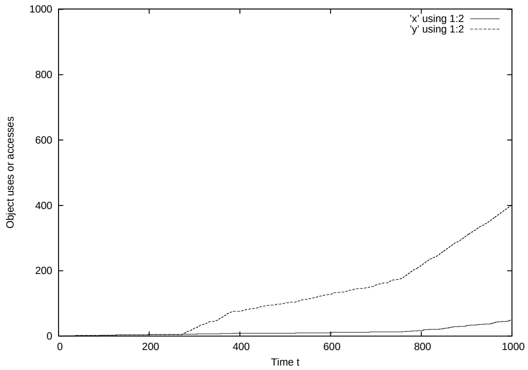

Figure 5.19: Right-random accesses.

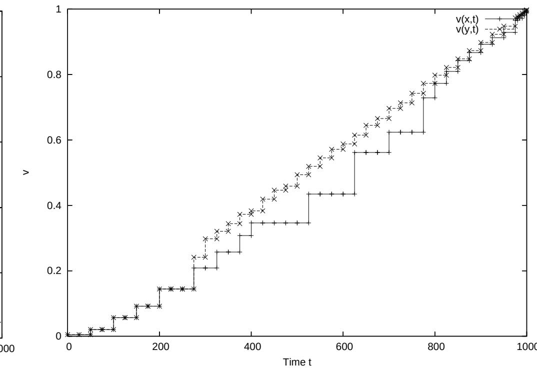

Figure 5.20: R-random: $v(x)$, $v(y)$.

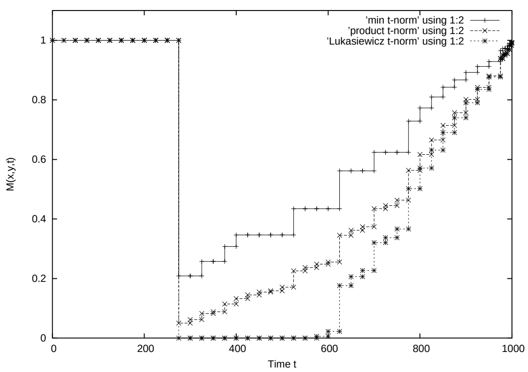

Figure 5.21: Right-random: M using minimum, product, Lukasiewicz.

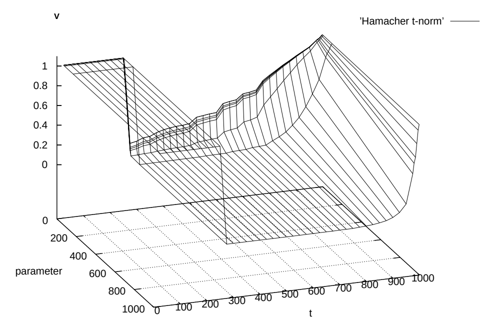

Figure 5.22: Right-random: M using Hamacher.

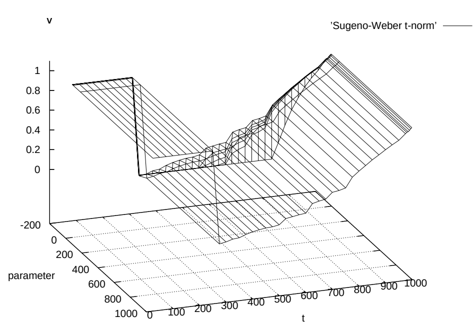

Figure 5.23: Right-random: M using Sugeno-Weber.

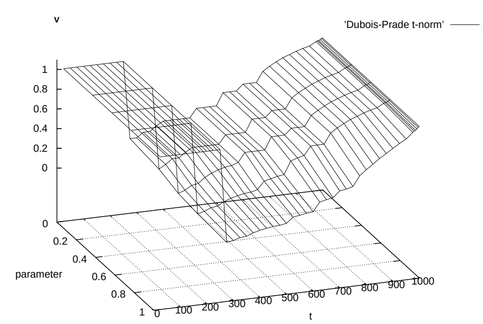

Figure 5.24: Right-random: M using Dubois-Prade.

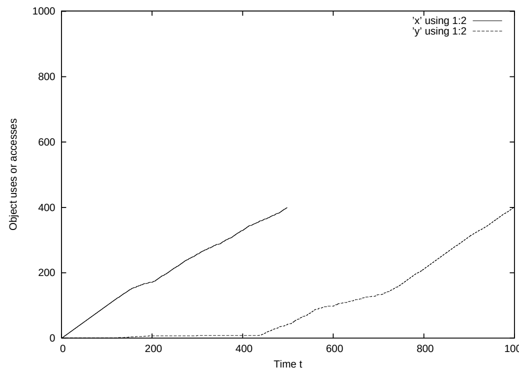

Figure 5.25: Opposite histories.

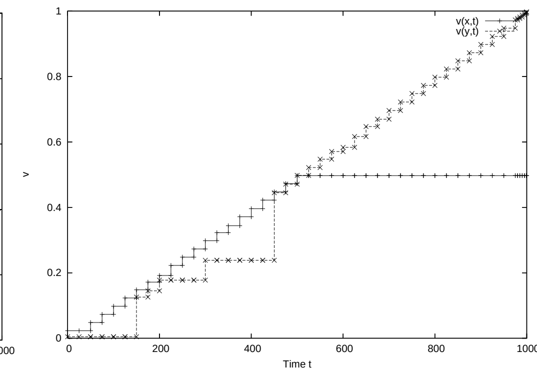

Figure 5.26: O. histories: $v(x)$, $v(y)$.

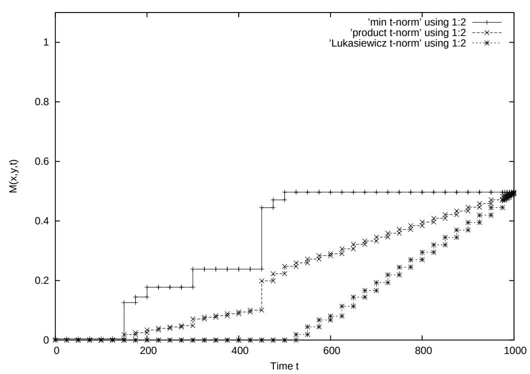

Figure 5.27: Opposite histories: M using minimum, product, Lukasiewicz.

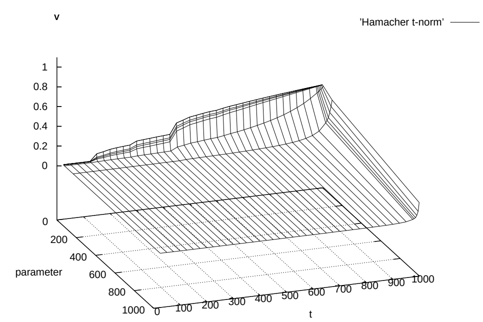

Figure 5.28: Opposite histories: M using Hamacher.

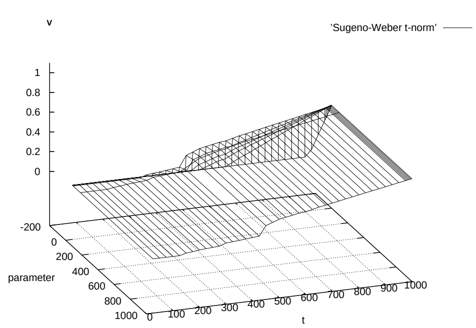

Figure 5.29: Opposite histories: M using Sugeno-Weber.

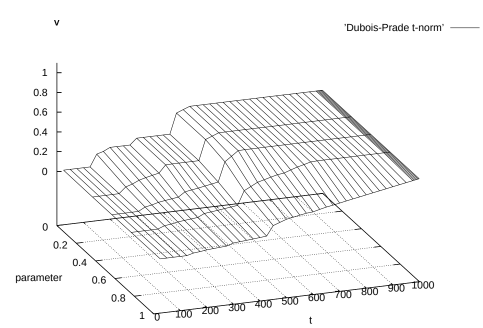

Figure 5.30: Opposite histories: M using Dubois-Prade.

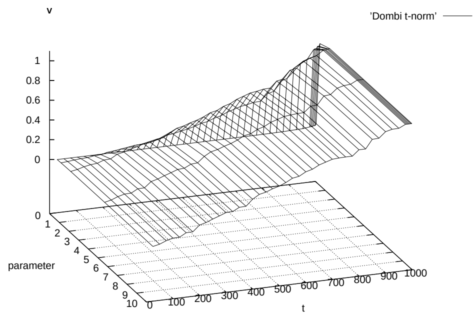

Figure 5.31: Random: Dombi.

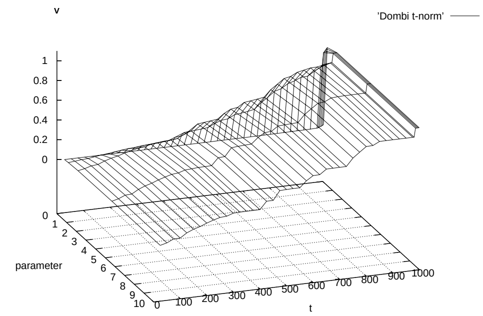

Figure 5.32: Left-Random: Dombi.

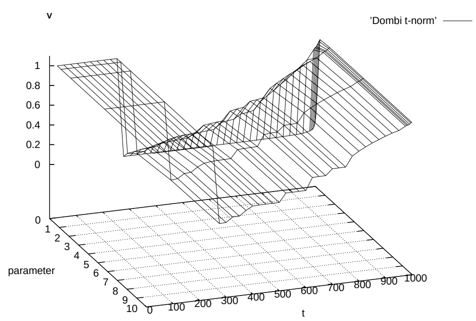

Figure 5.33: Right-Random: Dombi.

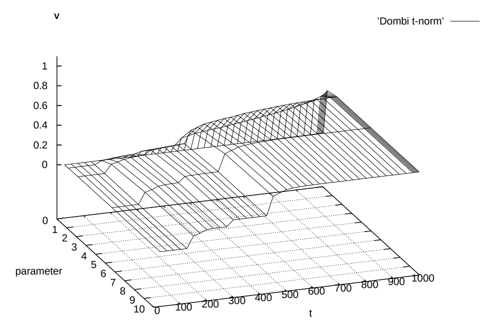

Figure 5.34: Opposite histories: Dombi.

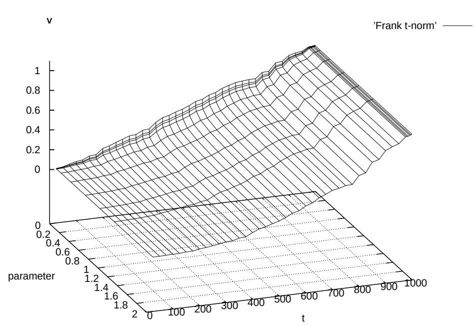

Figure 5.35: Random: Frank.

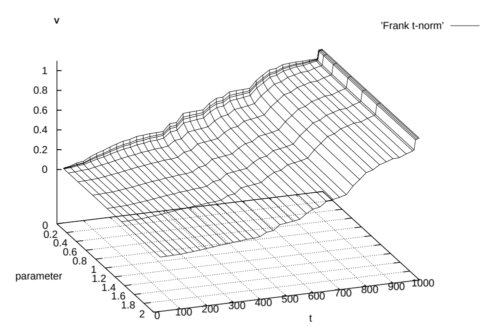

Figure 5.36: Left-Random: Frank.

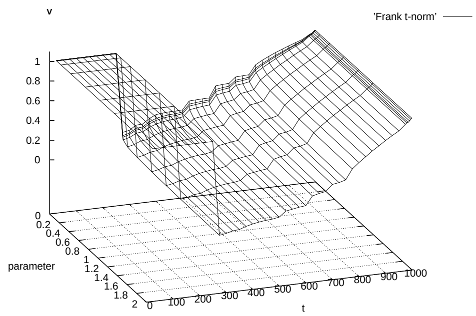

Figure 5.37: Right-Random: Frank.

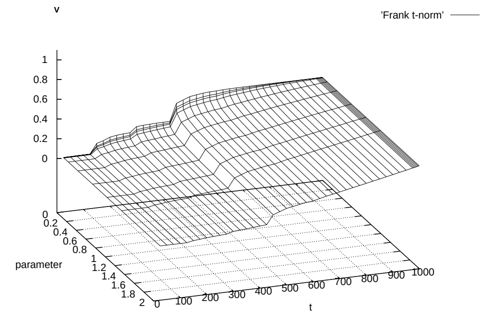

Figure 5.38: Opposite histories: Frank.

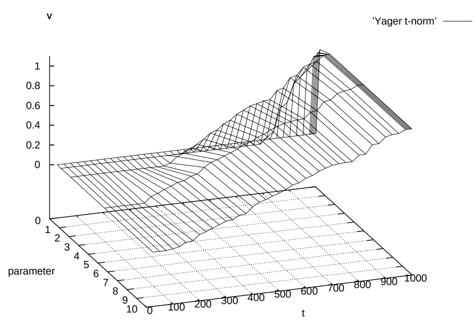

Figure 5.39: Random: Yager.

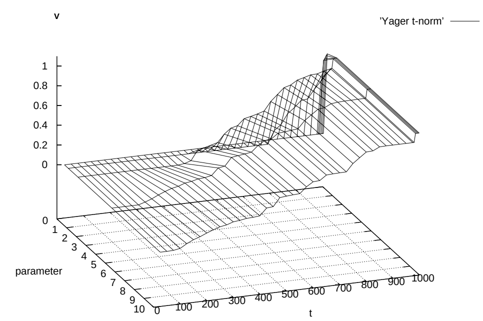

Figure 5.40: Left-Random: Yager.

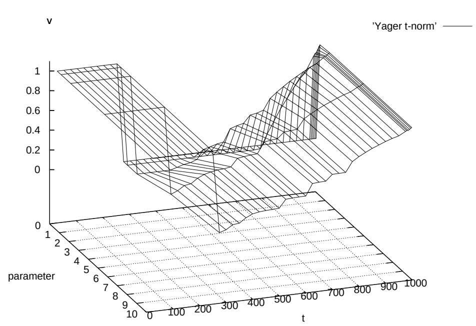

Figure 5.41: Right-Random: Yager.

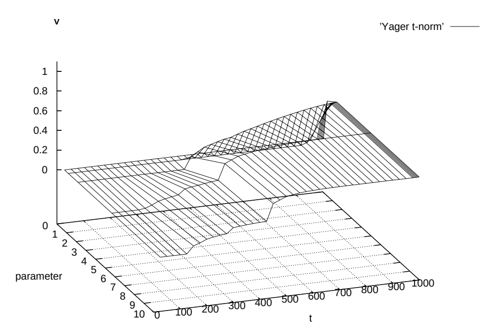

Figure 5.42: Opposite histories: Yager.

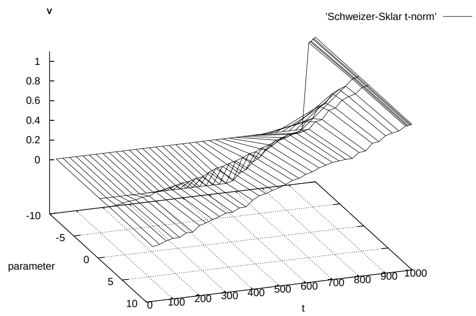

Figure 5.43: Random: Schweizer-Sklar.

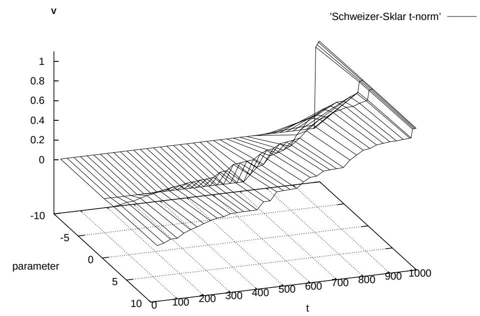

Figure 5.44: Left-Random: Schweizer-Sklar.

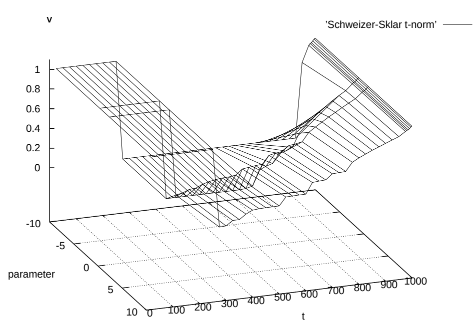

Figure 5.45: Right-Random: Schweizer-Sklar.

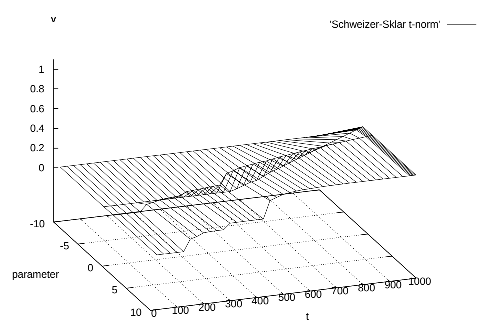

Figure 5.46: Opposite histories: Schweizer-Sklar.

5.3.3 Conclusions

Obviously, to take advantage of the history of accesses evaluation, we need at least an history that is representative of the regular uses of the element.

With the use of M we can model situations where the value of v is not high enough but the comparison of histories computed until a given moment t tells that access histories are close to each other. If we are certain that one of these histories presented optimal performance, this information can lead to update strategy tunning.

Here, as in [41] the fuzzy metric construction selected (in [41] $(M, *)$ is stationary while in our case it is not) is able to model a computer science problem.

Comparing this technique with the one presented in [31], notice how, even when both techniques are prepared for fast calculation and adaptability, while authors there assumed a statistical distribution of data accesses, here we base our model on the comparison of objects with others for which we know how the system should behave for optimal usage.

Further conclusions will be drawn in Section 5.4.3 after we show the intuitionistic fuzzy metric space based extension.

5.4 Intuitionistic fuzzy metric space extension

For our next step, we will use the fact that the notion of an intuitionistic fuzzy metric space is a natural generalization of a fuzzy metric space. This fact allows us to adapt the idea of an intuitionistic fuzzy set presented by Atanassov in [3] so that we can measure the degree of closeness and the degree of non-closeness between two access histories using an intuitionistic fuzzy metric space (see [47]). In our case, it is necessary to use the generalization found in Definition 2.2.11 defined by Alaca et al in [1], a study that was later enriched by Romaguera and Tirado work in [55].

Recall that Definition 2.2.11 defines a 5-tuple $(X, M, N, *, \diamond)$. In order to define the metric N we need to find a new function v' that allows us to compare two elements in a given moment t between 0 and T and a set of t-conorms so that:

$$N(x, y, t) = v'(x, t) \diamond v'(y, t)$$

While fuzzy sets introduce the concept of membership degree m_μ, intuitionistic fuzzy sets introduce the concept of non-membership n_μ and uncertainty p_i degrees in such a way that $p_i = 1 - m_\mu - n_\mu$.

The most immediate way to obtain N and build the intuitionistic fuzzy metric would be to use the fuzzy set $N = 1 - M_*$. This set is known as "crisp" set and gives no more information than what we already have.

Instead of using the "crisp" approach, we will use the following function v' that is a suitable candidate to take advantage of the intuitionistic fuzzy spaces properties and to complement the representation of localized accesses histories:

$v'(x, 0) = 1,$

$$v'(x, t) = \prod_{j=1}^{k(x,t)} \frac{T - t_{j(x)}}{T}$$

if $0 < t \leq T$, and $v'(x, t) = 0$ if $t > T$.

It is clear that for each $x \in X$, $v'(x\_)$ is left continuous and nonincreasing (i.e., $v'(x, t) \geq v'(x, s)$ whenever $t \leq s$).

Similarly to Proposition 5.3.1 we obtain the following general result.

Proposition 5.4.1. *Let X be a nonempty set, $v' : X \times [0, \infty) \to (0, 1]$ be a function such that for each $x \in X$ $v'(x, \_)$ is left continuous and nonincreasing* *and $\diamond$ be a t-conorm. Then $(X, N_\diamond, \diamond)$ verifies conditions (6)-(9) and (11) of Definitions 2.2.9-2.2.10 where $N_\diamond$ is defined by:*

(a) $N_\diamond(x, y, 0) = 1$,

(b) $N_\diamond(x, x, t) = 0$ *for each $t > 0$ and each $x \in X$.*

(c) $N_\diamond(x, y, t) = v'(x, t) \diamond v'(y, t)$ *for each $t > 0$ and each $x, y \in X$ with $x \neq y$.*

Proof:

Condition (6) in Definition 2.2.9 follows from (a).

On the other hand, since for each $x, y \in X$ with $x \neq y$, and each $t > 0$,

$$N_\diamond(x, y, t) = v'(x, t) \diamond v'(y, t) = v'(y, t) \diamond v'(x, t) = N_\diamond(y, x, t),$$

we deduce that condition (11) in Definition 2.2.10 also holds.

Now suppose that $N_\diamond(x, y, t) = 0$ for each $t > 0$. If $x \neq y$ we obtain

$$N_\diamond(x, y, t) = v'(x, t) \diamond v'(y, t) \geq v'(x, t) \vee v'(y, t) > 0,$$

a contradiction. Therefore condition (7) in Definition 2.2.9 holds.

Now let x, y, z three different elements in X and $t, s > 0$. Then

$$\begin{aligned} N_\diamond(x, z, t + s) &= v'(x, t + s) \diamond v'(z, t + s) \leq v'(x, t) \diamond v'(z, t) \\ &\leq v'(x, t) \diamond v'(y, t) \diamond v'(z, s) \diamond v'(y, s) \\ &= N_\diamond(x, y, t) \diamond N_\diamond(z, y, s) \end{aligned}$$

so condition (8) in Definition 2.2.9 is satisfied.

Finally, for each $x, y \in X$, $N_\diamond(x, y, \_) : [0, \infty) \to [0, 1]$ is left continuous because $v'(x, \_)$ and $v'(y, \_)$ are left continuous and $\diamond$ is continuous.

□

Remark. *For our v' choice, the triangular inequality (8) means that:*

$$v'(x,t) = \prod_{j=1}^{k(x,t)} \frac{T - t_{(j)(x,t)}}{T} \geq \prod_{j=1}^{k(x,t+s)} \frac{T - t_{(j)(x,t+s)}}{T} = v'(x,t+s)$$

and

$$v'(x,s) = \prod_{j=1}^{k(x,s)} \frac{T - t_{(j)(x,s)}}{T} \geq \prod_{j=1}^{k(x,t+s)} \frac{T - t_{(j)(x,t+s)}}{T} = v'(x,t+s)$$

so that:

$$v'(x,t+s) \diamond v'(y,t+s) \leq v'(x,t) \diamond v'(y,s) \leq v'(x,t) \diamond v'(y,s) \diamond (v'(z,t) \diamond v'(z,s))$$

In [6] we showed empirical results based on experiments using a variety of continuous t-conorms and t-conorm families.

Note that if we consider the 5-tuple $(X, M_*, N_\diamond, *, \diamond)$, where $M_*, N_\diamond$, are constructed as in Propositions 5.3.1 and 5.4.1 above, then condition (1) in Definition 2.2.11 does not hold in general so that $(X, M_*, N_\diamond, *, \diamond)$ is not an intuitionistic quasi-metric space, as the following example shows:

Example 5.4.1. *Let $X = \{x, y\}$, $x \neq y$, let $T = 1000$, $k(x) = 1$, $t_{1(x)} = 1$, $k(y) = 1$, $t_{1(y)} = 2$. Take $* = \wedge$ and $\diamond$ the Lukasiewicz dual, i.e., $a \diamond b = (a + b) \wedge 1$ for all $a, b \in [0, 1]$.*

If $t = T$, then:

$v(x,t) = 0.0005;$
$v(y,t) = 0.001;$
$v'(x,t) = 0.999;$
$v'(y,t) = 0.998.$

Hence

$$\begin{aligned} M(x,y,t) + N(x,y,t) &= v(x,t) * v(y,t) + v'(x,t) \diamond v'(x,t) \\ &= (0.0005 \wedge 0.001) + ((0.999 + 0.998) \wedge 1) > 1. \end{aligned}$$

However, this disarrangement is easily corrected taking functions $v/2$ and $v'/2$, respectively, instead of v and v'. Thus $(X, M_*, N_\diamond, *, \diamond)$ is an intuitionistic fuzzy metric space for any t-norm $*$ and any t-conorm $\diamond$.

5.4.1 Evaluated t-conorms

Our source of continuous t-conorms is again Dubois and Prade [13]. After discarding too complex t-conorms, we have taken into account:

- Minimum dual – Maximum: $\max(x, y) := \begin{cases} y & \text{if} \quad x \leq y \\ x & \text{if} \quad y < x \end{cases}$
- Product dual – Probabilistic sum: $\prod'(x, y) := x + y - xy$.
- Lukasiewicz dual: W'$(x, y) := \min\{x + y, 1\}$.

We have also included figures showing the results for:

- Hamacher'$(x, y) := \dfrac{x + y + (\beta - 1)xy}{1 + \beta xy}$ where $\beta \geq -1$. It is the Hamacher family dual.
- Sugeno-Weber'$(x, y) := \min\{x + y + \lambda xy, 1\}$ where $\lambda > -1$. It is the Sugeno-Weber family dual.

In order to see how their behaviour is very similar to the traditional t-norm duals.

T-conorms order

- $\max(x, y) \leq \prod'(x, y) \leq$ W'(x, y).
- $\max(x, y) \leq \prod'(x, y) \leq$ Sugeno-Weber'$_{<=0}(x, y) \leq$ W'$(x, y) \leq$ Sugeno-Weber'$_{>0}(x, y)$.

- Hamacher'(x, y) order depends on the parameter value β and on x and y:
  - Comparison with W'(x, y):
    - If $x + y < 1$ then in order that Hamacher'$(x, y) \leq$ W'(x, y):

      $$\frac{x + y + (\beta - 1)xy}{1 + \beta xy} \leq x + y$$

      The following condition should happen $\beta xy(1-(x+y)) \leq xy$. This is possible if, for instance $xy \neq 0$ then $\beta < \frac{1}{2}$. This means that there is not a definitive order between both t-conorms.
    - If $x + y \geq 1$ then for Hamacher'$(x, y) \leq$ W'(x, y) to happen:

      $$\frac{x + y + (\beta - 1)xy}{1 + \beta xy} \leq 1$$

      This is always true. Then if $x + y \geq 1$ we have that

      $$\text{Hamacher'}(x, y) \leq \text{W'}(x, y).$$
  - Comparison with $\prod'(x, y)$. In order that *hamacherco* $\leq \prod'(x, y)$:

    $$\frac{x + y + (\beta - 1)xy}{1 + \beta xy} \leq x + y - xy$$

    $\beta xy(x + y - xy - 1) \geq 1$. As $\prod'(x, y) \leq 1$, $xy \leq 1$ and it would be necessary that $1 \leq \beta < 0$. This is impossible and then Hamacher'$(x, y) \geq \prod'(x, y)$.
  - Comparison with Sugeno-Weber'(x, y):
    - If $x+y+\lambda xy \geq 1$ then for *hamacherco* $\leq$ Sugeno-Weber'(x, y) to happen:

      $$\frac{x + y + (\beta - 1)xy}{1 + \beta xy} \leq 1$$

      This is the same case than Hamacher'(x, y) and W'(x, y) comparison for $x + y \geq 1$. For this case Hamacher'$(x, y) \leq$ Sugeno-Weber'(x, y).

∗ If $x+y+\lambda xy \leq 1$ then for Hamacher'$(x, y) \leq$ Sugeno-Weber'(x, y) to happen:

$$\frac{x + y + (\beta - 1)xy}{1 + \beta xy} \leq x + y + \lambda xy$$

$xy(\beta - \lambda) \leq \beta xy(x + y + \lambda xy - xy)$. This is possible but not always true. For example, for $xy \neq 0$ and $\beta \neq 0$:

$$\text{Sugeno-Weber'}(x, y) - xy \geq 1 - \frac{\lambda}{\beta}$$

Then, there is not an order between the t-conorms in this case.

5.4.2 Empirical results: Comparison

The following figures illustrate the results obtained using the intuitionistic fuzzy metric for the same experiments than Section 5.3.2. It will be worth revisiting Figures 5.7, 5.13, 5.19 and 5.25 while studying the next ones.

Random variations – Random

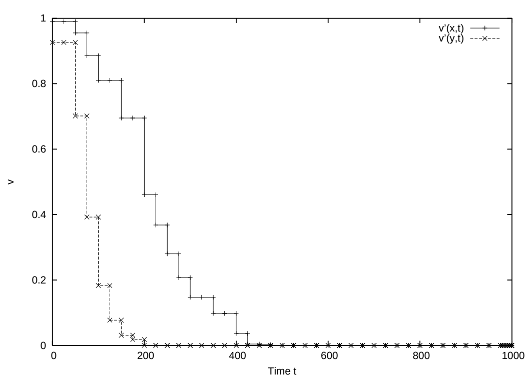

Figure 5.47: Random: $v'(x)$ and $v'(y)$.

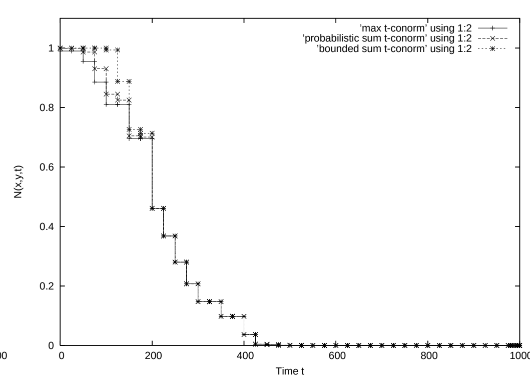

Figure 5.48: Random: $N(x, y, t)$.

For the random access cases (Figures 5.47, 5.48, 5.49, 5.50), it can be seen how the t-conorms ordering prevails and how fast M values decrease due to $v'(x, t)$ construction.

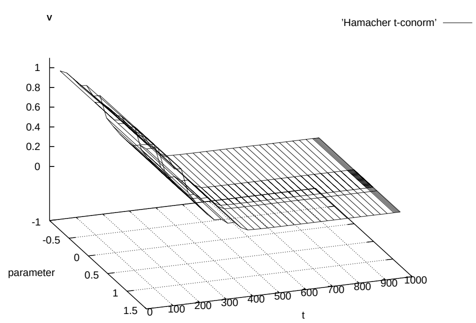

Figure 5.49: Random: N using Hamacher'.

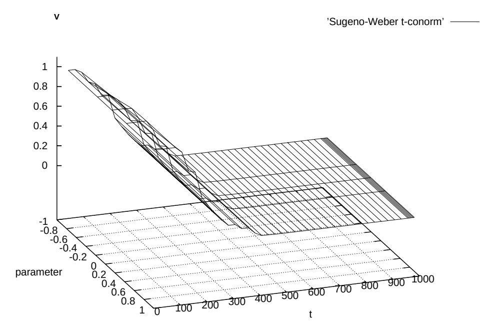

Figure 5.50: Random: N using Sugeno-Weber'.

Left-random variations – Left random

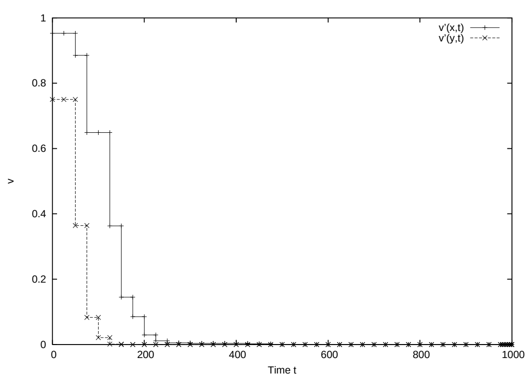

Figure 5.51: L-Random: $v'(x)$, $v'(y)$.

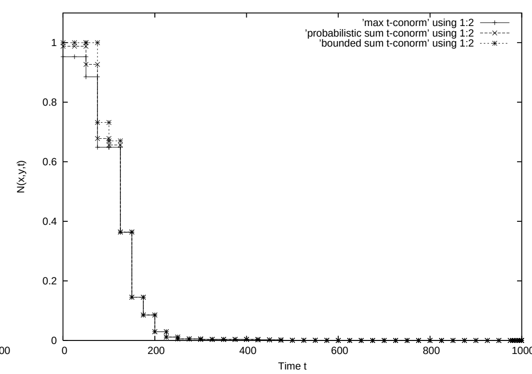

Figure 5.52: Left-Random: $N(x, y, t)$.

The experiments for left-random accesses (Figures 5.51, 5.52, 5.53, 5.54), display how N identifies the differences between element histories concentrated near to the starting point. From a given moment on, both histories show an identical access pattern and N returns very close to 0 values. This characteristic is reinforced by the fact that $k(x)$ and $k(y)$ are big enough for the product to tend to 0. As soon as we come across a situation like this, we must consider that histories distance is measured by v' more than by the t-conorm.

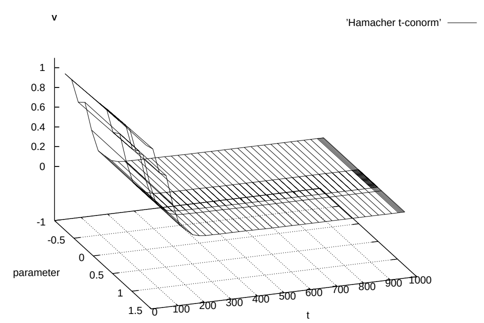

Figure 5.53: Left-Random: N using Hamacher'.

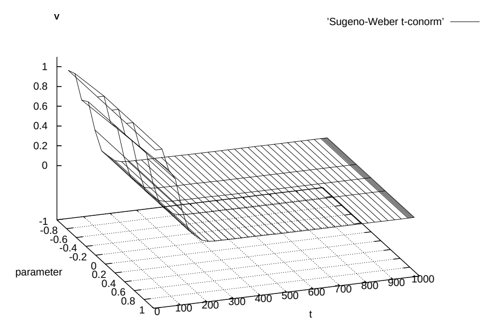

Figure 5.54: Left-Random: N using Sugeno-Weber'.

Right-random variations – Right random

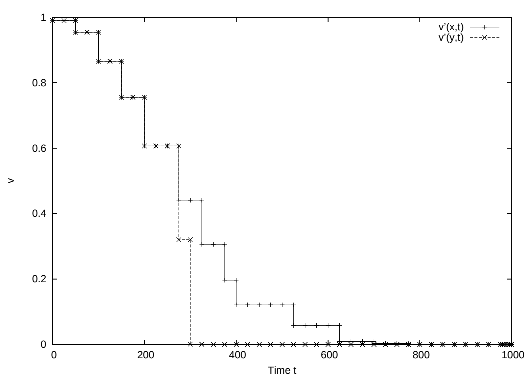

Figure 5.55: R-random: $v'(x)$, $v'(y)$.

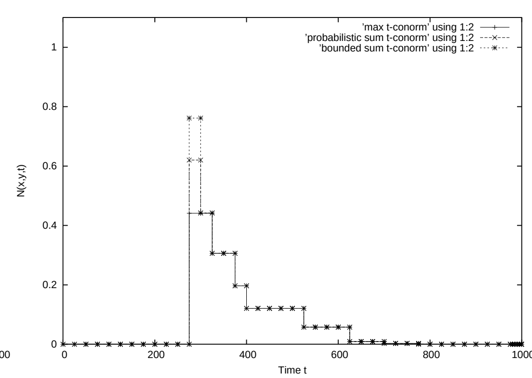

Figure 5.56: Right-random: $N(x, y, t)$.

For the right-random experiments histories (Figures 5.55, 5.56, 5.57, 5.58), divergence starts at the moment $T/4$. In that same moment N starts displaying differences until it turns 0 again due to the amount of accesses already performed (similar to the left-random accesses case).

Two independent histories – Opposite histories

When we compare the opposite histories experiments (Figures 5.59, 5.60, 5.61, 5.62), with the ones from the right-random accesses, N decrement for

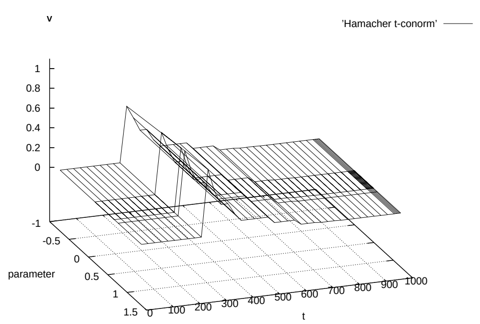

Figure 5.57: Right-random: N using Hamacher'.

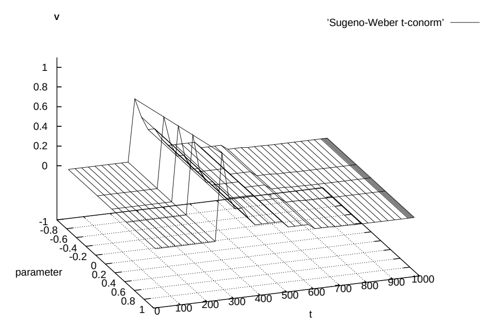

Figure 5.58: Right-random: N using Sugeno-Weber'.

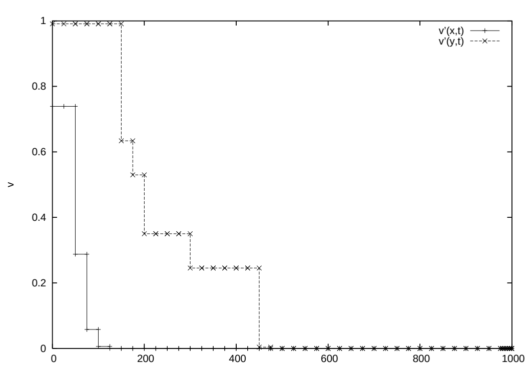

Figure 5.59: Opposite histories: $v'(x)$, $v'(y)$.

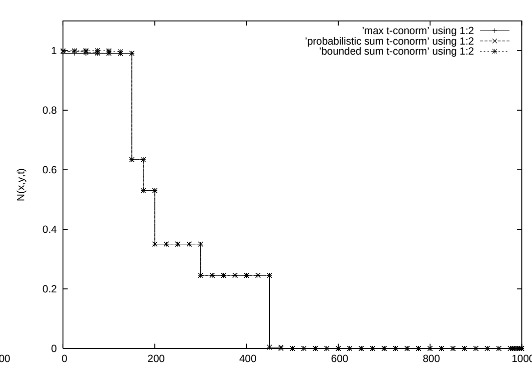

Figure 5.60: Opposite histories: $N(x, y, t)$.

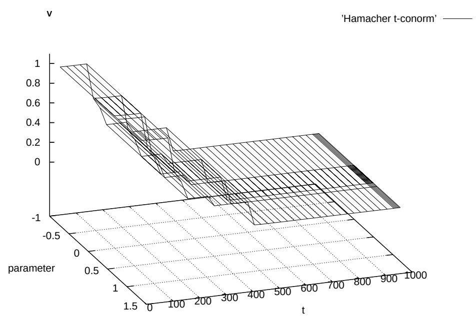

Figure 5.61: Opposite histories: N using Hamacher'.

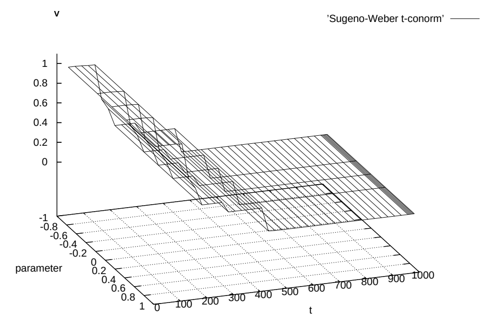

Figure 5.62: Opposite histories: N using Sugeno-Weber'.

the opposite histories case is softer than for the right-random case, as one might have expected because differences already appear far from the end of the histories.

As in the rest of the cases, history differences towards the end of the history are not compared as explicitly as the comparisons performed towards the beginning of the history due to the amount of already performed accesses.

5.4.3 Conclusions

We have shown that the optimization of accesses in systems based on locality can be achieved using a mathematical framework based on intuitionistic fuzzy metric spaces. For that means, we have also presented experimental results representing best, average and worst cases for a variety of elections in the fuzzy constructions we can build. Combination of the different t-norm and t-conorms allows us to model the general case of accesses locality.

The results for the experiments obtained with v' and the t-conorms are shown in the following figures on top of the ones obtained with v and the t-norms to show the intuitionistic fuzzy metric behaviour (applying the correction factor $v/2$ and $v'/2$ in order to fulfill all the intuitionistic fuzzy metric properties).

Figures 5.63, 5.64, 5.65 and 5.66 show the results of M and N obtained for the previous four scenarios. We have only shown the results for traditional t-norms because they act as boundaries (being the minimum the upper bound of all of them) for the considered families. Families are useful if we need a finer tuning or if our system behaves adaptively and the family parameter is able to introduce subtle improvements in the predictions. This also confirms that for a general scenario there is no such thing as an optimal parameter.

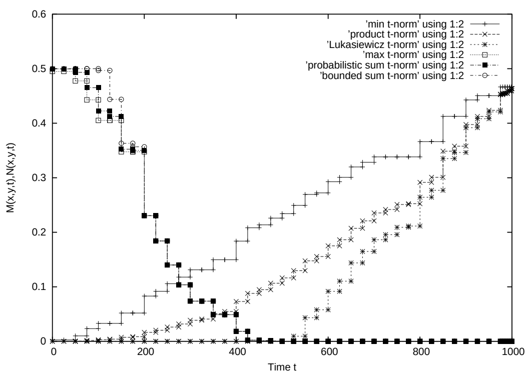

Figure 5.63: Random: M and N.

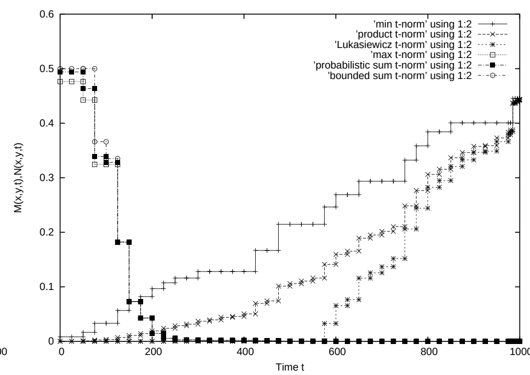

Figure 5.64: Left-Random: M and N.

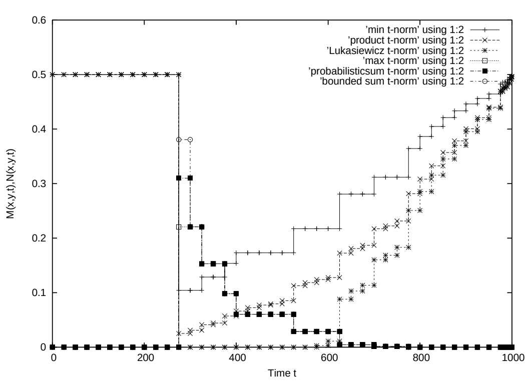

Figure 5.65: Right-Random: M and N.

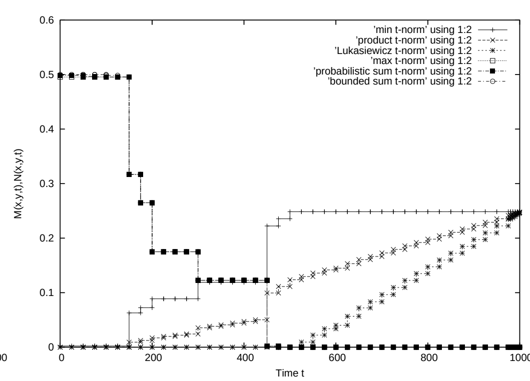

Figure 5.66: Opposite histories: M and N.

Indeed, M represents the closeness degree and N represents the remoteness degree. The intuitionistic fuzzy metric allows us to use two thresholds: one for "goodness" and another one for "badness" respectively.

For each scenario the intersection between a function M defined by a t-norm and a function N defined by a t-conorm (not necessarily the dual one) expresses the moment when a history is not "distant" anymore and becomes "close".

Notice that allowing a degree of uncertainty makes N useful. Otherwise it would not offer more information than the one we already had with M.

Initially, if the uncertainty degree is "reasonable":

(i) If $M(x, y, t) > N(x, y, t)$ then we can say that both elements belong to the same class.

(ii) If $N(x, y, t) > M(x, y, t)$ then we can say that they belong to different classes.

(iii) If $N(x, y, t) = M(x, y, t)$ then we cannot confirm anything about class membership solely with the result of the intuitionistic fuzzy metric.

For the latter case, we can always choose to decide optimistically (consider them from the same class) or pessimistically (consider them from different classes).

In general, our results show that traditional continuous t-norms are the constructs we need as basic elements in order to build the metric. We find specially outstanding the results obtained for the minimum and Lukasiewicz t-norms which are fast to compute and discriminate history results better than the rest. T-norm families rather than introducing complexity, introduce a very interesting possibility for finer tuning.

This makes our model extremely configurable and suitable for a range of possible future applications much greater than our initial study [7] for replicated database systems.

From here on we could try to find optimal thresholds for classification into classes and decisions regarding the “goodness” and “badness” of those.

One possibility is to make use of the multiple variations we can get by combining different t-norms and t-conorms. Another one would be to change the definition of v and v' while maintaining their definition in the scope of the intuitionistic fuzzy metric. Special care has to be taken for the election of v'. For instance, our current election is built using a product and its results decrease very abruptly. Other elections for v and v' shall be designed according to the desired behaviour one expects from the classification threshold.

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