Nontensorial Transformation Optics

C. García-Meca$^{1,*}$ and C. Barceló$^2$

$^1$Nanophotonics Technology Center, Universitat Politècnica de València, 46022 Valencia, Spain
$^2$Istituto de Astrofísica de Andalucía (CSIC), Glorieta de la Astronomía, 18008 Granada, Spain

(Received 7 January 2016; revised manuscript received 9 March 2016; published 17 June 2016)

We present an alternative version of transformation optics that allows us to mold the flow of light without rotating or scaling the electromagnetic fields. The resulting media experience unusual force densities, are nonreciprocal, and exhibit loss or gain. Because of these singular features, a variety of effects and devices unreachable by standard transformation optics can be achieved, including reflectionless light compression, optical modes with arbitrary in-plane polarization, and special isolators.

DOI: 10.1103/PhysRevApplied.5.064008

I. INTRODUCTION

Following the general principle of relativity, physical laws must have the same form in all admissible reference frames. Electrodynamics provides a paradigmatic example of this feature (if only one understands the metric as an abstract object). In fact, Maxwell’s equations admit a tensorial (T) representation that preserves its form under any coordinate transformation [1]. A recently developed branch of physics known as transformation optics (TO) exploits this form independence to find connections between the electromagnetic fields supported by different media. As a particular case, in an isotropic medium with permittivity $\varepsilon$ and permeability $\mu$, the time-harmonic Faraday’s and Ampère’s laws can be written in arbitrary spatial coordinates [e.g., arising from a transformation $\mathbf{x} = \mathbf{f}(\mathbf{s})$ of the Cartesian coordinates $\mathbf{s}$] as

$$
\varepsilon^{ij} \frac{\partial E_i}{\partial x_j} = -i\omega\mu_0 \sqrt{\gamma^{ij}} H_j, \quad (1)
$$

$$
\varepsilon^{ij} \frac{\partial H_i}{\partial x_j} = i\omega\varepsilon_0 \sqrt{\gamma^{ij}} E_j. \quad (2)
$$

Here $\omega$ is the angular frequency, $\varepsilon^{ij}$ the Levi-Civita symbol, $E_i$ and $H_i$ the components of the electric and magnetic fields in the chosen coordinate system, $\gamma^{ij}$ the spatial metric associated with such a system, and $\gamma$ its determinant [2]. Note that the other two Maxwell’s equations (Gauss’s laws) are automatically satisfied for $\omega \neq 0$, our case throughout this work. TO amounts to interpreting the coordinate-transformed version of Maxwell’s equations, now on just Eqs. (1) and (2), as representing a different medium with different fields living in, which written in the original coordinate system (Cartesian in our case) are taken to be

$$
E_i(x) = \Lambda^i_k E_{i0} [\mathbf{f}^{-1}(x)], \quad (3)
$$

$$
H_i(x) = \Lambda^i_k H_{i0} [\mathbf{f}^{-1}(x)], \quad (4)
$$

$$
\varepsilon^{ij}/\varepsilon_0 = \mu^{ij}/\mu_0 = \sqrt{\gamma^{ij}}. \quad (5)
$$

where $\Lambda^i_k = \partial x^i / \partial x^k$ and $E_0$ and $H_0$ are the electric and magnetic fields in the original untransformed equations.

The fact that the fields $E_0$ and $H_0$ supported by the original medium $\varepsilon_0$, $\mu_0$ (so-called virtual medium) and the fields $E$ and $H$ supported by the transformed medium $\varepsilon^{ij}$, $\mu^{ij}$ (so-called physical medium) are connected by a coordinate transformation, has allowed the design of unprecedented devices, such as invisibility cloaks (see Fig. 1 for an example) [3,4], optical wormholes [5], and many others [2,6,7].

According to Eqs. (3)–(5), each element in the new set of reinterpreted equations is obtained from its original counterpart through the proper T transformation, which is linked to its assumed physical nature; the permittivity and permeability are transformed as tensor densities and the fields as covectors. Although several extensions of the original TO technique have been proposed, for example, to work with complex coordinates [8,9] or in Fourier space [10], the fact that fields and medium must transform following Eqs. (3)–(5) under coordinate changes, has been invariably accepted. This implies that the fields are rotated and scaled inhomogeneously, depending on the applied mapping, as it would happen after a change of reference frame [Fig. 1(b)]. Although this effect is usually desired or irrelevant, a different, nontensorial (NT), transformation rule might be of interest in some cases. Actually, the process of transforming the equations to find connections between different field distributions is not subjected to relativity or any physical law. The only requirement is that the transformed fields are the solution of Maxwell’s equations for a given medium. Therefore, it is reasonable to explore the use of unphysical (NT) transformation laws within the TO context, as it may lead to novel physical effects. This is the purpose of this work. As discussed...
To this end, we start by expressing the new fields in terms of the old ones in the transformed Faraday’s law
\[
e^{ij\hat{k}} \frac{\partial (M^k_i E_{0k})}{\partial \hat{x}^j} = -i \omega \mu_0 \sqrt{g^j g^i} \Lambda^k_j H_{0k}. \tag{8}
\]

The question now is whether this equation can be made to correspond to Faraday’s law for \( E_{0k} \) and \( H_{0k} \) when \( \hat{x}^i \) are interpreted as the original coordinates. Remarkably, we have identified a quite general situation in which this is the case: a conformal transformation of the \( xy \) plane, in which the electric or magnetic field lies, of an initially \( z \)-invariant problem. This kind of mapping, together with quasiconformal mappings that effectively behave as conformal ones, constitutes a wide class of transformations that has been extensively used in TO, proving to have many applications \([3,7,12–22]\). The Jacobian of this transformation is of the form
\[
\Lambda^i_j = \begin{pmatrix} a & b & 0 \\ -b & a & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{9}
\]

This implies that \( \partial E_{0k}/\partial x^3 = 0 \), since \( \Lambda^1_1 = \Lambda^2_2 = 0 \), and \( \partial E_{0k}/\partial x^3 = 0 \), due to the \( z \) invariance of the initial problem. Given that \( e^{ij\hat{k}} M^k_i \partial E_{0k}/\partial \hat{x}^j \), the left-hand member of Eq. (8) is just \( e^{ij\hat{k}} M^k_i \partial E_{0k}/\partial \hat{x}^j \). Using Eq. (9),
\[
e^{ij\hat{k}} M^k_i \frac{\partial E_{0k}}{\partial \hat{x}^j} = \Lambda^j_1 \Lambda^j_1 \frac{\partial E_{03}}{\partial \hat{x}^2} - \Lambda^j_2 \Lambda^j_2 \frac{\partial E_{03}}{\partial \hat{x}^2} + \Lambda^j_3 \Lambda^j_3 \left( \frac{\partial E_{01}}{\partial \hat{x}^1} + \frac{\partial E_{02}}{\partial \hat{x}^2} \right). \tag{10}\]

The last term in brackets simplifies to \(-i \omega \mu_0 \Lambda^1_1 H_{03} \), since \( \frac{\partial E_{01}}{\partial \hat{x}^1} + \frac{\partial E_{02}}{\partial \hat{x}^2} = \nabla \cdot \mathbf{E}_0 = 0 \) and \( \frac{\partial E_{02}}{\partial \hat{x}^1} - \frac{\partial E_{01}}{\partial \hat{x}^2} = (\nabla \times \mathbf{E}_0)_3 \). Thus, defining \( M_i = \text{diag}(1,1,\Lambda_i^j) \), Eq. (10) can be expressed as
\[
e^{ij\hat{k}} M^k_i \frac{\partial E_{0k}}{\partial \hat{x}^j} = M_i^j e^{ij\hat{k}} \frac{\partial E_{0k}}{\partial \hat{x}^j} - \Lambda^1_3 \Lambda^1_3 \frac{\partial E_{01}}{\partial \hat{x}^2} \frac{\partial E_{02}}{\partial \hat{x}^1} \tag{11}\]

Finally, from Eqs. (8) and (11), we obtain
\[
e^{ij\hat{k}} \frac{\partial E_{0k}}{\partial \hat{x}^j} = -i \omega \mu_0 H_{0k}, \tag{12}\]
\[
\mu^{ij\hat{k}} = \mu_0 (M^{-1})^i_j \sqrt{g^j g^i} \Lambda^k_j - \delta_j^k \Lambda^j_3 \Lambda^j_3. \tag{13}\]

Proceeding in a similar manner with Ampère’s law yields analogous results, with \( e^{ij\hat{k}} E_0 = \mu^{ij\hat{k}}/\mu_0 \). Therefore, Eq. (13) represents the medium that implements the transformation.

Below, the formalism we will develop here around this idea has potential applications in high-energy electrodynamics, lensing, microscopy, particle acceleration, nanophotonics, and optical communications.

II. THEORY

Specifically, we focus on the possibility of achieving transformation media supporting the following transformation for the fields:
\[
E_k(x) = E_{0k}[f^{-1}(x)], \tag{6}
\]
\[
H_k(x) = H_{0k}[f^{-1}(x)]. \tag{7}
\]

That is, in this case, each field component would be transformed independently as if it were a scalar. This implies that the fields are simply transported, without being scaled or rotated as in standard TO [see Fig. 1(c)], and in contrast with previous field transformations based either on tensorial complex coordinate transformations \([8]\) or on direct reciprocal linear transformations of the fields \([11]\).
given by Eqs. (6) and (7), which after some manipulations can be expressed as

\[
e^{ij}_\mu/\epsilon_0 = \mu^{ij}/\mu_0 = \begin{pmatrix} \Lambda_1^1 & -\Lambda_1^2 & 0 \\ \Lambda_2^1 & \Lambda_1^1 & 0 \\ 0 & 0 & \Lambda_1^1 \end{pmatrix} \text{.} \tag{14}
\]

This result uncovers a whole class of media unreachable by the standard formalism. In particular, it shows that for any conformal mapping there is an alternative medium (different from the one prescribed by the standard theory) that also implements such a mapping, but without rotating or scaling the electromagnetic fields. This feature opens the door to a variety of effects and functionalities unattainable with the standard theory. To see this, compare the medium given by Eq. (14) with that prescribed by the standard formula, Eq. (5), which for the considered conformal mapping reduces to

\[
e^{ij}/\epsilon_0 = \mu^{ij}/\mu_0 = \text{diag}[1, 1, (\Lambda_1^1)^2 + (\Lambda_1^2)^2] \text{.} \tag{15}
\]

A first difference is related to the time-average force density \( F \) experienced by each medium under plane-wave illumination. Although the correct expression for \( F \) in media is still a subject of debate [23], one of the main candidates, which has been used to estimate the force density undergone by lossy particles [24], as well as transformation media [25], is the Chu force [23]

\[
F = \frac{1}{2} \text{Re}\{\epsilon_0(\nabla \cdot \mathbf{E}) \mathbf{E}^* + \mu_0(\nabla \cdot \mathbf{H}) \mathbf{H}^* - i\omega(\epsilon - \epsilon_0)\mathbf{E} \times \mu_0 \mathbf{H}^* + i\omega(\mu - \mu_0)\mathbf{H} \times \epsilon_0 \mathbf{E}^*\} \text{.}
\tag{16}
\]

For real-valued permittivity and permeability distributions satisfying the relation \( e^{ij}_\mu/\mu_0 = \mu^{ij}/\epsilon_0 \), as those given by Eqs. (14) and (15), the force due to the last two terms is zero. We calculate the contribution of the first two terms for TM waves (\( z \)-polarized magnetic field). Analogous results are obtained for the TE case. Because of the \( z \) invariance of the problem, \( \nabla \cdot \mathbf{H} = 0 \). Without loss of generality, the electric field in virtual space can be expressed as

\( \mathbf{E}_0 = e^{ikx}\mathbf{\hat{x}} \). Hence, for the standard T medium we have

\( \mathbf{E}_T = (\Lambda_1^1\mathbf{\hat{x}} + \Lambda_1^2\mathbf{\hat{y}})e^{ik(\rho/f_i)}(x,y) \), which upon substitution in Eq. (16) yields \( F = \partial \Lambda_1^1/\partial x(\Lambda_1^2\mathbf{\hat{x}} + \Lambda_1^1\mathbf{\hat{y}}) \). In the NT case, \( \mathbf{E}_{NT} = e^{ikx}\mathbf{\hat{x}}, \mathbf{\hat{y}} \), and we find that \( F = 0 \), i.e., unlike standard transformation media, NT transformation media are not subjected to any stress under the assumption of the Chu force (see Fig. 1). In general, T and NT transformations will give rise to different results for any version of \( F \). This could be important for high-energy applications, in which the electromagnetic force might be large enough to damage the device.

Another difference is that the T medium satisfies the reciprocity conditions \( e^{ij}_\mu = e^{ij}_\mu, \mu^{ij}_\mu = \mu^{ij}_\mu \), and the lossless conditions \( e^{ij}_\mu = (e^{ij})^*, \mu^{ij}_\mu = (\mu^{ij})^* \) [26], while the NT medium does not. This is to be expected, as we can, for example, squeeze a certain field distribution without changing its amplitude. Achieving this effect for all propagation directions requires asymmetric absorption-amplification. For example, if a wave incident from the left is squeezed without increasing its intensity, and hence absorbed, the device must equally amplify a field coming from the right so that it is expanded without decreasing its amplitude. The presence of such an asymmetric loss or gain could certainly complicate implementation as compared with standard transformation media. In this respect, metamaterials may provide a feasible solution [9]. A promising example was introduced in Ref. [27], in which a metamaterial with an antisymmetric permittivity tensor having off-diagonal components with both imaginary and real parts, while requiring no bias magnetic field, was developed.

Let us now show how these special features can give rise to interesting effects and applications not attainable with standard TO.

### III. APPLICATIONS

A first application is related to the compression-expansion of electromagnetic fields, for which TO provides an obvious solution [16,28,29]. Unfortunately, to transfer any light alteration from the transformed medium to the background, the transformation must be discontinuous at their boundary, and it is well known that such discontinuities introduce reflections [16,28].

Consider, for instance, an optical squeezer based on the quasiconformal mapping shown in Fig. 2(a). The device

![Figure 2](image-url)

**FIG. 2.** Optical squeezers based on a quasiconformal mapping. (a) Employed mapping (\( F = 2 \)). (b) Reflection coefficient. (c), (d) Norm of the electric field for the T and NT implementations (calculated with COMSOL Multiphysics). The source is a right-propagating Gaussian beam.
gradually compresses a beam entering from the left, delivering a squeezed version at the right exit. Intuitively, in a standard TO implementation, energy density is increased at the squeezer’s right end (the amplitude of the fields increases) and, since energy density is lower in the background medium, a reflected wave must appear to conserve power flux density. This can be clearly seen in Fig. 2(c), where reflections at the output boundary generate a standing wave in a squeezer based on Eq. (15).

In order to minimize these reflections, the so-called impedance-tunable technique is proposed [30]. However, the achieved reflection reduction is not complete and is frequency and polarization dependent. Remarkably, in our NT version of TO, the amplitude of the fields and, consequently, energy density, do not change, which suggests that it could be used to build reflectionless squeezers.

In fact, no reflections appear when the medium given by Eq. (14) is employed to build the squeezer [Fig. 2(d)]. From [16], for a conformal transformation and a homogeneous isotropic exterior medium, the squeezer’s reflection coefficient \( r \) fulfills

\[
|r| = \sqrt[\sqrt{x^2 + (k_0^2 - k^2)^2} - x^2 - (k_0^2 - k^2)^2] \rangle^2 - \sqrt[\sqrt{x^2 + (k_0^2 - k^2)^2} - x^2 - (k_0^2 - k^2)^2] \rangle^2}
\]

for both polarizations, where \( \xi^j \) is equal to the \( e^j/\epsilon_0 \) associated with a transformation \( \bar{x} = x/F, \bar{y} = y/F, F \) being the compression factor at the output boundary. Using Eqs. (14) and (15), we can obtain the values of \( r \) as a function of \( k_0 \) (oblique incidence) for the T and NT squeezers [Fig. 2(b)]. It can be seen that the former always introduces reflections, while the latter is close to zero in a wide range of incidence angles (reflections appear at high angles to satisfy phase matching). In particular, at normal incidence \( |r| = 1 - F/(1 + F) \) for the T squeezer, while \( r = 0 \) for the NT one, regardless of the frequency or polarization.

This kind of reflectionless squeezer might provide an enhanced performance in applications such as hyperlensing [16], fiber-to-chip coupling [31], or even surface plasmon excitation [32]. Note that, for continuous transformations along the propagation path, NT media exhibit balanced loss-gain, since they do not alter the outer fields and thus do not supply or absorb energy [e.g., if we compress and expand space as in the inset of Fig. 2(a)].

A second unusual effect is related to uniform rotations of space \( \bar{x} = x \cos \phi - y \sin \phi, \bar{y} = x \sin \phi + y \cos \phi \). In standard TO, such rotations do not have any physical consequence, that is, we obtain \( \epsilon^j/\epsilon_0 = \mu^j/\mu_0 = \delta^j/ \) from Eq. (15). This can be easily understood by looking at Figs. 3(a) and 3(b). Since every vector in virtual space is rotated, including the fields and wave vector \( \mathbf{k} \) associated with any fundamental mode, the transformation can be seen just as a change of perspective. However, in the NT case, the fields keep their orientation while \( \mathbf{k} \) rotates, modifying the in-plane polarization of the original modes [Fig. 3(c)]. In fact, it can be easily checked that the NT medium obtained with Eq. (14) is different from the original one. This effect can be used, for example, to engineer the modes supported by dielectric waveguides, as shown in Fig. 3(d). In this way, the original quasitransverse in-plane polarization can be rotated, even achieving quasilongitudinal electric or magnetic modes. Moreover, simulations show that a narrow air channel can be created at the center of the waveguide without significantly altering the fields. This special field directionality could have applications in particle acceleration and optical tweezing, among others [33].

As a final example, consider the nonuniform rotation of space induced by the complex exponential function \( \bar{z} = \exp(z) \) (Fig. 4). This transformation is approximately continuous (reflectionless) at the bottom boundary. With a T implementation, the device basically rotates the fields as they go through it [Figs. 4(a) and 4(b)]. Therefore, electromagnetic modes are also very similar at both sides of the left boundary, so the device acts just as a reciprocal bend [7]. However, with an NT implementation, the fields...
(c) Reflected and transmitted waves at the device left boundary for wave vector using (b) standard TO and (c) NT TO. (d),(e) Incident, (TM polarization). (b),(c) Transformed grid, electric field, and electric field (red), and wave vector (blue) in virtual space.

Proceeding analogously for the reverse direction, we obtain a transmitted wave with double amplitude [Fig. 4(f)]. It is worth mentioning that this special kind of isolator is not gyrotropic (i.e., the off-diagonal components of the constitutive parameters have no imaginary part), as Eq. (14) shows. Instead, its nonreciprocal behavior arises from the device loss-gain, and thus represents an alternative to typical Faraday isolators. In addition, note that nonreciprocal devices can also be achieved with \( PT \)-symmetric transformation media [9]. However, such devices present balanced loss and gain, and cannot provide the particular behavior of the proposed NT bend. Likewise, a certain kind of unidirectional propagation based on mode conversion can be achieved using reciprocal media resulting from gauge field transformations [34]. Nonetheless, such devices are not true isolators as the proposed NT bend, in which the input and output modes are the same [35].

IV. CONCLUDING REMARKS

Our results show that a completely different class of transformation media displaying unusual effects can be obtained by assigning nonstandard transformation rules to the elements of the electromagnetic equations, opening different pathways and applications in the field of transformation optics. Besides the aforementioned potential utilities of the designed squeezer and waveguides, these optical elements could provide improvements in applications such as solid immersion microscopy, nanolithography, and optical data storage [16,33,36]. In addition, the alternative route for the creation of isolators that we have presented might be of particular interest, as this kind of device plays an essential role in microwave and nanophotonic technology, including laser protection, as well as the suppression of undesired crosstalk, signal routing, or interferences in a communication system [27,35].

Moreover, the proposed idea provides additional perspectives to the general transformation concept, as it could be applied to other branches of physics in which tensorial or vectorial quantities, such as flux densities, are involved.

ACKNOWLEDGMENTS

C.G.-M. acknowledges support from Generalitat Valenciana through the VALi+d postdoctoral program (APOSTD/2014/044). Financial support for C.B. was provided by the Spanish Ministry through Projects No. FIS2011-30145-C03-01 and No. FIS2014-54800-C2-1 (with FEDER contribution), and by the Junta de Andalucía through Project No. FQM219.