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GEOMETRY AND GÂTEAUX SMOOTHNESS IN SEPARABLE BANACH SPACES

P. HÁJEK, V. MONTESINOS AND V. ZIZLER

Abstract. It is a classical fact, due to Day, that every separable Banach space admits an equivalent Gâteaux smooth renorming. In fact, it admits an equivalent uniformly Gâteaux smooth norm, as was shown later by Day, James, Swaminathan, and independently by the third named author. It is therefore rather unexpected that the existence of Gâteaux smooth renormings satisfying various quantitative estimates on the directional derivative has rather strong structural and geometrical implications for the space. For example, by a result of Vanderwerff, if the directional derivatives satisfy a p-estimate, where p varies arbitrarily with respect to the point and the direction in question, then the Banach space must be an Asplund space. In the present survey paper, we discuss the interplay between various types of Gâteaux differentiability of norms and extreme points with the geometry of separable Banach spaces. In particular, we present various characterizations of Asplund, reflexive, superreflexive, and other classes of separable Banach spaces, via smooth as well as rotund renormings. We also include open problems of various levels of difficulty, which may foster research in the area of smoothness and renormings of Banach spaces.

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