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Projective limits of weighted (LB)-spaces of holomorphic functions

PhD Dissertation

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CERTIFICA que la presente memoria **Projective limits of weighted (LB)-spaces of holomorphic functions** ha sido realizado bajo mi dirección en el Departamento de Matemática Aplicada de la Universidad Politécnica de Valencia, por SVEN-AKE WEGNER y constituye su tesis para optar al grado de Doctor en Ciencias Matemáticas.

Y para que así conste, en cumplimiento con la legislación vigente, presentamos ante el Departamento de Matemáticas de la Universidad Politécnica de Valencia, la referida Tesis Doctoral, firmando el presente certificado.

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Fdo. José Bonet Solves.

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Resumen

Los límites projectivos de límites inductivos de espacios de Banach, también llamados espacios-(PLB), surgen de forma natural en el análisis matemático. Por ejemplo el espacio de distribuciones, el espacio de funciones real analíticas y varios espacios de funciones ultradiferenciables y de ultradistribuciones son de este tipo. En esta tesis estudiaremos espacios-(PLB), cuyos bloques de contrucción son espacios de Banach de funciones holomorfas definidas por normas supremo ponderadas. El estudio de estos espacios extiende la investigación de Agethen, Bierstedt, Bonet quienes han considerado recientemente espacios-(PLB) ponderados de funciones continuas. Desde otra perspectiva, extiende la investigación de límites inductivos ponderados de espacios de Banach de funciones holomorfas, los cuales han sido analizados intensamente por varios autores los últimos años.

Nuestro propósito es estudiar las propiedades localmente convexas de los espacios descritos arriba. En particular, investigamos cuando son ultrabornológicos o tonelados. Como punto de partida en la definición de los espacios que investigamos tenemos una sucesión doble de funciones (pesos) estrictamente positivas y continuas, nuestro objetivo es caracterizar las propiedades mencionadas antes en términos de esta sucesión. Además, investigamos bajo qué circunstancias se pueden intercambiar el límite proyectivo y el inductivo y por lo tanto el espacio-(PLB) coincide con el límite inductivo de espacios de Fréchet definidos por la misma sucesión; espacios de este último tipo has sido investigados por Bierstedt, Bonet.

Probamos condiciones necesarias para las propiedades de los espacios antes mencionadas y para que los límites inductivo y projectivo sean intercambiables bajo hipótesis muy poco restrictivas. En cuanto a condiciones suficientes usamos métodos homológicos, cuya exploración fue iniciada por Palamodov al final de los sesenta y continuada por Vogt, Wengenroth y otros a lo largo de los últimos 40 años. Por razones técnicas los métodos que acabamos de mencionar no se aplican a todos los casos que queremos estudiar. Por consiguiente, presentamos un criterio para decidir si los espacios son tonelados adaptado a estas situaciones. No obstante, parece ser inevitable descomponer funciones holomorfas para probar cualquier resultado relativo a a las condiciones suficientes. Por lo tanto introducimos varios contextos en los cuales lo último es posible. Dentro de estos contextos conseguimos la descomposición de diferentes formas; es decir, por descomposición de polinomios (en el disco y en el espacio), un método conectado con la teoría de proyecciones de Bergman, dos tipos de representaciones del espacio de sucesiones y el método $\overline{\partial}$ de Hörmander. Bajo algunas hipótesis adicionales (satisfechas -como mostramos- por muchos ejemplos) finalmente damos en casi todos los contextos mencionados anteriormente unas caracterizaciones completas de cuando el espacio es ultrabornológico, cuando es tonelado y cuando los límites inductivo y projectivo son intercambiables.

Para finalizar nuestra investigación de espacios-(PLB) ponderados, presentamos dos resultados (uno para funciones continuas y otro para holomorfas) los cuales muestran que espacios de este tipo se pueden escribir en algunos casos como el producto tensorial de un espacio de Fréchet y un espacio-(DF). Combinado con los resultados en espacios-(PLB) ponderados, el resultado en funciones continuas

esá conectado con el trabajo de Grothendieck, el cual estudió cuando este tipo de producto tensorial era ultrabornológico. El segundo resultado en representaciones de productos tensoriales muestra que algunos espacios de ultradistribuciones (introducidos recientemente por Schmets y Valdivia) resultan ser espacios-(PLB) ponderados de funciones holomorfas.

Resum

Els límits projectius de límits inductius d'espais de Banach, també anomenats espais-(PLB), sorgeixen de forma natural a l'anàlisi matemàtica. Per exemple l'espai de distribucions, l'espai de funcions real analítiques, i diversos espais de funcions ultradiferenciables i ultradistribucions són d'aquest tipus. En aquesta tesi estudiem espais-(PLB), els blocs de construcció els quals són espais de Banach de funcions hol.lomorfes definides per normes suprem ponderades. La investigació d'aquests espais extén la recerca d'Agethen, Bierstedt, Bonet, els quals han estudiat recentment espais-(PLB) ponderats de funcions contínues. Des d'altra perspectiva, extén l'estudi de límits inductius ponderats d'espais de Banach de funcions hol.lomorfes, els quals han estat estudiats intensament per diversos autors els darrers anys.

El nostre propòsit és estudiar les propietats localment convexes dels espais descrits abans. En particular, investiguem quan són ultrabornològics o tonellats. Com a punt de partida en la definició dels espais que investiguem tenim una successió doble de funcions (pesos) estrictament positives i contínues. El nostre objectiu és caracteritzar les propietats mencionades abans en termes d'aquesta successió. A més, investiguem sota quines circumstàncies es poden intercanviar el límit projectiu i l'inductiu i per tant l'espai-(PLB) coincideix amb el límit inductiu ponderat d'espais de Fréchet definits per la mateixa successió; espais d'aquest darrer tipus han estat investigats per Bierstedt, Bonet.

Provem condicions necessàries per a les propietats abans mencionades dels espais i per a que els límits inductiu i projectiu siguen intercanviables sota hipòtesis molt poc restrictives. En quant a condicions suficients usem mètodes homològics, l'exploració dels quals va iniciar Palamodov al final dels seixanta i van continuar Vogt, Wengenroth i altres al llarg dels darrers 40 anys. Per raons tècniques els mètodes que acabem de mencionar no s'apliquen a tots els casos que volem estudiar. Conseqüentment, presentem un criteri per a decidir si els espais són tonellats adaptat a aquestes situacions. Tanmateix, sembla ser inevitable descompondre funcions hol.lomorfes per a provar qualsevol resultat relatiu a les condicions suficients. Per tant introduïm diversos contextos als quals el darrer és possible, dins d'aquests contextos aconseguim la descomposició de maneres diferents, és a dir, per descomposició de polinomis (en el disc i en el pla), un mètode connectat amb la teoria de projeccions de Bergman, dos tipus de representacions de l'espai de successions i el $\overline{\partial}$-mètode de Hörmander. Sota algunes hipòtesis addicionals (que satisfan -com mostrem- molts exemples) finalment donem en quasi tots els contextos mencionats anteriorment unes caracteritzacions completes de quan l'espai és ultrabornològic, quan és tonellat i quan els límits inductiu i projectiu són intercanviables.

Per finalitzar la nostra investigació d'espais-(PLB) ponderats, presentem dos resultats (un per funcions contínues i altre per a hol.lomorfes) els quals mostren que espais d'aquest tipus es poden escriure en alguns casos com el producte tensorial d'un espai de Fréchet i un espai-(DF). Combinat amb els resultats en espais-(PLB) ponderats, el resultat en funcions contínues està connectat amb el treball de Grothendieck, el qual va estudiar quan aquest tipus de producte tensorial era ultrabornològic. El segon resultat en representacions de productes tensorials mostra

que alguns espais d'ultradistribucions (introduïts recentment per Schmets i Valdivia) resulten ser espais-(PLB) ponderats de funcions hol.lomorfes.

Summary

Projective limits of inductive limits of Banach spaces, so-called (PLB)-spaces, arise naturally in analysis. For instance the space of distributions, the space of real analytic functions and several spaces of ultradifferentiable functions and ultradistributions are of this type. In this thesis we study (PLB)-spaces whose building blocks are Banach spaces of holomorphic functions defined by a weighted sup-norm. The investigation of these spaces extends research of Agethen, Bierstedt, Bonet who recently studied weighted (PLB)-spaces of continuous functions. From another perspective, it extends the study of weighted inductive limits of Banach spaces of holomorphic functions, which have been studied intensely during the last years by several authors.

Our aim concerning the spaces described above is to study locally convex properties like ultrabornologicity or barrelledness. As the starting point in the definition of the spaces under investigation is a double sequence of strictly positive and continuous functions (weights), our aim is to characterize the forementioned properties in terms of this sequence. In addition, we investigate under which circumstances projective and inductive limit can be interchanged and therefore the (PLB)-space coincides with the weighted inductive limit of Fréchet spaces defined by the same sequence; spaces of the latter type have been investigated by Bierstedt, Bonet.

We prove necessary conditions for the forementioned properties of the spaces and for the interchangeability of projective and inductive limit under rather mild assumptions. Concerning sufficient conditions we make use of homological methods, whose exploration was started by Palamodov in the late sixties and carried on by Vogt, Wengenroth and many others during the last 40 years. For technical reasons the methods just mentioned do not apply to all cases which we want to study. Thus, we first present a criterion for barrelledness adjusted to these situations. However, it seems to be inevitable to decompose holomorphic functions to prove any result concerning sufficient conditions at all. Therefore we introduce several settings in which the latter is possible; within these settings the decomposition is achieved in different ways, namely by the decomposition of polynomials (on the disc and on the plane), a method connected with the theory of Bergman projections, two types of sequence space representations and Hörmander's $\overline{\partial}$-method. Under some additional assumptions (which are – as we show – satisfied in many examples) we finally provide in almost all settings mentioned above a full characterization of ultrabornologicity, barrelledness and interchangeability of projective and inductive limit.

To accomplish our investigation of weighted (PLB)-spaces, we present two results (one for continuous and one for holomorphic functions) which show that spaces of this type can sometimes be written as a tensor product of a Fréchet space with a (DF)-space. Combined with results on weighted (PLB)-spaces the result on continuous functions is connected to work of Grothendieck, who studied ultrabornologicity of such kinds of tensor products. The second result on tensor product representations exhibits that some of the so-called mixed spaces of ultradistributions (introduced recently by Schmets and Valdivia) happen to be weighted (PLB)-spaces of holomorphic functions.

Contents

1 Introduction

In this thesis we investigate the structure of spaces of holomorphic functions defined on an open subset of $\mathbb{C}^N$ that can be written as a countable intersection of countable unions of weighted Banach spaces of holomorphic functions where the latter are defined by weighted sup-norms. The spaces we are interested in are examples of (PLB)-spaces, i.e. countable projective limits of countable inductive limits of Banach spaces. Spaces of this type arise naturally in analysis, for instance the space of distributions, the space of real analytic functions and several spaces of ultradifferentiable functions and ultradistributions are of this type. In particular, some of the so-called mixed spaces of ultradistributions (studied recently by Schmets, Valdivia [68, 69, 70]) appear to be weighted (PLB)-spaces of holomorphic functions (see section 15). In fact, all the forementioned spaces are even (PLS)-spaces that is the linking maps in the inductive spectra of Banach spaces are compact and some of them even appear to be (PLN)-spaces (i.e. the linking maps are nuclear). During the last years the theory of (PLS)-spaces has played an important role in the application of abstract functional analytic methods to several classical problems in analysis. We refer to the survey article [38] of Domański for applications, examples and further references.

The applications reviewed by Domański [38] are based on two abstract tools, namely sequence space representations and the theory of the so-called first derived projective limit functor; in fact our investigations in this thesis are also based on these two methods. The latter method has its origin in the application of homological algebra to functional analysis. The research on this subject was started by Palamodov [62, 61] in the late sixties and carried on since the mid eighties by Vogt [77] and many others. We refer to the book of Wengenroth [84], who laid down a systematic study of homological tools in functional analysis and in particular presents many ready-for-use results concerning concrete analytic problems. In particular, [84, section 5] illustrates that for the splitting theory of Fréchet or more general locally convex spaces, the consideration of (PLB)-spaces which are not (PLS)-spaces is indispensable.

A major application of the theory of the derived projective limit functor Proj^1 is the connection between its vanishing on a countable projective spectrum of (LB)-spaces and locally convex properties of the projective limit of the spectrum (e.g. being ultrabornological or barrelled). This connection was firstly noticed by Vogt [77, 79], see [84, 3.3.4 and 3.3.6], who also gave complete characterizations of the vanishing of Proj^1 and the forementioned properties in the case of sequence spaces, cf. [79, section 4]. A natural extension of Vogt's work is to study the case of continuous functions, which was the subject of the thesis of Agethen [1]. Recently, an extended and improved version of her results was published by Agethen, Bierstedt, Bonet [2]. In addition to the study of the projective limit functor, Agethen, Bierstedt, Bonet studied the interchangeability of projective and inductive limit, i.e. the question when the (PLB)-spaces are equal to the weighted (LF)-spaces of continuous functions studied for the first time by Bierstedt, Bonet [18]. Moreover, the work of Agethen, Bierstedt, Bonet exhibits that certain spaces of linear and continuous operators between Köthe echelon spaces as well as certain tensor products of a Köthe echelon with a coechelon space happen to be weighted

(PLB)-spaces of continuous functions, see [2, section 4]. In view of the results of [2], it is a natural objective to extend the investigation on weighted (PLB)-spaces of holomorphic functions, having in mind the same type of questions.

As in the cases of sequence spaces and continuous functions the starting point in the definition of weighted (PLB)-spaces of holomorphic functions is a double sequence of strictly positive and continuous functions (weights). According to the above, our first aim is the characterization of locally convex properties of the spaces in terms of this sequence. Secondly, we study the interchangeability of projective and inductive limit, which is of course closely connected with weighted (LF)-spaces of holomorphic functions, defined and studied recently by Bierstedt, Bonet [19]. A main concern in the research on the latter subject was the so-called projective description problem, whose study was initiated in the seminal article of Bierstedt, Meise, Summers [27] (for (LB)-spaces) and carried on by many others; we refer to the survey article [12] of Bierstedt for historical notes, further references and a summary of the state-of-the-art concerning projective description. Weighted (LB)-spaces of holomorphic functions as such but also results on projective description (which provides a characterization of the weighted inductive topology in terms of weighted sup-seminorms when answered positively) play an important role for this work, since the steps in the projective spectra under investigation are spaces of this type.

First, we establish definitions and terminology for the study of the weighted (PLB)-spaces of holomorphic functions $AH(G)$ and $(AH)_0(G)$ under O- respectively o-growth conditions in section 2. In section 3 we return to the case of continuous functions, where we give a review of the main results of Agethen, Bierstedt, Bonet. We supplement these results by presenting a tensor product representation which extends those of [2, section 4] in the way that the Köthe (co-)echelon spaces are replaced by a weighted Fréchet and a weighted (LB)-space of continuous functions, respectively. Involving the linear topological invariants (DN) and (Ω), introduced by Vogt [75] and Vogt, Wagner [81] and studied by many others, we deduce a criterion for the latter tensor products to be ultrabornological. The general question of determining locally convex properties of the tensor product of a Fréchet and a (DF)-space was raised by Grothendieck in the last section of [45]. In section 4 we firstly present a necessary condition for the vanishing of Proj^1 under the assumption that all the weighted Banach spaces are contained in some "big" space (for the spaces under investigation this is the case since here we are concerned with subspaces of the space of all holomorphic funtions). Secondly, we present an inheritance property of (quasi-)barrelledness, see 4.4, which is an abstract version of a well-known method used by Bierstedt, Bonet [16]. We conclude this rather abstract section with a criterion for bornologicity of projective limits of inductive limits of normed spaces (cf. 4.10) which will turn out to be very useful for finding sufficient conditions for barrelledness in the case of o-growth conditions. After these supplementary and preparatory sections we start the study of $AH(G)$ and $(AH)_0(G)$ in section 5 with results on necessary conditions for the vanishing of Proj^1 and for barrelledness of the spaces. With the help of 4.4 we are able to prove that the same weight condition is necessary for barrelledness of $AH(G)$ and $(AH)_0(G)$ within a setting of rather mild assumptions on the domain G and the double sequence, which is motivated by the article [20] of Bierstedt, Bonet, Galbis. In the second part of section 5, we discuss the special cases of so-called essential weights (see Taskinen [72]) and consequences of condition (Σ) which is a generalization of condition (V) of Bierstedt, Meise, Summers [27]. In sections 6-10 we study sufficient conditions for the vanishing of Proj^1 and for barrelledness of $(AH)_0(G)$, where we use for the latter the criterion 4.10 of section 4. For the application of the homological methods and also to use 4.10 we have to decompose holomorphic functions. Since there is up to now no method available to do this in broad generality (as it is possible in the case of continuous functions, cf. [2, 3.5]), we have to restrict ourselves to special situations. Thus, in sections 6 and 7 we study spaces over the unit disc and the complex plane where a decomposition of holomorphic functions can be achieved by a decomposition method for polynomials. These two settings trace back to work of Bierstedt, Bonet [19] and Bierstedt, Bonet, Taskinen [22]; the setting in section 6 relies on results of Lusky [53, 54]. In section 8 we study another special setting for the unit disc where the decomposition method is connected with the theory of Bergman projections and goes back to the article [32] of Bonet, Engliš, Taskinen. In section 9 we use two different methods (based on Meise [56] and Domański, Vogt [41]) to obtain a sequence space representation of $AH(G)$ which allows us to use the results on continuous functions in order to get sufficient conditions for the vanishing of Proj^1. By the special assumptions of that section the space $AH(G)$ is a priori a (PLN)-space. The last special setting (section 10) deals with non-radial weights and the decomposition is based on Hörmanders $\overline{\partial}$-methods in the variant developped by Meise, Taylor [57]. Before we study the interchangeability of projective and inductive limit in section 12, we discuss relations between the weight conditions and also abstract conditions (in particular of the condition (B1) which arised in the bornologicity criterion 4.10) used in the earlier sections. We also examine the special cases of $AH(G)$ and $(AH)_0(G)$ being Fréchet or (LB)-spaces. In section 13 we revisit condition (Σ) and present several corollaries under this additional assumption, which is in some sense rather natural as it is satisfied by many of the examples presented in section 14. Finally, in the appendix (section 15) we show that several mixed spaces of ultradistributions (cf. [68, 69, 70]) can be regarded as weighted (PLB)-spaces of holomorphic functions.

2 Preliminaries

Let G be an open subset of $\mathbb{C}^d$ and $d \geqslant 1$. By $H(G)$ we denote the space of all holomorphic functions on G. A *weight* a on G is a strictly positive and continuous function on G. For a weight a we define

$$\begin{aligned} Ha(G) &:= \{\, f \in H(G)\,;\ \|f\|_a := \sup_{z\in G} a(z)|f(z)| < \infty \,\},\\ Ha_0(G) &:= \{\, f \in H(G)\,;\ a|f| \text{ vanishes at } \infty \text{ on } G \,\}. \end{aligned}$$

Recall that a function $g\colon G \to \mathbb{R}$ is said to vanish at infinity on G if for each $\varepsilon > 0$ there is a compact set K in G such that $|g(z)| < \varepsilon$ for all $z \in G\backslash K$. The space $Ha(G)$ is a Banach space for the norm $\|\cdot\|_a$ and $Ha_0(G)$ is a closed subspace of $Ha(G)$. In the first case we speak of *O-growth conditions* and in the second of *o-growth conditions*.

In order to define the projective spectra we are interested in, we consider a double sequence $\mathcal{A} = ((a_{N,n})_{n\in\mathbb{N}})_{N\in\mathbb{N}}$ of weights on G which is decreasing in n and increasing in N, i.e.

$$\forall\, N,\, n \in \mathbb{N} \;:\; a_{N,n+1} \leqslant a_{N,n} \leqslant a_{N+1,n}.$$

This condition will be assumed on the double sequence $\mathcal{A}$ in the rest of this work. We define the norms $\|\cdot\|_{N,n} := \|\cdot\|_{a_{N,n}}$ and hence we have

$$\forall\, N,\, n \in \mathbb{N} \;:\; \|\cdot\|_{N,n+1} \leqslant \|\cdot\|_{N,n} \leqslant \|\cdot\|_{N+1,n}.$$

Accordingly, $Ha_{N,n}(G) \subseteq Ha_{N,n+1}(G)$ and $H(a_{N,n})_0(G) \subseteq H(a_{N,n+1})_0(G)$ holds with continuous inclusions for all N and n and we can define for each $N \in \mathbb{N}$ the weighted inductive limits

$$\mathcal{A}_N H(G) := \operatorname{ind}_n Ha_{N,n}(G) \;\text{ and }\; (\mathcal{A}_N)_0 H(G) := \operatorname{ind}_n H(a_{N,n})_0(G).$$

We denote by $B_{N,n}$ the closed unit ball of the Banach space $Ha_{N,n}(G)$, i.e.

$$B_{N,n} := \{\, f \in H(G)\,;\; \|f\|_{N,n} \leqslant 1 \,\}.$$

By Bierstedt, Meise, Summers [27, end of the remark after Theorem 1.13] (cf. also Bierstedt, Meise [25, Proposition 3.5.(2)]) we know that $\mathcal{A}_N H(G)$ is a complete, hence regular (LB)-space. We will assume without loss of generality by multiplying by adequate scalars, that every bounded subset B of $\mathcal{A}_N H(G)$ is contained in $B_{N,n}$ for some n.

The weighted inductive limits $(\mathcal{A}_N)_0 H(G)$ need not to be regular. The closed unit ball of the Banach space $H(a_{N,n})_0(G)$ is denoted by

$$B^{\circ}_{N,n} := \{\, f \in H(a_{N,n})_0(G)\,;\; \|f\|_{N,n} \leqslant 1 \,\}.$$

For each $N \in \mathbb{N}$ we have $\mathcal{A}_{N+1}H(G) \subseteq \mathcal{A}_N H(G)$ and $(\mathcal{A}_{n+1})_0 H(G) \subseteq (\mathcal{A}_N)_0 H(G)$ with continuous inclusions. $\mathcal{A}H := (\mathcal{A}_N H(G))_N$ and $\mathcal{A}_0 H := ((\mathcal{A}_N)_0 H(G))_N$ are projective spectra of (LB)-spaces with inclusions as linking maps. We can then form the following projective limits, called *weighted (PLB)-spaces of holomorphic functions*

$$AH(G) := \operatorname{proj}_N \mathcal{A}_N H(G),$$
$$(AH)_0(G) := \operatorname{proj}_N (\mathcal{A}_N)_0 H(G),$$

which are the object of our study in this work. From the universal property of the inductive limit it follows that $(\mathcal{A}_N)_0 H(G) \subseteq \mathcal{A}_N H(G)$ holds with continuous inclusion for each $N \in \mathbb{N}$. Hence the same is true for the projective limits, i.e. $AH(G) \subseteq (AH)_0(G)$ holds with continuous inclusion.

Our objectives concerning these spaces are the following:

a. Investigation of the structure of $AH(G)$ and $(AH)_0(G)$: In particular finding necessary or sufficient conditions for the spaces having "nice" locally convex properties, e.g. being (ultra-)bornological or barrelled, which are formulated in terms of $\mathcal{A}$.

b. Investigation of (homological) properties of the projective spectra $\mathcal{A}H$ and $\mathcal{A}_0H$: In particular finding necessary or sufficient conditions for the vanishing of the derived functor of the projective limit functor on these spectra.

c. Investigation of the commutativity of projective and inductive limit: In particular finding necessary or sufficient conditions for the interchangeability.

An important tool to handle weighted spaces of holomorphic functions is the technique of associated weights or growth conditions mentioned by Anderson and Duncan [3], studied for the first time in a systematic way by Bierstedt, Bonet, Taskinen [21] and used in many articles dealing with weighted spaces of holomorphic functions. For a given weight a we call $w := \frac{1}{a}$ the corresponding *growth condition* and define [21, Definition 1.1]

$$\tilde{w} = (\tfrac{1}{a})^{\sim}\colon G \to \mathbb{R},\;\; z \mapsto \sup_{\substack{g\in H(G),\\ |g|\leqslant w}} |g(z)| \;=\; \sup_{g\in B_a} |g(z)|.$$

In [21, previous to Observation 1.12], Bierstedt, Bonet, Taskinen put $\tilde{a} := \frac{1}{\tilde{w}}$ and called $\tilde{a}$ the *weight associated with* a. Since this notation is a bit subtle (to get $\tilde{a}$ we cannot just replace w with a in the above formula) we have to be careful and always distinguish weights and growth conditions. Note that by the above remarks $\big(\frac{1}{a}\big)^{\sim} = \frac{1}{\tilde{a}}$ holds. However, we will in most cases stick to the first notation (cf. the weight conditions below). Bierstedt, Bonet, Taskinen (cf. [21, 4.B after 1.12]) introduced as well an associated weight for the case of o-growth conditions, i.e. $\tilde{w}_0 = (\frac{1}{a})^{\sim}_0$, where

$$\tilde{w}_0 = (\tfrac{1}{a})^{\sim}_0\colon G \to \mathbb{R},\;\; z \mapsto \sup_{\substack{g\in H(G),\, |g|\leqslant w\\ \text{and } a|g| \text{ vanish-}\\ \text{es at } \infty \text{ on } G}} |g(z)| \;=\; \sup_{g\in B_a^{\circ}} |g(z)|,$$

but we will see that in a rather general setting both notions coincide.

In [80] Vogt introduced the conditions (Q) and (wQ). In the case of weighted (PLB)-spaces one can reformulate these conditions in terms of the weights as follows. We say that the sequence $\mathcal{A}$ satisfies condition (Q) if

$$\forall\, N\; \exists\, M \geqslant N,\, n\; \forall\, K \geqslant M,\, m,\, \varepsilon > 0\; \exists\, k,\, S > 0 : \tfrac{1}{a_{M,m}} \leqslant \max\big(\tfrac{\varepsilon}{a_{N,n}}, \tfrac{S}{a_{K,k}}\big),$$

we say that it satisfies (wQ) if

$$\forall\, N\; \exists\, M \geqslant N,\, n\; \forall\, K \geqslant M,\, m\; \exists\, k,\, S > 0 : \tfrac{1}{a_{M,m}} \leqslant \max\big(\tfrac{S}{a_{N,n}}, \tfrac{S}{a_{K,k}}\big).$$

It is clear that condition (Q) implies condition (wQ). Bierstedt, Bonet gave in [18] an example of a sequence which satisfies (wQ) but not (Q). We define the following conditions by the use of the associated weights, where the quantifiers are always those of (wQ) or (Q) resp. and the estimates are the following:

$$(\mathrm{Q})^{\sim}_{\mathrm{in}} \;:\; \big(\tfrac{1}{a_{M,m}}\big)^{\sim} \leqslant \max\big(\big(\tfrac{\varepsilon}{a_{N,n}}\big)^{\sim}, \big(\tfrac{S}{a_{K,k}}\big)^{\sim}\big)$$

$$\begin{array}{rcl} (\mathrm{Q})^{\sim}_{\mathrm{out}} & : & \big(\frac{1}{a_{M,m}}\big)^{\sim} \leqslant \big(\max\big(\frac{\varepsilon}{a_{N,n}}, \frac{S}{a_{K,k}}\big)\big)^{\sim} \\ (\mathrm{wQ})^{\sim}_{\mathrm{in}} & : & \big(\frac{1}{a_{M,m}}\big)^{\sim} \leqslant S \max\big(\big(\frac{1}{a_{N,n}}\big)^{\sim}, \big(\frac{1}{a_{K,k}}\big)^{\sim}\big) \\ (\mathrm{wQ})^{\sim}_{\mathrm{out}} & : & \big(\frac{1}{a_{M,m}}\big)^{\sim} \leqslant S\big(\max\big(\frac{1}{a_{N,n}}, \frac{1}{a_{K,k}}\big)\big)^{\sim} \end{array}$$

By Bierstedt, Bonet, Taskinen [21, Proposition 1.2.(vii)] $\frac{1}{a}, \frac{1}{b} \leqslant \max\big(\frac{1}{a}, \frac{1}{b}\big)$ implies $\big(\frac{1}{a}\big)^{\sim}, \big(\frac{1}{b}\big)^{\sim} \leqslant \max\big(\frac{1}{a}, \frac{1}{b}\big)^{\sim}$ and hence $\max\big(\big(\frac{1}{a}\big)^{\sim}, \big(\frac{1}{b}\big)^{\sim}\big) \leqslant \max\big(\frac{1}{a}, \frac{1}{b}\big)^{\sim}$ for all weights a and b in the above sense. That is, condition $(\mathrm{Q})^{\sim}_{\mathrm{in}}$ implies $(\mathrm{Q})^{\sim}_{\mathrm{out}}$ and $(\mathrm{wQ})^{\sim}_{\mathrm{in}}$ implies $(\mathrm{wQ})^{\sim}_{\mathrm{out}}$ in general. Moreover, by [21, Proposition 1.2.(vii)] condition (wQ) implies $(\mathrm{wQ})^{\sim}_{\mathrm{out}}$ and condition (Q) implies $(\mathrm{Q})^{\sim}_{\mathrm{out}}$.

In [76, Satz 1.1] Vogt introduced the following condition to characterize Fréchet spaces between which all continuous linear mappings are bounded. According to Vogt, but reformulated for our setting, we say that a sequence $\mathcal{A}$ as above satisfies condition (B) if

$$\forall\, (n(N))_{N\in\mathbb{N}} \subseteq \mathbb{N}\; \exists\, m\; \forall\, M\; \exists\, L,\, c > 0\colon a_{M,m} \leqslant c \max_{N=1,\dots,L} a_{N,n(N)}.$$

Condition $(\mathrm{B})^{\sim}$ is defined by the same quantifiers as above and the estimate replaced by

$$\tilde{a}_{M,m} \leqslant c\big(\max_{N=1,\dots,L} a_{N,n(N)}\big)^{\sim}.$$

As above, [21, Proposition 1.2.(vii)] provides that (B) implies $(\mathrm{B})^{\sim}$.

3 Weighted (PLB)-spaces of continuous functions: Summary of known and some supplementary results

3.1 Properties of the spaces AC(X) and $(\mathbf{AC})_0(\mathbf{X})$

The continuous analoga of the spaces $AH(G)$ and $(AH)_0(G)$ have been defined and investigated in the thesis of Agethen [1]. Recently her results have been extended, reorganized and published in Agethen, Bierstedt, Bonet [2]. This is the main reference for this section. Let in the sequel X be a locally compact and σ-compact topological space. The definition of $AC(X)$ and $(AC)_0(X)$ is completely analogous to that of $AH(G)$ and $(AH)_0(G)$ given in section 2: We simply replace H with C and G with X everywhere. Since by Bierstedt, Bonet [17, Section 1] the spaces $\mathcal{A}_N C(X)$ are always complete we may – as in the holomorphic case – assume that every bounded set in $\mathcal{A}_N C(X)$ is contained in $B_{N,n}$ for some n. Also as in the holomorphic case, $(\mathcal{A}_N)_0 C(X)$ needs not to be regular. By [27, Theorem 2.6] it is regular if and only if it is complete and this is equivalent to the fact that the sequence $\mathcal{A}_N := (a_{N,n})_{n\in\mathbb{N}}$ is regularly decreasing (see [27, Definition 2.1 and Theorem 2.6] and 3.18). However, $(\mathcal{A}_N)_0 C(X) \subseteq \mathcal{A}_N C(X)$ is a topological subspace for each $N \in \mathbb{N}$ (cf. [27, Corollary 1.4.(a)]) and hence $(AC)_0(X)$ is a topological subspace of $AC(X)$. Let us denote by $\mathcal{A}C = (\mathcal{A}_N C(X))_N$ and $\mathcal{A}_0 C = ((\mathcal{A}_N)_0 C(X))_N$ as in the holomorphic case the projective spectra of (LB)-spaces. Then $\mathcal{A}_0 C$ is reduced in the sense that $(AC)_0(X)$ is dense in every step (cf. [2, section 2]). We have the following two results concerning homological properties of the spectra $\mathcal{A}C$ and $\mathcal{A}_0C$ and the locally convex properties of $AC(X)$ and $(AC)_0(X)$.

Theorem A. ([2, Theorem 3.7]) The following are equivalent.

(i) $\operatorname{Proj}^1 \mathcal{A}_0C = 0$.
(ii) $(AC)_0(X)$ is ultrabornological.
(iii) $(AC)_0(X)$ is barrelled.
(iv) $\mathcal{A}$ satisfies condition (wQ).

Theorem B. ([2, Theorems 3.5 and 3.8]) We have (i)⇔(ii)⇒(iii)⇒(iv)⇒(v), where

(i) $\mathcal{A}$ satisfies condition (Q),
(ii) $\operatorname{Proj}^1 \mathcal{A}C = 0$,
(iii) $AC(X)$ is ultrabornological,
(iv) $AC(X)$ is barrelled,
(v) $\mathcal{A}$ satisfies condition (wQ).

In the sequel we complement the above by considering the special situation that the domain X is the product of two topological spaces X_1 and X_2 and the sequence $\mathcal{A}$ is the product of an increasing sequence $(a^1_N)_{N\in\mathbb{N}}$ defined on X_1 and a decreasing sequence $((a^2_n)^{-1})_{n\in\mathbb{N}}$ defined on X_2. In this special setting we can associate a weighted Fréchet space to each of the sequences $(a^1_N)_{N\in\mathbb{N}}$ and $(a^2_n)_{n\in\mathbb{N}}$ and thus draw the line between properties of the (PLB)-space and the invariants (DN) and (Ω) for Fréchet spaces. In order to do this we have to characterize (DN) and (Ω) in terms of the weights.

3.2 Supplementary results: (DN) and (Ω) vs. (wQ)

Vogt [75] and Vogt, Wagner [81] introduced the following conditions, which are topological invariants of Fréchet spaces. We say that a Fréchet space E with a fundamental sequence of seminorms $(\|\cdot\|_n)_{n\in\mathbb{N}}$ satisfies condition (DN), if

$$\exists\, n\;\forall\, m\geqslant n,\ 0<\theta<1\;\exists\, k\geqslant m,\ C>0\;\forall\, x\in E:\ \|x\|_m\leqslant C\|x\|_k^{\theta}\|x\|_n^{1-\theta}.$$

By Meise, Vogt [60, 29.10] the latter statement is equivalent to the original formulation [60, Definition on p. 359] of (DN). According to [60, Definition on p. 367] we say that a Fréchet space E with a fundamental sequence of seminorms $(\|\cdot\|_n)_{n\in\mathbb{N}}$ satisfies condition (Ω), if

$$\forall N\;\exists M\geqslant N\;\forall K\geqslant M\;\exists D>0,\ 0<\theta<1\;\forall y\in E':\ \|y\|^{\star}_M\leqslant D(\|y\|^{\star}_K)^{\theta}(\|y\|^{\star}_N)^{1-\theta},$$

where $\|y\|^{\star}_n:=\sup_{\|x\|_n\leqslant 1}|y(x)|$. It is well-known (and an immediate consequence of [60, 29.13]) that replacing the above estimate by the inclusion $U_M\subseteq r\,U_N+D\,r^{1-1/\theta}\,U_K$ (required for each $r>0$ and where $U_n:=\{x\in E\,;\,\|x\|_n\leqslant 1\}$ for $n\in\mathbb{N}$) yields an equivalent formulation of (Ω).

3.2.1 Weighted Fréchet spaces of continuous functions

Definition 3.1. Let $A=(a_n)_{n\in\mathbb{N}}$ be an increasing sequence of weights on a Hausdorff locally compact and σ-compact topological space X. We say that A satisfies condition $(\mathrm{DN})_{\mathrm{w}}$ if

$$\exists\, n\ \forall\, m \geqslant n,\ 0<\theta<1\ \exists\, k \geqslant m,\ C>0 : a_m \leqslant C a_k^{\theta} a_n^{1-\theta}.$$

We say that A satisfies condition $(\Omega)_{\mathrm{w}}$ if

$$\forall\, N\ \exists\, M \geqslant N\ \forall\, K \geqslant M\ \exists\, D>0,\ 0<\theta<1 : \tfrac{1}{a_M} \leqslant D\big(\tfrac{1}{a_K}\big)^{\theta}\big(\tfrac{1}{a_N}\big)^{1-\theta}.$$

Lemma 3.2. Let $A=(a_n)_{n\in\mathbb{N}}$ be an increasing sequence of weights on a Hausdorff locally compact and σ-compact topological space X. The Fréchet space $CA_{(0)}(X)=\operatorname{proj}_n C(a_n)_{(0)}(X)$ satisfies condition (DN) if and only if the sequence A satisfies $(\mathrm{DN})_{\mathrm{w}}$.

Proof. "$\Leftarrow$" We choose n as in $(\mathrm{DN})_{\mathrm{w}}$. For given $m \geqslant n$, $0<\theta<1$ we choose $k \geqslant m$ and $C>0$ as in $(\mathrm{DN})_{\mathrm{w}}$. For an arbitrary $f \in CA_0(X)$ we obtain

$$\begin{aligned}\|f\|_m &= \sup_{x\in X} a_m(x)|f(x)| \\ &\leqslant C \sup_{x\in X} a_k^{\theta}(x) a_n^{1-\theta}(x)|f(x)|^{\theta+1-\theta} \\ &\leqslant C \sup_{x\in X}(a_k(x)|f(x)|)^{\theta} \sup_{x\in X}(a_n(x)|f(x)|)^{1-\theta} \\ &= C\|f\|_k^{\theta}\|f\|_n^{1-\theta}\end{aligned}$$

and hence we have shown (DN).

"$\Rightarrow$" Let $x_0 \in X$ be fixed. Since X is locally compact, there is a neighborhood filter $(K_\beta)_{\beta\in B}$ for x_0 consisting of compact sets. Since X is Hausdorff, $\cap_{\beta\in B} K_\beta = \{x_0\}$. We choose for each K_β a function $f_\beta \in C_c(X)$, that is in the space of continuous functions with compact supportspaces]$C_c(X)$, with $f_\beta(x_0)=1$, $0 \leqslant f_\beta \leqslant 1$ and $\operatorname{supp} f_\beta \subseteq K_\beta$. Thus we have $\cap_{\beta\in B} \operatorname{supp} f_\beta = \{x_0\}$. Now we consider the net $(\sup_{x\in X} a(x)|f_\beta(x)|^{\gamma})_{\beta\in B} \subseteq \mathbb{R}$ for some fixed weight a and $\gamma>0$ and obtain

$$\sup_{x\in X} a(x)|f_\beta(x)|^{\gamma} = \sup_{x\in \operatorname{supp} f_\beta} a(x)|f_\beta(x)|^{\gamma} \leqslant \sup_{x\in \operatorname{supp} f_\beta} a(x) \ \longrightarrow\ a(x_0)$$

and $\sup_{x\in X} a(x)|f_\beta(x)|^{\gamma} \geqslant a(x_0)|f_\beta(x_0)|^{\gamma} = a(x_0)$.

Now we select n as in (DN). For given $m \geqslant n$ and $0<\theta<1$ we choose $k \geqslant m$ and $C>0$ as in (DN). Now we fix some $x_0 \in X$ and consider the f_β defined above. We have

$$\sup_{x\in X} a_m(x)|f_\beta(x)| \leqslant C \sup_{x\in X} a_k^{\theta}(x)|f_\beta(x)|^{\theta} \sup_{x\in X} a_n^{1-\theta}(x)|f_\beta(x)|^{1-\theta}$$

for each $\beta \in B$. Taking limits on each side yields the desired inequality, which finishes the proof since x_0 was arbitrary. ■

Lemma 3.3. Let $A=(a_n)_{n\in\mathbb{N}}$ be an increasing sequence of weights on a Hausdorff locally compact and σ-compact topological X. The Fréchet space $CA_{(0)}(X) = \operatorname{proj}_n C(a_n)_{(0)}(X)$ satisfies condition (Ω) if and only if the sequence A satisfies $(\Omega)_{\mathrm{w}}$.

Proof. "⇐" We put $a := \frac{1}{a_N}$ and $b := \frac{1}{a_K}$ and use (cf. [60, proof of 29.13]) that

$$\min_{s>0}(sa + s^{1-1/\theta}\, b) = \theta^{-1}(\theta^{-1}-1)^{\theta-1}a^{1-\theta}\, b^\theta$$

holds for arbitrary a and $b > 0$. We put $C := \theta^{-1}(\theta^{-1}-1)^{\theta-1} > 0$ and thus get

$$a^{1-\theta}\, b^\theta \leqslant \tfrac{1}{C}\,(sa + s^{1-1/\theta}\, b)$$

for each $s > 0$. Hence we obtain

$$D\,\big(\tfrac{1}{a_K}\big)^\theta\big(\tfrac{1}{a_N}\big)^{1-\theta} \leqslant \tfrac{D}{C}\big(s\,\tfrac{1}{a_N} + s^{1-1/\theta}\,\tfrac{1}{a_K}\big) \leqslant \max\big(\,2\,\tfrac{D}{C}\,s\,\tfrac{1}{a_N},\, 2\,\tfrac{D}{C}\,s^{1-1/\theta}\,\tfrac{1}{a_K}\,\big)$$

for each $s > 0$. Now we define $r := 4\,\frac{D}{C}\,s$ and $D' := \frac{4D}{C}(\frac{C}{4D})^{1-1/\theta} > 0$, hence $s = \frac{C}{4D}r$ and therefore $2\frac{D}{C}s^{1-1/\theta} = \frac{4D}{2C}(\frac{C}{4D})^{1-1/\theta}\,r^{1-1/\theta} = \frac{D'}{2}\,r^{1-1/\theta}$. Replacing D' with D we get

$$(\star)\ \forall N\ \exists M \geqslant N\ \forall K \geqslant M\ \exists D > 0,\ 0 < \theta < 1\ \forall r > 0 : \tfrac{1}{a_M} \leqslant \max\big(\tfrac{r}{2a_N},\ \tfrac{D\,r^{1-1/\theta}}{2a_K}\big).$$

Now we show (Ω) in the second formulation mentioned at the beginning of this section. Let N be given. We choose $M \geqslant N$ as in $(\star)$. For given $K \geqslant M$ we select $D > 0$ and $0 < \theta < 1$ as in $(\star)$ and take an arbitrary $r > 0$. Let $f \in U_M^{(\circ)}$ be fixed, i.e. $|f| \leqslant \frac{1}{a_M} \leqslant \max\big(\frac{r}{2a_N},\ \frac{D\,r^{1-1/\theta}}{2a_K}\big)$. By [2, Lemma 3.4] there exist $\varphi_1, \varphi_2 \in C(X)$ with $0 \leqslant \varphi_1, \varphi_2 \leqslant 1$, $\varphi_1 + \varphi_2 = f$ such that $|\varphi_1| \leqslant \frac{r}{a_N}$, $|\varphi_2| \leqslant \frac{Dr^{1-1/\theta}}{a_K}$, i.e. $\varphi_1 \in rU_N^{(\circ)}$, $\varphi_2 \in Dr^{1-1/\theta}U_K^{(\circ)}$ and thus $f \in rU_N^{(\circ)} + Dr^{1-1/\theta}U_K^{(\circ)}$, where

$$U_N^{(\circ)} = \{f \in C(A)_{(0)}(X)\,;\ \sup_{x\in X} a_N(x)|f(x)| \leqslant 1\}.$$

"⇒" For a fixed N and for $x_0 \in X$ we consider $\delta_{x_0} \colon C(a_n)_0(X) \to \mathbb{C}$, $\delta_{x_0}(f) := f(x_0)$. Then we have

$$\|\delta_{x_0}\|_N^\star = \sup_{f\in U_N} |\delta_{x_0}(f)| = \sup_{f\in U_N} |f(x_0)| \leqslant \tfrac{1}{a_N(x_0)}.$$

We choose $\varphi \in C_c(X)$ with $\varphi(x_0) = 1$, $0 \leqslant \varphi \leqslant 1$ on X and put $f_0 := \frac{\varphi}{a_N}$. Then $f_0 \in C(a_N)_0(X)$ and

$$\sup_{x\in X} a_N(x)|f_0(x)| = \sup_{x\in X} a_N(x)\tfrac{\varphi(x)}{a_N(x)} = \sup_{x\in X}\varphi(x) = 1,$$

i.e. $f_0 \in U_N^{(\circ)}$. $\delta_{x_0}(f_0) = f_0(x_0) = \frac{\varphi(x_0)}{a_N(x_0)} = \frac{1}{a_N(x_0)}$ implies $\|\delta_{x_0}\|_N^\star = \frac{1}{a_N(x_0)}$. $(\Omega)_{\mathrm{w}}$ is the special case of (Ω) where we choose the functional to be δ_x for arbitrary $x \in X$. ■

3.2.2 (DN) and (Ω) vs. (wQ)

After these preparations we are ready to investigate the situation we mentioned at the end of section 3.1: For $i \in \{1,2\}$ let X_i denote a locally compact and σ-

compact Hausdorff topological space. Moreover, let $A^i = (a_n^i)_{n\in\mathbb{N}}$ be an increasing sequence of weights on X_i, i.e. $a_n^i \leqslant a_{n+1}^i$ for all $n \in \mathbb{N}$. We define the double sequence $\mathcal{A} = ((a_{N,n})_{N\in\mathbb{N}})_{n\in\mathbb{N}}$ by setting

$$a_{N,n} : X_1 \times X_2 \to \mathbb{R},\ \ (x_1,x_2) \mapsto a_{N,n}(x_1,x_2) := \big[a_N^1 \otimes \tfrac{1}{a_n^2}\big](x_1,x_2) = \tfrac{a_N^1(x_1)}{a_n^2(x_2)}.$$

Thus, $\mathcal{A}$ satisfies the estimates $a_{N,n+1}(x_1,x_2) \leqslant a_{N,n}(x_1,x_2) \leqslant a_{N+1,n}(x_1,x_2)$ for all N, n and $(x_1,x_2) \in X_1 \times X_2$. In the sequel we refer to a sequence of the latter form by $\mathcal{A} = A^1 \otimes (A^2)^{-1}$. To simplify notation we put $X := X_1 \times X_2$ and consider the (PLB)-space $(AC)_0(X)$ in the notation established at the beginning of section 3. In view of 3.A we want to investigate if there is some relation between the conditions (DN) and (Ω) for the Fréchet spaces $C(A^i)_0(X_i)$ and the (PLB)-space $(AC)_0(X)$. According to the results above we can consider the weight conditions $(\mathrm{DN})_{\mathrm{w}}$, $(\Omega)_{\mathrm{w}}$ and (wQ).

The following result was proved by the author using different versions of $(\mathrm{DN})_{\mathrm{w}}$ and $(\Omega)_{\mathrm{w}}$. This original proof was contained in a first version of [83] and was inspired by that of [78, Theorem 5.1] of Vogt, who showed that $\mathrm{Ext}^1(E,F) = 0$ holds for a nuclear Fréchet space E satisfying (DN) and a Fréchet space F satisfying (Ω), see also [84, Corollary 5.2.8]. The versions of $(\mathrm{DN})_{\mathrm{w}}$ and $(\Omega)_{\mathrm{w}}$ aswell as the following proof (which is much simpler then the original one) are based on suggestions of the referee of [83].

Lemma 3.4. Let A^1 resp. A^2 be an increasing sequence of weights on X_1 resp. X_2. Assume that A^1 satisfies $(\Omega)_{\mathrm{w}}$ and that A^2 satisfies $(\mathrm{DN})_{\mathrm{w}}$. Then the sequence $\mathcal{A} = A^1 \otimes (A^2)^{-1}$ on $X_1 \times X_2$ satisfies (wQ).

Proof. In our special setting we have to show the following

$$\forall\, N\ \exists\, M \geqslant N,\ n\ \forall\, K \geqslant M,\ m\ \exists\, k,\ S > 0\ \forall\,(x_1,x_2) \in X_1 \times X_2 :$$

$$\tfrac{a_m^2(x_2)}{a_M^1(x_1)} \leqslant S \max\big(\tfrac{a_n^2(x_2)}{a_N^1(x_1)}, \tfrac{a_k^2(x_2)}{a_K^1(x_1)}\big).$$

In order to do this, let N be given. We select $M \geqslant N$ as in $(\Omega)_{\mathrm{w}}$ and n as in $(\mathrm{DN})_{\mathrm{w}}$. For given $K \geqslant M$ there exist $D > 0$ and $0 < \theta < 1$ with the estimate in $(\Omega)_{\mathrm{w}}$. For arbitrary m and the same θ there exist $k \geqslant m$ and $C > 0$ with the estimate in $(\mathrm{DN})_{\mathrm{w}}$. We put $S := 2CD$ and multiply the estimates in $(\mathrm{DN})_{\mathrm{w}}$ and $(\Omega)_{\mathrm{w}}$ to get

$$\begin{aligned}\tfrac{a_m^2(x_2)}{a_M^1(x_1)} &\leqslant CD\big(\tfrac{a_k^2(x_2)}{a_K^1(x_1)}\big)^{\theta}\big(\tfrac{a_n^2(x_2)}{a_N^1(x_1)}\big)^{1-\theta}\\ &\leqslant CD\big(\tfrac{a_k^2(x_2)}{a_K^1(x_1)} + \tfrac{a_n^2(x_2)}{a_N^1(x_1)}\big)\\ &\leqslant S\max\big(\tfrac{a_k^2(x_2)}{a_K^1(x_1)}, \tfrac{a_n^2(x_2)}{a_N^1(x_1)}\big)\end{aligned}$$

for each $(x_1,x_2) \in X_1 \times X_2$. ∎

Proposition 3.5. Let A^1 resp. A^2 be an increasing sequence of weights on X_1 resp. X_2. Consider the sequence $\mathcal{A} = A^1 \otimes (A^2)^{-1}$ on $X = X_1 \times X_2$ and assume that $C(A^1)_0(X_1)$ satisfies (Ω) and that $C(A^2)_0(X_2)$ satisfies (DN). Then $(AC)_0(X)$ is ultrabornological.

Proof. This follows immediately with 3.2, 3.3, 3.4 and 3.1.A. ■

Example 3.6. In general the assumptions of 3.5 do not imply that $\mathcal{A}$ satisfies condition (B). Let s be the Fréchet space of rapidly decreasing sequences. Then for $A = (a_k)_{k\in\mathbb{N}}$ with $a_k(j) = j^k$ we have (cf. [60, 28.16])

$$s = \{\, x \in \mathbb{K}^{\mathbb{N}} \;;\; \forall\, k \in \mathbb{N} : \lim_{j\to\infty} j^k|x_j| = 0 \,\} = c_0(A) = CA_0(\mathbb{N}).$$

In particular, s satisfies (DN) and (Ω) (cf. Meise, Vogt [60, III.29]). Now we consider – exactly as in [82, Example 5.14] – $\mathcal{A} = ((a_N \otimes \frac{1}{a_n})_{n\in\mathbb{N}})_{N\in\mathbb{N}}$ on $\mathbb{N}\times\mathbb{N}$, i.e.

$$(AC)_0(\mathbb{N}\times\mathbb{N}) = \operatorname{proj}_N \operatorname{ind}_n C(a_N \otimes \tfrac{1}{a_n})_0(\mathbb{N}\times\mathbb{N})$$

For completing our example it is enough, to show that $\mathcal{A}$ does not satisfy (B). But this one can find in [82, Example 5.14], where the above example was stated to show that condition (wQ) does not imply condition (B) in general. In [82] for the above space condition (wQ) was checked "by hand" – as we have seen, now 3.5 provides a much more convient way to conclude this. For consequences of the latter see section 3.3.

3.2.3 $(\underline{\text{DN}})$ and $(\overline{\overline{\Omega}})$ vs. (wQ)

In this section we show that 3.5 remains true if in the assumptions we change (DN) into the weaker condition $(\underline{\text{DN}})$ and on the other hand replace (Ω) with the stronger condition $(\overline{\overline{\Omega}})$. In order to do this, we have to go through the preceding proofs and assure that all our arguments apply to the new situation.

As Meise, Vogt [60, Definition previous to 29.11] and e.g. Bonet, Domański [29, 30, 31], we define the invariants $(\underline{\text{DN}})$ and $(\overline{\overline{\Omega}})$: Let E be a Fréchet space with an increasing fundamental system of seminorms $(\|\cdot\|_n)_{n\in\mathbb{N}}$. We say that E satisfies condition $(\underline{\text{DN}})$ if

$$\exists\, n\, \forall\, m \geqslant n\, \exists\, k \geqslant m,\ 0<\theta<1,\ C>0\, \forall\, x \in E : \|x\|_m \leqslant C\|x\|_k^{\theta}\|x\|_n^{1-\theta}.$$

We say that E satisfies condition $(\overline{\overline{\Omega}})$, if

$$\forall N\ \exists M \geqslant N \forall K \geqslant M,\ 0<\theta<1 \exists D>0\, \forall y \in E' : \|y\|_M^{\star} \leqslant D(\|y\|_K^{\star})^{\theta}(\|y\|_N^{\star})^{1-\theta}.$$

Definition 3.7. Let $A = (a_n)_{n\in\mathbb{N}}$ be an increasing sequence of weights on a Hausdorff locally compact and σ-compact space X. We say that A satisfies condition $(\underline{\text{DN}})_{\text{w}}$ if

$$\exists\, n\, \forall\, m \geqslant n\, \exists\, k \geqslant m,\ \alpha>0,\ C>0 : a_m \leqslant C a_k^{\theta} a_n^{1-\theta}.$$

We say that A satisfies condition $(\overline{\overline{\Omega}})_{\text{w}}$ if

$$\forall\, N\, \exists\, M \geqslant N\, \forall\, K \geqslant M,\ \alpha>0\, \exists\, D>0 : \tfrac{1}{a_M} \leqslant D\big(\tfrac{1}{a_K}\big)^{\theta}\big(\tfrac{1}{a_N}\big)^{1-\theta}.$$

The following two lemmas can be proved in complete analogy to 3.2 and 3.3.

Lemma 3.8. Let $A = (a_n)_{n\in\mathbb{N}}$ be an increasing sequence of weights on a Hausdorff locally compact and σ-compact topological X. The Fréchet space $CA_{(0)}(X) = \operatorname{proj}_n C(a_n)_{(0)}(X)$ satisfies $(\underline{\mathrm{DN}})$ if and only if the sequence A satisfies $(\underline{\mathrm{DN}})_{\mathrm{w}}$.

Lemma 3.9. Let $A = (a_n)_{n\in\mathbb{N}}$ be an increasing sequence of weights on a Hausdorff locally compact and σ-compact topological X. The Fréchet space $CA_{(0)}(X) = \operatorname{proj}_n C(a_n)_{(0)}(X)$ satisfies $(\overline{\overline{\Omega}})$ if and only if the sequence A satisfies $(\overline{\overline{\Omega}})_{\mathrm{w}}$.

Analogously to 3.4 one may prove the following.

Lemma 3.10. Let A^1 resp. A^2 be an increasing sequence of weights on X_1 resp. X_2. Assume that A^1 satisfies $(\overline{\overline{\Omega}})_{\mathrm{w}}$ and that A^2 satisfies $(\underline{\mathrm{DN}})_{\mathrm{w}}$. Then the sequence $\mathcal{A} = A^1 \otimes (A^2)^{-1}$ on $X_1 \times X_2$ satisfies (wQ).

Finally, 3.8, 3.9 and 3.10 imply the desired analog of 3.5.

Proposition 3.11. Let A^1 resp. A^2 be an increasing sequence of weights on X_1 resp. X_2. Consider the sequence $\mathcal{A} = A^1 \otimes (A^2)^{-1}$ on $X = X_1 \times X_2$ and assume that $C(A^1)_0(X_1)$ satisfies $(\overline{\overline{\Omega}})$ and that $C(A^2)_0(X_2)$ satisfies $(\underline{\mathrm{DN}})$. Then $(AC)_0(X)$ is ultrabornological.

3.2.4 Tensor product representation

The constructions in the earlier sections already suggest the question, wether the space $(AC)_0(X)$ with an underlying sequence $\mathcal{A} = A^1 \otimes (A^2)^{-1}$ can be realized as the tensor product of a Fréchet and an (LB)-space. In this section we deduce a representation of this kind, which finally will enable us (see 3.17) to utilize 3.5 to prove a criterion for the ultrabornologicity of an ε-tensor product of a weighted Fréchet space of continuous functions and a weighted (LB)-space of continuous functions.

The underlying general question of determining topological properties of the tensor product of a Fréchet space and a (DF)-space was raised by Grothendieck in the last section of his thèse [45]. He investigated the case of π-tensor products of echelon and coechelon spaces of order one, see [45, Chapitre II, §4, No. 3, Theorem 15]. His results inspired many authors to further studies; see Varol [74, Section 0] for references and a generalization of Grothendiecks original result to the case of π-tensor products of a Köthe coechelon space of order one and arbitrary Fréchet spaces ([74, Theorem 2.1]). Varol [74, Theorem 2.7] investigated in addition the case of ε-tensor products of Köthe coechelon spaces of order zero and arbitrary Fréchet spaces. Note that also the classical result [45, Chapitre II, §4, No. 3, Corollaire 2] (see also Bonet, Pérez Carreras [63, Proposition 11.6.13]) of Grothendieck on the ultrabornologicity of $s' \hat{\otimes}_\pi s$ is a result on an ε-tensor product of a Fréchet space and a (DF)-space due to the nuclearity of s.

In this section we will again consider a sequence $\mathcal{A} = A^1 \otimes (A^2)^{-1}$ on $X =$

$X_1 \times X_2$ of the type explained earlier. We put $\mathcal{V}^2 := (A^2)^{-1} = ((a_N^2)^{-1})_{N\in\mathbb{N}}$, which gives rise to the weighted (LB)-space of continuous functions $\mathcal{V}_0^2 C(X_2) = \operatorname{ind}_N C((a_N^2)^{-1})_0(X_2)$.

In order to prove the desired result on the tensor product representation 3.16 below and for re-use in the appendix (15.4), we first present an abstract result, which roughly speaking states that given two (PLB)-spaces $E = \operatorname{proj}_N \operatorname{ind}_n E_{N,n}$ and $F = \operatorname{proj}_N \operatorname{ind}_n F_{N,n}$ with $E_{N,n} \cong F_{N,n}$ for all N and n in a way that these isomorphisms are “compatible” with the linking maps of the spectra we get $E \cong F$. In view of this rough formulation (of a plausible statement) the notation we introduce now for the proof seems rather disproportionate, but without this technical notation, it seems to be hard to give a detailed and complete proof.

Let $E_{N,n}$ be Banach spaces for n, $N \in \mathbb{N}$. Assume that we are given continuous and injective maps $i_n^{N,N+1}\colon E_{N+1,n} \to E_{N,n}$ and $i_{n+1,n}^N\colon E_{N,n} \to E_{N,n+1}$. We put $i_n^{N,N} = i_{n,n}^N = \operatorname{id}_{E_{N,n}}$ and $i_{m,n}^N = i_{m,m-1}^N \circ \cdots \circ i_{n+1,n}^N$ for $m > n$ and get $i_{k,m}^N \circ i_{m,n}^N = i_{k,n}^N$ for arbitrary N and $k \geqslant m \geqslant n$. Thus, $\mathcal{E}_N = (E_{N,n}, i_{m,n}^N)$ is an inductive spectrum of Banach spaces for each N (see e.g. Vogt [77, section 2, p. 12]). We put $E_N = \operatorname{ind}_n E_{N,n}$, denote by $i_n^N\colon E_{N,n} \to E_N$ the canonical maps. Moreover, we assume that $i_{m,n}^N \circ i_n^{N,N+1} = i_m^{N,N+1} \circ i_{m,n}^{N+1}$ holds for all N and $m \geqslant n$. By the latter assumption we get for each N from the universal property of the inductive limit E_{N+1} a map $i_{N,N+1}\colon E_{N+1} \to E_N$ which satisfies

$$(1) \qquad i_{N,N+1} \circ i_n^{N+1} = i_n^N \circ i_n^{N,N+1}$$

for each n. We put $i_{N,N} = \operatorname{id}_{E_N}$ and $i_{M,N} = i_{N,N+1} \circ \cdots \circ i_{M-1,M}$ for $M \geqslant N$ and get $i_{K,M} \circ i_{M,N} = i_{K,N}$ for all $N \geqslant M \geqslant K$. Therefore, $\mathcal{E} = (E_N, i_{N,M})$ is a projective spectrum of (LB)-spaces. We put $E = \operatorname{proj}_N E_N$ and denote by $i_N\colon E \to E_N$ the canonical maps.

Proposition 3.12. Let $E = \operatorname{proj}_N \operatorname{ind}_n E_{N,n}$ and $F = \operatorname{proj}_N \operatorname{ind}_n F_{N,n}$ be (PLB)-spaces where we replace for the case of F in the above notation E with F, $\mathcal{E}$ with $\mathcal{F}$ and i with j. Assume that we are given a system of isomorphisms $T_{N,n}\colon F_{N,n} \to E_{N,n}$ with the properties

$$(2) \qquad T_{N,n+1} \circ j_{n+1,n}^N = i_{n+1,n} \circ T_{N,n}$$
$$(3) \qquad T_{N,n} \circ j_n^{N,N+1} = i_n^{N,N+1} \circ T_{N+1,n}.$$

Then the spectra $\mathcal{E}$ and $\mathcal{F}$ are equivalent and in particular $E \cong F$ holds.

Proof. 1. Let us first show that $\mathcal{E}_N \sim \mathcal{F}_N$ for each $N \in \mathbb{N}$. For fixed N we put $\alpha_n^N = T_{N,n}$ and $\beta_n^N = j_{n+1,n}^N \circ T_{N,n}^{-1}$. Then we have

$$\beta_n^N \circ \alpha_n^N = j_{n+1,n}^N \circ T_{N,n}^{-1} \circ T_{N,n} = j_{n+1,n}^N \quad \text{and}$$
$$\alpha_{n+1}^N \circ \beta_n^N = T_{N,n+1} \circ j_{n+1,n}^N \circ T_{N,n}^{-1} \overset{(2)}{=} i_{n+1,n}^N \circ T_{N,n} \circ T_{N,n}^{-1} = i_{n+1,n}^N$$

that is the diagram

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[dr] & E_{N,n} \arrow[r, "i^N_{n+1,n}"] \arrow[dr, "\beta^N_n"] & E_{N,n+1} \arrow[r, "i^N_{n+2,n+1}"] \arrow[dr, "\beta^N_{n+1}"] & E_{N,n+2} \arrow[r] \arrow[dr] & \cdots \\
\cdots \arrow[r] & F_{N,n} \arrow[r, "j^N_{n+1,n}"'] \arrow[u, "\alpha^N_n"'] & F_{N,n+1} \arrow[r, "j^N_{n+2,n+1}"'] \arrow[u, "\alpha^N_{n+1}"'] & F_{N,n+2} \arrow[r] \arrow[u, "\alpha^N_{n+2}"'] & \cdots
\end{tikzcd}$$

is commutative and we thus have $\mathcal{E}_N \sim \mathcal{F}_N$.

2. The above equivalence induces for each $N \in \mathbb{N}$ an isomorphism $T_N \colon F_N \to E_N$ with

$$(4) \qquad T_N \circ j^N_n = i^N_n \circ T_{N,n}$$

for each $n \in \mathbb{N}$.

3. Now we claim $\mathcal{E} \sim \mathcal{F}$. We put $\alpha_N = T_N$ and $\beta_N = j_{N-1,N} \circ T_N^{-1}$. Then we have

$$\beta_N \circ \alpha_N = j_{N-1,N} \circ T_N^{-1} \circ T_N = j_{N-1,N} \quad \text{and}$$
$$\alpha_N \circ \beta_{N+1} = T_N \circ j_{N,N+1} \circ T_{N+1} \overset{(5)}{=} i_{N,N+1} \circ T_{N+1} \circ T_{N+1}^{-1} = i_{N+1,N}$$

where

$$(5) \qquad T_N \circ j_{N,N+1} = i_{N,N+1} \circ T_{N+1}$$

is obtained as follows. By the universal property of the inductive limit it is enough to show that

$$T_N \circ j_{N,N+1} \circ j^{N+1}_n = i_{N,N+1} \circ T_{N+1} \circ j^{N+1}_n$$

holds for each $n \in \mathbb{N}$. Thus, let n be fixed. Then

$$\begin{aligned}
T_N \circ j_{N,N+1} \circ j^{N+1}_n &\overset{(1)}{=} T_N \circ j^N_n \circ j^{N,N+1}_n \\
&\overset{(4)}{=} i^N_n \circ T_{N,n} \circ j^{N,N+1}_n \\
&\overset{(3)}{=} i^N_n \circ i^{N,N+1}_n \circ T_{N+1,n} \\
&\overset{(1)}{=} i_{N,N+1} \circ i^{N+1}_n \circ T_{N+1,n} \\
&\overset{(4)}{=} i_{N,N+1} \circ T_{N+1} \circ j^{N+1}_n
\end{aligned}$$

which yields (5). Thus, the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & F_{N+1} \arrow[r, "j_{N,N+1}"] \arrow[d, "\alpha_{N+1}"'] & F_N \arrow[r, "j^N_{N-1,N}"] \arrow[d, "\alpha_N"'] & F_{N-1} \arrow[r] & \cdots \\
\cdots \arrow[r] \arrow[ur] & E_{N+1} \arrow[r, "i_{N,N+1}"'] \arrow[ur, "\beta_{N+1}"] & E_N \arrow[r, "j_{N-1,N}"'] \arrow[ur, "\beta_N"] & E_{N-1} \arrow[r] \arrow[u, "\alpha_{N-1}"'] \arrow[ur] & \cdots
\end{tikzcd}$$

commutes and hence $\mathcal{E} \sim \mathcal{F}$. By Wengenroth [84, 3.1.7] this implies in particular $E \cong F$. ■

Remark 3.13. Since the formulation and the proof of 3.12 is somewhat technical let us mention that if we regard $E_{N+1,n} \subseteq E_{N,n} \subseteq E_{N,n+1}$ as linear subspaces and assume that the maps $i_n^{N,N+1}$ and $i_{n+1,n}^N$ are just the inclusion maps (and the same for F and j), the conditions (2) and (3) reduce to

$$T_{N,n+1}|_{F_{N,n}} = T_{N,n} \quad \text{and} \quad T_{N,n}|_{F_{N+1,n}} = T_{N+1,n}$$

which is in many cases very easy to see.

Let us add the following consequence of the proof of 3.12.

Scholium 3.14. Let $\mathcal{E} = (E_N, i_{M,N})$ and $\mathcal{E} = (F_N, j_{M,N})$ be projective spectra of locally convex spaces with injective linking maps. Assume that we are given a sequence of isomorphisms $T_N \colon F_N \to E_N$ with

$$(6) \qquad T_N \circ j_{N,N+1} = i_{N,N+1} \circ T_{N+1}$$

for each $N \in \mathbb{N}$. Then $\mathcal{E} \sim \mathcal{F}$ and in particular $E \cong F$.

Proof. Since (6) is exactly equation (5) in the proof of 3.12 we may obtain the equivalence as in the third part of the latter proof. $E \cong F$ follows again from [84, 3.1.7]. ■

Remark 3.15. Analogously to 3.13 let us note that the condition (6) can be written as

$$T_N|_{F_{N+1}} = T_{N+1}$$

if we regard $E_{N+1} \subseteq E_N$ as linear subspaces and assume that the $i_{N+1,N}$ are just the inclusion maps (and the same for F and j).

With the above preparations we can prove the result already announced by putting several well-known isomorphisms together.

Proposition 3.16. Let A^1 resp. A^2 be an increasing sequence of weights on X_1 resp. X_2. Consider the double sequence $\mathcal{A} = A^1 \otimes (A^2)^{-1}$ on $X = X_1 \times X_2$ and assume that $\mathcal{V}^2 = (A^2)^{-1}$ is regularly decreasing. Then we have the isomorphism

$$C(A^1)_0(X_1) \,\check{\otimes}_\varepsilon\, \mathcal{V}_0^2 C(X_2) \cong (AC)_0(X),$$

where $C(A^1)_0(X_1) = \operatorname{proj}_N C(a_N^1)_0(X_1)$ resp. $\mathcal{V}_0^2 C(X_2) = \operatorname{ind}_n C((a_n^2)^{-1})_0(X_2)$ is a weighted Fréchet resp. (LB)-space of continuous functions.

Proof. We compute

$$\begin{aligned} C(A^1)_0(X_1) \,\check{\otimes}_\varepsilon\, \mathcal{V}_0^2 C(X_2) &\overset{\text{dfn}}{=} \big(\operatorname{proj}_N C(a_N^1)_0(X_1)\big) \,\check{\otimes}_\varepsilon\, \big(\operatorname{ind}_n C((a_n^2)^{-1})_0(X_2)\big) \\ &\overset{(1)}{=} \operatorname{proj}_N \big[C(a_N^1)_0(X_1) \,\check{\otimes}_\varepsilon\, \operatorname{ind}_n C((a_n^2)^{-1})_0(X_2)\big] \\ &\overset{(2)}{=} \operatorname{proj}_N \operatorname{ind}_n \big[C(a_N^1)_0(X_1) \,\check{\otimes}_\varepsilon\, C((a_n^2)^{-1})_0(X_2)\big] \\ &\overset{(3)}{=} \operatorname{proj}_N \operatorname{ind}_n \big[C(a_N^1 \otimes (a_n^2)^{-1})_0(X_1 \times X_2)\big] \\ &\overset{\text{dfn}}{=} (AC)_0(X). \end{aligned}$$

The isomorphy (1) is true in general, see e.g. Jarchow [50, 16.3.2].

(2) can be seen as follows. We have

a. [48, Theorem 4.1]: $C(a^1_N)_0(X_1)$ is an ε-space by Hollstein [48, Proposition 2.3],
b. [48, Theorem 4.1.(ii)]: $\mathcal{V}^2_0 C(X_2)$ is quasi-complete, compact-regular (see Bierstedt, Meise, Summers [27, Corollary 2.7]) and all the $C((a^2_n)^{-1})_0(X_2)$ have the approximation property (see Bierstedt [8, Theorem 5.5.(3)]),
c. [48, Proposition 4.4.(1)]: $C(a^1_N)_0(X_1)$ is Banach and $\mathcal{V}^2_0 C(X_2)$ is compact-regular (see b.),
d. [48, Theorem 4.1., 2nd part]: $\mathcal{V}^2_0 C(X_2)$ and all the $C((a^2_n)^{-1})_0(X_2)$ are complete (see b.).

Therefore by Hollstein [48, Theorem 4.1] we have an isomorphism

$$\operatorname{ind}_n \big[C(a^1_N)_0(X_1)\,\check{\otimes}_\varepsilon\, C((a^2_n)^{-1})_0(X_2)\big] \xrightarrow{T_N} C(a^1_N)_0(X_1)\,\check{\otimes}_\varepsilon\, \operatorname{ind}_n C((a^2_n)^{-1})_0(X_2)$$

for each $N \in \mathbb{N}$, which is just the mapping induced by the maps $\mathrm{id}_{C(a^1_N)_0(X_1)} \otimes i_n$ via the universal property of the inductive limit on the left hand side (note that by the properties we stated above, Köthe [52, §44, 2.(5)] yields that we have the equalities $C(a^1_N)_0(X_1)\,\check{\otimes}_\varepsilon\, C((a^2_n)^{-1})_0(X_2) = C(a^1_N)_0(X_1) \otimes_\varepsilon C((a^2_n)^{-1})_0(X_2)$ and $C(a^1_N)_0(X_1)\,\check{\otimes}_\varepsilon\, \operatorname{ind}_n C((a^2_n)^{-1})_0(X_2) = C(a^1_N)_0(X_1) \otimes_\varepsilon \operatorname{ind}_n C((a^2_n)^{-1})_0(X_2)$), cf. [48, remarks previous to Prop. 4.4]. Therefore the maps T_N satisfy the condition of 3.15 and we get the desired isomorphism (2) by 3.14.

Finally, (3) follows from Bierstedt [9, 1.2] and 3.12 since

$$T_{N,n}\colon C(a^1_N)_0(X_1)\,\check{\otimes}_\varepsilon\, C((a^2_n)^{-1})_0(X_2) \to C(a^1_N \otimes (a^2_n)^{-1})_0(X_1 \times X_2),$$

$$\sum_{i=1}^{j} f_j \otimes g_i \mapsto \Big[(x_1,x_2) \mapsto \sum_{i=1}^{j} f_i(x_1) g_i(x_2)\Big]$$

(Köthe [52, §44, 2.(5)] implies $C(a^1_N)_0(X_1)\,\check{\otimes}_\varepsilon\, C((a^2_n)^{-1})_0(X_2) = C(a^1_N)_0(X_1) \otimes_\varepsilon C((a^2_n)^{-1})_0(X_2)$ since both spaces are complete and $C((a^2_n)^{-1})_0(X_2)$ has the approximation property by a.) satisfies the conditions in 3.13 where we regard

$$\begin{aligned} C(a^1_{N+1})_0(X_1)\,\check{\otimes}_\varepsilon\, C((a^2_n)^{-1})_0(X_2) &\subseteq C(a^1_N)_0(X_1)\,\check{\otimes}_\varepsilon\, C((a^2_n)^{-1})_0(X_2) \\ &\subseteq C(a^1_N)_0(X_1)\,\check{\otimes}_\varepsilon\, C((a^2_{n+1})^{-1})_0(X_2) \end{aligned}$$

as linear subspaces via the maps $i_{N,N+1} \otimes \mathrm{id}_{C((a^2_n)^{-1})_0(X_2)}$ and $\mathrm{id}_{C(a^1_N)_0(X_1)} \otimes j_{n+1,n}$ where $i_{N,N+1}\colon C(a^1_{N+1})_0(X_1) \to C(a^1_N)_0(X_1)$ and $j_{n+1,n}\colon C((a^2_n)^{-1})_0(X_2) \to C((a^2_{n+1})^{-1})_0(X_2)$ are the inclusion maps. ■

In the special case of sequence spaces, the above tensor product is of the type $\lambda^0(A)\,\check{\otimes}_\varepsilon\, k^0(B)$, i.e. it is the tensor product of a Köthe echelon and Köthe coechelon space and thus 3.16 can be regarded as an extension of [2, Lemma 4.3] to continuous functions.

The space considered in 3.6, $(AC)_0(\mathbb{N}\times\mathbb{N})$ with $\mathcal{A}=((a_N\otimes a_n^{-1})_{N\in\mathbb{N}})_{n\in\mathbb{N}}$ and $a_k(j)=j^k$ is by 3.16 isomorphic to the space $s\,\check{\otimes}_\varepsilon\,k^0(B)$ where $B=(j^{-k})_{j,k\in\mathbb{N}}$ (in this case the regularly decreasing condition [26, Definition 3.1] can easily be verified).

Corollary 3.17. Let A^1 resp. A^2 be an increasing sequence on X_1 resp. X_2. We consider a double sequence $\mathcal{A}=A^1\otimes(A^2)^{-1}$ and assume that $\mathcal{V}^2=(A^2)^{-1}$ is regularly decreasing, that $C(A^1)_0(X_1)$ satisfies (Ω), resp. $(\overline{\overline{\Omega}})$, and that $C(A^2)_0(X_2)$ satisfies (DN), resp. $(\underline{\mathrm{DN}})$,. Then the ε-tensor product $C(A^1)_0(X_1)\,\check{\otimes}_\varepsilon\,\mathcal{V}_0^2C(X_2)$ of a Fréchet space and a (DF)-space is ultrabornological.

Proof. This follows directly from 3.16 and 3.5. ■

Let us explain the meaning of 3.17 in the case of sequence spaces: Let $A=(a_n)_{n\in\mathbb{N}}$ be a Köthe matrix and $B=(b_n)_{n\in\mathbb{N}}$ be a decreasing sequence of strictly positive functions on $\mathbb{N}$ which is regularly decreasing. We put $B^{-1}=(b_n^{-1})_{n\in\mathbb{N}}$ and assume that $\lambda^0(A)$ satisfies (Ω) and $\lambda^0(B^{-1})$ satifies (DN) or that $\lambda^0(A)$ satisfies $(\overline{\overline{\Omega}})$ and $\lambda^0(B^{-1})$ satisfies $(\underline{\mathrm{DN}})$. Then 3.17 implies that the space $\lambda^0(A)\,\check{\otimes}_\varepsilon\,k^0(B)$ is ultrabornological.

We note that B is regularly decreasing if and only if $\lambda^0(B^{-1})$ is quasinormable and that this is equivalent to condition (wS) of Bierstedt, Meise, Summers, see [26, Proposition on p. 48 and Proposition 3.2].

Let us mention, that the latter statement follows also from the results in [2, Section 4] and Proposition 3.4, since in [2, Section 4] ultrabornologicity of the space $\lambda^0(A)\,\check{\otimes}_\varepsilon\,k^0(B)$ was characterized via condition (wQ). Moreover, the results 3.16 and 3.17 should be compared with results of Piszczek [66, Theorem 9 and Theorem 6] and Domański [39, Corollary 5.6] who studied tensor products of (nuclear) (PLS)-spaces.

3.3 Interchangeability of projective and inductive limit

In [18] Bierstedt, Bonet investigated the spaces $\mathcal{V}C(X)=\mathrm{ind}_n\,\mathrm{proj}_N\,Cv_{n,N}(X)$ and $\mathcal{V}_0C(X)=\mathrm{ind}_n\,\mathrm{proj}_N\,C(v_{n,N})_0(X)$ and called this setting the (LF)-case of $\mathcal{V}C(X)$ (resp. $\mathcal{V}_0C(X)$) in analogy to the (LB)-case studied by Bierstedt, Meise, Summers in [27], where the underlying system of weights $\mathcal{V}=((v_{n,N})_{n\in\mathbb{N}})_{N\in\mathbb{N}}$ was assumed to satisfy $v_{n+1,N}\leqslant v_{n,N}\leqslant v_{n,N+1}$ in contrast to our definition of $\mathcal{A}$. Since we want to study under which conditions projective and inductive limit in the definition of $AC(X)$ and $(AC)_0(X)$ can be interchanged, we have to consider the (LF)-spaces

$$\mathcal{V}C(X)=\mathrm{ind}_n\,\mathrm{proj}_N\,Ca_{N,n}(X)\quad\text{and}\quad\mathcal{V}_0C(X)=\mathrm{ind}_n\,\mathrm{proj}_N\,C(a_{N,n})_0(X),$$

that is in the notation of [18] we put $v_{n,N}:=a_{N,n}$. According to [18] we will in the sequel denote the steps of the latter inductive limits with

$$CV_n(X):=\mathrm{proj}_N\,Ca_{N,n}(X)\text{ and }C(V_n)_0(X):=\mathrm{proj}_N\,C(a_{N,n})_0(X).$$

The following theorem summarizes the results of Agethen, Bierstedt, Bonet [2] on the commutativity of projective and inductive limit, i.e. with the notation we just established the question if $AC(X) = \mathcal{V}C(X)$, resp. $(AC)_0(X) = \mathcal{V}_0C(X)$ holds.

Theorem C. ([2, 3.10, 3.11]) For the (PLB)- and (LF)-spaces $AC(X)$ and $\mathcal{V}C(X)$ (resp. $(AC)_0(X)$ and $\mathcal{V}_0C(X)$) the following statements are true.

(1) $\mathcal{V}C(X) \subseteq AC(X)$ and $\mathcal{V}_0C(X) \subseteq (AC)_0(X)$ holds in general with continuous inclusions. $AC(X) = \mathcal{V}C(X)$ holds algebraically if and only if the sequence $\mathcal{A}$ satisfies condition (B).

(2) If the sequence $\mathcal{A}$ satisfies condition (B), then the space $(AC)_0(X)$ equals $\mathcal{V}_0C(X)$ algebraically. If each $(\mathcal{A}_N)_0(X)$ is complete, then the converse is also true.

(3) If all $(\mathcal{A}_N)_0C(X)$ are complete, then $(AC)_0(X) = \mathcal{V}_0C(X)$ holds algebraically and topologically if and only if the sequence $\mathcal{A}$ satisfies the conditions (B) and (wQ).

(4) If $\mathcal{A}$ satisfies the conditions (B) and (Q), then $AC(X) = \mathcal{V}C(X)$ holds algebraically and topologically. If $AC(X) = \mathcal{V}C(X)$ holds algebraically and topologically, then $\mathcal{A}$ satisfies the conditions (wQ) and (B).

Remark 3.18. Note that the completeness of the steps $(\mathcal{A}_0)_NC(X)$ of the (PLB)-space $(AC)_0(X)$ can be characterized by the following condition of Bierstedt, Meise, Summers. Let $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ be a decreasing sequence of weights on X. According to [27, Definition 2.1], $\mathcal{V}$ is said to be *regularly decreasing* if

$$\forall\, n\; \exists\, m \geqslant n\; \forall\, \varepsilon > 0,\; k \geqslant m\; \exists\, \delta > 0\; \forall\, x \in X\colon v_m(x) \geqslant \varepsilon v_n(x) \Rightarrow v_k(x) \geqslant \delta v_n(x).$$

In [27, Theorem 2.6.(a)], Bierstedt, Meise, Summers showed that $\mathcal{V}_0C(X)$ is complete if and only if $\mathcal{V}$ is regularly decreasing. In particular we may replace the completeness assumptions in 3.3.C by requiring $\mathcal{A}_N = (a_{N,n})_{n\in\mathbb{N}}$ to be regular decreasing for each N.

To end this section, let us mention the paper [65] by Piszczek, who studied questions related to the above in the setting of arbitrary (PLS)- resp. (LFS)-spaces and power series spaces.

4 Generalities on projective limits of inductive limits of normed and Banach spaces

For the study of the spaces $AH(G)$ und $(AH)_0(G)$ we will make use of homological methods. For notation on this subject we refer to Wengenroth [84] and Vogt [77]. This section provides some refinements of results of the general theory which are adjusted to the situations we are dealing with.

4.1 A necessary condition for the vanishing of Proj^1

Let $\mathcal{X} = (X_N, \rho_M^N)$ be a projective spectrum of (LB)-spaces $X_N = \operatorname{ind}_n X_{N,n}$. We denote by $X = \operatorname{proj}_N X_N = \operatorname{proj}_N \operatorname{ind}_n X_{N,n}$ its limit and assume that $X_N = \cup_{n\in\mathbb{N}} B_{N,n}$ where $B_{N,n}$ denotes the unit ball of the Banach space $X_{N,n}$. The X_N are tacitly assumed to be separated.

For many (PLB)-spaces which arise in nature, all the Banach spaces $X_{N,n}$ are contained as linear subspaces in some "big" space. For example this is true for $AH(G)$ and $(AH)_0(G)$: All the $Ha_{N,n}(G)$ and $H(a_{N,n})_0(G)$ are subspaces of $H(G)$. Our first result abstracts exactly the situation of $AH(G)$; in this case (under some mild assumptions, see section 5) the balls $B_{N,n}$ are compact in $H(G)$ if we endow $H(G)$ with the compact open topology. For the proof we need the following well known fact.

Lemma 4.1. Let X, Y and Z be topological spaces, $i\colon X \to Z$, $j\colon Z \to Y$ and $k\colon X \to Y$ be maps such that $j \circ i = k$. If j is injective and moreover j and k are continuous then i has closed graph.

Proof. We consider

$$\operatorname{gr}(i) \subseteq X \times Z \stackrel{\operatorname{id}_X \times j}{\longrightarrow} X \times Y \supseteq \operatorname{gr}(k).$$

Then, $\operatorname{id}_X \times j$ is continuous w.r.t. the product topologies on $X \times Z$ and $X \times Y$. We claim that $(\operatorname{id}_X \times j)^{-1}(\operatorname{gr}(k)) = \operatorname{gr}(i)$. Since $\operatorname{gr}(k) \subseteq X \times Y$ is closed as k is continuous and $\operatorname{id}_X \times j$ is continuous this will finish the proof.

"$\supseteq$" We have $(\operatorname{id}_X \times j)(x, i(x)) = (x, j(i(x))) = (x, k(x))$ and therefore $(x, i(x)) \in (\operatorname{id}_X \times j)^{-1}(x, k(x))$.

"$\subseteq$" Let $(x, y) \in (\operatorname{id}_X \times j)^{-1}(\operatorname{gr}(k))$. Then by definition $(\operatorname{id}_X \times j)(x, y) = (x, j(y)) \in \operatorname{gr}(k)$. That means $j(y) = k(x) = (j \circ i)(x)$ and since j is injective, we have $y = i(x)$ which implies that $(x, y) \in \operatorname{gr}(i)$ holds. ∎

Theorem 4.2. Assume that there exists a locally convex space (Y, τ) and a sequence $(i_N)_{N\in\mathbb{N}}$, where $i_N\colon X_N \to (Y, \tau)$ is continuous and injective, such that the compatibility condition $i_N \circ \rho_M^N = i_M$ for all $M \geqslant N$ is satisfied. If $\operatorname{Proj}^1 \mathcal{X} = 0$ then

$$\forall\, N\; \exists\, M \geqslant N,\, n\; \forall\, K \geqslant M,\, m\; \exists\, k,\, S > 0 : \rho_M^N B_{M,m} \subseteq S(\rho_K^N B_{K,k} + B_{N,n}).$$

Proof. $\operatorname{Proj}^1 \mathcal{X} = 0$ yields by a result of Retakh [67] (see Palamodov [61, Theorem 5.4], Wengenroth [84, 3.2.9]) that there exists a sequence $(B_N)_{N\in\mathbb{N}}$, where $B_N \subseteq X_N$ is a Banach disc for each N, such that

(α) $\rho_M^N(B_M) \subseteq B_N$ for $N \leqslant M$,
(β) $\forall\, N\; \exists\, M \geqslant N\; \forall\, K \geqslant M : \rho_M^N X_M \subseteq \rho_K^N X_K + B_N$.

Let $N \in \mathbb{N}$ be given. We choose $M \geqslant N$ as in (β). Since B_N is a Banach disc in the (LB)-space $X_N = \operatorname{ind}_n X_{N,n}$ there exists n such that $B_N \subseteq X_{N,n}$ is already a

Banach disc and hence there exists $S' > 1$ with $B_N \subseteq S'B_{N,n}$. Now let $K \geqslant M$ be given. By the above and by (β) we have

$$\begin{aligned}\rho_M^N X_M \subseteq \rho_K^N X_K + S'B_{N,n} &= \rho_K^N\big(\textstyle\bigcup_{k'\in\mathbb{N}} B_{K,k'}\big) + S'B_{N,n}\\ &= \textstyle\bigcup_{k'\in\mathbb{N}} \rho_K^N B_{K,k'} + S'B_{N,n}\\ &\subseteq \textstyle\bigcup_{k'\in\mathbb{N}} (S'\rho_K^N B_{K,k'} + S'B_{N,n})\\ &= \textstyle\bigcup_{k'\in\mathbb{N}} S'(\rho_K^N B_{K,k'} + B_{N,n}).\end{aligned}$$

We put $C_{k'} := S'(\rho_K^N B_{K,k'} + B_{N,n}) \subseteq X_N$ and note that $i_N(C_{k'}) \subseteq (Y,\tau)$ is a Banach disc, since linear continuous images of Banach discs aswell as sums of Banach discs are again Banach discs (for the latter see [63, proof of 3.2.6]). Let us denote the associated Banach space by $Z_{k'}$. We have

$$C_{k'} = S'(\rho_K^N B_{K,k'} + B_{N,n}) \subseteq S'(\rho_K^N B_{K,k'+1} + B_{N,n}) = C_{k'+1},$$

thus $i_N(C_{k'}) \subseteq i_N(C_{k'+1})$ and hence $Z_{k'} \subseteq Z_{k'+1}$ with continuous inclusion. We define $Z = \operatorname{ind}_{k'} Z_{k'}$, i.e. algebraically we have

$$i_N(\rho_M^N X_M) \subseteq i_N\big(\textstyle\bigcup_{k'\in\mathbb{N}} S'(\rho_K^N B_{K,k'} + B_{N,n})\big) \subseteq Z.$$

Now let m be given. We have $X_{M,m} \subseteq X_M$, i.e. $\rho_M^N X_{M,m} \subseteq \rho_M^N X_M$ and thus $i_N\rho_M^N X_{M,m} \subseteq Z$. We claim that $i := i_N \circ \rho_M^N \circ i_{M,n}\colon X_{M,m} \to Z$ is continuous, where $i_{M,n}$ denotes the inclusion map $X_{M,m} \hookrightarrow X_M$. Since the $i_N(C_{k'})$ are Banach discs in (Y,τ), we have continuous inclusions $j_{k'}\colon Z_{k'} \to (Y,\tau)$ and hence by the universal property of the inductive limit $Z = \operatorname{ind}_{k'} Z_{k'}$ we get that the inclusion $j\colon Z \to (Y,\tau)$ is continuous. On the other hand, the composition $k := i_M \circ i_{M,n}\colon X_{M,m} \to (Y,\tau)$ is continuous and injective. Since j is just the identity we have $j \circ i = j \circ i_N \circ \rho_M^N \circ i_{M,m} = i_M \circ i_{M,m} = k$. By 4.1, i has closed graph and since Z is webbed and $X_{M,m}$ is even Banach, i has to be continuous.

By Grothendieck's factorization theorem (e.g. [60, 24.33]) there exists k such that $i(X_{M,m}) \subseteq Z_k$ and $i\colon X_{M,m} \to Z_k$ is continuous. Since $X_{M,m}$ and Z_k are Banach spaces the image of the unit ball under i has to be bounded, that is there exists $S'' > 0$ such that $i_N \circ \rho_M^N B_{M,m} = i_N \circ \rho_M^N \circ i_{M,m} B_{M,m} = i(B_{M,m}) \subseteq S'' \cdot i_N C_k$ ($i_{M,m}$ is the identity). Since i_N is injective this yields $\rho_M^N \subseteq SC_k$ and we finally obtain the desired condition by using the definition of C_k and selecting $S := S' \cdot S''$. ∎

Remark 4.3. The condition in 4.2 is exactly the condition (P_2) of Braun, Vogt [36]. They showed that a (DFS)-spectrum $\mathcal{X}$ is reduced and satisfies (P_2) if and only if $\operatorname{Proj}^1 \mathcal{X} = 0$.

4.2 An inheritance property of barrelledness

The following abstract results generalizes a method used by Bierstedt, Bonet [16, Proof of "(ii)⇒(iii)" of Theorem 3.10] for vector valued sequence spaces. Bierstedt, Bonet showed that the projective hull of a Köthe coechelon space of order zero,

$K_0(\overline{V}, E)$ is (quasi-)barrelled, if the projective hull of the Köthe coechelon space of order infinity for the same sequence of weights is barrelled by proceeding exactly as we will do in the proof of 4.4. The same method was also applied by Agethen, Bierstedt, Bonet [2, Proof of Theorem 3.8.(2)]. In particular it applies to the inclusion mapping $(AH)_0(G) \subseteq AH(G)$ as we will see in 6.2.

Lemma 4.4. Let X and X_0 be locally convex spaces and $J\colon X_0 \to X$ be a linear and continuous map. Assume that there exists an equicontinuous net $(S_\alpha)_{\alpha\in A} \subseteq L(X, X_0)$ such that $S_\alpha(J(x)) \to x$ holds for each $x \in X_0$. If X is barrelled, then X_0 is quasibarrelled.

Proof. Let T_0 be a bornivorous barrel in X_0. We put

$$T := \big\{\, x \in X \,;\, \forall\, \alpha \in A\colon S_\alpha x \in T_0 \,\big\} = \bigcap_{\alpha\in A} S_\alpha^{-1}(T_0).$$

Since the S_α are linear and continuous, T has to be absolutely convex and closed. We claim that it is absorbing and hence a barrel. Let $y \in X$ be given. Consider the set $B_y := \{\, S_\alpha y \in X \,;\, \alpha \in A \,\} \subseteq X_0$. Let $(p_\lambda)_{\lambda\in L}$ and $(q_\mu)_{\mu\in M}$ be fundamental systems of seminorms for X_0, resp. X. For arbitrary $\lambda \in L$ there exists $C > 0$ and $\mu_1, \ldots, \mu_n$ such that $p_\lambda(S_\alpha y) \leqslant C \max_{k=1,\ldots,n} q_{\mu_k}(y)$ for all $\alpha \in A$, since $(S_\alpha)_{\alpha\in A}$ is equicontinuous (e.g. Horvath [49, 3.4.5]). Hence

$$\sup_{x\in B_y} p_\lambda(x) = \sup_{\alpha\in A} p_\lambda(S_\alpha y) \leqslant C \max_{k=1,\ldots,n} q_{\mu_k}(y) < \infty,$$

i.e. B_y is bounded in X_0. Since T_0 is bornivorous, there exists $\beta > 0$ such that $B_y \subseteq \beta T_0$, hence we have $\frac{1}{\beta} S_\alpha y \in T_0$ for all $\alpha \in A$, i.e. $\frac{1}{\beta} y \in T$ and finally $y \in \beta T$, which establishes our claim. Since X is barrelled, T has to be a neighborhood of zero. We show $J^{-1}(T) \subseteq T_0$ which provides that T_0 is a neighborhood of zero and thus finishes the proof. Let $x \in J^{-1}(T)$. That is $J(x) \in T$ and by definition $S_\alpha(J(x)) \in T_0$. But since $S_\alpha(J(x)) \to x$ and T_0 is closed this yields $x \in T_0$. ∎

Remark 4.5. Note that the assumptions of 4.4 imply that the mapping J is even injective: If $J(x) = 0$ then $0 = S_\alpha(0) = S_\alpha(J(x)) \to x$ and hence $x = 0$.

The next statement is useful for checking that “canonical candidates” for nets $(S_\alpha)_{\alpha\in A}$ have the properties needed to apply 4.4. As we will see later, in the holomorphic setting the S_α can choosen to be the maps which send a holomorphic function on its α-th Cesàro mean ($A = \mathbb{N}$ in this case).

Let $X_0 = \operatorname{proj}_N X_N^0$, $X_N^0 = \operatorname{ind}_n X_{N,n}^0$ and $X = \operatorname{proj}_N X_N$, $X_N = \operatorname{ind}_n X_{N,n}$ be (PLB)-spaces where we assume that all linking maps are just inclusions, $P \subseteq X_0$ be a linear space included in each Banach space and (Y, τ) be a locally convex space such that all the spaces considered above are contained in Y and their topologies are stronger than τ. Moreover, let $X_{N,n}^0 \subseteq X_{N,n}$ be a topological subspace for all N, $n \in \mathbb{N}$. Let $(S_\alpha)_{\alpha\in A} \subseteq L(Y, P)$ be a net of maps. In view of the assumptions above we can restrict the S_α to each of the Banach spaces $X_{N,n}$, to each of the (LB)-spaces X_N and to the (PLB)-space X. Moreover we can consider it as a map into each Banach space $X^0_{N,n}$, into each (LB)-space X^0_N and into the (PLB)-space X_0. To simplify notation we will write for all these maps just S_α.

Remark 4.6. If in the situation above, $(S_\alpha)_{\alpha\in A} \subseteq L(X_{N,n}, X^0_{N,n})$ is equicontinuous for each N, $n \in \mathbb{N}$ then $(S_\alpha)_{\alpha\in A}$ is equicontinuous in $L(X, X_0)$.

Proof. We fix $N \in \mathbb{N}$ and claim that $(S_\alpha)_{\alpha\in A} \subseteq L(X_N, X^0_N)$ is equicontinuous. By Horváth [49, Prop. 3.4.5] it is enough to show that $(S_\alpha)_{\alpha\in A} \subseteq L(X_{N,n}, X^0_N)$ is equicontinuous for each $n \in \mathbb{N}$. For fixed n let $V \subseteq X^0_N$ be a 0-neighborhood. Then $V \cap X^0_{N,n}$ is a 0-neighborhood in $X^0_{N,n}$. By our assumptions there exists a 0-neighborhood U in $X_{N,n}$ such that $S_\alpha(U) \subseteq V \cap X^0_{N,n} \subseteq V$ for each $\alpha \in A$, which establishes the claim.

Now let V be a 0-neighborhood in X_0. Then there exists $N \in \mathbb{N}$ and a 0-neighborhood V' in X^0_N such that $V = V' \cap X_0$. By the above there exists a 0-neighborhood U' in X_N such that $S_\alpha(U') \subseteq V'$ for each $\alpha \in A$. We put $U := U' \cap X$, which is a 0-neighborhood in X and obtain $S_\alpha(U) = S_\alpha(U' \cap X) \subseteq V' \cap X_0 = V$ for each $\alpha \in A$ and are done. ∎

Remark 4.7. In the above abstract setting we can even show that the mapping J is nearly open in the sense of Pták (cf. Köthe [52, p. 24]), i.e.

$$\forall\, U \in \mathcal{U}_0(X_0) : \overline{J(U)}^X \in \mathcal{U}_0(\overline{\operatorname{im} J}^X)$$

holds.

Proof. We fix $U \in \mathcal{U}_0(X_0)$. Since $(S_\alpha)_{\alpha\in A}$ is equicontinuous there exists $V \in \mathcal{U}_0(X)$ such that $S_\alpha(V) \subseteq U$ for each $\alpha \in A$. We claim that $V \cap \operatorname{im} J \subseteq J(\overline{U}^{X_0})$. Let $y \in V \cap \operatorname{im} J$. Then there exists $x \in X_0$ such that $J(x) = y$ and for all $\alpha \in A$ we have $U \ni S_\alpha(J(x)) = S_\alpha(y) \to x$, hence $x \in \overline{U}^{X_0}$, i.e. $J(x) = y \in J(\overline{U}^{X_0})$, which establishes the claim. Since $J(\overline{U}^{X_0}) \subseteq \overline{J(U)}^X$ we have $V \cap \operatorname{im} J \subseteq \overline{J(U)}^X$ and hence $V \cap \overline{\operatorname{im} J}^X \subseteq \overline{J(U)}^X$. Thus, we have shown that for arbitrary $U \in \mathcal{U}_0(X_0)$ there exists $V \in \mathcal{U}_0(X)$ such that $V \cap \operatorname{im} J \subseteq \overline{J(U)}^X$, i.e. for arbitrary $U \in \mathcal{U}_0(X_0)$, $\overline{J(U)}^X$ is a 0-neighborhood in $\overline{\operatorname{im} J}^X$. ∎

Remark 4.8. (a) The situation of 4.4 applies to weighted (PLB)-spaces of continuous functions as introduced in section 2, by taking for the equicontinuous net the mappings

$$S_\alpha \colon AC(X) \to (AC)_0(X), \quad S_\alpha(f)(x) := \alpha(x) \cdot f(x),$$

where $A := \{\, \alpha \in C_c(X)\, ;\, 0 \leqslant \alpha \leqslant 1 \,\}$ and $\alpha \leqslant \beta :\Leftrightarrow \alpha(x) \leqslant \beta(x)$ for each $x \in X$. Then [2, 8.3.(2)] follows directly from 4.4.

(b) If we put $X = \mathbb{N}$ we are in the case of sequence spaces, i.e. in the usual notation we consider an infinite matrix $(a_{j;N,n})_{j,N,n\in\mathbb{N}}$ with

$$a_{j;N,n} > 0 \text{ and } a_{j;N,n+1} \leqslant a_{j;N,n} \leqslant a_{j;N+1,n}$$

and denote the spaces $AC(\mathbb{N})$ and $(AC)_0(\mathbb{N})$ by

$$X = \operatorname{proj}_N \operatorname{ind}_n X_{N,n} \quad \text{and} \quad X_0 = \operatorname{proj}_N \operatorname{ind}_n X^0_{N,n}$$

where

$$X_{N,n} = \left\{ x = (x_1, x_2, \dots) \, ; \, \|x\|_{N,n} = \sup_{j\in\mathbb{N}} a_{j;N,n}|x_j| < \infty \right\}$$

and

$$X^0_{N,n} = \left\{ x = (x_1, x_2, \dots) \, ; \, \lim_{j\to\infty} a_{j;N,n}|x_j| = 0 \right\}$$

with the induced norm. The space d of finite sequences is contained in all spaces defined above. Moreover, all the spaces are contained in the space $\mathbb{K}^{\mathbb{N}}$ of all sequences. For $t \in \mathbb{N}$ we define

$$S_t \colon \mathbb{K}^{\mathbb{N}} \to d, \; S_t(x) = (x_1, x_2, \dots, x_t, 0, \dots).$$

The sequence $(S_t)_{t\in\mathbb{N}}$ is equicontinuous as a subset of $L(X_{N,n}, X^0_{N,n})$ for all $N, n \in \mathbb{N}$ and we have $S_t x \to x$ in X^0.

Thus, (i), 4.6 and 4.4 imply that X^0 is (quasi-)barrelled if X is barrelled.

(c) As we have seen so far, the setting considered in 4.4 applies to the inclusion $(AC)_0(X) \subseteq AC(X)$. Hence 4.7 yields that this inclusion mapping is nearly open. However, in the situation of $(AC)_0(X) \subseteq AC(X)$ we already know that the embedding map is even open. It is not clear if the abstract situation allows to prove the latter. In 5.5 we will see that the same situation occurs in the case of holomorphic functions.

Proof. (a) Using 4.6 it is enough to show that the net $(S_\alpha)_{\alpha\in A}$ is equicontinuous as a subset of $L(Ca_{N,n}(X), C(a_{N,n})_0(X))$. But this follows from

$$\|S_\alpha f\|_{N,n} = \sup_{x\in X} a_{N,n}(x)|\alpha(x)f(x)| \leqslant \sup_{x\in X} a_{N,n}(x)|f(x)| = \|f\|_{N,n}$$

for arbitrary $\alpha \in A$.

Let $f \in (AC)_0(X)$ and let N be arbitrary. Then there exists n such that $f \in C(a_{N,n})_0(X)$. As $S_\alpha f$ has compact support, we have $S_\alpha f \in C(a_{N,n})_0(X)$. We claim that $S_\alpha f \to f$ in $C(a_{N,n})_0(X)$. Let $\varepsilon > 0$ be given. Then there exists $K \subseteq X$ compact such that $a_{N,n}|f| < \varepsilon$ on $X\backslash K$. Since X is locally compact there exists $\beta \in C_c(X)$ with $\beta|_K \equiv 1$. Let $\alpha \geqslant \beta$ be in A. Then

$$\|S_\alpha f - f\|_{N,n} = \sup_{x\in X} a_{N,n}(x)|S_\alpha f(x) - f(x)| = \sup_{x\in X} a_{N,n}(x)|f(x)|(1-\alpha(x)).$$

$(1-\alpha(x))$ is zero for all x with $\beta(x) = 1$, i.e. in particular $(1-\alpha)|_K \equiv 0$. Moreover, $0 \leqslant 1-\alpha \leqslant 1$ holds on X. Thus

$$\|S_\alpha f - f\|_{N,n} = \sup_{x\in X\backslash K} a_{N,n}(x)|f(x)|(1-\alpha(x)) \leqslant \sup_{x\in X\backslash K} a_{N,n}(x)|f(x)| \leqslant \varepsilon,$$

and we have shown the claim. Hence $S_\alpha f \to f$ in $(\mathcal{A}_N)_0C(X)$ and since N was arbitrary $S_\alpha f \to f$ in $(AC)_0(X)$. ■

4.3 Bornologicity of projective limits of inductive limits of normed spaces

In the following sections we want to study the space $(AH)_0(G)$. To find sufficient conditions for the barrelledness of this space we will first study the space of polynomials endowed with a weighted topology. More precisely we will endow this space with the projective topology of a spectrum of spaces which are countable inductive limits of normed spaces. Hence the resulting space is a priori not a (PLB)-space. For its investigation we will need the following results.

Let $\mathcal{X} = (X_N, \rho_M^N)$ be a projective spectrum of inductive limits of normed spaces $X_N = \operatorname{ind}_n X_{N,n}$, where the ρ_M^N are inclusions of linear subspaces. We denote by $X = \operatorname{proj}_n \operatorname{ind}_n X_{N,n}$ its limit and assume that $X_N = \cup_{n\in\mathbb{N}} B_{N,n}$, where $B_{N,n}$ denotes the closed unit ball of the normed space $X_{N,n}$. For all $n \in \mathbb{N}$, we will assume that for each bounded set $B \subseteq X_N$ there exists $n \in \mathbb{N}$ such that $B \subseteq B_{N,n}$. If the spaces X_N are regular inductive limits the latter can be assumed without loss of generality.

Lemma 4.9. Assume that

$$\text{(B1)} \qquad \forall\, N\ \exists\, M\ \forall\, m\ \exists\, n : B_{M,m} \subseteq \bigcap_{k\in\mathbb{N}} (B_{N,n} \cap X + \tfrac{1}{k} B_{N,n})$$

holds for the spectrum $\mathcal{X}$. Let $T \subseteq X$ be an absolutely convex set such that

$$\text{(B2)} \qquad \exists\, N\ \forall\, n\ \exists\, S > 0 : B_{N,n} \cap X \subseteq ST$$

holds. Then T is a 0-neighborhood in X.

Proof. We select N as in (B2). For this N, we select M as in (B1). For $n \in \mathbb{N}$, we put

$$T_n := \bigcap_{k\in\mathbb{N}} (T + \tfrac{1}{k} B_{N,n}).$$

Since T and $B_{N,n}$ are absolutely convex, T_n is absolutely convex for each $n \in \mathbb{N}$. Moreover, since $T \subseteq X = \cap_{N'\in\mathbb{N}} X_{N'}$ we have $T \subseteq X_N$ by definition and $B_{N,n} \subseteq X_N$ holds for each $n \in \mathbb{N}$. Thus we have $T_n \subseteq X_N$ for all $n \in \mathbb{N}$. We claim that $T_n \subseteq T_{n+1}$ holds for each n. Let $t \in T_n = \cap_{k\in\mathbb{N}} T + \frac{1}{k} B_{N,n}$. For each k there exists $t' \in T$ and $b \in B_{N,n}$ such that $t = t' + \frac{b}{k}$. Since $b \in B_{N,n} \subseteq B_{N,n+1}$ we get that $t \in T_{n+1}$. Hence, $\cup_{n\in\mathbb{N}} T_n$ is absolutely convex. We put $T_0 := X_M \cap (\cup_{n\in\mathbb{N}} T_n)$. Clearly, T_0 is absolutely convex in X_M.

We claim that T_0 absorbs $B_{M,m}$ for each $m \in \mathbb{N}$. To see this, we fix $m \in \mathbb{N}$ and select n as in (B1), i.e.

$$(\circ) \qquad B_{M,m} \subseteq \bigcap_{k\in\mathbb{N}} (B_{N,n} \cap X + \tfrac{1}{k} B_{N,n}).$$

By (B2) for the above n there exists $S > 0$ such that $B_{N,n} \cap X \subseteq ST$. For arbitrary

$k \in \mathbb{N}$ we add $\frac{1}{k}B_{N,n}$ on both sides and obtain

$$B_{N,n} \cap X + \tfrac{1}{k}B_{N,n} \subseteq ST + \tfrac{1}{k}B_{N,n} = ST + \tfrac{S}{Sk}B_{N,n} = S\big(T + \tfrac{1}{Sk}B_{N,n}\big).$$

Since k was arbitrary, this yields

$$\bigcap_{k\in\mathbb{N}}\big(B_{N,n} \cap X + \tfrac{1}{k}B_{N,n}\big) \subseteq S \bigcap_{k\in\mathbb{N}}\big(T + \tfrac{1}{Sk}B_{N,n}\big) \subseteq ST_n.$$

The latter combined with ($\circ$) yields $B_{M,m} \subseteq ST_n$. With $B_{M,m} \subseteq X_M$ we obtain

$$B_{M,m} = B_{M,m} \cap X_M \subseteq ST_n \cap X_M \subseteq S(T_n \cap X_M) \subseteq ST_0$$

and have established the claim. Since X_M is bornological as it is an inductive limit of normed spaces, and we assumed that the $B_{M,m}$ form a fundamental system of bounded sets for X_M the above yields that $T_0 \in \mathcal{U}_0(X_M)$. Thus, $T_0 \cap X$ is a 0-neighborhood in X. Now we claim that $T_0 \cap X \subseteq 2T$. Let $t \in T_0 \cap X$ be given, i.e. $t \in X$ and $t \in T_n$ for some n. For this n we apply (B2) to get $S > 0$ such that $B_{N,n} \cap X \subseteq ST$. We select $k > S$. By the definition of $T_n = \cap_{k\in\mathbb{N}}(T + \frac{1}{k}B_{N,n})$, t has to be in $T + \frac{1}{k}B_{N,n}$, i.e. $t = t_k + \frac{1}{k}b_k \in$, where $t \in X$, $t_k \in T \subseteq X$ and $b_k \in B_{N,n}$. Thus, $b_k = k(t - t_k)$ has to be in $X \cap B_{N,n} \subseteq ST$. Moreover, $b_k \in ST$, hence $\frac{1}{k}b_k \in \frac{S}{k}T$ and since $\frac{S}{k} \leqslant 1$, we have $b_k \in T$. Finally

$$x = t_k + \tfrac{1}{k}b_k \in T + T \subseteq 2T \quad\text{and}\quad \tfrac{1}{2}(T_0 \cap X) \subseteq T$$

and since $\frac{1}{2}(T_0 \cap X)$ is a 0-neighborhood in X, T has to be a 0-neighborhood in X. ■

Theorem 4.10. Let $\mathcal{X}$ as in 4.9 satisfy (B1). Then X is bornological if and only if condition (B2) holds for each absolutely convex and bornivorous set $T \subseteq X$.

Proof. "$\Rightarrow$" If X is bornological then T is a 0-neighborhood. By definition there exists $N \in \mathbb{N}$ and $V \in \mathcal{U}_0(X_N)$ such that that $V \cap X \subseteq T$. Now we fix $n \in \mathbb{N}$ and consider $B_{N,n}$ which is bounded in X_N. Hence there exists $S > 0$ such that $B_{N,n} \subseteq SV$ and thus $B_{N,n} \cap X \subseteq ST$.

"$\Leftarrow$" By 4.9, T is a 0-neighborhood in this case. ■

4.4 Remarks on condition (B1)

Let us make some remarks on the conditions established in the latter section.

Remark 4.11. (a) If T is a 0-neighborhood in X, then (B2) is satisfied without any other assumption.

(b) If the space spectrum $\mathcal{X} = (X_N, \subseteq_{N+1}^N)$ satisfies (B1), then it is reduced in the sense
$$\forall\, N\; \exists\, M \geqslant N \colon X_M \subseteq \overline{X}^{X_N}.$$

(c) If the $X_{N,n}$ are all Banach spaces and $\operatorname{Proj}^1 \mathcal{X} = 0$ holds, then (B1) is satisfied.

Proof. (a) We showed this in the proof of 4.10.

(b) For given N we choose $M \geqslant N$ as in (B1). Let $x \in X_M$ be given. Then there exists m such that $x \in X_{M,m}$ and $\rho > 0$ with $\rho x \in B_{M,m}$. For this m by (B1) there exists n with

$$B_{M,m} \subseteq \bigcap_{k\in\mathbb{N}} (B_{N,n} \cap X + \tfrac{1}{k} B_{N,n}) \subseteq \overline{B_{N,n} \cap X}^{X_{N,n}}$$

and hence $\rho x \in \overline{B_{N,n} \cap X}^{X_{N,n}}$. Thus there exists $(x_j)_{j\in\mathbb{N}} \subseteq B_{N,n} \cap X$ with $x_j \to \rho x$ for $j \to \infty$ w.r.t. $\|\cdot\|_{N,n}$ and hence in particular w.r.t. the inductive topology of X_N. Hence $\rho x \in \overline{X}^{X_N}$ and since the latter is a linear space, $x \in \overline{X}^{X_N}$.

(c) We may assume w.l.o.g. that $(B_{N,n})_{n\in\mathbb{N}}$ is a fundamental system of Banach discs in each of the (LB)-spaces X_N. In the proof of Wengenroth [84, 3.3.4] it is shown that $\operatorname{Proj}^1 \mathcal{X} = 0$ implies

$$\forall\, N\ \exists\, M\ \forall\, D \in \mathcal{BD}(X_M)\ \exists\, A \in \mathcal{BD}(X_N) : D \subseteq \overline{A \cap X}^{(X_N)_A}$$

where $\mathcal{BD}(X_N)$ is the system of all Banach discs in X_N and $(X_N)_A$ is the Banach space associated to the Banach disc A. Now we may replace the Banach disc A by $B_{M,m}$ for some m, resp. D by $B_{N,n}$ for some n and thus the above condition transforms into

$$\forall\, N\ \exists\, M\ \forall\, m\ \exists\, n : B_{M,m} \subseteq \overline{B_{N,n} \cap X}^{X_{N,n}}$$

where we used that $(X_N)_{B_{N,n}}$ is just the Banach space $X_{N,n}$. But then we have

$$\overline{B_{N,n} \cap X}^{X_{N,n}} \subseteq \bigcap_{\varepsilon>0} (B_{N,n} \cap X + \varepsilon B_{N,n}) \subseteq \bigcap_{k\in\mathbb{N}} \big(B_{N,n} \cap X + \tfrac{1}{k} B_{N,n}\big)$$

by observing that for a fixed $x \in \overline{B_{N,n} \cap X}^{X_{N,n}}$ for each $\varepsilon > 0$ there exists $x_\varepsilon \in B_{N,n} \cap X$ such that $\|x - x_\varepsilon\|_{N,n} < \varepsilon$ that is $x \in U_\varepsilon(x_\varepsilon) = x_\varepsilon + U_\varepsilon(0) = x_\varepsilon + \varepsilon B_{N,n}$, where $U_\varepsilon(y)$ denotes the ball with radius ε and centre y in the Banach space $X_{N,n}$. ■

Remark 4.12. Let us recall and discuss several (different) notions of reducedness introduced in the literature. Let $\mathcal{X} = (X_N, \rho_M^N)$ be a projective spectrum of locally convex spaces and X its limit. By $\rho^N : X \to E_N$ we denote the canonical maps.

(a) The "classical" notion (e.g. Floret, Wloka [42, p. 143]) is the following. $\mathcal{X}$ is *reduced* if and only if
$$\forall\, N : \overline{\rho^N(X)}^{X_N} = X_N.$$
If the ρ^N are just inclusions (as it is the case in all spectra which we consider) then the latter means that the limit space is dense in every step. In the sequel let us call this property *classical reducedness*.

(b) In [84, 3.2.17] Wengenroth called $\mathcal{X}$ if and only if

$$\forall\, N\ \exists\, M \geqslant N\ \forall\, K \geqslant M : \rho_M^N(X_M) \subseteq \overline{\rho_K^N(X_K)}^{X_N}$$

holds. This notion is also the one used by Braun, Vogt [36, Definition 4].

(c) Moreover, Wengenroth [84, 3.5.5] called $\mathcal{X}$ *strongly reduced* if and only if

$$\forall\, N\ \exists\, M \geqslant N\colon \rho_N^M(X_M) \subseteq \overline{\rho^N(X)}^{X_N}.$$

In the notation above strong reducedness (in the sense of (c)) implies reducedness (in the sense of (b)), cf. Wengenroth [84, remarks previous to 3.3.8] and classical reducedness (in the sense of (a)) implies strong reducedness.

Moreover, Wengenroth [84, remarks previous to 3.3.8] mentioned that for a spectrum $\mathcal{X}$ of separated (LB)-spaces $\operatorname{Proj}^1 \mathcal{X} = 0$ implies that $\mathcal{X}$ is strongly reduced and that for a spectrum $\mathcal{X}$ of Banach spaces reducedness, strongly reducedness and the vanishing of Proj^1 are equivalent.

Proof. "(c) $\Rightarrow$ (b)" Let $\mathcal{X}$ satisfy the condition in (c). In order to show (b) let N be given. We choose M according to (c). Let $K \geqslant M$ be given. Then $\rho_N^M(X_M) \subseteq \overline{\rho^N(X)}^{X_N} = \overline{\rho_k^N \circ \rho^K(X)}^{X_N} \subseteq \overline{\rho_K^N(X_K)}^{X_N}$.

"(a) $\Rightarrow$ (c)" Let $\mathcal{X}$ satisfy the condition in (a). In oder to show (c) let N be given. Choose $M = N$. Since ρ_N^N is the identity, (a) yields exactly the inclusion required in (c). ■

After these first (abstract) observations, which we will extend in 4.17, let us investigate the meaning of condition (B1) in the case of weighted (PLB)-spaces of continuous functions.

Remark 4.13. (a) $\mathcal{A}_0C$ always satisfies condition (B1).

(b) If $\mathcal{A}C$ satisfies (B1), then $\mathcal{A}$ satisfies ($\underline{\mathrm{Q}}$) that is

$$\forall\, N\ \exists\, M\ \forall\, m\ \exists\, n\ \forall\, K,\, \varepsilon > 0\ \exists\, k,\, S > 0 : \tfrac{1}{a_{M,m}} \leqslant \max\left(\tfrac{\varepsilon}{a_{N,n}}, \tfrac{S}{a_{K,k}}\right).$$

and clearly (Q) implies ($\underline{\mathrm{Q}}$).

(c) $\mathcal{A}_0C$ satisfying (B1) does not imply that the sequence $\mathcal{A}$ satisfies (wQ).

(d) If $\mathcal{A}$ satisfies (Q) then the spectrum $\mathcal{A}C$ satisfies condition (B1).

Proof. (a) We claim $B_{N,n}^\circ = \overline{B_{N,n}^\circ \cap (AC)_0(X)}^{C(a_{N,n})_0(X)}$.

"$\supseteq$" Trivial.

"$\subseteq$" Let $f \in B_{N,n}^\circ$ that is $a_{N,n}|f|$ vanishes at ∞ and $a_{N,n}|f| \leqslant 1$ on X. Let S_α and A be defined as in 4.8.(a). We consider $(S_\alpha f)_{\alpha\in A}$. Then we have $S_\alpha f \in C(a_{N,n})_0(X)$ and since $a_{N,n}|S_\alpha f| \leqslant a_{N,n}|f| \leqslant 1$ we have $S_\alpha f \in B_{N,n}^\circ \cap (AC)_0(X)$. By the proof of 4.8.(a) we have $S_\alpha f \to f$ w.r.t. $\|\cdot\|_{N,n}$ and hence f belongs to $\overline{B_{N,n}^\circ \cap (AC)_0(X)}^{C(a_{N,n})_0(X)}$, which establishes the claim.

To check (B1) let N be given. We put $M := N$ and for given m we put $n := m$. Then we have by the above

$$B_{N,n}^\circ \subseteq \overline{B_{N,n}^\circ \cap (AC)_0(X)}^{C(a_{N,n})_0(X)} = \bigcap_{k\in\mathbb{N}} B_{N,n}^\circ \cap (AC)_0(X) + \tfrac{1}{k} B_{N,n}^\circ.$$

(b) For given N we select M as in (B1) and for given m we select n as in (B1). Then

$$\begin{aligned} B_{M,m} &\subseteq \bigcap_{k\in\mathbb{N}} \big(B_{N,n} \cap AC(X) + \tfrac{1}{k}B_{N,n}\big) \\ &\subseteq \bigcap_{k\in\mathbb{N}} \big(AC(X) + \tfrac{1}{k}B_{N,n}\big) \\ &\subseteq \bigcap_{\varepsilon>0} \big(AC(X) + \varepsilon B_{N,n}\big). \end{aligned}$$

For the last inclusion let $f \in \cap_{k\in\mathbb{N}}(AC(X)+\frac{1}{k}B_{N,n})$ and $\varepsilon > 0$ be given. We select $k \in \mathbb{N}$ sucht that $\frac{1}{k} \leqslant \varepsilon$. Then exist $f_1 \in AC(X)$ and $f_2 \in \frac{1}{k}B_{N,n}$ such that $f = f_1+f_2$. According to our choice of k we get $f_2 \in \varepsilon B_{N,n}$ and thus $f_1+f_2 \in AC(X)+\varepsilon B_{N,n}$. Since epsilon was arbitrary this yields $f \in \cap_{\varepsilon>0}(AC(X) + \varepsilon B_{N,n})$.

Now we fix $\varepsilon > 0$. Since $\frac{1}{a_{M,m}} \in B_{M,m}$ the above yields $\frac{1}{a_{M,m}} \in AC(X) + \frac{\varepsilon}{2}B_{N,n}$. Thus there exist f and g such that $\frac{1}{a_{M,m}} = f + \frac{\varepsilon}{2}g$ with $f \in AC(X)$ and $g \in B_{N,n}$. That is, for each K there exists k and $\lambda > 0$ with $|f| \leqslant \frac{\lambda}{a_{K,k}}$ and $|g| \leqslant \frac{1}{a_{N,n}}$ and we may compute

$$\tfrac{1}{a_{M,m}} = |f + \varepsilon g| \leqslant |f| + \tfrac{\varepsilon}{2}|g| \leqslant \tfrac{\lambda_K}{a_{K,k}} + \tfrac{\varepsilon}{2a_{N,n}} \;\leqslant \max\big(\tfrac{2\varepsilon}{2a_{N,n}}, \tfrac{2\lambda}{a_{K,k}}\big)$$

to obtain finally condition $(\underline{\mathrm{Q}})$ by setting $S := 2\lambda$. In view of the quantifiers, it is clear that $(\underline{\mathrm{Q}})$ is a specialization of (Q).

(c) There are examples of sequences $\mathcal{A}$ which do not satisfy (wQ) (cf. [82, Example 5.12]) but (B1) is always satisfied (by (a)).

(d) By 3.B, (Q) is equivalent to $\operatorname{Proj}^1 \mathcal{A}C = 0$. Thus, 4.11.(c) yields that (B1) holds. ■

If $\mathcal{A}$ satisfies condition (Q) it follows from 4.13.(d) that $\mathcal{A}C$ satisfies (B1) and (cf. section 2) that $\mathcal{A}$ satisfies (wQ). In view of 3.1.B – which provides a characterization of $\operatorname{Proj}^1 \mathcal{A}C = 0$ (via (Q)) but no characterization of ultrabornologicity or barrelledness of $AC(X)$ – it is not clear if (wQ) is also sufficient for these properties. Unfortunately, we cannot solve this problem but (which is even worse) show (see 4.15) that the methods developed in the previous section (in particular 4.10) cannot be used to make any progress in this direction. For the proof of 4.14 we need the following notation which is of (independent) interest in view of our discussion after 4.17.

We say that $\mathcal{A}$ satisfies condition $(\overline{\mathrm{wS}})$ if

$$\forall\, M \;\exists\, M' \;\forall\, m \;\exists\, m' \;\forall\, \varepsilon > 0 \;\exists\, \overline{a} \in \overline{A}\colon \tfrac{1}{a_{M',m}} \leqslant \overline{a} + \tfrac{\varepsilon}{a_{M,m'}},$$

where $\overline{A} := \{\, \overline{a}\colon X \to\,]0,\infty[\;;\; \overline{a} \in C(X) \text{ and } \forall\, N \;\exists\, n\colon \sup_{x\in X} a_{N,n}(x)\overline{a}(x) < \infty \,\}$.

Observation 4.14. The following are equivalent.

(i) $\mathcal{A}$ satisfies condition (wQ) and $\mathcal{A}_0C$ satisfies (B1).

(ii) $\mathcal{A}$ satisfies condition (Q).

Proof. "(ii)⇒(i)" Clearly (Q) implies (wQ) and by 4.13.(d), (Q) implies also (B1).

"(i)⇒(ii)" Condition (B1) clearly implies

$$\forall\, M\ \exists\, M'\ \forall\, m\ \exists\, m'\ \forall\, \varepsilon>0\colon\ B_{M',m}\subseteq AC(X)+\varepsilon B_{M,m'}.$$

We show that $\mathcal{A}$ satisfies $(\overline{\mathrm{wS}})$. For given M we select M' and for given m we select m' as in the condition above. Let $\varepsilon>0$ be given. To show the estimate in $(\star)$, we consider $\frac{1}{a_{M',m}}\in B_{M',m}$ that is by the latter condition there exist $a'\in AC(X)$ and $f\in B_{N,n'}$ such that $\frac{1}{a_{M',m}}=a'+\varepsilon f$ and hence $\frac{1}{a_{M',m}}=\big|\frac{1}{a_{M',m}}\big|\leqslant|a'|+\varepsilon|f|\leqslant \overline{a}+\frac{\varepsilon}{a_{N,n'}}$ since $f\in B_{N,n'}$ and by selecting $\overline{a}:=|a'|$.

Let us now write (wQ) in the following way

$$\forall\, N\ \exists\, M,\, n\ \forall\, K\ m'\ \exists\, k,\, S>0\colon\ \tfrac{1}{a_{M,m'}}\leqslant S\big(\tfrac{1}{a_{N,n}}+\tfrac{1}{a_{K,k}}\big)$$

and claim (Q) in the notation

$$\forall\, N\ \exists\, M',\, n\ \forall\, K,\, m\,,\varepsilon>0\ \exists\, k,\, S'>0\colon\ \tfrac{1}{a_{M',m}}\leqslant\tfrac{\varepsilon}{a_{N,n}}+\tfrac{S'}{a_{K,k'}}.$$

Let $N\in\mathbb{N}$ be given. We choose M and n as in (wQ). We put M into $(\overline{\mathrm{wS}})$ and obtain M'. Let K, m and $\varepsilon>0$ be given. We put m into $(\overline{\mathrm{wS}})$ and obtain m'. We put m', K and $\varepsilon>0$ into (wQ) and obtain k and $S>0$. Finally, we put $\frac{\varepsilon}{S}$ into $(\overline{\mathrm{wS}})$ and obtain $\overline{a}$. Now by (wQ) and $(\overline{\mathrm{wS}})$ we have the two estimates

$$(\circ)\quad \tfrac{1}{a_{M',m}}\leqslant\overline{a}+\tfrac{\varepsilon}{S}\tfrac{1}{a_{M,m'}}\qquad\text{and}\qquad(\circ\circ)\quad\tfrac{1}{a_{M,m'}}\leqslant\tfrac{S}{a_{N,n}}+\tfrac{S}{a_{K,k}}.$$

Since $\overline{a}\in AC(X)$ we in particular have $\overline{a}\in\mathcal{A}_KC(X)$ and hence there exists k' and $\lambda>0$ such that

$$(\circ\circ\circ)\quad a_{K,k'}\,\overline{a}\leqslant\lambda$$

and we clearly may choose $k'\geqslant k$. Hence

$$\begin{aligned}\tfrac{1}{a_{M',m}}&\stackrel{(\circ)}{\leqslant}B+\tfrac{\varepsilon}{S}\tfrac{1}{a_{M,m'}}\\&\stackrel{(\circ\circ)}{\leqslant}\overline{a}+\tfrac{\varepsilon}{S}\big(\tfrac{S}{a_{N,n}}+\tfrac{S}{a_{K,k}}\big)\leqslant\overline{a}+\tfrac{\varepsilon}{a_{N,n}}+\tfrac{\varepsilon}{a_{K,k}}\\&\stackrel{(\circ\circ\circ)}{\leqslant}\tfrac{\lambda}{a_{K,k'}}+\tfrac{\varepsilon}{a_{N,n}}+\tfrac{\varepsilon}{a_{N,n}}+\varepsilon\tfrac{1}{a_{K,k}}\leqslant\tfrac{\lambda+\varepsilon}{a_{K,k'}}+\tfrac{\varepsilon}{a_{N,n}}.\end{aligned}$$

Now we put $S':=(\lambda+\varepsilon)$ and have $\frac{1}{a_{M',m}}\leqslant\frac{S'}{a_{K,k'}}+\frac{\varepsilon}{a_{N,n}}$ as desired. ■

Remark 4.15. As already mentioned, 4.14 emphazises that we cannot use the techniques developed in section 4.3 to find sufficient conditions for bornologicity of $AC(X)$ which are strictly weaker than (Q) since in 4.10 we assume (B1) and bornologicity (or even barrelledness) implies (wQ) by 3.B, hence by the above we already have (Q).

Scholium 4.16. If the spectrum $\mathcal{A}C$ satisfies (B1),

$(\overline{\mathrm{B1}})\quad \forall\, N\ \exists\, M>N\ \forall\, m\ \exists\, n\ \forall\,\varepsilon>0\ \exists\, B\subseteq AC(X)$ bounded: $B_{M,m}\subseteq B+\varepsilon B_{N,n}$

holds.

Proof. In the proof of 4.14 we showed that (B1) implies $(\overline{\mathrm{wS}})$, which we may write in the following way

$$\forall\, N \,\exists\, M > N \,\forall\, m \,\exists\, n \,\forall \varepsilon > 0 \,\exists\, \overline{a} \in \overline{A}\colon \tfrac{1}{a_{M,m}} \leqslant \overline{a} + \tfrac{\varepsilon}{a_{N,n}}.$$

To show $(\overline{\mathrm{B1}})$, let N be given. We select M as in $(\overline{\mathrm{wS}})$. For given m we select n as in $(\overline{\mathrm{wS}})$. Let $\varepsilon > 0$ be given. We put $\frac{\varepsilon}{4}$ into $(\overline{\mathrm{wS}})$ and select $\overline{a}$ as in $(\overline{\mathrm{wS}})$. We put $B := \{\, f \in AC(X)\,;\, |f| \leqslant 4\overline{a}\,\}$. Now we have to show the inclusion in $(\overline{\mathrm{B1}})$: Let $f \in B_{M,m}$ that is $a_{M,m}|f| \leqslant 1$, i.e. $|f| \leqslant \frac{1}{a_{M,m}} \leqslant \overline{a} + \frac{\varepsilon}{2a_{N,n}} \leqslant 2\max(\overline{a}, \frac{\varepsilon}{4a_{N,n}}) = \max(2\overline{a}, \frac{\varepsilon}{2a_{N,n}})$. According to [2, Lemma 3.5] there exist $f_1,\ f_2 \in C(X)$ with $f = f_1 + f_2$ and $|f_1| \leqslant 2\cdot 2\overline{a}$, $|f_2| \leqslant 2\cdot\frac{\varepsilon}{2a_{N,n}}$. That is $f_1 \in B$ and $f_2 \in \varepsilon B_{N,n}$, i.e. $f \in B + \varepsilon B_{N,n}$. ■

For a further interpretation of (B1) we compare $(\overline{\mathrm{B1}})$ with the following two conditions of Braun, Vogt [36]: Let $\mathcal{X}$ be as in section 4.3. Let us write the conditions of [36, 4.] in the following way. We say that $\mathcal{X}$ satisfies (P_2) if

$$\forall\, N \,\exists\, M,\, n \,\forall\, K,\, m' \,\exists\, k,\, S > 0\colon B_{M,m'} \subseteq S(B_{N,n} + B_{K,k}).$$

We say that $\mathcal{X}$ satisfies $(\overline{\mathrm{P}_2})$ if

$$\forall\, N \,\exists M',\, n \,\forall\, K,\, m,\, \varepsilon > 0 \,\exists\, k',\, S' > 0\colon B_{M',m} \subseteq \varepsilon B_{N,n} + S' B_{K,k'}.$$

Braun, Vogt [36] proved that $\mathrm{Proj}^1 \mathcal{X} = 0$ holds if $\mathcal{X}$ satisfies $(\overline{\mathrm{P}_2})$, where $\mathcal{X}$ is an arbitrary projective spectrum of (LB)-spaces.

Proposition 4.17. Let $\mathcal{X}$ be as above and assume that the X_N are regular (LB)-spaces. If $\mathcal{X}$ satisfies (P_2) and $(\overline{\mathrm{B1}})$ then $\mathcal{X}$ satisfies $(\overline{\mathrm{P}_2})$.

Proof. $(\overline{\mathrm{B1}})$ can be written as follows

$$\forall\, M \,\exists\, M' \,\forall\, m \,\exists\, m' \,\forall\, \varepsilon > 0 \,\exists\, B \subseteq X \text{ bounded}\colon B_{M',m} \subseteq B + \varepsilon B_{M,m'}.$$

We show $(\overline{\mathrm{P}_2})$ in the way it is stated above. Let $N \in \mathbb{N}$ be given. We choose M and n as in (P_2) and put M into $(\overline{\mathrm{B1}})$ to obtain M'. Let K, m and $\varepsilon > 0$ be given. We put m into $(\overline{\mathrm{B1}})$ and obtain m'. We put m', K and $\varepsilon > 0$ into (P_2) and obtain k and $S > 0$. Finally, we put $\frac{\varepsilon}{S}$ into $(\overline{\mathrm{B1}})$ and get a bounded set $B \subseteq X$. Now we have by $(\overline{\mathrm{B1}})$ and (P_2) the two inclusions

$$(\circ)\quad B_{M',m} \subseteq B + \tfrac{\varepsilon}{S} B_{M,m'} \qquad \text{and} \qquad (\circ\circ)\quad B_{M,m'} \subseteq S B_{N,n} + S B_{K,k}.$$

Since B is bounded in X, B is also bounded in the (LB)-spaces X_K and this space is regular, i.e. there exists k' and $\lambda > 0$ such that

$$(\circ\circ\circ)\quad B \subseteq \lambda B_{K,k'}$$

and we clearly may choose $k' \geqslant k$. Hence

$$\begin{aligned} B_{M',m} &\overset{(\circ)}{\subseteq} B + \tfrac{\varepsilon}{S} B_{M,m'} \\ &\overset{(\circ\circ)}{\subseteq} B + \tfrac{\varepsilon}{S}(SB_{N,n} + SB_{K,k}) \subseteq B + \varepsilon B_{N,n} + \varepsilon B_{K,k} \\ &\overset{(\circ\circ\circ)}{\subseteq} \lambda B_{K,k'} + \varepsilon B_{N,n} + \varepsilon B_{N,n} + \varepsilon B_{K,k} \subseteq (\lambda+\varepsilon) B_{K,k'} + \varepsilon B_{N,n}. \end{aligned}$$

Now we put $S' := (\lambda+\varepsilon)$ and have $B_{M',m} \subseteq S' B_{K,k'} + \varepsilon B_{N,n}$. ■

Remark 4.18. The implication in 4.16, that is "(B1) $\Rightarrow$ ($\overline{\text{B1}}$)", is not true in general (cf. 4.22). This follows from an investigation of the following special case. Assume $X_{N,n} = X_{N,n+1} =: X_N$ for all $n \in \mathbb{N}$ and w.l.o.g. $B_{N+1} \subseteq B_{N+1}$ that is $X = \operatorname{proj}_n X_N$. We assume the X_N to be Banach spaces. Then X is a Fréchet space. In this case condition (B1) reduces to

$$\forall\, N\; \exists\, M > N\colon B_M \subseteq \bigcap_{k\in\mathbb{N}} B_N \cap X + \tfrac{1}{k} B_N$$

and ($\overline{\text{B1}}$) reduces to

$$\forall\, N\; \exists\, M > N\; \forall\, \varepsilon > 0\; \exists\, B \subseteq X \text{ bounded}\colon B_M \subseteq B + \varepsilon B_N$$

which implies

$$\forall\, N\; \exists\, M > N\; \forall\, \varepsilon > 0\; \exists\, B \subseteq X \text{ bounded}\colon B_M \cap X \subseteq B + \varepsilon(B_N \cap X).$$

The latter condition is exactly the definition of quasinormability, which was invented by Grothendieck [44, Definition 4, p. 106 and Lemma 6, p. 107](cf. [60, Definition after Proposition 26.12]).

Remark 4.19. As we just mentioned, by Grothendieck a Fréchet space E is called quasinormable if

$$\forall\, U \in \mathcal{U}_0(E)\; \exists\, V \in \mathcal{U}_0(E)\; \forall\, \varepsilon > 0\; \exists\, B \subseteq E \text{ bounded}\colon V \subseteq B + \varepsilon U.$$

The latter definition generalizes the Schwartz spaces: A Fréchet spaces is Schwartz if and only if the above condition holds with a finite set B, cf. [60, Remark previous to 26.13].

Let us moreover remark that Grothendieck's definition of quasinormability is equivalent to the following condition, cf. [60, Lemma 26.14]

$$\forall\, p\; \exists\, q\; \forall\, k,\, \varepsilon > 0\; \exists\, \lambda > 0\colon U_q \subseteq \lambda U_k + \varepsilon U_p,$$

where $(U_n)_{n\in\mathbb{N}}$ denotes a basis of 0-neighborhoods in E. The above condition is due to Bonet [28, (2) on p. 301] who pointed out that the latter is equivalent to quasinormability by observing that the above condition is equivalent to the fact that the Fréchet space E satisfies (2) in Meise, Vogt [59, Theorem 7]: According to the article [59, section (iii) and (iv)] we put

$$\mathbb{M} := \{\, \varphi\,;\ \varphi\colon\ ]0,\infty[\, \to\,]0,\infty[\ \text{ is strictly increasing}\,\}.$$

Then the Fréchet space E is said to satisfy condition (Ω_φ) for some $\varphi \in \mathbb{M}$, if

$$\forall\, p\; \exists\, q\; \forall\, k\; \exists\, C > 0\; \forall\, r > 0\colon U_q \subseteq C\varphi(r)U_k + \tfrac{1}{r}U_p$$

holds. The result [59, Theorem 7] now in particular states that E is quasinormable if and only if there exists $\varphi \in \mathbb{M}$ such that E satisfies (Ω_φ).

Note, that Bonet [28] also gives an alternative proof for the equivalence of quasinormability and his condition, which is independent of the result [59, Theorem 7] and in particular "less involved" than the proof of the result of Meise and Vogt.

Proposition 4.20. If $\mathcal{X} = (X_M)_{M\in\mathbb{N}}$ is a projective spectrum of Banach spaces with inclusions as linking maps and $X = \operatorname{proj}_N X_N$ is the corresponding Fréchet space, we have (i)$\Rightarrow$(ii)$\Leftrightarrow$(iii) where

(i) Condition $(\overline{\text{B1}})$ holds,

(ii) $\mathcal{X}$ is reduced in the sense $\forall\, N\; \exists\, M > N\colon X_M \subseteq \overline{X}^{X_N}$,

(iii) Condition (B1) holds.

In particular, "$(\overline{\text{B1}}) \Rightarrow$ (B1)" holds for projective spectra of Banach spaces with inclusions as linking maps.

Proof. "(i)$\Rightarrow$(ii)" Let $(\overline{\text{B1}})$ be satified that is

$$\forall\, N\; \exists\, M > N\; \forall\, \varepsilon > 0\; \exists\, B \subseteq X \text{ bounded}\colon B_M \subseteq B + \varepsilon B_N.$$

To show that $\mathcal{X}$ is reduced, we fix $N \in \mathbb{N}$ and choose M as in the condition above. Then $(\overline{\text{B1}})$ implies in particular that $B_M \subseteq X + \varepsilon B_N$ holds for each $\varepsilon > 0$ that is $B_M \subseteq \overline{X}^{X_N}$ and thus $X_M \subseteq \overline{X}^{X_N}$.

"(ii)$\Rightarrow$(iii)" For given N we choose $M > N$ such that $X_M \subseteq \overline{X}^{X_N}$. Let $x \in B_M$. Since $B_M \subseteq X_M \subseteq \overline{X}^{X_N}$ we have $x \in \overline{X}^{X_N}$. Since $B_M \subseteq B_N$ we also have $x \in B_N$. Hence $x \in B_N \subseteq B_N \cap \overline{X}^{X_N}$. Now we claim $x \in \overline{B \cap X}^{X_N}$. If x is in the interior of B_N we can choose a sequence $(x_j)_{j\in\mathbb{N}} \subseteq X$ with $x_j \to x$ in X_N. Since x is in the interior of B_N there exists $J \in \mathbb{N}$ such that $x_j \in B_N$ for all $j \geqslant J$. Hence $(x_j)_{j\geqslant J} \subseteq B_N \cap X$ with $x_j \to x$ in X_N and thus $x \in \overline{B_N \cap X}^{X_N}$. If otherwise $\|x\|_N = 1$ let $(x_j)_{j\in\mathbb{N}} \subseteq X$ with $x_j \to x$ in X_N. Put $y_j := \frac{x_j}{\|x_j\|_N}$. Then $(y_j)_{j\in\mathbb{N}} \subseteq B_N \cap X$ and $y_j \to \frac{x}{\|x\|_N} = \frac{x}{1} = x$ and hence $x \in \overline{B_N \cap X}^{X_N}$.

"(iii)$\Rightarrow$(ii)" This is true even in the (PLB)-case, see 4.11.(b).

The last statement is now clear. ∎

For the rest of this section we will study the case that the spaces $AC(X)$ and $(AC)_0(X)$ are Fréchet spaces. That is we put $a_{N,n} = 2^n a_N$ for some increasing sequence $(a_N)_{N\in\mathbb{N}}$. Alternatively, we may simply define $AC(X) = \operatorname{proj}_N Ca_N(X)$ and $(AC)_0(X) = \operatorname{proj}_N C(a_N)_0(X)$.

Before we present results on the above spaces for rather general, that is Hausdorff locally compact and σ-compact, X (cf. 4.23 and 4.24) let us study the case $X = \mathbb{N}$. In this situation, the spaces under consideration turn out to be the well-known Köthe echelon spaces $\lambda^\infty(A)$ and $\lambda^0(A)$ where the Köthe matrix A is given by

$A = (a_N)_{N\in\mathbb{N}}$ (in the notation of [26, Definition 1.2]).

The following observations are easy; they all refer to the case that the spaces $AC(X)$ and $(AC)_0(X)$ are Fréchet spaces and that $X = \mathbb{N}$.

(a) The system $\overline{A}$ introduced in the proof of 4.14 is just the Köthe set

$$\overline{V} = \big\{\overline{a}\colon \mathbb{N} \to \ ]0,\infty[\,;\ \forall\, N\colon \sup_{i\in\mathbb{N}} a_N(i)\overline{a}(i) < \infty\big\}$$

of Bierstedt, Meise, Summers [26, Definition 1.4].

(b) Condition $(\overline{\text{wS}})$ of the proofs of 4.14 and 4.16 reduces to

$$\forall\, N\ \exists\, M > N\ \forall\, \varepsilon > 0\ \exists\, \overline{a} \in \overline{A}\,:\ \tfrac{1}{a_M} \leqslant \overline{a} + \tfrac{\varepsilon}{a_N}$$

which is equivalent to condition

$$\text{(wS)}\qquad \begin{array}{c} \forall\, N\ \exists\, M > N\ \forall\, \varepsilon > 0\ \exists\, \overline{a} \in \overline{A}\ \ \forall\, i \in \mathbb{N}\colon \\ \tfrac{1}{a_M(i)} \leqslant \tfrac{\varepsilon}{a_N(i)}\ \text{ whenever }\ \overline{a}(i) < \tfrac{1}{a_M(i)} \end{array}$$

of Bierstedt, Meise, Summers [26, Proposition 3.2].

(c) The conditions (Q) and $(\underline{\text{Q}})$ both are equivalent to

$$\forall\, N\ \exists\, M\ \forall\, K,\, \varepsilon > 0\ \exists\, S > 0\colon \tfrac{1}{a_M} \leqslant \tfrac{\varepsilon}{a_N} + \tfrac{S}{a_K}.$$

Let us now review some well-known results on the spaces $\lambda^\infty(A)$ and $\lambda^0(A)$, which should be compared with 4.23 and 4.24. In the following remark we denote by $\mathcal{A}L$ and $\mathcal{A}_0L$ the natural spectra corresponding with $\lambda^\infty(A)$ and $\lambda^0(A)$.

Remark 4.21. (Bierstedt, Meise, Summers [26, Proposition on p. 48, Proposition 3.2, Corollary 3.5 and Example 3.11], Vogt [78, last Remark on page 167] and Meise, Vogt [60, 27.20]) Let A be a Köthe matrix.

(a) The following are equivalent.
   (i) $\mathcal{A}L$ is reduced.
   (ii) $\lambda^\infty(A)$ is quasinormable.
   (iii) A satisfies condition (wS).
   (iv) A satisfies condition (Q).
   (v) A satisfies condition $(\underline{\text{Q}})$.

(b) The spectrum $\mathcal{A}_0L$ is always reduced. Moreover, the following are equivalent.
   (i) $\lambda^0(A)$ is quasinormable.
   (ii) A satisfies condition (wS).
   (iii) A satisfies condition (Q).
   (iv) A satisfies condition $(\underline{\text{Q}})$.

(c) There exists a Köthe matrix A which does not satisfy condition (wS), that is the spectrum $\mathcal{A}_0L$ is reduced but $\lambda^0(A)$ is not quasinormable.

From 4.21.(c) we get immediately the following.

Remark 4.22. (a) The implication "(ii)$\Rightarrow$(i)" in 4.20 cannot be true in general.

(b) The implication "(B1)$\Rightarrow$ ($\overline{\text{B1}}$)" cannot be true in general: We showed in 4.20 that for projective spectra of Banach space with inclusions as linking maps (B1) is equivalent to reducedness and we remarked in 4.18 that in this case ($\overline{\text{B1}}$) implies quasinormability.

As promised let us now treat the Fréchet spaces $AC(X)$ and $(AC)_0(X)$ for a Hausdorff locally compact and σ-compact space X. Since we just explained what conditions ($\overline{\text{wS}}$), (Q) and ($\underline{\text{Q}}$) look like in the Fréchet case we can start right away. Note that in section 6 we will obtain similar results for holomorphic functions.

Proposition 4.23. In the Fréchet case under O-growth conditions the following are equivalent.

(i) $AC(X)$ is quasinormable.
(ii) $\mathcal{A}C$ is reduced.
(iii) $\mathcal{A}C$ satisfies (B1).
(iv) $\mathcal{A}C$ satisfies ($\overline{\text{B1}}$).
(v) $\mathcal{A}$ satisfies (Q).
(vi) $\mathcal{A}$ satisfies ($\underline{\text{Q}}$).
(vii) $\mathcal{A}$ satisfies condition ($\overline{\text{wS}}$).

Proof. "(i)$\Rightarrow$(ii)" This is 4.20.

"(ii)$\Rightarrow$(iii)" This is 4.20.

"(iii)$\Rightarrow$(iv)" This is 4.16.

"(iv)$\Rightarrow$(i)" As we noted in 4.18, for projective spectra of Banach spaces with inclusions as linking maps ($\overline{\text{B1}}$) implies quasinormability.

"(v)$\Leftrightarrow$(vi)" As we noted previous to 4.21, in the Fréchet case (Q) and ($\underline{\text{Q}}$) coincide.

"(v)$\Rightarrow$(iii)" This is 4.13.(d)

"(iii)$\Rightarrow$(v)" In the Fréchet case condition (wQ) reduces to

$$\forall\, N\ \exists\, M \geqslant N\ \forall\, K \geqslant M\ \exists\, S > 0 : \tfrac{1}{a_M} \leqslant S \max\big(\tfrac{1}{a_N}, \tfrac{1}{a_K}\big)$$

and is always satisfied: Let N be given. We choose $M := N$. Let $K \geqslant M$ be given. We put $S := 1$. Then the estimate $\frac{1}{a_N} \leqslant \max(\frac{1}{a_N}, \frac{1}{a_K})$ is trivial. Hence, 4.14 yields the desired implication.

"(i)$\Leftrightarrow$(vii)" This follows from Bierstedt, Meise [24, Proof of Proposition 5.8]. ∎

Let us sum up the information we have concerning o-growth conditions in the Fréchet case.

Proposition 4.24. In the Fréchet case under o-growth conditions the following statements are true.

(a) $\mathcal{A}_0C$ is always reduced.
(b) (wQ) is always satisfied.
(c) For $\mathcal{A}_0C$, condition (B1) is always satisfied.
(d) $(AC)_0(X)$ fails to be quasinormable in general. Thus, condition (B1) and ($\overline{\text{B1}}$) are not equivalent for $\mathcal{A}_0C$.

Proof. (a) This follows from Agethen, Bierstedt, Bonet [2, section 2].

(b) See the proof of "(iii)⇒(v)" in 4.23.

(c) By 4.20, (B1) is equivalent to the reducedness of $(AC)_0(X)$. Hence the assertion follows from (a).

(d) This follows from 4.21.(c). Now, it is enough to recall that for projective spectra of Banach spaces with inclusions as linking maps condition ($\overline{\text{B1}}$) implies the definition of quasinormability. ■

At the end of section 11 we will organize the results above and the corresponding results on the holomorphic (PLB)- and Fréchet spaces in a comprehensive way by drawing schemes of implications arranged in a table which allows a direct comparison of the different settings and cases (cf. 11.8).

Remark 4.25. In addition to the Fréchet cases of the (PLB)-spaces we can also look at their (LB)-cases. That is, we assume $X_{N,n} = X_{N+1,n} =: X_n$ for all $N \in \mathbb{N}$ and w.l.o.g. $B_n \subseteq B_{n+1}$. In this case condition (B1) reduces to

$$\forall\, m \;\exists\, n\colon B_m \subseteq \bigcap_{k\in\mathbb{N}} B_n + \tfrac{1}{k} B_n.$$

and it is clear that this condition is satisfied in general. In view of our discussion concerning ($\underline{\text{Q}}$) it might look convenient to change the quantifiers in (B1) into the following stronger condition $(\text{B1})^\star$

$$\forall\, N \;\exists\, M,\, n \;\forall\, m\colon B_{M,m} \subseteq \bigcap_{k\in\mathbb{N}} B_{N,n} \cap X + \tfrac{1}{k} B_{N,n}$$

since then the proof of 4.13.(b) would yield that $(\text{B1})^\star$ implies (Q). But, in the (LB)-case $(\text{B1})^\star$ reduces to

$$\exists\, n \;\forall\, m\colon B_m \subseteq \bigcap_{k\in\mathbb{N}} B_n + \tfrac{1}{k} B_n$$

which implies that there exists n such that for each m the inclusion $B_m \subseteq \overline{B_n}^{X_n} = B_n$ is valid and hence $X_m \subseteq X_n$ holds. But this implies that $X = X_n$ is a Banach space. This shows that $(\text{B1})^\star$ would be a much too strong condition.

5 Weighted (PLB)-spaces of holomorphic functions: Results for arbitrary and for balanced domains

After the abstract results of the last section we start with the investigation of the spaces introduced in section 2. In what follows we establish necessary conditions for barrelledness of the spaces $AH(G)$ and $(AH)_0(G)$. This is possible under rather mild assumptions: In [20] Bierstedt, Bonet, Galbis studied the following setting: G is balanced, all considered weights are radial (i.e. for each weight a they assume $a(z) = a(\lambda z)$ for every $\lambda \in \mathbb{C}$ with $|\lambda| = 1$), the Banach space topologies are stronger than co and the polynomials are contained in all the considered spaces.

They remark that for bounded G the latter is equivalent to requiring that each weight $a_{N,n}$ extends continuously to $\overline{G}$ with $a_{N,n}|_{\partial G} = 0$, while for $G = \mathbb{C}^d$ the assumption means exactly that each weight $a_{N,n}$ is rapidly decreasing at ∞ (cf. [20, remark previous to 1.2]). In this setting (which we will in the sequel call the *balanced setting*) we have $\overline{B_a^\circ}^{\mathrm{co}} = B_a$ (cf. [20, 1.5.(c)]) and each step $(\mathcal{A}_N)_0H(G)$ is even a topological subspace of $\mathcal{A}_NH(G)$ (see Bierstedt, Bonet, Galbis [20, 1.6.(d)]) and hence this is also true for the (PLB)-spaces. Bierstedt, Bonet, Taskinen [21, 1.13] showed that $\tilde{w} = \tilde{w}_0$ if $Ha_0(G)'' = Ha(G)$ holds isometrically. By [20, 1.5.(d)] the latter is the case in the balanced setting. One of the crucial techniques used by Bierstedt, Bonet, Galbis is based on the existence of a Taylor series representation about zero for each $f \in H(G)$,

$$f(z) = \sum_{k=0}^{\infty} p_k(z) \quad \text{for } z \in \mathbb{D},$$

where p_k is a k-homogeneous polynomial for $k = 0, 1, \dots$. The series converges to f uniformly on each compact subset of G. The *Cesàro means* of the partial sums of the Taylor series of f are denoted by $S_n(f)$, $n = 0, 1, \dots$, that is,

$$[S_n(f)](z) = \tfrac{1}{n+1} \sum_{l=0}^{n} \Big(\sum_{k=0}^{l} p_k(z) \Big) \quad \text{for } z \in G.$$

Each $S_n(f)$ is a polynomial of degree less or equal to n and $S_n(f) \to f$ uniformly on every compact subset of G (cf. [20, section 1]).

5.1 Reducedness

Proposition 5.1. Assume that we are in the balanced setting. Then $\mathcal{A}_0H$ is reduced (in the sense of 4.12.(a)).

Proof. By the definition of the balanced setting we have $\mathbb{P} \subseteq H(a_{N,n})_0(G)$ for all N, $n \in \mathbb{N}$, where $\mathbb{P}$ denotes the space of polynomials. Moreover, each $Ha_{N,n}(G)$ has a topology stronger than co. Thus, by Bierstedt, Bonet, Galbis [20, 1.6] it follows that $\mathbb{P}$ is dense in $(\mathcal{A}_N)_0H(G)$ for each $N \in \mathbb{N}$. Thus, the polynomials are contained in the projective limit $(AH)_0(G)$, and hence $(\mathcal{A}_N)_0H(G) \supseteq \overline{(AH)_0(G)}^{(\mathcal{A}_N)_0H(G)} \supseteq \overline{\mathbb{P}}^{(\mathcal{A}_n)_0H(G)} = (\mathcal{A}_N)_0H(G)$ holds for each $N \in \mathbb{N}$, i.e. the projective limit is dense in every step, that is, $\mathcal{A}_0H$ is reduced. ■

Scholium 5.2. In the balanced setting, the space of all polynomials $\mathbb{P}$ on G is dense in every step of the projective spectrum $\mathcal{A}_0H$ and also dense in its limit $(AH)_0(G)$.

Note, that it is open if $\mathcal{A}H$ is reduced under the assumptions of the balanced setting or even under the stronger assumptions of the subsequent sections.

5.2 Necessary conditions for the vanishing of Proj ^{1}

Let us first state an immediate consequence of section 4.1. In the sequel we will show that the following result can even be improved.

Corollary 5.3. (of 4.2) Assume that we are in the balanced setting. If $\operatorname{Proj}^1 \mathcal{A}H = 0$ then $(\mathrm{wQ})^{\sim}_{\mathrm{out}}$ holds for the sequence $\mathcal{A}$.

Proof. In the balanced setting we know that all the considered Banach spaces are continuously included in $(H(G), \mathrm{co})$. Hence, 4.2 yields

$$\forall\, N\ \exists\, M \geqslant N, n\ \forall\, K \geqslant M, m\ \exists\, k, S > 0\colon B_{M,m} \subseteq S(B_{K,k} + B_{N,n}).$$

In order to show $(\mathrm{wQ})^{\sim}_{\mathrm{out}}$, we put $B := \{\, g \in H(G)\ ;\ a_{M,m}|g| \leqslant 1 \,\}$ and observe

$$\begin{aligned} B &= \{\, g \in H(G)\ ;\ g \in B_{M,m} \,\} \\ &\subseteq \{\, g \in H(G)\ ;\ g \in S(B_{K,k} + B_{N,n}) \,\} \\ &= \{\, g \in H(G)\ ;\ g = S(g_1 + g_2),\ g_1 \in B_{K,k} \text{ and } g_2 \in B_{N,n} \,\} \\ &= \{\, g \in H(G)\ ;\ g = S(g_1 + g_2),\ a_{K,k}|g_1| \leqslant 1 \text{ and } a_{N,n}|g_2| \leqslant 1 \,\} \\ &= \{\, g \in H(G)\ ;\ g = S(g_1 + g_2),\ |g_1| \leqslant \tfrac{1}{a_{K,k}} \text{ and } |g_2| \leqslant \tfrac{1}{a_{N,n}} \,\} \\ &\subseteq \{\, g \in H(G)\ ;\ g = S(g_1 + g_2),\ |g| \leqslant S(|g_1| + |g_2|) \leqslant S(\tfrac{1}{a_{K,k}} + \tfrac{1}{a_{N,n}}) \,\} \\ &= \{\, g \in H(G)\ ;\ (\tfrac{1}{a_{K,k}} + \tfrac{1}{a_{N,n}})^{-1}|g| \leqslant S \,\} =: C. \end{aligned}$$

Since the quantifiers in the above condition coincide with those in $(\mathrm{wQ})^{\sim}_{\mathrm{out}}$, it is enough to show the estimate

$$\begin{aligned} \big(\tfrac{1}{a_{M,m}}\big)^{\sim}(z) &= \sup\{\, |g(z)|\ ;\ g \in B \,\} \leqslant \sup\{\, |g(z)|\ ;\ g \in C \,\} \\ &= S \sup\{\, |g(z)|\ ;\ g \in H(G),\ (\tfrac{1}{a_{K,k}} + \tfrac{1}{a_{N,n}})^{-1}|g| \leqslant 1 \,\} \\ &= S\big(\tfrac{1}{a_{K,k}} + \tfrac{1}{a_{N,n}}\big)^{\sim}(z) \leqslant 2S\big(\max\big(\tfrac{1}{a_{K,k}}, \tfrac{1}{a_{N,n}}\big)\big)^{\sim}(z). \end{aligned}$$

Hence we obtain condition $(\mathrm{wQ})^{\sim}_{\mathrm{out}}$ by selecting the last constant to be $2S$. ∎

As we noted in section 2, among the "tilded" conditions $(\mathrm{wQ})^{\sim}_{\mathrm{out}}$ and $(\mathrm{wQ})^{\sim}_{\mathrm{in}}$, the latter is the stronger one. We will show in 5.8 that $\operatorname{Proj}^1 \mathcal{A}H = 0$ implies $(\mathrm{wQ})^{\sim}_{\mathrm{in}}$, which improves the above result. In order to do this, we have to investigate the o-growth case first.

Theorem 5.4. Assume that we are in the balanced setting. Then we have the implications (i)⇒(ii)⇒(iii)⇒(iv), where

(i) $\operatorname{Proj}^1 \mathcal{A}_0 H = 0$,
(ii) $(AH)_0(G)$ is ultrabornological,
(iii) $(AH)_0(G)$ is barrelled,
(iv) $\mathcal{A}$ satisfies condition $(\mathrm{wQ})^{\sim}_{\mathrm{in}}$.

Proof. "(i)⇒(ii)⇒(iii)" The first implication holds in general, see Wengenroth [84, 3.3.4] (cf. Vogt [77, 5.7]). The second also holds in general.

"(iii)⇒(iv)" Barrelledness and reducedness imply by [84, 3.3.6] (cf. [77, 5.10]), condition ($\mathrm{P}_2^\star$), that is

$$\forall\, N\; \exists\, M \geqslant N,\, n\; \forall\, K \geqslant M,\, m\; \exists\, k,\, S>0\; \forall\, \varphi \in (\mathcal{A}_N)_0 H(G)'\colon$$

$$\|\varphi\|^\star_{M,m} \leqslant S(\|\varphi\|^\star_{N,n} + \|\varphi\|^\star_{K,k})$$

where $\|\varphi\|^\star_{N,n} = \sup_{f\in B^\circ_{N,n}} |\varphi(f)|$ denotes the dual norm. For arbitrary $z\in G$ we consider the special case $\varphi = \delta_z$ with $\delta_z(f) := f(z)$ and compute

$$\|\delta_z\|^\star_{n,k} = \sup_{f\in B^\circ_{N,n}} |\delta_z(f)| = \big(\tfrac{1}{a_{N,n}}\big)^\sim_0(z) = \big(\tfrac{1}{a_{N,n}}\big)^\sim(z).$$

Thus, and since the sum in the above condition can be estimated by two times the maximum we get $(\mathrm{wQ})^\sim_{\mathrm{in}}$ immediately. ∎

In the proof above the reducedness of the spectrum $\mathcal{A}_0 H$ is an essential ingredient. Since we do not know if the projective spectrum in the O-growth case is reduced we have to work a little harder in this case: We will make use of 4.6 to see that the family $(S_j)_{j\in\mathbb{N}}$ is equicontinuous. Then we can apply 4.4 to transfer our results from the o-growth to the O-growth case.

Lemma 5.5. Assume that we are in the balanced setting. The family $(S_j)_{j\in\mathbb{N}}$ of the Cesàro means of the partial sums of the Taylor series is an equicontinuous net the space $L(AH(G),(AH)_0(G))$ which satisfies $S_j(J(f)) \to f$ for each $f \in (AH)_0(G)$, where $J\colon (AH)_0(G) \to AH(G)$ is the inclusion mapping.

Proof. In [20, 1.2.(b)] Bierstedt, Bonet, Galbis showed that the sequence $(S_j)_{j\in\mathbb{N}} \subseteq L(Ha_{N,n}(G), H(a_{N,n})_0(G))$ is an equicontinuous net for all N and $n\in\mathbb{N}$. 4.6 yields the equicontinuity in $L(AH(G),(AH)_0(G))$.

Let $f\in (AH)_0(G)$ and $N\in\mathbb{N}$ be arbitrary. Then there exists n such that $f\in H(a_{N,n})_0H(G)$. Since S_jf is a polynomial we have $S_jf \in H(a_{N,n})_0H(G)$ and by Bierstedt, Bonet, Galbis [20, 1.2.(e)] $S_jf\to f$ in $H(a_{N,n})_0H(G)$. Hence $S_jf \to f$ holds in $(\mathcal{A}_N)_0H(G)$. Since N was arbitrary, we obtain $S_jf\to f$ in $(AH)_0(G)$. ∎

Corollary 5.6. Assume that we are in the balanced setting and assume $AH(G)$ to be barrelled. Then $(AH)_0(G)$ is barrelled.

Proof. By 4.4 and 5.5 barrelledness of $AH(G)$ implies that $(AH)_0(G)$ is quasibarrelled. By Vogt [79, 3.1] for $(AH)_0(G)$ quasibarrelledness is equivalent to barrelledness, since the projective spectrum in the o-growth case is reduced thanks to 5.1. ∎

Remark 5.7. As we have mentioned already in 4.8, 5.5 together with 4.7 yield that the inclusion $(AH)_0(G) \hookrightarrow AH(G)$ is nearly open in the balanced setting. But as in the case of continuous functions we know already that this inclusion is even open in this setting (cf. section 2 and 3.1).

Now we are able to prove the promised improvement of 5.3 in the O-growth case.

Theorem 5.8. Assume that we are in the balanced setting. Then we have the implications (i)⇒(ii)⇒(iii)⇒(iv), where

(i)	$\operatorname{Proj}^1 \mathcal{A}H = 0$,	(iii)	$\mathcal{A}H(G)$ is barrelled,
(ii)	$\mathcal{A}H(G)$ is ultrabornological,	(iv)	$\mathcal{A}$ satisfies condition $(\mathrm{wQ})^{\sim}_{\mathrm{in}}$.

Proof. "(i)⇒(ii)⇒(iii)" The first implication holds in general, see Wengenroth [84, 3.3.4] (cf. Vogt [77, 5.10]). The second also holds in general.

"(iii)⇒(iv)" By 5.6 barrelledness of $\mathcal{A}H(G)$ implies that $(\mathcal{A}H)_0(G)$ is barrelled and hence 5.4 yields condition $(\mathrm{wQ})^{\sim}_{\mathrm{in}}$. ■

5.3 Remarks on associated weights

The results obtained so far substantially represent all our results on necessary conditions for barrelledness of the spaces $\mathcal{A}H(G)$ and $(\mathcal{A}H)_0(G)$ which involve only the sequence $\mathcal{A}$: Only in section 9 we will derive different necessary conditions by the use of sequence space representations. All other "improvements" of the results 5.4 and 5.8 which will be discussed later do not extend the latter, but are due to the fact that under the assumptions of the special settings we will discuss in sections 6–10 (which we need to derive sufficient conditions) condition $(\mathrm{wQ})^{\sim}_{\mathrm{in}}$ might turn out to be equivalent to certain a priori stronger conditions; in particular in several situations it will be possible to ommit the $\sim$'s from the conditions and therefore get much more accessible results. However, in view of the above proofs it seems to be impossible to show e.g. that (wQ) is necessary for barrelledness under the rather general assumptions of this sections.

A general setting in which the latter is possible is that of so-called essential weights, which we will now explain. In the terminology of Taskinen [72] a weight is called *essential* if there exists $C > 0$ such that

$$(+) \qquad \big(\tfrac{1}{a}\big)^{\sim} \leqslant \tfrac{1}{a} \leqslant C \cdot \big(\tfrac{1}{a}\big)^{\sim}$$

holds on G. We have the following result which is even true without the assumptions of the balanced setting.

Proposition 5.9. Assume that every weight $a \in \mathcal{A}$ satisfies (+). Then (i) – (iii) resp. (iv) – (vi) are equivalent, where

(i)	$\mathcal{A}$ satisfies condition (wQ),	(iv)	$\mathcal{A}$ satisfies condition (Q),
(ii)	$\mathcal{A}$ satisfies condition $(\mathrm{wQ})^{\sim}_{\mathrm{in}}$,	(v)	$\mathcal{A}$ satisfies condition $(\mathrm{Q})^{\sim}_{\mathrm{in}}$,
(iii)	$\mathcal{A}$ satisfies condition $(\mathrm{wQ})^{\sim}_{\mathrm{out}}$,	(vi)	$\mathcal{A}$ satisfies condition $(\mathrm{Q})^{\sim}_{\mathrm{out}}$.

Proof. In the sequel let $C_{N,n} > 0$ be such that $\big(\frac{1}{a_{N,n}}\big)^{\sim} \leqslant \frac{1}{a_{N,n}} \leqslant C_{N,n} \cdot \big(\frac{1}{a_{N,n}}\big)^{\sim}$ for all N, $n \in \mathbb{N}$.

"(i)⇒(iii)" This follows directly from Bierstedt, Bonet, Taskinen [21, 1.2.(vii)] and is true in general, cf. section 2.

"(iii)⇒(ii)" Let $(\mathrm{wQ})^{\sim}_{\mathrm{out}}$

$$\forall\, N\ \exists\, M \geqslant N,\ n\ \forall\, K \geqslant M,\ m\ \exists\, k,\ S' > 0\colon \big(\tfrac{1}{a_{M,m}}\big)^{\sim} \leqslant S' \max\big(\tfrac{1}{a_{N,n}}, \tfrac{1}{a_{K,k}}\big)^{\sim}$$

be satisfied. In order to show $(\mathrm{wQ})^{\sim}_{\mathrm{in}}$ let N be given. We select M and n as in $(\mathrm{wQ})^{\sim}_{\mathrm{out}}$. For given K and m we select k and $S' > 0$ as in $(\mathrm{wQ})^{\sim}_{\mathrm{out}}$ and put $S := S' \max(C_{N,n}, C_{K,k})$. Then

$$\begin{aligned}\big(\tfrac{1}{a_{M,m}}\big)^{\sim} &\leqslant S' \max\big(\tfrac{1}{a_{N,n}}, \tfrac{1}{a_{K,k}}\big)^{\sim} \leqslant S' \max\big(\tfrac{1}{a_{N,n}}, \tfrac{1}{a_{K,k}}\big)\\ &\leqslant S' \max\big(C_{N,n}\big(\tfrac{1}{a_{N,n}}\big)^{\sim}, C_{K,k}\big(\tfrac{1}{a_{K,k}}\big)^{\sim}\big) \leqslant S \max\big(\big(\tfrac{1}{a_{N,n}}\big)^{\sim}, \big(\tfrac{1}{a_{K,k}}\big)^{\sim}\big)\end{aligned}$$

and we are done.
"(ii)⇒(i)" Let $(\mathrm{wQ})^{\sim}_{\mathrm{in}}$

$$\forall\, N\ \exists\, M \geqslant N,\ n\ \forall\, K \geqslant M,\ m\ \exists\, k,\ S' > 0\colon \big(\tfrac{1}{a_{M,m}}\big)^{\sim} \leqslant S' \max\big(\big(\tfrac{1}{a_{N,n}}\big)^{\sim}, \big(\tfrac{1}{a_{K,k}}\big)^{\sim}\big)$$

be given. To show (wQ) let N be given. We select M and n as in $(\mathrm{wQ})^{\sim}_{\mathrm{in}}$. For given K and m we select k and $S' > 0$ as in $(\mathrm{wQ})^{\sim}_{\mathrm{in}}$ and put $S := S' C_{M,m}$. Then

$$\tfrac{1}{a_{M,m}} \leqslant C_{M,m}\big(\tfrac{1}{a_{M,m}}\big)^{\sim} \leqslant C_{M,m} S' \max\big(\big(\tfrac{1}{a_{N,n}}\big)^{\sim}, \big(\tfrac{1}{a_{K,k}}\big)^{\sim}\big) \leqslant S \max\big(\tfrac{1}{a_{N,n}}, \tfrac{1}{a_{K,k}}\big)$$

and we are done.

"(iv)⇒(vi)" This follows directly from Bierstedt, Bonet, Taskinen [21, 1.2.(vii)] and is true in general, see section 2.
"(vi)⇒(v)": Let $(\mathrm{Q})^{\sim}_{\mathrm{out}}$

$$\forall\, N\ \exists\, M \geqslant N,\ n\ \forall\, K \geqslant M,\ m,\ \varepsilon > 0\ \ \exists\, k,\ S' > 0\colon \big(\tfrac{1}{a_{M,m}}\big)^{\sim} \leqslant \max\big(\tfrac{\varepsilon'}{a_{N,n}}, \tfrac{S'}{a_{K,k}}\big)^{\sim}$$

be given. To show $(\mathrm{Q})^{\sim}_{\mathrm{in}}$ let N be given. We select M and n as in $(\mathrm{Q})^{\sim}_{\mathrm{out}}$. Let K, m and $\varepsilon > 0$ be given. We select k and $S' > 0$ according to $(\mathrm{Q})^{\sim}_{\mathrm{out}}$ w.r.t. K, m and $\varepsilon' := \frac{\varepsilon}{\max(C_{N,n}, C_{K,k})}$ and put $S := S' \max(C_{N,n}, C_{K,k})$. Then

$$\begin{aligned}\big(\tfrac{1}{a_{M,m}}\big)^{\sim} &\leqslant \max\big(\tfrac{\varepsilon'}{a_{N,n}}, \tfrac{S'}{a_{K,k}}\big)^{\sim}\\ &\leqslant \max\big(\tfrac{\varepsilon'}{a_{N,n}}, \tfrac{S'}{a_{K,k}}\big)\\ &\leqslant \max\big(\big(\tfrac{\varepsilon' C_{N,n}}{a_{N,n}}\big)^{\sim}, \big(\tfrac{S' C_{K,k}}{a_{K,k}}\big)^{\sim}\big)\\ &\leqslant \max\big(\big(\tfrac{\varepsilon' \max(C_{N,n}, C_{K,k})}{a_{N,n}}\big)^{\sim}, \big(\tfrac{S' \max(C_{N,n}, C_{K,k})}{a_{K,k}}\big)^{\sim}\big)\\ &= \max\big(\big(\tfrac{\varepsilon}{a_{N,n}}\big)^{\sim}, \big(\tfrac{S}{a_{K,k}}\big)^{\sim}\big).\end{aligned}$$

"(v)⇒(iv)" Let $(\mathrm{Q})^{\sim}_{\mathrm{in}}$

$$\forall\, N\ \exists\, M \geqslant N,\ n\ \forall\, K \geqslant M,\ m,\ \varepsilon > 0\ \ \exists\, k,\ S' > 0\colon$$
$$\big(\tfrac{1}{a_{M,m}}\big)^{\sim} \leqslant \max\big(\big(\tfrac{\varepsilon'}{a_{N,n}}\big)^{\sim}, \big(\tfrac{S'}{a_{K,k}}\big)^{\sim}\big)$$

be satisfied. In order to show (wQ) let N be given. We select M and n as in $(\mathrm{Q})^{\sim}_{\mathrm{in}}$. Let K, m and $\varepsilon > 0$ be given. We select k and $S' > 0$ according to $(\mathrm{Q})^{\sim}_{\mathrm{in}}$ w.r.t. K,

m and $\varepsilon' := \frac{\varepsilon}{C_{M,m}}$. We put $S := S'C_{M,m}$. Then

$$\begin{aligned} \frac{1}{a_{M,m}} &\leqslant C_{M,m}\big(\frac{1}{a_{M,m}}\big)^{\sim} \leqslant C_{M,m} \max\big(\big(\frac{\varepsilon'}{a_{N,n}}\big)^{\sim}, \big(\frac{S'}{a_{K,k}}\big)^{\sim}\big) \\ &\leqslant C_{M,m} \max\big(\frac{\varepsilon'}{a_{N,n}}, \frac{S'}{a_{K,k}}\big) \leqslant \max\big(\frac{\varepsilon' C_{M,m}}{a_{N,n}}, \frac{S' C_{M,m}}{a_{K,k}}\big) \\ &= \max\big(\frac{\varepsilon}{a_{N,n}}, \frac{S}{a_{K,k}}\big) \end{aligned}$$

which finishes the proof. ■

5.4 Condition (Σ)

In [18, section 5] Bierstedt, Bonet introduced a condition which they called (Σ) for weighted (LF)-spaces of continuous functions. This condition is a generalisation of condition (S) (or (V)) of Bierstedt, Meise, Summers [27], for (LB)-spaces (cf. also 15.2). Moreover, it is the canonical extension of a condition for sequence spaces introduced by Vogt [80, 5.17]. We reformulate (Σ) for the (PLB)-setting: A double sequence $\mathcal{A} = ((a_{N,n})_{n\in\mathbb{N}})_{N\in\mathbb{N}}$ satisfies condition (Σ) if

$$\forall\, N\ \exists\, K \geqslant N\ \forall\, k\ \exists\, n \geqslant k\colon \frac{a_{N,n}}{a_{K,k}} \text{ vanishes at } \infty \text{ on } G.$$

As we have seen in the previous sections, in many situations we have to replace the weights by their associated weights. Thus, we say that a sequence $\mathcal{A}$ as above satisfies condition $(\Sigma)^{\sim}$ if

$$\forall\, N\ \exists\, K \geqslant N\ \forall\, k\ \exists\, n \geqslant k\colon \frac{\tilde{a}_{N,n}}{\tilde{a}_{K,k}} \text{ vanishes at } \infty \text{ on } G.$$

Note that by Bierstedt, Bonet, Taskinen [21] $\tilde{a}_{N,n} := \frac{1}{\tilde{w}_{N,n}}$ where $w_{N,n}$ is the growth condition assigned to $a_{N,n}$, i.e. $w_{N,n} = \frac{1}{a_{N,n}}$. Now we can prove the following.

Proposition 5.10. If $\mathcal{A}$ satisfies condition (Σ) or $(\Sigma)^{\sim}$ then $AH(G) = (AH)_0(G)$ holds algebraically.

Proof. Assume that $(\Sigma)^{\sim}$ holds. Let $f \in AH(G)$ and $N \in \mathbb{N}$ be given. We choose $K \geqslant N$ according to (Σ) and select k such that $f \in Ha_{K,k}(G)$. Then there exists $b_k > 0$ such that $a_{K,k}|f| \leqslant b_k$, i.e. $|f| \leqslant \frac{b_k}{a_{K,k}}$ on G. By Bierstedt, Bonet, Taskinen [21, 1.2.(iii) and (vi)] this implies $|f| \leqslant b_k\big(\frac{1}{a_{K,k}}\big)^{\sim} = b_k\tilde{w}_{K,k}$. Now we select $n \geqslant k$ according to $(\Sigma)^{\sim}$ and compute $\frac{1}{\tilde{w}_{N,n}}|f| \leqslant b_k\frac{\tilde{w}_{K,k}}{\tilde{w}_{N,n}}$. Since by ($\Sigma$) the right hand side vanishes at ∞ on G, this has also to be true for $\frac{1}{\tilde{w}_{N,n}}|f|$. Finally, we have by [21, 1.2.(i)] $\tilde{w}_{N,n} \leqslant w_{N,n}$ i.e. $a_{N,n}|f| = \frac{1}{w_{N,n}}|f| \leqslant \frac{1}{\tilde{w}_{N,n}}|f|$ and therefore $a_{N,n}|f|$ has also to vanish at ∞ on G, i.e. we have shown that for each N there exists n such that $f \in H(a_{N,n})_0(G)$ holds. That is $f \in (AH)_0(G)$.
The above proof clearly shows that the statement concerning (Σ) is valid, too. ■

Corollary 5.11. Assume that we are in the balanced setting. If $\mathcal{A}$ satisfies condition (Σ) or $(\Sigma)^{\sim}$ then $AH(G) = (AH)_0(G)$ holds algebraically and topologically.

Proof. Since in the balanced setting $(AH)_0(G)$ is a topological subspace of $AH(G)$, this follows immediately with 5.10. ■

Remark 5.12. The proof of 5.10 even shows that the spectra $\mathcal{A}_0H$ and $\mathcal{A}H$ are equivalent in the sense of Wengenroth [84, 3.1.6] if we assume $\mathcal{A}$ to satisfy (Σ) or $(\Sigma)^{\sim}$.

Proof. In the proof of 5.10 we obtained a sequence $(K(N))_{N\in\mathbb{N}}$ with $K(N) \geqslant N$ for each N such that $\mathcal{A}_{K(N)}H(G) \subseteq (\mathcal{A}_N)_0H(G)$ holds for each $N \in \mathbb{N}$. We denote the inclusion mappings with $i_{N,K(N)}$ and obtain the following diagram

$$\begin{tikzcd}
\cdots \arrow[r] & (\mathcal{A}_{K^2(1)})_0H(G) \arrow[r] \arrow[d, "“\subseteq”"'] & \cdots \arrow[r] & (\mathcal{A}_{K(1)})_0H(G) \arrow[r] \arrow[d, "“\subseteq”"'] & \cdots \arrow[r] & (\mathcal{A}_1)_0H(G) \arrow[d, "“\subseteq”"'] \\
\cdots \arrow[r] & \mathcal{A}_{K^2(1)}H(G) \arrow[r] \arrow[urr, "i_{K^2(1),K(1)}"] & \cdots \arrow[r] & \mathcal{A}_{K(1)}H(G) \arrow[r] \arrow[urr, "i_{K(1),1}"] & \cdots \arrow[r] & \mathcal{A}_1H(G)
\end{tikzcd}$$

where the vertical arrows are the linking maps of the projective spectra, i.e. compositions of the inclusions of the steps, e.g.

$$(\mathcal{A}_{K(1)})_0H(G) \xrightarrow{\subseteq^{K(1)-1}_{K(1)}} \cdots \xrightarrow{\subseteq^{1}_{2}} (\mathcal{A}_1)_0H(G).$$

We define $(k(N))_{N\in\mathbb{N}}$ and $(l(N))_{N\in\mathbb{N}}$ by $k(N) = l(N) = K^N(1)$. For each N the map $\alpha_N\colon (\mathcal{A}_{k(N)})_0H(G) \to \mathcal{A}_{l(N)}H(G)$ is the inclusion map $(\mathcal{A}_{K^N(1)})_0H(G) \subseteq \mathcal{A}_{K^N(1)}H(G)$ and $\beta_N\colon (\mathcal{A}_{l(N)})_0H(G) \to \mathcal{A}_{k(n-1)}H(G)$ by $\beta_N = i_{K^N(1),K^{N-1}(1)}$, where we have $N \leqslant K^N(1) \leqslant K^{N+1}(1)$ that is $N \leqslant l(N) \leqslant k(N) \leqslant l(N+1)$. Moreover, the above diagram is commutative and hence $\mathcal{A}H$ and $\mathcal{A}_0H$ are equivalent. ■

Consequence 5.13. Let $\mathcal{A}$ satisfy condition (Σ) or $(\Sigma)^{\sim}$. Then by 5.12 and Wengenroth [84, 3.1.7] $\operatorname{Proj}^1 \mathcal{A}H = \operatorname{Proj}^1 \mathcal{A}_0H$ holds (as linear spaces). In particular in this case, $\operatorname{Proj}^1 \mathcal{A}H = 0$ if and only if $\operatorname{Proj}^1 \mathcal{A}_0H = 0$.

Observation 5.14. If we reformulate the result [18, 5.2] of Bierstedt, Bonet for the (PLB)-setting, the latter means exactly that condition (Σ) implies that (wQ) and (Q) are equivalent.

In addition to the above observation, we may prove a similar result for $(\Sigma)^{\sim}$, $(\mathrm{wQ})^{\sim}_{\mathrm{in}}$ and $(\mathrm{Q})^{\sim}_{\mathrm{in}}$ by a slight modification of the proof of [18, 5.2].

Lemma 5.15. Let $\mathcal{A}$ satisfy condition $(\Sigma)^{\sim}$. Then the following are equivalent.

(i) $\mathcal{A}$ satisfies $(\mathrm{Q})^{\sim}_{\mathrm{in}}$,

(ii) $\forall\, N\; \exists\, M \geqslant N,\, m,\, \varepsilon > 0\; \forall\, K\; \exists\, k,\, C \subseteq G$ compact $\forall\, z \in G\backslash C$:

$$\tilde{w}_{M,m}(z) \leqslant \varepsilon \max\big(\tilde{w}_{N,n}(z), \tilde{w}_{K,k}(z)\big),$$

(iii) $\forall\, N\; \exists\, N \geqslant M,\, m\; \forall\, K\; \exists\, k,\, C \subseteq G$ compact $\forall\, z \in G\backslash C$:

$$\tilde{w}_{M,m} \leqslant \max\big(\tilde{w}_{N,n}(z), \tilde{w}_{K,k}(z)\big),$$

(iv) $\mathcal{A}$ satisfies $(\mathrm{wQ})_{\mathrm{in}}^{\sim}$.

Proof. "(i)⇒(iv)" Trivial.
"(ii)⇒(iii)" Trivial.
"(ii)⇒(i)" For given N we choose $M \geqslant N$ and n as in (ii). For given $K \geqslant M$, m and $\varepsilon > 0$ we select k and C as in (ii). Now we put

$$S := \max\big(\varepsilon, \sup_{z\in C} \tfrac{\tilde{w}_{M,m}(z)}{\tilde{w}_{K,k}(z)}\big)$$

where the supremum is less than infinity since C is compact. Now let $z \in G$. If $z \in G\backslash C$, then $\tilde{w}_{M,m}(z) \leqslant \varepsilon \max\big(\tilde{w}_{N,n}(z), \tilde{w}_{K,k}(z)\big) \leqslant S \max\big(\tilde{w}_{N,n}(z), \tilde{w}_{K,k}(z)\big)$ holds by (ii). Otherwise, for $z \in C$, the definition of S yields $S \geqslant \frac{\tilde{w}_{M,m}(z)}{\tilde{w}_{K,k}(z)}$ and hence with (ii), $\tilde{w}_{M,m}(z) \leqslant S\tilde{w}_{K,k}(z) \leqslant \max\big(\varepsilon\tilde{w}_{N,n}(z), S\tilde{w}_{K,k}(z)\big)$.
"(iii)⇒(iv)" This can be proved analogously to the above.
"(iv)⇒(ii)" Assume $(\Sigma)^{\sim}$ in the form

$$\forall\, M\ \exists M' \geqslant M\ \forall\, m\ \exists\, m' \geqslant m : \ \tfrac{\tilde{w}_{M',m}}{\tilde{w}_{M,m'}} \text{ vanishes at } \infty \text{ on } G.$$

In order to check (ii), we fix $N \in \mathbb{N}$ and select $M \geqslant N$ and n as in $(\mathrm{wQ})_{\mathrm{in}}^{\sim}$. For this M we choose $M' \geqslant M$ as in $(\Sigma)^{\sim}$. Given k, m and $0 \leqslant \varepsilon \leqslant 1$ we take $m' \geqslant m$ from $(\Sigma)^{\sim}$, i.e. such that $\frac{\tilde{w}_{M',m}}{\tilde{w}_{M,m'}}$ vanishes at ∞ on G and apply $(\mathrm{wQ})_{\mathrm{in}}^{\sim}$ w.r.t. K and m' to get k and $S > 0$ with $\tilde{w}_{M,m'} \leqslant S \max\big(\tilde{w}_{N,n}, \tilde{w}_{K,k}\big)$. Now we put

$$C := \big\{\, z \in G\,;\ \tilde{w}_{M',m}(z) \geqslant \tfrac{\varepsilon}{S}\, \tilde{w}_{M,m'}(z) \,\big\} = \big\{\, z \in G\,;\ \tfrac{\tilde{w}_{M',m}(z)}{\tilde{w}_{M,m'}(z)} \geqslant \tfrac{\varepsilon}{S} \,\big\}.$$

Since $\frac{\tilde{w}_{M',m}}{\tilde{w}_{M,m'}}$ vanishes at ∞ on G there exists a compact set $C' \subseteq G$ such that $\frac{\tilde{w}_{M',m}}{\tilde{w}_{M,m'}} < \frac{\varepsilon}{S}$ on $G\backslash C'$ and hence $C \subseteq C'$. Moreover all weights are continuous and therefore C is closed, hence compact. Now let $z \in G\backslash C$. By the definition of C and the estimate we deduced at the beginning $\tilde{w}_{M',m}(z) \leqslant \frac{\varepsilon}{S}\, \tilde{w}_{M,m'}(z) \leqslant \varepsilon \max\big(\tilde{w}_{N,n}, \tilde{w}_{K,k}\big)$, which is the desired estimate. For $\varepsilon > 1$ the assertion follows immediately. ∎

As mentioned at the beginning of section 5.3, our results on necessary conditions of section 5.2 can be strengthend via $(\Sigma)^{\sim}$ only in the sense that now $(\mathrm{wQ})_{\mathrm{in}}^{\sim}$ is equivalent to $(\mathrm{Q})_{\mathrm{in}}^{\sim}$.

Corollary 5.16. (of 5.8) Assume that we are in the balanced setting. If $\mathcal{A}$ satisfies condition $(\Sigma)^{\sim}$ then $(AH)_0(G) = AH(G)$ holds algebraically and topologically, $\operatorname{Proj}^1 \mathcal{A}_0 H = \operatorname{Proj}^1 \mathcal{A}H$ and we have (i)⇒(ii)⇒(iii)⇒(iv)⇔(v) where

(i) $\operatorname{Proj}^1 \mathcal{A}_{(0)} H = 0$,
(ii) $(AH)_{(0)}(G)$ is ultrabornological,
(iii) $(AH)_{(0)}(G)$ is barrelled,
(iv) $\mathcal{A}$ satisfies condition $(\mathrm{wQ})_{\mathrm{in}}^{\sim}$,
(v) $\mathcal{A}$ satisfies condition $(\mathrm{Q})_{\mathrm{in}}^{\sim}$.

In the last section we discussed the case of essential weights. Let us extend this discussion to $(\Sigma)^{(\sim)}$. Indeed, if all weights in $\mathcal{A}$ are essential, we get that (Σ) and $(\Sigma)^{\sim}$ are equivalent. This follows from the following lemma.

Lemma 5.17. Let a and b be weights on G and assume that there exists C_a, $C_b > 0$ such that $\big(\frac{1}{a}\big)^{\sim} \leqslant \frac{1}{a} \leqslant C_a\big(\frac{1}{a}\big)^{\sim}$ and $\big(\frac{1}{b}\big)^{\sim} \leqslant \frac{1}{b} \leqslant C_b\big(\frac{1}{b}\big)^{\sim}$ holds on G. Then a/b vanishes at infinity on G if and only if the same is true for $\tilde{a}/\tilde{b}$.

Proof. "⇒" Let a/b vanish at infinity. To show that the same holds for $\tilde{a}/\tilde{b}$ let $\varepsilon > 0$ be given. We put $\varepsilon' := \frac{\varepsilon}{C_a}$. Then there exists $K \subseteq G$ compact such that $a/b \leqslant \varepsilon'$ on $G\backslash K$ hence on $G\backslash K$ we have

$$\frac{\big(\frac{1}{b}\big)^{\sim}}{\big(\frac{1}{a}\big)^{\sim}} \leqslant \frac{\frac{1}{b}}{\frac{1}{a}\frac{1}{C_a}} = C_a \tfrac{a}{b} \leqslant C_a\varepsilon' = \varepsilon.$$

"⇐" Let $\tilde{a}/\tilde{b}$ vanish at infinity and let $\varepsilon > 0$ be given. We put $\varepsilon' := \frac{\varepsilon}{C_b}$ Then there exists $K \subseteq G$ compact such that $a/b \leqslant \varepsilon'$ on $G\backslash K$ and hence

$$\tfrac{a}{b} = \frac{\frac{1}{b}}{\frac{1}{a}} \leqslant \frac{C_b\big(\frac{1}{b}\big)^{\sim}}{\big(\frac{1}{a}\big)^{\sim}} \leqslant C_b\varepsilon' = \varepsilon$$

on $G\backslash K$. ∎

By 5.17 we get the following formulation of 5.16.

Corollary 5.18. (of 5.16) Assume that we are in the balanced setting and that all weights in $\mathcal{A}$ are essential. If $\mathcal{A}$ satisfies condition (Σ) then $(AH)_0(G) = AH(G)$ holds algebraically and topologically, $\operatorname{Proj}^1 \mathcal{A}_0H = \operatorname{Proj}^1 \mathcal{A}H$ and we have (i)⇒(ii)⇒(iii)⇒(iv)⇔(v) where

(i) $\operatorname{Proj}^1 \mathcal{A}_{(0)}H = 0$,
(ii) $(AH)_{(0)}(G)$ is ultrabornological,
(iii) $(AH)_{(0)}(G)$ is barrelled,
(iv) $\mathcal{A}$ satisfies condition (wQ),
(v) $\mathcal{A}$ satisfies condition (Q).

6 A special setting for the unit disc: The class $\mathcal{W}$

To find sufficient conditions for the vanishing of $\operatorname{Proj}^1 \mathcal{A}H$ and for barrelledness of $(AH)_0(G)$, we need to decompose holomorphic functions. In the case of the unit disc, a decomposition suitable for our purposes is possible if we assume that our defining sequence $\mathcal{A}$ belongs to some set of weights W which is assumed to be of *class* $\mathcal{W}$ defined by Bierstedt, Bonet [19]. That is, we assume that W consists of radial weights and further that each $w \in W$ satisfies $\lim_{r\nearrow 1} w(r) = 0$ and is non-increasing if restricted to $[0,1[$. We assume W to be stable under multiplication with strictly positive scalars and under the formation of finite minima. Next, we assume that there exists a sequence of linear and continuous operators $(R_n)_{n=1,2,\ldots}$, $R_n\colon (H(\mathbb{D}),\mathrm{co}) \to (H(\mathbb{D}),\mathrm{co})$ such that for $n = 1,2,\ldots$ the image of R_n is a finite dimensional subspace of the space $\mathbb{P}$ of polynomials on $\mathbb{D}$. Further we assume that for each $p \in \mathbb{P}$ there exists n with $R_np = p$ and that for arbitrary n, $m = 1,2,\ldots$ $R_n \circ R_m = R_{\min(n,m)}$ holds. Moreover, we require that there is $c > 0$ such that for each $n \in \mathbb{N}$, $r \in]0,1[$ and $p \in \mathbb{P}$ the estimate $\sup_{|z|=r} |[R_np](z)| \leqslant c \sup_{|z|=r} |p(z)|$

holds. By setting $R_0 := 0$ and $r_n := 1 - 2^{-n}$ for $n = 0, 1, 2, \ldots$ we get a system $(R_n, r_n)_{n=0,1,2,\ldots}$ which is assumed to satisfy the following two conditions

(P1) $\exists\, C \geqslant 1\ \forall\, v \in W,\ p \in \mathbb{P}$:

$$\tfrac{1}{C} \sup_{n\in\mathbb{N}} \big(v(r_n) \sup_{|z|=r_n} \big|[(R_{n+2} - R_{n-1})p](z)\big| \big) \leqslant \sup_{z\in\mathbb{D}} v(z)|p(z)|,$$

$$\sup_{z\in\mathbb{D}} v(z)|p(z)| \leqslant C \sup_{n\in\mathbb{N}} \big(v(r_n) \sup_{|z|=r_n} \big|[(R_{n+1} - R_n)p](z)\big| \big).$$

(P2) $\forall\, v \in W\ \exists\, D(v) \geqslant 1\ \forall\, (p_n)_{n\in\mathbb{N}} \subseteq \mathbb{P}$, $p_n \neq 0$ only for finitely many n:

$$\sup_{z\in\mathbb{D}} v(z) \big| \sum_{n=1}^{\infty} [(R_{n+1} - R_n)p_n](z) \big| \leqslant D(v) \sup_{k\in\mathbb{N}} \big(v(r_k) \sup_{|z|=r_k} |p_k(z)| \big).$$

Note that for a system of weights in W the requirements of the balanced setting are automatically satisfied. Moreover, Theorem's 3.1 and 4.1 of Bierstedt, Bonet [19] and the results of Bierstedt, Meise, Summers [27] imply that for $\mathcal{A} \subseteq W$, $\mathcal{A}_N H(\mathbb{D}) \subseteq \mathcal{A}_N C(\mathbb{D})$ and $(\mathcal{A}_N)_0 H(\mathbb{D}) \subseteq (\mathcal{A}_N)_0 C(\mathbb{D})$ are all topological subspaces.

6.1 Sufficient conditions for the vanishing of $\mathrm{Proj}^1 \mathcal{A}H$

Our investigation in section 5.2 has shown that finding necessary conditions for the vanishing of Proj^1 the o-growth case was easier to handle than the O-growth case, since the spectrum $\mathcal{A}_0 H$ is reduced. However, also in the O-growth case the balanced setting allowed to prove “the same” result for the o-growth case: In both situations, $(\mathrm{wQ})^{\sim}_{\mathrm{in}}$ is necessary for barrelledness. As we will see in the sequel for sufficient conditions the situation is the other way round, that is the O-growth case is the easier one. But the situation is not symmetric: We are not able to prove sufficient conditions for $\mathrm{Proj}^1 \mathcal{A}_0 H = 0$ at all.

In this whole section we will need that our defining sequence of weights $\mathcal{A}$ is contained in some set W which is of class $\mathcal{W}$, since we cannot decompose holomorphic functions without such an assumption and as we will see decomposition is the crucial point in all the following proofs.

Let us moreover remark that in what follows it would clearly be possible to replace condition $(\mathrm{Q})^{\sim}_{\mathrm{out}}$ or $(\mathrm{Q})^{\sim}_{\mathrm{in}}$ by (Q), since we noted already in section 2 that (Q) and $(\mathrm{Q})^{\sim}_{\mathrm{in}}$ both imply $(\mathrm{Q})^{\sim}_{\mathrm{out}}$. But since the necessary conditions we obtained in the latter section had to be formulated with associated weights, we will proceed in the same way concerning the sufficient conditions. Moreover, $(\mathrm{Q})^{\sim}_{\mathrm{out}}$ a priori is a weaker (albeit less accessible) condition than (Q).

Theorem 6.1. Let $\mathcal{A}$ be a sequence in W and assume that $\mathcal{A}$ satisfies condition $(\mathrm{Q})^{\sim}_{\mathrm{out}}$. Then $\mathrm{Proj}^1 \mathcal{A}H = 0$.

Proof. In order to show that $\mathrm{Proj}^1 \mathcal{A}H = 0$ we use Braun, Vogt [36, Theorem 8] (which was independently obtained by Frerick, Wengenroth [43]). That is, we have to show condition $(\overline{\mathrm{P}_2})$

$$\forall\, N\; \exists\, M,\, n\; \forall\, K,\, m,\, \varepsilon > 0\; \exists\, k,\, S > 0 :\; B_{M,m} \subseteq \varepsilon B_{N,n} + S B_{K,k}.$$

For given N we select M and n as in $(\mathrm{Q})^{\sim}_{\mathrm{out}}$. For given K, m, $\varepsilon > 0$ we put $\varepsilon' := \frac{\varepsilon}{(D_1+2c^2)C}$ and choose K and $S' > 0$ according to $(\mathrm{Q})^{\sim}_{\mathrm{out}}$ w.r.t. ε' and put $S := S'(2c^2 + D_2)$. Now we fix an arbitrary $f \in B_{M,m}$ and consider $S_t f$. We have $a_{M,m}|S_t f| \leqslant a_{M,m}|f| \leqslant 1$, i.e. $|S_t f| \leqslant \frac{1}{a_{M,m}}$. With Bierstedt, Bonet, Taskinen [21, Proposition 1.2.(iii)] it follows $|S_t f| \leqslant (\frac{1}{a_{M,m}})^{\sim}$ and by the estimate in $(\mathrm{Q})^{\sim}_{\mathrm{out}}$ we obtain $|S_t f| \leqslant \max(\varepsilon'(\frac{1}{a_{N,n}}), S'(\frac{1}{a_{K,k}}))^{\sim} \leqslant \max(\frac{\varepsilon'}{a_{N,n}}, \frac{S'}{a_{K,k}})$ where the last estimate follows from Bierstedt, Bonet, Taskinen [21, Proposition 1.2.(i)]. We put $u_1 := \frac{a_{N,n}}{\varepsilon'}$, $u_2 := \frac{a_{K,k}}{S'}$ and $u := \min(u_1, u_2)$. Then the above transforms to $|S_t f| \leqslant \max(\frac{1}{u_1}, \frac{1}{u_2}) = \frac{1}{u}$, i.e. $u|S_t f| \leqslant 1$. As W is closed under the formation of finite minima and under multiplication with positive scalars $u \in W$ holds.

From now on we will use the decomposition method invented by Bierstedt, Bonet [19], which was used successfully also by Wolf [85, 86]: We can decompose $S_t f = R_1 S_t f + \sum_{\nu=1}^{\infty}(R_{\nu+1} - R_\nu) S_t f$ where both summands are polynomials.

Let us study the first summand: By the estimate previous to (P1) there exists $c > 0$ such that $\sup_{|z|=r_1} |[R_1 S_t f](z)| \leqslant c \sup_{|z|=r_1} |S_t f(z)|$. We multiply with $u(r_1)$ and use $u|S_t f| \leqslant 1$ to get $u(r_1)\sup_{|z|=r_1} |[R_1 S_t f](z)| \leqslant c u(r_1) \sup_{|z|=r_1} |S_t f(z)| \leqslant 1$. By the definition of u we have $u(r_1) = \min(u_1(r_1), u_2(r_2))$. Let $i \in \{1, 2\}$ such that $u(r_1) = u_i(r_1)$. Now we can use the second inequality of (P1) to obtain the following estimate. There exists $C \geqslant 1$ such that

$$\begin{aligned}
\sup_{z\in\mathbb{D}} u_i(z)|R_1 S_t f(z)| &\leqslant C \sup_{n\in\mathbb{N}} \big(u_i(r_n) \sup_{|z|=r_n} |[(R_{n+1} - R_n) R_1 S_t f](z)|\big) \\
&= C u(r_1) \sup_{|z|=r_1} |[(R_2 - R_1) R_1 S_t f](z)| \\
&\leqslant 2cC u(r_1) \sup_{|z|=r_1} |R_1 S_t f(z)| \\
&\leqslant 2c^2 C.
\end{aligned}$$

By the definition of the u_i and the choice of i we get $\sup_{z\in\mathbb{D}} a_{N,n}(z)|R_1 S_t f(z)| \leqslant 2c^2 C\varepsilon'$ or $\sup_{z\in\mathbb{D}} a_{K,k}(z)|R_1 S_t f(z)| \leqslant 2c^2 CS$, i.e. $R_1 S_t f \in 2c^2 C\varepsilon' B_{N,n}$ or $R_1 S_t f \in 2c^2 CS B_{K,k}$.

Now we consider $S_t f - R_1 S_t f = \sum_{\nu=1}^{\infty}(R_{\nu+1} - R_\nu) S_t f$. We may use the first inequality in (P1) for u and $S_t f$ to get with the same $C \geqslant 1$ as in the last step

$$\tfrac{1}{C} \sup_{\nu\in\mathbb{N}} (u(r_\nu) \sup_{|z|=r_\nu} |[(R_{\nu+2} - R_{\nu-1}) S_t f](z)|) \leqslant \sup_{z\in\mathbb{D}} u(z)|S_t f(z)| \leqslant 1$$

i.e. for each $\nu \in \mathbb{N}$ we have $u(r_\nu)\sup_{|z|=r_\nu} |[(R_{\nu+2} - R_{\nu-1}) S_t f](z)| \leqslant C$. Now we write $\mathbb{N} = J_1 \,\dot\cup\, J_2$ such that $u(r_j) = u_1(r_j)$ for $j \in J_1$ and $u(r_j) = u_2(r_j)$ for $j \in J_2$. For $i \in \{1, 2\}$ we put

$$g_i := \sum_{\nu\in J_i} (R_{\nu+1} - R_\nu) S_t f \quad \text{and} \quad p_\nu^i := \begin{cases} (R_{\nu+2} - R_{\nu-1}) S_t f & \text{for } n \in J_i \\ 0 & \text{otherwise.} \end{cases}$$

Then we obtain by construction $S_tf - R_1S_tf = g_1 + g_2$ and the properties of class $\mathcal{W}$ yield $g_i = \sum_{\nu\in J_i}(R_{\nu+1} - R_\nu)S_tf = \sum_{\nu\in J_i}(R_{\nu+1} - R_\nu)(R_{\nu+2} - R_{\nu-1})S_tf = \sum_{\nu\in J_i}(R_{\nu+1} - R_\nu)p^i_\nu$. Since $(p^i_\nu)_{\nu\in\mathbb{N}} \subseteq \mathbb{P}$ with only finitely many $p^i_\nu \neq 0$ we can apply (P2) and get $D(u_i) =: D_i \geqslant 1$ such that

$$\begin{aligned}
\sup_{z\in\mathbb{D}} u_i(z)|g_i(z)| &= \sup_{z\in\mathbb{D}} \Big|\sum_{\nu=1}^{\infty}[(R_{\nu+1} - R_\nu)p^i_\nu](z)\Big| \\
&\leqslant D_i \sup_{\nu\in\mathbb{N}} \big(u_i(r_\nu \sup_{|z|=r_\nu} |p^i_\nu(z)|\big) \\
&= D_i \sup_{\nu\in J_i} \big(u_i(r_\nu) \sup_{|z|=r_\nu} |p^i_\nu(z)|\big) \\
&= D_i \sup_{\nu\in J_i} \big(u_i(r_\nu) \sup_{|z|=r_\nu} |[(R_{\nu+2} - R_{\nu-1})S_tf](z)|\big) \\
&\leqslant D_i \sup_{\nu\in J_i} \big(u(r_\nu) \sup_{|z|=r_\nu} |[(R_{\nu+2} - R_{\nu-1})S_tf](z)|\big) \\
&\leqslant D_iC.
\end{aligned}$$

This yields $g_1 \in D_1C\varepsilon' B_{N,n}$ and $g_2 \in D_2CS'B_{K,k}$. Thus, $S_tf = R_1S_tf + g_1 + g_2 \in \varepsilon'(2c^2+D_1)CB_{N,n} + S'(2c^2+D_2)B_{K,k} = \varepsilon B_{N,n} + SB_{K,k}$ for each $t \in \mathbb{N}$. Since $B_{N,n}$ and $B_{K,k}$ are both co-compact and $S_tf \to f$ w.r.t. co we obtain $f \in \varepsilon B_{N,n} + SB_{K,k}$ and hence $(\overline{\text{P}_2})$. By Braun, Vogt [36, Theorem 8] it follows $\operatorname{Proj}^1 \mathcal{A}H = 0$. ■

6.2 Barrelledness of $(\mathbf{AH})_0(\mathbf{G})$

In the proof of the last result we used in the final step that the balls $B_{N,n}$ are co-compact. For the balls $B^\circ_{N,n}$ this cannot be true: If $B^\circ_{N,n}$ is co-closed for all N, $n \in \mathbb{N}$, then we get $B^\circ_{N,n} = \overline{B^\circ_{N,n}}^{\text{co}} = B_{N,n}$ where the last set is co-compact. The equality $B^\circ_{N,n} = B_{N,n}$ yields $Ha_{N,n}(G) = H(a_{N,n})_0(G)$ which implies (using $H(a_{N,n})_0(G)'' = Ha_{N,n}(G)$, see Bierstedt, Bonet, Galbis [20, Theorem 1.5.(d)]) that $H(a_{N,n})_0(G)$ is reflexive. By Bonet, Wolf [35, Corollary 2] this implies that the space is finite dimensional. But this is a contradiction to $\mathbb{P} \subseteq H(a_{N,n})_0(G)$ and $\dim \mathbb{P} = \infty$, which we have at least in the balanced setting and in particular in the setting where $\mathcal{A} \subseteq W$ which we need to decompose.

Unfortunately the above obstacles do not permit us to use the methods we utilized for O-growth conditions to get sufficient conditions for the vanishing of $\operatorname{Proj}^1 \mathcal{A}_0H$. But by some detour we will find a sufficient condition for $(AH)_0(\mathbb{D})$ being barrelled under the assumptions of class $\mathcal{W}$.

To get the latter, we have to consider the space of polynomials $\mathbb{P}$ endowed with two different topologies. Algebraically all the spaces which we will introduce now, are the same. We will state the following in the setting of (LF)-spaces, although it would be enough to consider only (LB)-space in view of the application we have in mind. Hence we consider a definining sequence $\mathcal{V} = ((v_{n,N})_{N\in\mathbb{N}})_{n\in\mathbb{N}}$ with $v_{n+1,N} \leqslant v_{n,N} \leqslant v_{n,N+1}$. Finally we will of course need the assumptions of class $\mathcal{W}$. However, let us state the next definitions in the most possible generality – in fact we will use the results independent of class $\mathcal{W}$ again in section 7. We put

$$P(v_{n,N})_0(G) := (\mathbb{P}, \|\cdot\|_{n,N}), \quad P(V_n)_0(G) := \operatorname{proj}_N P(v_{n,N})_0(G)$$

and

$$\mathcal{V}_0 P(G) := \operatorname{ind}_n P(V_n)_0(G).$$

Clearly, the o-growth notation seems to be artificial, but the space just defined is the smallest in the chain

$$\mathcal{V}_0 P(G) \subseteq \mathcal{V}_0 H(G) \subseteq \mathcal{V} H(G)$$

where we of course stick to the balanced setting to assure that the polynomials are contained in the spaces of holomorphic functions. In the sequel we will write $\mathbb{P}$ for the space of polynomials endowed with the topology induced by $\mathcal{V}_0 H(G)$ and $\mathcal{V}_0 P(G)$ for the same space endowed with the inductive topology defined above. The following lemma provides that the polynomials form a so-called limit subspace of $\mathcal{V}_0 H(\mathbb{D})$.

To prove the following lemma we use several results on the so-called method of "projective description", which was invented and studied by Bierstedt, Meise, Summers [27] for (LB)-spaces of continuous and holomorphic functions and since then has been extended and improved by several autors. We refer to the survey article [12] of Bierstedt for historical notes, further references and a summary of the "state of the art" concerning this subject. At this point we just give the definition of the so-called "projective hull" for a weighted (LF)-space of holomorphic function and explain what we understand under projective description. For this purpose let $\mathcal{V}$ be a sequence as above. We put

$$\overline{V} := \big\{\overline{v} \in C(G)\,;\, \overline{v} \geqslant 0 \text{ and } \forall\, n\, \exists\, N(n),\, \alpha_n > 0\colon \overline{v} \leqslant \alpha_n v_{n,N(n)} \text{ on } G\big\}$$

and define the *projective hulls*

$$H\overline{V}(G) := \operatorname{proj}_{\overline{v}\in\overline{V}} H\overline{v}(G) \quad \text{and} \quad H\overline{V}_0(G) := \operatorname{proj}_{\overline{v}\in\overline{V}} H\overline{v}_0(G)$$

of the (LF)-spaces $\mathcal{V}H(G)$ and $\mathcal{V}_0(G)$, respectively. It is easy to see that both spaces are contained in their projective hulls with continuous inclusions and indeed in the first case both spaces coincide algebraically. We say that *projective description holds* if $\mathcal{V}H(G) = H\overline{V}(G)$ holds topologically and $\mathcal{V}_0 H(G) \subseteq H\overline{V}_0(G)$ is a topological subspace, respectively. In this case, the inductive topology of the (LF)-spaces can be given by a system of weighted sup-seminorms and therefore the topology becomes much more accessible for concrete computations. In fact, the setting of class $\mathcal{W}$ was invented by Bierstedt, Bonet [19] in order to prove that projective description holds for weighted (LF)-spaces with weights within this class, see the proof of 6.2 for detailed references.

Lemma 6.2. Let $\mathcal{V}$ be in W. Then $\mathcal{V}_0 P(\mathbb{D}) \subseteq \mathcal{V}_0 H(\mathbb{D})$ is a topological subspace.

Proof. In the case of projective spectra a subspectrum yields a subspace in general. Hence, $P(V_n)_0(\mathbb{D}) \subseteq H(V_n)_0(\mathbb{D})$ is a topological subspace for each $n \in \mathbb{N}$.

Moreover, the inclusion $\mathcal{V}_0P(\mathbb{D}) \subseteq \mathcal{V}_0H(\mathbb{D})$ is continuous by the universal property of the inductive limit. Hence we have to show

$$\forall\, U \in \mathfrak{U}_0(\mathcal{V}_0P(\mathbb{D}))\ \exists\, V \in \mathfrak{U}_0(\mathbb{P}) :\ V \subseteq U.$$

By [19, 3.1] the topology of $\mathbb{P}$ can be described by the seminorms $\|\cdot\|_{\overline{v}}$, $\overline{v} \in \overline{V}$. We proceed analogously to the proof of Bierstedt, Bonet:

We fix an absolutely convex 0-neighborhood U in $\mathcal{V}_0P(\mathbb{D})$. For each n we choose $N(n)$ and $\varepsilon_n > 0$ such that

$$U_n = \{\, f \in P(V_n)_0(\mathbb{D})\,;\ \|f\|_{n,N(n)} \leqslant \varepsilon_n \,\} \subseteq 2^{-n}U.$$

This is possible since $P(V_n)_0(\mathbb{D}) \hookrightarrow \mathcal{V}_0P(\mathbb{D})$ is continuous. For each n let us denote by $D_n \geqslant 1$ the constant $D(v_{n,N(n)})$ of condition (P2) and let $\lambda_n := CD_n\varepsilon_n^{-1}$, where C is the constant in condition (P1). For each m we define

$$w_m := \min_{\nu=1,\dots,m} \lambda_\nu v_{\nu,N(\nu)}.$$

According to our assumption on the set W we have $w_m \in W$ for each m. Moreover, the sequence $(w_m)_{m\in\mathbb{N}}$ is decreasing. Now we put

$$W_m := \{\, g \in C(V_m)_0(\mathbb{D})\,;\ \|g\|_{w_m} \leqslant 1 \,\}.$$

Since $(w_m)_{m\in\mathbb{N}}$ is decreasing, we have $W_m \subseteq W_{m+1}$ for each m and W_m is a 0-neighborhood in $C(V_m)_0(\mathbb{D})$, since

$$\{\, g \in C(V_m)_0(\mathbb{D})\,;\ \sup_{z\in\mathbb{D}} v_{m,N(m)}(z)|f(z)| < \lambda_m^{-1} \,\} \subseteq W_m.$$

By the above, $W := \cup_{m\in\mathbb{N}} W_m$ is an absolutely convex 0-neighborhood in the space $\mathcal{V}_0C(\mathbb{D}) = \operatorname{ind}_m C(V_m)_0(\mathbb{D})$. By Bierstedt, Meise, Summers [27, 1.3] there exists $\overline{v} \in \overline{V}$ such that

$$\{\, g \in \mathcal{V}_0C(\mathbb{D})\,;\ \sup_{z\in\mathbb{D}} \overline{v}(z)|g(z)| \leqslant 1 \,\} \subseteq W.$$

We put

$$W_0 := \{\, g \in H\overline{V}_0(\mathbb{D})\,;\ \sup_{z\in\mathbb{D}} \overline{v}(z)|g(z)| \leqslant 1 \,\}.$$

W_0 is a 0-neighborhood in the projective hull, hence by [19, 3.1] and the definition of $\mathbb{P}$, i.e. $\mathbb{P} \subseteq \mathcal{V}_0H(\mathbb{D}) \subseteq H(\overline{V})_0(\mathbb{D})$ are all topological subspaces, $W_0 \cap \mathbb{P}$ is a 0-neighborhood in $\mathbb{P}$. We put $V := (c^2+1)^{-1}(W_0 \cap \mathbb{P})$ which is a 0-neighborhood in $\mathbb{P}$ and claim that $V \subseteq U$. Here, $c \geqslant 0$ is the constant defined previous to (P1).

Let $q \in V$. We put $p := (c^2+1)q$. Then $p \in W_0 \cap \mathbb{P}$ and we have to show that $p \in (c^2+1)U$. Since $p \in W_0 \cap \mathcal{V}_0C(\mathbb{D}) \subseteq W$, there is m such that $w_m|p| \leqslant 1$ on $\mathbb{D}$. We have

$$p = \sum_{n=0}^{\infty}(R_{n+1} - R_n)p = R_1p + \sum_{n=1}^{\infty}(R_{n+1} - R_n)p$$

and the sum is in fact finite. We first treat the term R_1p. By the condition previous to (P1) and the estimate on $w_m|p|$ we get

$$w_m(r_1) \sup_{|z|=r_1} |R_1 p(z)| \leqslant c w_m(r_1) \sup_{|z|=r_1} |p(z)| \leqslant 1.$$

We select $s \in \{1, \dots, m\}$ with $w_m(r_1) = \lambda_s v_{s,N(s)}(r_1)$. From the second inequality in (P1) applied to the polynomial $R_1 p$ and $v_{s,N(s)}$ and once more the condition previous to (P1), we conclude

$$\begin{aligned}
\sup_{z\in\mathbb{D}} v_{s,N(s)}(z)|R_1 p(z)| &\leqslant C \sup_{n\in\mathbb{N}} v_{s,N(s)}(r_n)\big(\sup_{|z|=r_n} |(R_{n+1}-R_n)R_1 p(z)|\big) \\
&= C v_{s,N(s)}(r_1) \sup_{|z|=r_1} |(R_2 - R_1)R_1 p(z)| \\
&= C\lambda_s^{-1} w_m(r_1) \sup_{|z|=r_1} |(R_2 - R_1)R_1 p(z)| \\
&\leqslant 2cC\lambda_s^{-1} w_m(r_1) \sup_{|z|=r_1} |R_1 p(z)| \\
&\leqslant 2c^2 C \lambda_s^{-1} \\
&\leqslant 2c^2 C D_2 \lambda_s^{-1} \\
&\leqslant 2c^2 \varepsilon_s,
\end{aligned}$$

which implies $R_1 p \in 2c^2 U_s \subseteq c^2 U$.

Now we treat $p - R_1 p = \sum_{n=1}^{\infty} (R_{n+1} - R_n)p$. We apply the inequality in (P1) for w_m and the estimate for $w_m|p|$ to get

$$(\star) \qquad w_m(r_n)\big(\sup_{|z|=r_n} |(R_{n+2} - R_{n-1})p(z)|\big) \leqslant C$$

for each $n \in \mathbb{N}$. Inductively we write $\mathbb{N}$ as a disjoint union $\cup_{s=1}^m J_s$ such that

$$w_m(r_j) = \lambda_s v_{s,N(s)}(r_j) \quad \text{for} \quad j \in J_s.$$

For $s = 1, \dots, m$ we put $g_s := \sum_{j\in J_s} (R_{j+1} - R_j)p$, which is a polynomial. Clearly $p - R_1 p = \sum_{s=1}^m g_s$. We fix $s \in \{1, \dots, m\}$ and put

$$p_n^s := \begin{cases} (R_{n+2} - R_{n-1})p & \text{for } n \in J_s \\ 0 & \text{otherwise.} \end{cases}$$

The properties of the sequence $(R_n)_{n\in\mathbb{N}}$ imply

$$g_s = \sum_{n\in J_s} (R_{n+1} - R_n)(R_{n+2} - R_{n-1})p = \sum_{n=1}^{\infty} (R_{n+1} - R_n)p_n^s$$

and all the sums are finite. Hence

$$\sup_{z\in\mathbb{D}} v_{s,N(s)}(z)|g_s(z)| = \sup_{z\in\mathbb{D}} v_{s,N(s)}(z)\Big|\sum_{n=1}^{\infty} (R_{n+1} - R_n)p_n^s\Big|.$$

Since only a finite number of the p_n^s are non-zero and all the weights belong to the set W, we can apply (P2) and the estimate ($\star$) to conclude

$$\begin{aligned}
\sup_{z\in\mathbb{D}} v_{s,N(s)}(z)|g_s(z)| &\leqslant D_s \sup_{z\in\mathbb{D}} \big(\sup_{|z|=r_n} |p_n^2(z)|\big) v_{s,N(s)}(r_n) \\
&\leqslant D_s \sup_{n\in J_s} \big(\sup_{|z|=r_n} |p_n^s(z)|\big) v_{s,N(s)}(r_n) \\
&= D_s \big(\sup_{|z|=r_n} |R_{n+2} - R_{n-1})p(z)|\big) v_{s,N(s)}(r_n) \\
&\leqslant D_s \lambda_s^{-1} \big(\sup_{|z|=r_n} |p_n^s(z)|\big) w_m(r_n) \\
&\leqslant D_s \lambda_s^{-1} C \\
&= \varepsilon_s,
\end{aligned}$$

which yields $g_s \in U_s \subseteq 2^{-s}U$.

Finally

$$p - R_1 p = \sum_{s=1}^{m} g_s \in \sum_{s=1}^{m} s^{-s}U \subseteq U,$$

i.e. $p \in (c^2+1)U$ as desired. ■

Now we consider the (LB)-case of 6.2, i.e. the inductive limit $\mathcal{V}_0P(\mathbb{D})$ and the (LB)-spaces $\mathcal{V}_0H(\mathbb{D})$ and $\mathcal{V}H(\mathbb{D})$, where $\mathcal{V} \subseteq W$. Then by 6.2, $\mathcal{V}_0P(\mathbb{D}) \subseteq \mathcal{V}_0P(\mathbb{D})$ is a topological subspace and $\mathcal{V}_0H(\mathbb{D}) \subseteq \mathcal{V}H(\mathbb{D})$ is a topological subspace even in the balanced setting. Moreover, the (LB)-space $\mathcal{V}H(\mathbb{D})$ is regular in this situation.

The next result (which is well-known) will exhibit a fundamental system of bounded sets in the (LB)-space $\mathcal{V}_0P(\mathbb{D})$, see 6.4.

Lemma 6.3. Let E and F be locally convex spaces, $i\colon E \to F$ be linear, continuous, open and injective. Let $(B_N)_{N\in\mathbb{N}}$ be a fundamental system of bounded sets in F. Then $(i^{-1}(B_N))_{N\in\mathbb{N}}$ is a fundmental system of bounded sets in E.

Proof. Let us first show that for each bounded set $B \subseteq E$ there exists N and $\rho > 0$ such that $\rho i^{-1}(B_N) \subseteq B$. For this purpose let $B \subseteq E$ be bounded. Then $i(B)$ is bounded in F. Since $(B_N)_{N\in\mathbb{N}}$ is a fundamental system of bounded sets in F, there exists N and $\rho > 0$ such that $\rho i(B) \subseteq B_N$, i.e.

$$(\circ) \qquad \forall\, x \in i(B)\ \exists\, b \in B_N : x = \rho b.$$

We claim that $B \subseteq \rho i^{-1}(B_N)$. Let $y \in B$ be given. Then $i(y) \in i(B)$. By ($\circ$), there exists $b \in B_N$ such that $i(y) = \rho b \in \rho B_N$. Hence $i(y) \in \rho B_N$ and thus $i(\frac{1}{\rho}y) = \frac{1}{\rho}i(y) \in B_N$. Since i is injective, $\frac{1}{\rho}y \in i^{-1}(B_N)$, i.e. $y \in \rho i^{-1}(B_N)$ which establishes the claim.

It remains to show that $i^{-1}(B)$ is bounded in E for each $N \in \mathbb{N}$. We fix $N \in \mathbb{N}$ and $U \in \mathcal{U}_0(E)$ and have to show that there exists $\rho > 0$ such that $\rho i^{-1}(B_N) \subseteq U$. Since i is open, $i(U)$ is open in F and since B_N is bounded, there exists $\rho > 0$ such that $\rho B_N \subseteq i(U)$. We claim that $\rho i^{-1}(B_N) \subseteq U$. Let $x \in \rho i^{-1}(B_N)$, i.e. $\frac{1}{\rho}x \in i^{-1}(B_N)$ and thus $i(\frac{1}{\rho}x) \in B_N$. Hence $i(x) \in \rho B_N \subseteq i(U)$ and since i is injective, we have $x \in U$ and are done. ■

Consequence 6.4. Let $\mathcal{V} \subseteq W$. Then the system of unit balls $(P_n^\circ)_{n\in\mathbb{N}}$, i.e.

$$P_n^\circ := \big\{p \in P(v_n)_0(\mathbb{D})\ ;\ \|f\|_n \leqslant 1\big\} = B_n^\circ \cap \mathbb{P} = B_n \cap \mathbb{P},$$

where $B_n^{(\circ)}$ denotes the unit ball of the Banach space $H(v_n)_{(0)}(\mathbb{D})$, is a fundamental system of bounded sets in the inductive limit $\mathcal{V}_0P(\mathbb{D})$.

Proof. Since $\mathcal{V}H(\mathbb{D})$ is regular, $(B_n)_{n\in\mathbb{N}}$ is a fundamental system of bounded sets in $\mathcal{V}H(\mathbb{D})$. 6.3 yields that the same is true for $(P_n)_{n\in\mathbb{N}}$ in the space $\mathcal{V}_0P(\mathbb{D})$. ■

For the proof of the next proposition we need the following technical lemma.

Lemma 6.5. Let $X = \operatorname{proj}_N \operatorname{ind}_n X_{N,n}$ with normed spaces $X_{N,n}$. Let $T \subseteq X$ be absolutely convex and bornivorous and $(n(N))_{N\in\mathbb{N}} \subseteq \mathbb{N}$ be arbitrary. Then there exists $N' \in \mathbb{N}$ such that $\cap_{N=1}^{N'} B_{N,n(N)}$ is absorbed by T.

Proof. Assume the contrary. Hence, for $N' \in \mathbb{N}$ fixed $\cap_{N=1}^{N'} B_{N,n(N)}$ is not absorbed by T. That is, for each $\rho > 0$ there exists $x \in \cap_{N=1}^{N'} B_{n,N(n)} \backslash \rho T$ and hence for each N there exists $x \in \cap_{N=1}^{N'} B_{n,N(n)} \backslash NT$. In particular we get $x \in \cap_{N=1}^{N'} B_{n,N(n)} \backslash N'T$. Hence we have for each N' some $x_{N'} \in \cap_{N=1}^{N'} B_{N,n(N)} \backslash N'T$ and may put $B := \{x_{N'}\ ;\ N' \in \mathbb{N}\}$. Now we claim that B is bounded in X. In order to show this, we fix $L \in \mathbb{N}$ and write

$$B = \{x_{N'}\ ;\ 1 \leqslant N' \leqslant L\} \cup \{x_{N'}\ ;\ N' \geqslant L\}$$

To show that $B \subseteq X$ is bounded it is enough to show the latter for $B' := \{x_{N'}\ ;\ N' \geqslant L\}$ since $\{x_{N'}\ ;\ 1 \leqslant N' \leqslant L\}$ is finite. We claim that $B' \subseteq X_L$ and that B' is bounded there. By definition each $x_{N'} \in B'$ lies in $\cap_{N=1}^{N'} B_{N,n(N)}$ and for $L \leqslant N'$ we have

$$\bigcap_{N=1}^{N'} B_{N,n(N)} \subseteq B_{L,n(L)}$$

and the latter set is bounded in X_L. Hence the same holds for B and we have established the claim. By our assumptions, T is bornivorous. Hence there exists $\lambda > 0$ such that $B \subseteq \lambda T$, since B is bounded. Now we choose $N' \geqslant \lambda$. Then $x_{N'} \notin N'T \supseteq \lambda T$, i.e. $x_N' \notin \lambda T$ which is a contradiction. ■

In the following proposition we have (for technical reasons) to assume that $\mathcal{W}$ is closed under finite maxima. 6.10 will provide that this is true for the main example of a set W of class $\mathcal{W}$.

Proposition 6.6. Let $\mathcal{A} \subseteq W$ and assume that W is closed under finite maxima. Let $\mathcal{A}$ satisfy condition (wQ). Then $(AP)_0(G) := \operatorname{proj}_N \operatorname{ind}_n P(a_{N,n})_0(\mathbb{D})$ is bornological.

Proof. By Bierstedt, Bonet [18], condition (wQ) implies condition (wQ)* that is

$$\exists\,(n(\sigma))_{\sigma\in\mathbb{N}}\subseteq\mathbb{N}\text{ increasing }\forall\,N\;\exists\,M\;\forall\,K,\,m\;\exists\,S>0,\,k\colon$$
$$\tfrac{1}{a_{M,m}}\leqslant S\max\big(\tfrac{1}{a_{K,k}},\min_{\sigma=1,\dots,N}\tfrac{1}{a_{\sigma,n(\sigma)}}\big).$$

We fix an absolutely convex and bornivorous set T in $(AP)_0(\mathbb{D})$. Since $(AP)_0(\mathbb{D})=P(a_{N,n})_0(\mathbb{D})$ holds algebraically for all N, n we may consider T as a subset of the latter space and claim that there exists N such that for each n the ball $P^\circ_{N,n}=B^{(\circ)}_{N,n}\cap\mathbb{P}$ is absorbed by T. We proceed by contradiction and hence assume

$$(\star)\qquad\forall\,M\;\exists\,m(M)\colon P^\circ_{M,m(M)}\text{ is not absorbed by }T.$$

By 6.5, there exists N such that $\cap_{\sigma=1}^N P^\circ_{\sigma,m(\sigma)}$ is absorbed by T. For the sequence $(n(\sigma))_{\sigma\in\mathbb{N}}$ and this N we choose M as in (wQ)*. By $(\star)$ there exists $m(M)$ such that for each K there exists $S_K>0$ and $k(K)$ such that $\frac{1}{a_{M,m(M)}}\leqslant S_K\max\big(\frac{1}{a_{K,k(K)}},\min_{\sigma=1,\dots,N}\frac{1}{a_{\sigma,n(\sigma)}}\big)$. We claim

$$(\circ)\qquad\forall\,K\colon\tfrac{1}{a_{M,m(M)}}\leqslant S'_K\max(u_K,w_N)$$

where we used the definitions $w_N:=\min_{\sigma=1,\dots,N}\frac{1}{a_{\sigma,n(\sigma)}}$, $u_K:=\min_{\mu=1,\dots,K}\frac{1}{a_{\mu,k(\mu)}}$ and $S'_K:=\max_{\mu=1,\dots,K}S_\mu$. To establish the claim let us fix K. Then for $\mu=1,\dots,K$ we have $\frac{1}{S'_K a_{M,m(M)}}\leqslant\frac{1}{S_\mu a_{M,m(M)}}\leqslant\max\big(\frac{1}{a_{\mu,k(\mu)}},w_N\big)$ by the very definition of S'_K and the estimate we deduced from (wQ)*. If $\frac{1}{S'_K a_{M,m(M)}}\leqslant w_N$ holds, we are done. Otherwise the above yields $\frac{1}{S'_K a_{M,m(M)}}\leqslant\frac{1}{a_{\mu,k(\mu)}}$ for $\mu=1,\dots,K$, i.e. $\frac{1}{S'_K a_{M,m(M)}}\leqslant\min_{\mu=1,\dots,K}\frac{1}{a_{\mu,k(\mu)}}=u_K$ and we are done as well.

Now we again make use of the decomposition method based on class $\mathcal{W}$ to show the following

$$\forall\,K\;\exists\,\tau_K>0\colon P^\circ_{M,m(M)}\subseteq\tau_K\big[\bigcap_{\sigma=1}^{N}P^\circ_{\sigma,n(\sigma)}+\bigcap_{\mu=1}^{K}P^\circ_{\mu,k(\mu)}\big].$$

We fix K. Let $p\in P^\circ_{M,m(M)}$, i.e. $a_{M,m(M)}|p|\leqslant 1$ hence $|p|\leqslant\frac{1}{a_{M,m(M)}}$ and by $(\circ)$ we get the estimate

$$|p|\leqslant S'_K\max(u_K,w_N)=\max\big(\min_{\sigma=1,\dots,N}\tfrac{S'_K}{a_{\sigma,n(\sigma)}},\min_{\mu=1,\dots,K}\tfrac{S'_K}{a_{\mu,k(\mu)}}\big),$$

may define $\frac{1}{u_1}:=\min_{\sigma=1,\dots,N}\frac{S'_K}{a_{\sigma,n(\sigma)}}$, $\frac{1}{u_2}:=\min_{\mu=1,\dots,K}\frac{S'_K}{a_{\mu,k(\mu)}}$ and thus obtain $u_1:=\max_{\sigma=1,\dots,N}\frac{a_{\sigma,n(\sigma)}}{S'_K}$, $u_2:=\max_{\mu=1,\dots,K}\frac{a_{\mu,k(\mu)}}{S'_K}\in W$ since W is closed under the formation of finite maxima. We put $u:=\min(u_1,u_2)$. Since W is closed under finite minima, $u\in W$ holds. Moreover, $\frac{1}{u}=\max(\frac{1}{u_1},\frac{1}{u_2})$ that is by the above $|p|\leqslant\frac{1}{u}$, i.e. $u|p|\leqslant 1$.

Now we may repeat the proof of 6.1, i.e. apply the decomposition method of Bierstedt, Bonet [19] to obtain for $p=R_1p+g_1+g_2$ the estimates $\sup_{z\in\mathbb{D}}u_1|R_1p|\leqslant 2c^2C$ or $\sup_{z\in\mathbb{D}}u_2|R_1p|\leqslant 2c^2C$, $\sup_{z\in\mathbb{D}}u_1|g_1|\leqslant D_1C$ and $\sup_{z\in\mathbb{D}}u_2|g_2|\leqslant D_2C$

that is $R_1p \in 2c^2CS'_K \cap_{\sigma=1}^N P^\circ_{\sigma,n(\sigma)}$ or $R_1p \in 2c^2CS'_K \cap_{\mu=1}^K P^\circ_{\mu,k(\mu)}$ as well as $g_1 \in D_1CS'_K \cap_{\sigma=1}^N P^\circ_{\sigma,n(\sigma)}$ and $g_2 \in D_2CS'_K \cap_{\mu=1}^K P^\circ_{\mu,k(\mu)}$. We put $\tau_K := CS'_K(2c^2 + \max(D_1, D_2))$ and obtain

$$p = R_1p + g_1 + g_2 \in (2c^2 + D_1)CS'_K \bigcap_{\sigma=1}^{N} P^\circ_{\sigma,n(\sigma)} + (2c^2 + D_2)CS'_K \bigcap_{\mu=1}^{K} P^\circ_{\mu,k(\mu)},$$

i.e. $p \in \tau_K\big[\cap_{\sigma=1}^N P^\circ_{\sigma,n(\sigma)} + \cap_{\mu=1}^K P^\circ_{\mu,k(\mu)}\big]$, which establishes the claim.

6.5 yields the existence of K' such that $\cap_{\mu=1}^{K'} P^\circ_{\mu,k(\mu)}$ is absorbed by T. But now we have in particular

$$P^\circ_{M,m(M)} \subseteq \tau_{K'}\big[\bigcap_{\sigma=1}^{N} P^\circ_{\sigma,n(\sigma)} + \bigcap_{\mu=1}^{K'} P^\circ_{\mu,k(\mu)}\big],$$

where the set on the left hand side is not absorbed by T unlike the set on the right hand side, a contradiction.

To finish the proof, we observe that our claim is exactly the statement (B2) in 4.9. Since statement (B1) of 4.9 is trivial for the projective spectrum under consideration (just put $M := N$ and $n := m$), and by 6.4, $(P^\circ_{N,n})_{n\in\mathbb{N}}$ is a fundamental system of bounded sets in $(\mathcal{A}_N)_0P(\mathbb{D})$ for each $N \in \mathbb{N}$, the conclusion follows from 4.10. ■

For the final result we need the following lemma, which we will also use in section 7, compare with Bonet, Pérez Carreras [63, 4.2.1].

Lemma 6.7. Let X and Y be locally convex spaces, $X \subseteq Y$ be a dense topological subspace. If X is bornological then Y is quasibarrelled.

Proof. Let $T \subseteq Y$ be a bornivorous barrel. We put $V := T \cap X$. Then V is absolutely convex and we claim that V is bornivorous. Let $B \subseteq X$ be bounded. Then $B \subseteq Y$ is bounded. Since T is bornivorous there exists $\rho > 0$ such that $B \subseteq \rho T$. Thus, $B = B \cap X \subseteq \rho T \cap X \subseteq \rho V$. Since X is bornological, V is a 0-neighborhood in X. Since $X \subseteq Y$ is dense, the same is true for $\overline{V}^Y$ in Y (see e.g. Jarchow [50, 3.4.1]). But since T is closed, $\overline{V}^Y \subseteq T$ and hence T is a 0-neighborhood. Therefore, Y is quasibarrelled. ■

Theorem 6.8. Assume $\mathcal{A} \subseteq W$ and that W is closed under finite maxima. Let $\mathcal{A}$ satisfy condition (wQ). Then $(AH)_0(\mathbb{D})$ is barrelled.

Proof. By 6.6, the space $(AP)_0(\mathbb{D})$ is bornological. 6.2 provides that $(AP)_0(\mathbb{D}) \subseteq (AH)_0(\mathbb{D})$ is a topological subspace and this subspace is dense by 5.2. By Lemma 6.7, $(AH)_0(\mathbb{D})$ is quasibarrelled. As we mentioned already in a previous proof, for $(AH)_0(G)$ quasibarrelledness is equivalent to barrelledness by Vogt [79, 3.1] since the projective spectrum in the o-growth case is reduced (cf. 5.1) and thus we are done. ■

6.3 Summary of results

Remark 6.9. Let us summarize the results of sections 5 and 6 in the following schemes, which in particular illustrate again the lines of the proofs for the results and the assumptions needed for each single implication.

$$
\begin{tikzcd}
(\mathrm{Q})^{\sim}_{\mathrm{in}} \arrow[dr, Rightarrow, "\text{i.g.}"] & & & & & & & \\
& (\mathrm{Q})^{\sim}_{\mathrm{out}} \arrow[r, Rightarrow, "G=\mathbb{D}", "\mathcal{A}\subseteq W"'] & \operatorname{Proj}{}^1\mathcal{A}H=0 \arrow[r, Rightarrow, "\text{i.g.}"] & \begin{array}{c} AH(G) \\ \text{ultra-} \\ \text{bornological} \end{array} \arrow[r, Rightarrow, "\text{i.g.}"] & \begin{array}{c} AH(G) \\ \text{barrelled} \end{array} \arrow[dr, Rightarrow, "\text{balanced setting}"'] & & (\mathrm{wQ})^{\sim}_{\mathrm{in}} \arrow[r, Rightarrow, "\text{i.g.}"] & (\mathrm{wQ})^{\sim}_{\mathrm{out}} \\
(\mathrm{Q}) \arrow[ur, Rightarrow, "\text{i.g.}"'] & & & & & \begin{array}{c} (AH)_0(G) \\ \text{barrelled} \end{array} \arrow[ur, Rightarrow, "\text{balanced setting}"'] & &
\end{tikzcd}
$$

Figure 1: Class $\mathcal{W}$: Scheme of implications for O-growth conditions.

$$
\begin{tikzcd}
\operatorname{Proj}{}^1\mathcal{A}_0H=0 \arrow[r, Rightarrow, "\text{i.g.}"] & \begin{array}{c} (AH)_0(G) \\ \text{ultra-} \\ \text{bornological} \end{array} \arrow[r, Rightarrow, "\text{i.g.}"] & \begin{array}{c} (AH)_0(G) \\ \text{barrelled} \end{array} \arrow[r, Rightarrow, "\text{balanced setting}"] & (\mathrm{wQ})^{\sim}_{\mathrm{in}} \arrow[r, Rightarrow, "\text{i.g.}"] & (\mathrm{wQ})^{\sim}_{\mathrm{out}} \\
& & \begin{array}{c} (AP)_0(G) \\ \text{bornological} \end{array} \arrow[u, Rightarrow, "G=\mathbb{D}, \ \mathcal{A}\subseteq W"] & & \\
& & (\mathrm{wQ}) \arrow[u, Rightarrow, "G=\mathbb{D},\ \mathcal{A}\subseteq W,\ W \text{ closed under finite maxima}"] \arrow[uurr, Rightarrow, "\text{i.g.}"'] & &
\end{tikzcd}
$$

Figure 2: Class $\mathcal{W}$: Scheme of implications for o-growth conditions.

6.4 The main example: $\mathcal{W}(\varepsilon_0,k_0)$

The main example for some system W which is of class $\mathcal{W}$ is the set of all radial weights w on $\mathbb{D}$ which satisfy $\lim_{r\nearrow 1} w(r) = 0$, are non increasing on $[0,1[$ and such that there are $\varepsilon_0 > 0$ and $k_0 \in \mathbb{N}$ which satisfy

$$
\text{(L1)} \qquad \inf_{k\in\mathbb{N}} \frac{w(r_{k+1})}{w(r_k)} \geqslant \varepsilon_0
$$

and

$$
\text{(L2)} \qquad \limsup_{k\to\infty} \frac{w(r_{k+k_0})}{w(r_k)} < 1-\varepsilon_0.
$$

In this case R_n can be choosen as the *convolution with the de la Vallée Poussin kernel* i.e. for a holomorphic function f on $\mathbb{D}$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$, we have

$$
[R_n f](z) = \sum_{k=0}^{2^n} a_k z^k + \sum_{k=2^n+1}^{2^{n+1}} \frac{2^{n+1}-k}{2^n} a_k z^k.
$$

That is, R_n is just the arithmetic mean of the partial sums of index $2^n, \ldots, 2^{n+1}-1$ of the Taylor series of f. The conditions (L1) and (L2) form a uniform version of the conditions introduced by Lusky in [53, 54] and they also appear in the sequence space representation for weighted (LB)-spaces studied by Mattila, Saksman, Taskinen [55]. Bierstedt, Bonet showed in [19] that this system W satisfies the axioms of class $\mathcal{W}$.

For the proof of 6.6 we assumed that W is closed under finite maxima. This is not included in the definition of class $\mathcal{W}$ given by Bierstedt, Bonet [19] but for the main example explained above the latter is true. To prevent confusion, from now on we will denote the above example by $\mathcal{W}(\varepsilon_0, k_0)$ whereas W always is some arbitrary set of class $\mathcal{W}$.

Observation 6.10. The set $\mathcal{W}(\varepsilon_0, k_0)$ is closed under finite maxima.

Proof. Clearly it is enough to check that the condition (L1) and (L2) are stable under finite maxima. Let w_1, $w_2 \in \mathcal{W}(\varepsilon_0, k_0)$.

To check condition (L1) we compute

$$\begin{aligned}\tfrac{\max(w_1(r_{k+1}),w_2(r_{k+1}))}{\max(w_1(r_k),w_2(r_k))} &= \max(w_1(r_{k+1}), w_2(r_{k+1})) \cdot \min(\tfrac{1}{w_1(r_k)}, \tfrac{1}{w_2(r_k)})\\ &= \min(\tfrac{\max(w_1(r_{k+1}),w_2(r_{k+1}))}{w_1(r_k)}, \tfrac{\max(w_1(r_{k+1}),w_2(r_{k+1}))}{w_2(r_k)})\\ &\geqslant \min(\tfrac{w_1(r_{k+1})}{w_1(r_k)}, \tfrac{w_2(r_{k+1})}{w_2(r_k)})\\ &\geqslant \varepsilon_0,\end{aligned}$$

for each $k \in \mathbb{N}$ since $\inf_{k\in\mathbb{N}} \frac{w_1(r_{k+1})}{w_1(r_k)}$, $\inf_{k\in\mathbb{N}} \frac{w_2(r_{k+1})}{w_2(r_k)} \geqslant \varepsilon_0$. Hence we have

$$\inf_{k\in\mathbb{N}} \tfrac{\max(w_1(r_{k+1}),w_2(r_{k+1}))}{\max(w_1(r_k),w_2(r_k))} \geqslant \varepsilon_0$$

which is (L1) for $\max(w_1, w_2)$.

To check condition (L2) we compute

$$\begin{aligned}\tfrac{\max(w_1(r_{k+1}),w_2(r_{k+1}))}{\max(w_1(r_k),w_2(r_k))} &= (\max(w_1(r_k), w_2(r_k)) \cdot \min(\tfrac{1}{w_1(r_{k+k_0})}, \tfrac{1}{w_2(r_{k+k_0})}))^{-1}\\ &= (\min(\tfrac{\max(w_1(r_k),w_2(r_k))}{w_1(r_{k+k_0})}, \tfrac{\max(w_1(r_k),w_2(r_k))}{w_2(r_{k+k_0})}))^{-1}\\ &\leqslant (\min(\tfrac{w_1(r_k)}{w_1(r_{k+k_0})}, \tfrac{w_2(r_k)}{w_2(r_{k+k_0})}))^{-1}\\ &= \max(\tfrac{w_1(r_{k+k_0})}{w_1(r_k)}, \tfrac{w_2(r_{k+k_0})}{w_2(r_k)})\end{aligned}$$

for each $k \in \mathbb{N}$ since $\limsup_k \frac{w_1(r_{k+k_0})}{w_1(r_k)}$, $\limsup_k \frac{w_1(r_{k+k_0})}{w_1(r_k)} < 1 - \varepsilon_0$ and thus

$$\begin{aligned}\limsup_{k\to\infty} \tfrac{\max(w_1(r_{k+1}),w_2(r_{k+1}))}{\max(w_1(r_k),w_2(r_k))} &\leqslant \limsup_{k\to\infty} \max(\tfrac{w_1(r_{k+k_0})}{w_1(r_k)}, \tfrac{w_2(r_{k+k_0})}{w_2(r_k)})\\ &\leqslant \max(\limsup_{k\to\infty} \tfrac{w_1(r_{k+k_0})}{w_1(r_k)}, \limsup_{k\to\infty} \tfrac{w_2(r_{k+k_0})}{w_2(r_k)})\\ &< 1 - \varepsilon_0,\end{aligned}$$

which is (L2) for $\max(w_1, w_2)$. ■

We discussed already at the beginning of sections 5.3 the appearence of associated weights in the weight conditions which we used to characterize properties of the spaces or the spectra. Clearly, the appearence of the associated weights is natural within the setting of holomorphic functions and – as we also noted already – concerning necessary conditions within the balanced setting they seem to be inevitable. But however, we promised that in several situations it will be possible to ommit all the $\sim$'s from the conditions. In fact for $\mathcal{A} \subseteq \mathcal{W}(\varepsilon_0, k_0)$ this is true, since weights in this set are essential automatically. The latter follows from Bierstedt, Bonet [19] who showed that the conditions (L1) and (L2) are equivalent to the conditions (U) and (L) of Shields and Williams [71] who showed that the above holds ([71, Lemma 1.(iv)]). A detailed proof is contained in Domański, Lindström [40]. Compare also Bierstedt, Bonet, Taskinen [21, Proposition 3.4].

Let us now formulate a collective and in view of concrete examples quite accessible version of the results obtained in this section.

Corollary 6.11. (of 6.1) Let $\mathcal{A}$ be contained in the set $\mathcal{W}(\varepsilon_0, k_0)$. Then we have (i)$\Rightarrow$(ii)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(v), where

(i) $\mathcal{A}$ satisfies condition (Q).

(ii) $\operatorname{Proj}^1 \mathcal{A}H = 0$,

(iii) $AH(\mathbb{D})$ is ultrabornological,

(iv) $AH(\mathbb{D})$ is barrelled,

(v) $\mathcal{A}$ satisfies condition (wQ).

Corollary 6.12. (of 6.8) Let $\mathcal{A}$ be contained in the set $\mathcal{W}(\varepsilon_0, k_0)$. Then $(AH)_0(\mathbb{D})$ is barrelled if and only if $\mathcal{A}$ satisfies condition (wQ).

7 A special setting for the complex plane: The class $(\mathrm{E})_{\mathrm{C,c}}$

In this section we study the case $G = \mathbb{C}$. Similar to the last section we need special assumptions on the weights which allow a decomposition of holomorphic functions to get sufficient conditions for $\operatorname{Proj}^1 \mathcal{A}H = 0$ resp. barrelledness of $(AH)_0(\mathbb{C})$. A convenient setting to do this was invented by Bierstedt, Bonet, Taskinen in [22]. Following their definition 2.1, we denote by $(E)_{C,c}$ for given constants C, $c > 0$ the set of all radial weights $a\colon \mathbb{C} \to \mathbb{R}_{>0}$ which are of the form

$$a(z) = b(|z|)e^{-c|z|}$$

where $b\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ is differentiable, strictly increasing and satisfies

$$\sup_{r\in[0,\infty[} \tfrac{rb'(r)}{cb(r)} \leqslant C.$$

The weights in the "class $(E)_{C,c}$" have the following important properties.

Theorem D. (Bierstedt, Bonet, Taskinen [22, 2.3]) For fixed C, $c > 0$ there exists a sequence $(T_n)_{n=1,2,\ldots}$ of linear mappings $T_n\colon \mathbb{P} \to \mathbb{P}$ of finite rank from the space of polynomials $\mathbb{P}$ into itself, which satisfies the following properties

(a) The operators T_n satisfy $T_n \circ T_m = 0$ if $|n-m| \geqslant 2$ and we have

$$T_n \circ T_{n+1} = T_{n+1} \circ T_n.$$

(b) For each $p \in \mathbb{P}$ we have $\sum_{n=1}^{\infty} T_n p = p$ and the sum is finite.

(c) There is a constant $D \geqslant 1$ such that for each $r > 1$ and every $p \in \mathbb{P}$ we have

$$\sup_{|z|=r} |T_n p(z)| \leqslant D \sup_{|z|=r} |p(z)|.$$

(d) There exist increasing positive sequences $(\rho_n)_{n=1,2,\dots}$ and $(\sigma_n)_{n=1,2,\dots}$, $\rho_n < \sigma_n$, such that for each weight $a \in (E)_{C,c}$, $a(z) = b(|z|)e^{-a|z|}$, there exists a constant $C(a) > 0$ such that for each $p \in \mathbb{P}$

$$\tfrac{1}{D} \sup_{n \geqslant 1} \sup_{\rho_n \leqslant |z| \leqslant \sigma_n} b(\rho_n) e^{-a|z|} |T_n p(z)| \leqslant \|p\|_a$$

and

$$\|p\|_a \leqslant C(a) \sup_{n \geqslant 1} \sup_{\rho_n \leqslant |z| \leqslant \sigma_n} b(\rho_n) e^{-a|z|} |T_n p(z)|,$$

where D is the constant of statement (c), which does not depend on the weight a.

(e) There exists a constant $0 < d \leqslant 1$, independent of the weight, such that (with the notation of (d)) $b(\rho_n) \geqslant d b(\rho_{n+1})$ for $n \geqslant 1$.

Note that for a system of weights in $(E)_{C,c}$ the requirements w.r.t. to the results of Bierstedt, Bonet, Galbis [20] which we explained above are automatically satisfied. Moreover, Theorem's 3.2 and 4.1 of Bierstedt, Bonet, Taskinen [22] imply that for $\mathcal{A}$ in $(E)_{C,c}$, $\mathcal{A}_n H(\mathbb{C}) \subseteq \mathcal{A}_n C(\mathbb{C})$ and $(\mathcal{A}_n)_0 H(\mathbb{C}) \subseteq H(\overline{A}_n)_0(\mathbb{C})$ are all topological subspaces (for each $n \in \mathbb{N}$). With the result of Bierstedt, Meise, Summers [27] we stated above it follows that in this case $(\mathcal{A}_n)_0 H(\mathbb{C}) \subseteq (\mathcal{A}_n)_0 C(\mathbb{C})$ is a topological subspace for each $n \in \mathbb{N}$.

In the sequel it will be less important to know the constants C and c and hence we will consider a system of weights E such that there exist C, $c > 0$ with $E = (E)_{C,c}$.

7.1 Sufficient conditions for the vanishing of $\mathrm{Proj}^1 \mathcal{A}\mathrm{H}$

The proof of the following result is based on the decomposition method developed by Bierstedt, Bonet, Taskinen [22] which was also used by Wolf [87].

Theorem 7.1. Let $\mathcal{A}$ be a sequence in E and assume that $\mathcal{A}$ satisfies condition $(\mathrm{Q})^{\sim}_{\mathrm{out}}$. Then $\mathrm{Proj}^1 \mathcal{A}H = 0$.

Proof. In order to show that $\mathrm{Proj}^1 \mathcal{A}H = 0$ we use Braun, Vogt [36, Theorem 8] (which was independently obtained by Frerick, Wengenroth [43]). That is, we have to show condition $(\overline{\mathrm{P}_2})$

$$\forall\, N\ \exists\, M,\ n\ \forall\, K,\ m,\ \varepsilon > 0\ \exists\, k,\ S > 0 :\ B_{M,m} \subseteq \varepsilon B_{N,n} + S B_{K,k}.$$

For given $N \in \mathbb{N}$ we select M and n as in $(\mathrm{Q})^{\sim}_{\mathrm{out}}$. For given K, m and $\varepsilon > 0$ we put $\varepsilon' := (3D^2d^{-1}C_1)^{-1}\varepsilon$ (where $C_1 := C(a_{N,n})$) and choose k and S' according to $(\mathrm{Q})^{\sim}_{\mathrm{out}}$ with respect to ε'. Finally we put $S := 3D^2d^{-1}C_2S'$ (where $C_2 := C(a_{K,k})$). The constants are those of Theorem 7.D. Now we fix an arbitrary $f \in B_{M,m}$ and consider S_tf. We have $a_{M,m}|S_tf| \leqslant a_{M,m}|f| \leqslant 1$, i.e. $|S_tf| \leqslant \frac{1}{a_{M,m}}$. With [20, 1.2.(iii)] it follows $|S_tf| \leqslant (\frac{1}{a_{M,m}})^{\sim}$ and by the estimate in $(\mathrm{Q})^{\sim}_{\mathrm{out}}$ we obtain $|S_tf| \leqslant \max\big(\frac{\varepsilon'}{a_{N,n}}, \frac{S'}{a_{K,k}}\big)^{\sim} \leqslant \max\big(\frac{\varepsilon'}{a_{N,n}}, \frac{S'}{a_{K,k}}\big)$. Now we put $u_1 := a_{N,n}$, $u_2 := a_{K,k}$, $\alpha_1 := \frac{1}{\varepsilon'}$, $\alpha_2 := \frac{1}{S'}$ and $u := \min(\alpha_1u_1, \alpha_2u_2)$ to obtain $|S_tf| \leqslant \max(\frac{1}{\alpha_1u_1}, \frac{1}{\alpha_2u_2}) = \min(\alpha_1u_1, \alpha_2u_2) = \frac{1}{u}$ on $\mathbb{C}$ and hence $u|S_tf| \leqslant 1$ on $\mathbb{C}$.

Since $\mathcal{A} \subseteq E$ we have $a_{N,n}(z) = b_{N,n}(|z|)e^{-c|z|}$ and $a_{K,k}(z) = b_{K,k}(|z|)e^{-c|z|}$. We put $s := \min(\alpha_1s_1, \alpha_2s_2)$ where $s_1 := b_{N,n}$ and $s_2 := b_{K,k}$. Then we may choose $M(J) \in \{1,2\}$ for each $J \in \mathbb{N}$ such that $s(\rho_J) = \alpha_{M(J)}s_{M(J)}(\rho_J)$, where $(\rho_J)_{J\in\mathbb{N}}$ is the sequence in 7.D.(d). We define the sets

$$\mathbb{N}_1 := \big\{J \in \mathbb{N}\,;\, M(J) = 1\big\} \text{ and } \mathbb{N}_2 := \big\{J \in \mathbb{N}\,;\, M(J) = 2\big\},$$

i.e. $\mathbb{N} = \mathbb{N}_1 \dot\cup \mathbb{N}_2$. For $M = 1,2$ we define $p_M := \sum_{J\in\mathbb{N}_M} T_J(S_tf)$, where the sum has in fact only finitely many non-zero terms by 7.D.(b). Now by 7.D.(a) we may compute

$$\begin{aligned} T_Jp_M &= \textstyle\sum_{j\in\mathbb{N}_M}[T_J \circ T_j](S_tf) \\ &= \big[\chi_{M,J-1}T_J \circ T_{J-1} + \chi_{M,J}T_J^2 + \chi_{M,J+1}T_J \circ T_{J+1}\big](S_tf) \\ &= \big[\chi_{M,J-1}T_{J-1} \circ T_J + \chi_{M,J}T_J^2 + \chi_{M,J+1}T_{J+1} \circ T_J\big](S_tf), \end{aligned}$$

where

$$\chi_{M,J} := \begin{cases} 1 & \text{if } J \in \mathbb{N}_M, \\ 0 & \text{otherwise.} \end{cases}$$

7.D.(d) and 7.D.(c) imply (with $C(u_1) = C_1$)

$$\begin{aligned} \|p_1\|_{N,n} &\overset{\text{7.D.(d)}}{\leqslant} C_1 \sup_{J\geqslant 1} \sup_{\rho_J\leqslant|z|\leqslant\sigma_J} b_{N,n}(\rho_J)e^{-c|z|}|T_Jp_M(z)| \\ &= C_1 \sup_{J\geqslant 1} \sup_{\rho_J\leqslant|z|\leqslant\sigma_J} b_{N,n}(\rho_J)e^{-c|z|}\big|\big[\chi_{M,J-1}T_{J-1} \circ T_J + \chi_{M,J}T_J^2 \\ &\qquad\qquad + \chi_{M,J+1}T_{J+1} \circ T_J\big](S_tf)\big| \\ &\leqslant C_1 \sup_{J\in\mathbb{N}_1} \Bigg[\sup_{\rho_{J+1}\leqslant r\leqslant\sigma_{J+1}} \Big(b_{N,n}(\rho_{J+1})e^{-cr} \sup_{|z|=r} |(T_J \circ T_{J+1})(S_tf)(z)| \Big) \\ &\qquad + \sup_{\rho_J\leqslant r\leqslant\sigma_J} \Big(b_{N,n}(\rho_J)e^{-cr} \sup_{|z|=r} |T_J^2(S_tf)(z)| \Big) \\ &\qquad + \sup_{\rho_{J-1}\leqslant r\leqslant\sigma_{J-1}} \Big(b_{N,n}(\rho_{J-1})e^{-cr} \sup_{|z|=r} |(T_J \circ T_{J-1})(S_tf)(z)| \Big) \Bigg] \\ &\overset{\text{7.D.(c)}}{\leqslant} C_1 \sup_{J\in\mathbb{N}_1} \Bigg[\sup_{\rho_{J+1}\leqslant r\leqslant\sigma_{J+1}} \Big(b_{N,n}(\rho_{J+1})e^{-cr} D \sup_{|z|=r} |T_{J+1}(S_tf)(z)| \Big) \end{aligned}$$

$$
\begin{aligned}
&\qquad + \sup_{\rho_J\leqslant r\leqslant\sigma_J}\Big(b_{N,n}(\rho_J)e^{-cr}D\sup_{|z|=r}|T_J(S_tf)(z)|\Big)\\
&\qquad + \sup_{\rho_{J-1}\leqslant r\leqslant\sigma_{J-1}}\Big(b_{N,n}(\rho_{J-1})e^{-cr}D\sup_{|z|=r}|T_{J-1}(S_tf)(z)|\Big)\Big]\\
&= DC_1\sup_{J\in\mathbb{N}_1}\Big[\sup_{\rho_{J+1}\leqslant|z|\leqslant\sigma_{J+1}}\Big(b_{N,n}(\rho_{J+1})e^{-c|z|}|T_{J+1}(S_tf)(z)|\Big)\\
&\qquad + \sup_{\rho_J\leqslant|z|\leqslant\sigma_J}\Big(b_{N,n}(\rho_J)e^{-c|z|}|T_J(S_tf)(z)|\Big)\\
&\qquad + \sup_{\rho_{J-1}\leqslant|z|\leqslant\sigma_{J-1}}\Big(b_{N,n}(\rho_{J-1})e^{-c|z|}|T_{J-1}(S_tf)(z)|\Big)\Big] =: (\circ).
\end{aligned}
$$

For $J\in\mathbb{N}_1$ we have $b_{N,n}(\rho_J)=\alpha_1^{-1}s(\rho_J)$ and hence by 7.D.(e)

$$
\begin{aligned}
d\alpha_1 b_{N,n}(\rho_{J+1}) &\leqslant \alpha_1 b_{N,n}(\rho_J)=s(\rho_J)\leqslant s(\rho_{J+1}),\\
\alpha_1 b_{N,n}(\rho_{J-1}) &\leqslant \alpha_1 b_{N,n}(\rho_J)=s(\rho_J)\leqslant \tfrac{1}{d}s(\rho_{J-1}).
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
b_{N,n}(\rho_{J+1}) &\leqslant \tfrac{\alpha_1^{-1}}{d}s(\rho_{J+1})=\tfrac{\varepsilon'}{d}s(\rho_{J+1}),\\
b_{N,n}(\rho_J) &\leqslant \alpha_1^{-1}s(\rho_J)=\varepsilon' s(\rho_J)\leqslant \tfrac{\varepsilon'}{d}s(\rho_J),\\
b_{N,n}(\rho_{J-1}) &\leqslant \tfrac{\alpha_1^{-1}}{d}s(\rho_{J-1})=\tfrac{\varepsilon'}{d}s(\rho_{J-1})
\end{aligned}
$$

and hence

$$
\begin{aligned}
(\circ) &\leqslant DC_1\tfrac{\varepsilon'}{d}\sup_{J\in\mathbb{N}_1}\Big[\sup_{\rho_{J+1}\leqslant|z|\leqslant\sigma_{J+1}}\Big(s(\rho_{J+1})e^{-c|z|}|T_{J+1}(S_tf)(z)|\Big)\\
&\qquad + \sup_{\rho_J\leqslant|z|\leqslant\sigma_J}\Big(s(\rho_J)e^{-c|z|}|T_J(S_tf)(z)|\Big)\\
&\qquad + \sup_{\rho_{J-1}\leqslant|z|\leqslant\sigma_{J-1}}\Big(s(\rho_{J-1})e^{-c|z|}|T_{J-1}(S_tf)(z)|\Big)\Big]\\
&\leqslant DC_1\tfrac{\varepsilon'}{d}\sup_{J\in\mathbb{N}_1}3\cdot\max\Big[\sup_{\rho_{J+1}\leqslant|z|\leqslant\sigma_{J+1}}\Big(s(\rho_{J+1})e^{-c|z|}|T_{J+1}(S_tf)(z)|\Big),\\
&\qquad \sup_{\rho_J\leqslant|z|\leqslant\sigma_J}\Big(s(\rho_J)e^{-c|z|}|T_J(S_tf)(z)|\Big),\\
&\qquad \sup_{\rho_{J-1}\leqslant|z|\leqslant\sigma_{J-1}}\Big(s(\rho_{J-1})e^{-c|z|}|T_{J-1}(S_tf)(z)|\Big)\Big]\\
&= \tfrac{3DC_1\varepsilon'}{d}\sup_{J\in\mathbb{N}_1}\max_{j=-1,0,1}\Big[\sup_{\rho_{J-j}\leqslant|z|\leqslant\sigma_{J-j}} s(\rho_{J-j})e^{-c|z|}|T_{J-j}(S_tf)(z)|\Big]\\
&\overset{\kappa:=J-j}{=} \tfrac{3DC_1\varepsilon'}{d}\sup_{j=-1,0,1}\sup_{\kappa+j\in\mathbb{N}_1}\sup_{\rho_\kappa\leqslant|z|\leqslant\sigma_\kappa} s(\rho_\kappa)e^{-c|z|}|T_\kappa(S_tf)(z)|
\end{aligned}
$$

$$\begin{aligned} &\overset{7.\mathrm{D.(d)}}{\leqslant} \frac{3DC_1\varepsilon'}{d} \cdot D \cdot \|S_t f\|_u \\ &\leqslant \quad 3D^2C_1\varepsilon' d^{-1}. \end{aligned}$$

Finally we get $\|p_1\|_{N,n} \leqslant 3D^2C_1\varepsilon' d^{-1}$, i.e. $p_1 \in 3D^2C_1\varepsilon' d^{-1}B_{N,n}$. Analogously, we obtain $\|p_2\|_{K,k} \leqslant 3D^2C_2S'd^{-1}$, i.e. $p_2 \in 3D^2C_2S'd^{-1}B_{K,k}$. By 7.D.(b) we get

$$S_t f = p_1 + p_2 \in 3D^2d^{-1}C_1\varepsilon' B_{N,n} + 3D^2d^{-1}C_2S'B_{K,k} = \varepsilon B_{N,n} + SB_{K,k}.$$

The last set is co-compact and we have $S_t f \to f$ for $t \to \infty$ with respect to co and hence $f \in \varepsilon B_{N,n} + SB_{K,k}$, which finishes the proof of $(\overline{\mathrm{P}_2})$. ■

7.2 Barrelledness of $(\mathrm{AH})_0(\mathrm{G})$

In order to find sufficient conditions for $(AH)_0(G)$ being barrelled we proceed as in section 6.2. Since 6.3 and 6.5 are independent of the special assumptions (class $\mathcal{W}$) of section 6.2, it is enough to prove analoga of 6.2, 6.4 and 6.6. Then we will be able to conclude as in 6.8 to get the desired result.

We use the notation established at the beginning of section 6.2.

Lemma 7.2. Let $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ be in E. Then (the (LB)-space) $\mathcal{V}_0P(\mathbb{C}) \subseteq \mathcal{V}_0H(\mathbb{C})$ is a topological subspace.

Proof. Since the identity $\mathcal{V}_0P(\mathbb{C}) \to \mathbb{P}$ is continuous, it is enough to show

$$\forall\, U \in \mathcal{U}_0(\mathcal{V}_0P(\mathbb{C}))\ \exists\, V \in \mathcal{U}_0(\mathbb{P})\colon V \subseteq U.$$

By Bierstedt, Bonet, Taskinen [22, 3.2] $\mathcal{V}_0H(\mathbb{C})$ is a topological subspace of $H\overline{V}_0(\mathbb{C})$ hence the topology of $\mathbb{P}$ is given by the seminorms $\|\cdot\|_{\overline{v}}$, $\overline{v} \in \overline{V}$. Let $U \in \mathcal{U}_0(\mathcal{V}_0P(\mathbb{C})$ be given. We may assume that $U = \Gamma(\cup_{k=1}^{\infty} \varepsilon_k P_k^{\circ})$ where $\varepsilon_k > 0$ is decreasing. In [22, 3.1], Bierstedt, Bonet, Taskinen defined $\overline{v} \in \overline{V}$ such that

$$V := \left\{\, p \in \mathbb{P}\,;\ \|p\|_{\overline{v}} \leqslant 1 \,\right\} \subseteq U$$

and by the above $V \in \mathcal{U}_0(\mathbb{P})$. ■

In the above setting we have

$$\mathcal{V}_0P(\mathbb{C}) \subseteq \mathcal{V}_0H(\mathbb{C}) \subseteq \mathcal{V}H(\mathbb{C})$$

where all inclusions are topological subspaces. Moreover, $\mathcal{V}H(\mathbb{C})$ is regular. Thus, 6.3 implies the following analog of 6.4 for the current setting.

Consequence 7.3. Let $\mathcal{V} \subseteq E$. Then $(P_n^{\circ})_{n\in\mathbb{N}}$ with $P_n^{\circ} := B_n^{\circ} \cap \mathbb{P} = B_n \cap \mathbb{P}$ is a fundamental system of bounded sets in the inductive limit $\mathcal{V}_0P(\mathbb{C})$, where $B_n^{(\circ)}$ denotes the unit ball of the Banach space $H(v_n)_{(0)}(\mathbb{C})$.

Let us now prove the analog of 6.6.

Proposition 7.4. Let $\mathcal{A} \subseteq E$ and assume that E is closed under the formation of finite maxima. Let $\mathcal{A}$ satisfy condition (wQ). Then $(AP)_0(G)$ is bornological.

Proof. We proceed as in 6.6: By Bierstedt, Bonet [18], condition (wQ) implies condition (wQ)* that is

$$\exists\,(n(\sigma))_{\sigma\in\mathbb{N}} \subseteq \mathbb{N} \text{ increasing } \forall\, N\, \exists\, M\, \forall\, K,\, m\, \exists\, S>0,\, k\colon$$
$$\tfrac{1}{a_{M,m}} \leqslant S \max\big(\tfrac{1}{a_{K,k}}, \min_{\sigma=1,\dots,N} \tfrac{1}{a_{\sigma,n(\sigma)}}\big).$$

We fix an absolutely convex and bornivorous set T in $(AP)_0(\mathbb{C})$. Since $(AP)_0(\mathbb{C}) = P(a_{N,n})_0(\mathbb{C})$ holds algebraically for all N, n we may consider T as a subset of the latter space and claim that there exists N such that for each n the ball $P^\circ_{N,n}$ is absorbed by T. We proceed by contradiction and hence assume

$$(\star) \qquad \forall\, M\, \exists\, m(M)\colon P^\circ_{M,m(M)} \text{ is not absorbed by } T.$$

By 6.5, there exists N such that $\cap_{\sigma=1}^N P^\circ_{\sigma,m(\sigma)}$ is absorbed by T. For the sequence $(n(\sigma))_{\sigma\in\mathbb{N}}$ and this N we choose M as in (wQ)*. By $(\star)$ there exists $m(M)$ such that for each K there exists $S_K>0$ and $k(K)$ such that $\frac{1}{a_{M,m(M)}} \leqslant S_K \max\big(\frac{1}{a_{K,k(K)}}, \min_{\sigma=1,\dots,N} \frac{1}{a_{\sigma,n(\sigma)}}\big)$ and thus we get as in the proof of 6.6

$$(\circ) \qquad \forall\, K\colon \tfrac{1}{a_{M,m(M)}} \leqslant S'_K \max\big(\min_{\mu=1,\dots,K} \tfrac{1}{a_{\mu,k(\mu)}}, \min_{\sigma=1,\dots,N} \tfrac{1}{a_{\sigma,n(\sigma)}}\big)$$

with $S'_K := \max_{\mu=1,\dots,K} S_\mu$.

Now we will again make use of the decomposition method based on class $(E)_{C,c}$ to show the following

$$\forall\, K\, \exists\, \tau_K>0\colon P^\circ_{M,m(M)} \subseteq \tau_K \big[\bigcap_{\sigma=1}^{N} P^\circ_{\sigma,n(\sigma)} + \bigcap_{\mu=1}^{K} P^\circ_{\mu,k(\mu)}\big].$$

We fix $K\in\mathbb{N}$. Let $p \in P^\circ_{M,m(M)}$, i.e. $a_{M,m(M)}|p| \leqslant 1$ hence $|p| \leqslant \frac{1}{a_{M,m(M)}}$ and by $(\circ)$ we get the estimate $|p| \leqslant \max\big(\min_{\sigma=1,\dots,N} \frac{S'_K}{a_{\sigma,n(\sigma)}}, \min_{\mu=1,\dots,K} \frac{S'_K}{a_{\mu,k(\mu)}}\big)$ and define $\frac{1}{u_1} := \min_{\sigma=1,\dots,N} \frac{1}{a_{\sigma,n(\sigma)}}$, $\frac{1}{u_2} := \min_{\mu=1,\dots,K} \frac{1}{a_{\mu,k(\mu)}}$ and $\alpha_1 := \alpha_2 := \frac{1}{S'_K}$ and obtain $u_1 = \max_{\sigma=1,\dots,N} a_{\sigma,n(\sigma)}$, $u_2 = \max_{\mu=1,\dots,K} a_{\mu,k(\mu)}$. As in the proof of 7.1 we put $u := \min(\alpha_1 u_1, \alpha_2 u_2)$ and get $\frac{1}{u} = \max(\frac{1}{\alpha_1 u_1}, \frac{1}{\alpha_2 u_2})$ that is by the above $|p| \leqslant \frac{1}{u}$, i.e. $u|p| \leqslant 1$.

$\mathcal{A} \subseteq E$ implies $a_{\sigma,n(\sigma)}(z) = b_{\sigma,n(\sigma)}(|z|)e^{-c|z|}$ and $a_{\mu,k(\mu)}(z) = b_{\mu,k(\mu)}(|z|)e^{-c|z|}$ for $z\in\mathbb{C}$, $1\leqslant \sigma \leqslant N$ and $1 \leqslant \mu \leqslant K$. We put $s_1(|z|) := \max_{\sigma=1,\dots,N} b_{\sigma,n(\sigma)}(|z|)$ and $s_2(|z|) := \max_{\mu=1,\dots,K} b_{\mu,k(\mu)}(|z|)$ and obtain

$$\begin{aligned} u_1(z) &= \max_{\sigma=1,\dots,N} \big(b_{\sigma,n(\sigma)}(|z|)e^{-c|z|}\big) = \big(\max_{\sigma=1,\dots,N} b_{\sigma,n(\sigma)}(|z|)\big)e^{-c|z|} = s_1(|z|)e^{-c|z|},\\ u_2(z) &= \max_{\mu=1,\dots,K} \big(b_{\mu,n(\mu)}(|z|)e^{-c|z|}\big) = \big(\max_{\mu=1,\dots,K} b_{\mu,n(\mu)}(|z|)\big)e^{-c|z|} = s_2(|z|)e^{-c|z|}. \end{aligned}$$

Since we assumed that E is closed under finite maxima, u_1 and $u_2 \in E$ and hence we may apply the results of 7.D to the functions s_1 and s_2. We define $s := \min(\alpha_1 s_1, \alpha_2 s_2)$ and get with the method of the proof of 7.1 (with $s_1(\rho_{J+1}) \leqslant \frac{S'_K}{d} s(\rho_{J+1})$, $s_1(\rho_J) \leqslant \frac{S'_K}{d} s(\rho_J)$ and $s_1(\rho_{J-1}) \leqslant \frac{S'_K}{d} s(\rho_{J-1})$) the inequality $\|p_1\|_{u_1} \leqslant 3D^2 C_1 S'_K d^{-1}$ and analogously $\|p_2\|_{u_2} \leqslant 3D^2 C_2 S'_K d^{-1}$ that is

$$p \in 3D^2 d^{-1} S'_K \big[C_1 \bigcap_{\sigma=1}^{N} B_{\sigma,n(\sigma)} + C_2 \bigcap_{\mu=1}^{K} B_{\mu,k(\mu)}\big] \subseteq \tau_K \big[\bigcap_{\sigma=1}^{N} P^\circ_{\sigma,n(\sigma)} + \bigcap_{\mu=1}^{K} P^\circ_{\mu,k(\mu)}\big]$$

by setting $\tau_K := 3D^2 d^{-1} S'_K \max(C_1, C_2)$. As in the proof of 6.6, statement (B1) of 4.9 is trivial and our claim is exactly the statement (B2). 6.4 provides that $(P^\circ_{N,n})_{n\in\mathbb{N}}$ is a fundamental system of bounded sets in $(\mathcal{A}_N)_0 P(\mathbb{D})$ for each $N \in \mathbb{N}$ and the conclusion follows from 4.10. ∎

As over the unit disc, 6.7 combined with the results above yields immediately the desired result on bornologicity of $(AH)_0(\mathbb{C})$.

Theorem 7.5. Assume $\mathcal{A} \subseteq E$ and assume that E is closed under the formation of finite maxima. Let $\mathcal{A}$ satisfy condition (wQ). Then $(AH)_0(\mathbb{C})$ is barrelled.

7.3 Summary of results

Remark 7.6. Analogously to 6.9, let us summarize the results of sections 5 and 7 in the following schemes, which in particular illustrate again the lines of the proofs for the results and the assumptions needed for each single implication.

$$\begin{tikzcd}
(\mathrm{Q})^\sim_{\mathrm{in}} \arrow[dr, Rightarrow, "\text{i.g.}"] & & & & & & & \\
 & (\mathrm{Q})^\sim_{\mathrm{out}} \arrow[r, Rightarrow, "G=\mathbb{C}", "\mathcal{A}\subseteq E"'] & \mathrm{Proj}^1 \mathcal{A}H=0 \arrow[r, Rightarrow, "\text{i.g.}"] & \begin{array}{c} AH(G) \\ \text{ultra-} \\ \text{bornological} \end{array} \arrow[r, Rightarrow, "\text{i.g.}"] & \begin{array}{c} AH(G) \\ \text{barrelled} \end{array} \arrow[dr, Rightarrow, "\begin{array}{c}\text{balanced}\\ \text{setting}\end{array}"'] & & (\mathrm{wQ})^\sim_{\mathrm{in}} \arrow[r, Rightarrow, "\text{i.g.}"] & (\mathrm{wQ})^\sim_{\mathrm{out}} \\
(\mathrm{Q}) \arrow[ur, Rightarrow, "\text{i.g.}"'] & & & & & \begin{array}{c} (AH)_0(G) \\ \text{barrelled} \end{array} \arrow[ur, Rightarrow, "\begin{array}{c}\text{balanced}\\ \text{setting}\end{array}"'] & &
\end{tikzcd}$$

Figure 3: Class $(\mathrm{E})_{\mathrm{C,c}}$: Scheme of implications for O-growth conditions.

$$\begin{tikzcd}
\mathrm{Proj}^1 \mathcal{A}_0 H=0 \arrow[r, Rightarrow, "\text{i.g.}"] & \begin{array}{c} (AH)_0(G) \\ \text{ultra-} \\ \text{bornological} \end{array} \arrow[r, Rightarrow, "\text{i.g.}"] & \begin{array}{c} (AH)_0(G) \\ \text{barrelled} \end{array} \arrow[r, Rightarrow, "\begin{array}{c}\text{balanced}\\ \text{setting}\end{array}"] & (\mathrm{wQ})^\sim_{\mathrm{in}} \arrow[r, Rightarrow, "\text{i.g.}"] & (\mathrm{wQ})^\sim_{\mathrm{out}} \\
 & & \begin{array}{c} (AP)_0(G) \\ \text{bornological} \end{array} \arrow[u, Rightarrow, "\begin{array}{c} G=\mathbb{D} \\ \mathcal{A}\subseteq W \end{array}"] & & \\
 & & (\mathrm{wQ}) \arrow[u, Rightarrow, "\begin{array}{c} G=\mathbb{C},\ \mathcal{A}\subseteq E,\ E \text{ closed} \\ \text{under finite maxima} \end{array}"] \arrow[uurr, Rightarrow, "\text{i.g.}"'] & &
\end{tikzcd}$$

Figure 4: Class $(\mathrm{E})_{\mathrm{C,c}}$: Scheme of implications for o-growth conditions.

In contrast to the last section it is not clear if the sets $(\mathrm{E})_{C,c}$ are closed under finite maxima in general. Moreover, it is also unclear, if the weights in $(\mathrm{E})_{C,c}$ are essential in general. However, in concrete cases the latter might be true and thus the following versions of our results are useful for applications.

Corollary 7.7. (of 7.1) Let $\mathcal{A}$ be in E and assume that all the weights in $\mathcal{A}$ are essential. Then we have (i)⇒(ii)⇒(iii)⇒(iv)⇒(v), where

(i) $\mathcal{A}$ satisfies condition (Q),

(ii) $\operatorname{Proj}^1 \mathcal{A}H = 0$,

(iii) $AH(\mathbb{C})$ is ultrabornological,

(iv) $AH(\mathbb{C})$ is barrelled,

(v) $\mathcal{A}$ satisfies condition (wQ).

Corollary 7.8. (of 7.5) Let $\mathcal{A}$ be in E and assume that E is closed under finite maxima and that all weights in $\mathcal{A}$ are essential. Then $(AH)_0(\mathbb{C})$ is barrelled if and only if $\mathcal{A}$ satisfies condition (wQ).

8 Another special setting for the unit disc: The condition (LOG)

In this section we present another set of assumptions which allows the decomposition of holomorphic functions defined on the unit disc. The definition of the so-called class (LOG) goes back to Bonet, Engliš, Taskinen [32, 4.1] and was used to prove a projective description for weighted (LB)-spaces of holomorphic functions. Moreover, it was applied by Wolf [88] to characterize weighted (LB)-spaces having the Dual Density Condition (for the latter notion see e.g. the articles of Bierstedt, Bonet [13, 14, 15]).

In this section all the considered weights are defined on the unit disc $\mathbb{D}$ of the complex plane. For every $\kappa \in \mathbb{N}$ we put $r_\kappa := 1 - 2^{-2^\kappa}$, $r_0 := 0$ and $I_\kappa := [r_\kappa, r_{\kappa+1}]$. We say that the sequence $\mathcal{A} = ((a_{N,n})_{N\in\mathbb{N}})_{n\in\mathbb{N}}$ satisfies condition (LOG) if each weight in the sequence is radial and approaches monotonically 0 as $r \nearrow 1$ and there exist constants $0 < a < 1 < A$ such that

(LOG 1) $A \cdot a_{N,n}(r_{\kappa+1}) \geqslant a_{N,n}(r_\kappa)$ and

(LOG 2) $a_{N,n}(r_{\kappa+1}) \leqslant a \cdot a_{N,n}(r_\kappa)$

holds for all N, n and $\kappa \in \mathbb{N}$.

The above assumptions imply that if a sequence $\mathcal{A}$ satisfies condition (LOG) it also satisfies the assumptions of the balanced setting (cf. the remarks at the beginning of section 5). Therefore, we get necessary conditions for the vanishing of Proj^1, ultrabornologicity and barrelledness from 5.4.

As in the previous sections we need in the final step of the proof of 8.2 that the balls $B_{N,n}$ are co-compact and thus we a priori are only able to handle the O-growth case, cf. also 8.3. However, the method of Bonet, Engliš, Taskinen is different from the methods used in the previous sections; it does not involve a decomposition of polynomials.

8.1 Sufficient conditions for the vanishing of $\mathrm{Proj}^1\mathcal{A}H$

In the sequel we will use the following well-known fact, for the sake of completeness we give a proof.

Remark 8.1. Let v be a radial weight which is decreasing on $[0,1[$. Assume that $(r_n)_{n\in\mathbb{N}} \subseteq [0,1[$ is a sequence with $r_n \nearrow 1$ as $n \to \infty$. Let $g \in Hv(\mathbb{D})$ and put $g_n(z) := g(r_n z)$ for $z \in \mathbb{D}$. Then $g_n \to g$ holds w.r.t. the compact open topology.

Proof. We note first that $g_n \in Hv(\mathbb{D})$ holds. For $K \subseteq \mathbb{D}$ compact we select $0 < R < 1$ such that $K \subseteq \overline{B}_R(0)$ and estimate

$$\begin{aligned}\sup_{z\in K}|g(z)-g(r_nz)| &\leqslant \sup_{z\in K}\max_{\xi\in[r_nz,z]}|g'(\xi)||z-r_nz|\\ &\leqslant (1-r_n)\sup_{z\in K}\max_{\xi\in[r_nz,z]}|g'(\xi)|\\ &\leqslant (1-r_n)\sup_{z\in\overline{B}_R(0)}|g'(z)| \stackrel{n\to\infty}{\longrightarrow} 0\end{aligned}$$

which yields the desired co-convergence. ■

The following proof was inspired by the method developed in [32, section 4] which was also used in [88].

Proposition 8.2. Let $\mathcal{A}$ satisfy condition (LOG) and assume that condition $(\mathrm{Q})^{\sim}_{\mathrm{out}}$ is satisfied. Then $\mathrm{Proj}^1\mathcal{A}H = 0$.

Proof. In order to show that $\mathrm{Proj}^1\mathcal{A}H = 0$ we use Braun, Vogt [36, Theorem 8] (which was independently obtained by Frerick, Wengenroth [43]). That is, we have to show condition $(\overline{\mathrm{P}_2})$

$$\forall\, N\ \exists\, M,\ n\ \forall\, K,\ m,\ \varepsilon>0\ \exists\, k,\ S>0:\ B_{M,m}\subseteq \varepsilon B_{N,n}+SB_{K,k}.$$

We denote by $0 < a < 1 < A$ the constants of (LOG 1) and (LOG 2) and put $B := \max\big(\sum_{\kappa=0}^{\infty}a^{\kappa},\ \sup_{\kappa>t+2}2^{-\kappa}A^{\kappa-t}2^{-2^{\kappa-1}}\big)$. Now we put $T := 2A^2(B+A^2)+4(A^2+2B)$.

For given N we select M and n as in $(\mathrm{Q})^{\sim}_{\mathrm{out}}$. For given K, m, $\varepsilon > 0$ we put $\varepsilon' := \frac{\varepsilon}{2T}$ and choose k and $S' > 0$ according to $(\mathrm{Q})^{\sim}_{\mathrm{out}}$ w.r.t. ε' and put $S := 2TS'$. Now we fix an arbitrary $f \in B_{M,m}$. We have $|f| \leqslant \frac{1}{a_{M,m}}$, i.e. with Bierstedt, Bonet, Taskinen [21, Proposition 1.2.(iii)] it follows $|f| \leqslant (\frac{1}{a_{M,m}})^{\sim}$. By the estimate in $(\mathrm{Q})^{\sim}_{\mathrm{out}}$ we obtain $|f| \leqslant \max(\varepsilon'(\frac{1}{a_{N,n}}),\ S'(\frac{1}{a_{K,k}}))^{\sim} \leqslant \max(\frac{\varepsilon'}{a_{N,n}},\ \frac{S}{a_{K,k}})$ where the last estimate follows from Bierstedt, Bonet, Taskinen [21, Proposition 1.2.(i)]. We put $u := \min(\frac{a_{N,n}}{\varepsilon'},\ \frac{a_{K,k}}{S'})$. Hence $\|f\|_u \leqslant 1$. By defining $u_0 := a_{N,n}$, $u_1 := a_{K,k}$, $a_0 := 1/\varepsilon'$ and $a_1 := 1/S'$ we get $u = \min(a_0u_0,\ a_1u_1)$. We put (according to Bonet, Engliš, Taskinen [32, Proof of 4.5]) $f_{r_\kappa}(z) := f(r_\kappa z)$. By 8.1 we have $f_{r_\kappa} \to f$ within the compact open topology.

Since all the weights in $\mathcal{A}$ are non-increasing, this is also true for u. Hence

$$(1) \qquad \inf_{|z|\in I_\kappa} u(z) = u(r_{\kappa+1}) \geqslant u(r_{\kappa+2}) = \inf_{|z|\in I_{\kappa+1}} u(z) \overset{\text{(LOG 1)}}{\geqslant} A^{-2}u(r_\kappa).$$

For every κ in $\mathbb{N}$ we pick $i(\kappa) \in \{0, 1\}$ such that

$$(2) \qquad u(r_\kappa) = a_{i(\kappa)}u_{i(\kappa)}(r_\kappa) = a_{i(\kappa)} \sup_{|z|\in I_\kappa} u_{i(\kappa)}(z).$$

For $\nu \in \mathbb{N}$ and $\ell \in \{0, 1\}$ we define $N_\ell := \{\, \kappa \in \mathbb{N}\, ;\, \kappa \leqslant \nu \text{ and } i(\kappa) = \ell\,\}$. For each $\kappa \geqslant 1$ we put $g_\kappa(z) := f(r_{\kappa+1}z) - f(r_\kappa z)$ and $g_0(z) := f(0)$ and finally for $\ell \in \{0, 1\}$ we define

$$h_\ell := \sum_{\kappa\in N_\ell} g_\kappa.$$

We have

$$\begin{aligned}
(h_0 + h_1 + g_0)(z) &= \sum_{\kappa\in N_1} g_\kappa(z) + \sum_{\kappa\in N_2} g_\kappa(z) + g_0(z)\\
&= \sum_{\substack{\kappa\leqslant\nu\\ i(\kappa)=0}} (f(r_{\kappa+1}z) - f(r_\kappa z)) + \sum_{\substack{\kappa\leqslant\nu\\ i(\kappa)=1}} (f(r_{\kappa+1}z) - f(r_\kappa z)) + g_0(z)\\
&= \sum_{\kappa=0}^{\nu} f(r_{\kappa+1}z) - f(r_\kappa z)\\
&= \sum_{\kappa=1}^{\nu+1} f(r_\kappa z) - \sum_{k=0}^{\nu} f(r_\kappa z)\\
&= f(r_0 z) + f(r_{\nu+1}z) - f(r_0 z)\\
&= f(r_{\nu+1}z)
\end{aligned}$$

for arbitrary z that is $f_{r_{\nu+1}} = g_0 + h_0 + h_1$. For the constant function g_0 we have

$$|g_0(z)| = |f(0)| = |f(r_0)| \leqslant a_{i(0)}^{-1} u_{i(0)}(0)^{-1}$$

that is $a_{N,n}(z)|g_0(z)| \leqslant \varepsilon' \leqslant \frac{\varepsilon}{2}$ (if $i(0) = 0$) or $a_{K,k}(z)|g_0(z)| \leqslant S' \leqslant \frac{S}{2}$ (if $i(0) = 1$).

Now we fix $\ell \in \{0,1\}$, pick $\kappa \in N_\ell$ and estimate $|g_\kappa(z)|$ for different z.

1. Assume first $|z| \geqslant r_{\kappa-1}$ (where we put $r_{\kappa-1} := r_0$ for $\kappa = 0$).
   a. Let $\kappa \geqslant 2$. Then we have

$$\begin{aligned}
|r_\kappa z| = |r_\kappa||z| \geqslant |r_\kappa||r_{\kappa-1}| &= (1 - 2^{-2^\kappa})(1 - 2^{-2^{\kappa-1}})\\
&= 1 - 2^{-2^{\kappa-1}} - 2^{-2^\kappa} + 2^{-2^\kappa} \cdot 2^{-2^{\kappa-1}}\\
&\geqslant 1 - 2^{-2^{\kappa-1}} - 2^{-2^\kappa}\\
&\geqslant 1 - 2^{-2^{\kappa-1}} - 2^{-2^{\kappa-1}}\\
&= 1 - 2 \cdot 2^{-2^{\kappa-1}}
\end{aligned}$$

$$\begin{aligned}&\geqslant 1-2^{-2^{\kappa-2}}\\&=r_{\kappa-2}.\end{aligned}$$

Since $r_\kappa \leqslant r_{\kappa+1}$ and $|z| \leqslant 1$ we get

$$r_{\kappa-2} \leqslant |r_\kappa z| \leqslant |r_{\kappa+1}z| \leqslant r_{\kappa+1}$$

for $\kappa \geqslant 2$. Since $\|f\|_u \leqslant 1$, we have $|f(z)| \leqslant u(z)^{-1}$ on $\mathbb{D}$. Thus we get by the above, since u is non-increasing and by (1)

$$\begin{aligned}|g_\kappa(z)| &\stackrel{\text{dfn}}{=} |f(r_{\kappa+1}) - f(r_\kappa)|\\
&\leqslant |f(r_\kappa z)| + |f(r_{\kappa+1}z)|\\
&\leqslant u(r_\kappa z)^{-1} + u(r_{\kappa+1}z)^{-1}\\
&\leqslant 2 \sup_{r_{\kappa-2}\leqslant r\leqslant r_{\kappa+1}} u(r)^{-1}\\
&= 2 \sup_{r\in I_{\kappa-2}\cup I_{\kappa-1}\cup I_\kappa} u(r)^{-1}\\
&= 2\max\big(\sup_{r\in I_{\kappa-2}} u(r)^{-1},\ \sup_{r\in I_{\kappa-1}} u(r)^{-1},\ \sup_{r\in I_\kappa} u(r)^{-1}\big)\\
&\leqslant 2u(r_{\kappa+1})^{-1}\\
&\leqslant 2A^2u(r_\kappa)^{-1}\\
&= 2A^2a_\ell^{-1}u_\ell(r_\kappa)^{-1}\end{aligned}$$

where the last equality follows since $u(r_\kappa) = a_{i(\kappa)}u_{i(\kappa)}(r_\kappa)$ and $\kappa \in N_\ell$ implies $i(\kappa) = \ell$ (cf. (2)).

b. Let $\kappa = 1$. In this case we have

$$\begin{aligned}|g_1(z)| = |f(r_2z) - f(r_1z)| &\leqslant |f(r_2z)| + |f(r_1z)|\\
&\leqslant u(r_2z)^{-1} + u(r_1z)^{-1}\\
&\leqslant 2 \sup_{r_0\leqslant r\leqslant r_2} u(r)^{-1}\\
&= 2 \sup_{r\in I_0\cup I_1} u(r)^{-1}\\
&= 2\max\big(\sup_{r\in I_0} u(r)^{-1},\ \sup_{r\in I_1} u(r)^{-1}\big)\\
&= 2u(r_2)^{-1}\\
&\stackrel{(1)}{\leqslant} 2A^2u(r_1)^{-1}\\
&= 2A^2a_\ell^{-1}u_\ell(r_1)^{-1}\end{aligned}$$

where the last equality follows as above.

c. Let $\kappa = 0$. Then we have $|g_\kappa(z)| = |f(0)|$ and $\|f\| \leqslant 1$ implies in particular $u(0)|f(0)| \leqslant 1$, i.e.

$$\begin{aligned}|g_\kappa(z)| = |f(0)| \leqslant u(0)^{-1} &= u(r_0)^{-1}\\
&= a_{i(0)}^{-1}u_{i(0)}(r_0)^{-1}\end{aligned}$$

$$\begin{aligned} &\leqslant 2A^2 a_{i(\kappa)}^{-1} u_{i(\kappa)}(r_\kappa)^{-1} \\ &= 2A^2 a_\ell^{-1} u_\ell(r_\kappa)^{-1} \end{aligned}$$

by (2), since $A > 1$ and by our selection $\kappa \in N_\ell$.

To sum up the results of the cases a., b. and c., we have

$$|g_\kappa(z)| \leqslant 2A^2 a_\ell^{-1} u_\ell(r_\kappa)^{-1}$$

for $|z| \geqslant r_{\kappa-1}$ and $\kappa \geqslant 0$.

3. Assume that $\kappa > t+1$ and $|z| \in I_t$, i.e. $r_t \leqslant |z| \leqslant r_{t+1}$. We have $|g_\kappa(z)| = |f(r_\kappa z) - f(r_{\kappa+1} z)|$ by definition. By the mean value theorem there exists ξ between $r_\kappa z$ and $r_{\kappa+1} z$ with

$$|f(r_\kappa z) - f(r_{\kappa+1})| = |f'(\xi)||r_\kappa z - r_{\kappa+1} z| \leqslant |f'(\xi)||r_\kappa - r_{\kappa+1}|.$$

Hence we may estimate

$$\begin{aligned} |g_\kappa(z)| &\leqslant \sup_{|r_\kappa z| \leqslant |\xi| \leqslant |r_{\kappa+1} z|} |f'(\xi)||r_\kappa - r_{\kappa+1}| \\ &\leqslant \sup_{r_\kappa r_t \leqslant |\xi| \leqslant r_{\kappa+1} r_{t+1}} |f'(\xi)| 2^{-2^\kappa}, \end{aligned}$$

since $|r_{\kappa+1} - r_\kappa| = 1 - 2^{-2^{\kappa+1}} - 1 + 2^{-2^\kappa} \leqslant 2^{-2^\kappa}$. $\kappa > t+1$, i.e. $t < \kappa - 1$ implies $|\xi| \leqslant r_{\kappa+1} r_{t+1} < r_{t+1} \leqslant r_\kappa$ and we thus may use the Cauchy formula

$$(3) \qquad |f'(\xi)| \leqslant \tfrac{1}{2\pi} \int_{|\eta| = r_\kappa} \tfrac{|f(\eta)|}{|\eta - \xi|^2} d\eta$$

to estimate $|f'(\xi)|$. We have $|f(\eta)| \leqslant u(\eta)^{-1} = u(r_\kappa)^{-1}$, since $\|f\|_u \leqslant 1$ and u is radial. Now we estimate $\frac{1}{|\eta - \xi|^2}$.

a. Let $\kappa > t+2$. That is, $\kappa \geqslant t+3$, i.e. $t \leqslant \kappa - 3$. Hence $|\xi| \leqslant r_{\kappa+1} r_{t+1} \leqslant r_{\kappa+1} r_{\kappa-2} \leqslant r_{\kappa-2}$. Now, $|\eta - \xi| \geqslant \big||\eta| - |\xi|\big| \geqslant |\eta| - |\xi| \geqslant r_\kappa - r_{\kappa-2} = 1 - 2^{-2^\kappa} - 1 + 2^{-2^{\kappa-2}} = 2^{-2^{\kappa-2}} - 2^{-2^\kappa}$. We claim that $2^{-2^{\kappa-2}} - 2^{-2^\kappa} \geqslant 2^{-1} 2^{-2^{\kappa-2}}$ holds. We clearly have $2^\kappa - 2^{\kappa-2} \geqslant 1$, i.e. $2^\kappa - 1 \geqslant 2^{\kappa-2}$ and thus $2^{2^\kappa - 1} \geqslant 2^{2^{\kappa-2}}$, therefore $2^{1-2^\kappa} \leqslant 2^{-2^{\kappa-2}}$ and thus $-2 \cdot 2^{2^\kappa} = -2^{1-2^\kappa} \geqslant -2^{-2^{\kappa-2}}$. This implies $2 \cdot 2^{-2^{\kappa-2}} - 2 \cdot 2^{-2^\kappa} \geqslant 2 \cdot 2^{-2^{\kappa-2}} - 2^{-2^{\kappa-2}} = 2^{-2^{\kappa-2}}$ which shows the claim. Thus we have $|\eta - \xi| \geqslant 2^{-1} 2^{-2^{\kappa-2}}$ hence $\frac{1}{|\eta - \xi|} \leqslant 2 \cdot 2^{2^{\kappa-2}}$ which yields $\frac{1}{|\eta - \xi|^2} \leqslant 2^2 \cdot 2^{2 \cdot 2^{\kappa-2}} = 4 \cdot 2^{2^{\kappa-1}}$. Now we get

$$|f'(\xi)| \leqslant \tfrac{2\pi r_\kappa}{2\pi} \cdot 4 \cdot 2^{2^{\kappa-1}} u(r_\kappa)^{-1} \leqslant 4 \cdot 2^{2^{\kappa-1}} u(r_\kappa)^{-1}$$

from (3) since $r_\kappa \leqslant 1$ and can continue the estimate of $|g_\kappa(z)|$, i.e.

$$\begin{aligned} |g_\kappa(z)| &\leqslant 4 \cdot 2^{2^{\kappa-1}} 2^{-2^\kappa} u(r_\kappa)^{-1} \\ &= 4 \cdot 2^{2^{\kappa-1} - 2^\kappa} u(r_\kappa)^{-1} \\ &= 4 \cdot 2^{2^{\kappa-1}(1-2^1)} u(r_\kappa)^{-1} \end{aligned}$$

$$= 4 \cdot 2^{-2^{\kappa-1}} u(r_\kappa)^{-1}$$
$$= 4 \cdot 2^{2^{\kappa-1}} a_\ell^{-1} u_\ell(r_\kappa)^{-1}$$

where the last equality is obtained as in the previous cases.

b. Let $\kappa = t+2$. That is $t = \kappa - 2$. Hence $|\xi| \leqslant r_{\kappa+1}r_{t+1} \leqslant r_{\kappa+1}r_{\kappa-1} \leqslant r_{\kappa-1}$. Similar to the above, $|\eta-\xi| \geqslant r_\kappa - r_{\kappa-1} = 1-2^{-2^\kappa}-1+2^{-2^{\kappa-1}} = 2^{-2^{\kappa-1}} - 2^{-2^\kappa}$ and we claim that $2^{-2^{\kappa-1}} - 2^{-2^\kappa} \geqslant 2^{-1}2^{-2^{\kappa-1}}$ holds. We clearly have $2^\kappa - 2^{\kappa-1} \geqslant 1$, i.e. $2^\kappa - 1 \geqslant 2^{\kappa-1}$ and thus $2^{2^\kappa-1} \geqslant 2^{2^{\kappa-1}}$, therefore $2^{1-2^\kappa} \leqslant 2^{-2^{\kappa-1}}$ and thus $-2\cdot 2^{2^\kappa} = -2^{1-2^\kappa} \geqslant -2^{-2^{\kappa-1}}$. This implies $2\cdot 2^{-2^{\kappa-1}} - 2\cdot 2^{-2^\kappa} \geqslant 2\cdot 2^{-2^{\kappa-1}} - 2^{-2^{\kappa-1}} = 2^{-2^{\kappa-1}}$ which shows the claim. Similar to the above, we get $|\eta-\xi| \geqslant 2^{-1}2^{-2^{\kappa-1}}$ hence $\frac{1}{|\eta-\xi|} \leqslant 2\cdot 2^{2^{\kappa-1}}$ which yields $\frac{1}{|\eta-\xi|^2} \leqslant 2^2 \cdot 2^{2\cdot 2^{\kappa-1}} = 4\cdot 2^{2^\kappa}$. We get

$$|f'(\xi)| \leqslant \tfrac{2\pi r_\kappa}{2\pi} \cdot 4 \cdot 2^{2^\kappa} u(r_\kappa)^{-1} \leqslant 4\cdot 2^{2^\kappa} u(r_\kappa)^{-1}$$

from (3) since $r_\kappa \leqslant 1$ and can also in this case continue the estimate of $|g_\kappa(z)|$, i.e.

$$|g_\kappa(z)| \leqslant 4\cdot 2^{2^\kappa} u(r_\kappa)^{-1} 2^{-2^\kappa} = 4u(r_\kappa)^{-1} = 4a_\ell^{-1}u_\ell(r_\kappa)^{-1}$$

by the choice $\kappa \in N_\ell$.

Now we use (LOG 1) $(\kappa - t)-$times to obtain

$$u_\ell(r_t) \leqslant Au_\ell(r_{\kappa+1}) \leqslant A^2 u_\ell(r_{t+2}) \leqslant \cdots \leqslant A^{\kappa-t}u_\ell(r_{t+\kappa-t}) \leqslant A^{\kappa-t}u_\ell(r_\kappa).$$

Since $|z| \geqslant r_t$, u_ℓ is radial and decreasing for $r \nearrow 1$ we have $u_\ell(r_t) \geqslant u_\ell(z)$ and thus we get $u_\ell(z) \leqslant u_\ell(r_t) \leqslant A^{\kappa-t}u_\ell(r_\kappa)$, which finally yields $u_\ell(r_\kappa)^{-1} \leqslant A^{\kappa-t}u_\ell(z)^{-1}$. We continue the estimates in a. and b.

c. Let $\kappa > t+2$. From the latter and our estimate in a. we get $|g_\kappa(z)| \leqslant 4a_\ell^{-1}u_\ell(z)^{-1}A^{\kappa-t}2^{-2^{\kappa-1}}$. By our selection of B we get $A^{\kappa-t}2^{-2^{\kappa-1}} \leqslant B2^{-\kappa}$ and therefore $|g_\kappa(z)| \leqslant 4\cdot 2^{-\kappa}Ba_\ell^{-1}u_\ell(z)^{-1}$.

d. Let $\kappa = t+2$. Then the above yields $|g_\kappa(z)| \leqslant 4a_\ell^{-1}u_\ell(z)^{-1}A^2$.

To sum up the results of 2., we have

$$|g_\kappa(z)| \leqslant 4a_\ell^{-1}u_\ell(z)^{-1} \begin{cases} 2^{-\kappa}B & \text{if } \kappa > t+2 \\ A^2 & \text{if } \kappa = t+2. \end{cases}$$

for $|z| \in I_t$ and κ as indicated above.

To complete the proof, let now $z \in \mathbb{D}$ be arbitrary. We select $t \in \mathbb{N}$ such that $|z| \in I_t = [r_t, r_{t+1}]$. Then

$$|h_\ell(z)| \overset{\text{dfn}}{=} \Big|\sum_{\kappa\in N_\ell} g_\kappa(z)\Big| \leqslant \sum_{\substack{\kappa\in N_\ell \\ \kappa\leqslant t+1}} |g_\kappa(z)| + \sum_{\substack{\kappa\in N_\ell \\ \kappa>t+1}} |g_\kappa(z)| =: G_\ell(z) + H_\ell(z).$$

(i) Consider $G_\ell(z)$, that is all occuring κ satisfy $0 \leqslant \kappa \leqslant t+1$ and $\kappa \in N_\ell$. Thus we have $\kappa - 1 \leqslant t$, hence $|z| \geqslant r_t \geqslant r_{\kappa-1}$ (remember that we defined

$r_{-1} := r_0 = 0$). By the estimate obtained in 1. we therefore have

$$G_\ell(z) \stackrel{\text{dfn}}{=} \sum_{\substack{\kappa \in N_\ell \\ \kappa \leqslant t+1}} |g_\kappa(z)| \leqslant \sum_{\substack{\kappa \in N_\ell \\ \kappa \leqslant t+1}} 2A^2 a_\ell^{-1} u_\ell(r_\kappa)^{-1}.$$

(LOG 2) implies $u_\ell(r_{\kappa+1}) \leqslant a u_\ell(r_\kappa)$, i.e. $u_\ell(r_\kappa)^{-1} \leqslant a u_\ell(r_{\kappa+1})^{-1}$ for arbitrary κ. Iterating this estimate $t-\kappa$ times for a fixed $\kappa \leqslant t$ we get

$$u_\ell(r_\kappa)^{-1} \leqslant a u_\ell(r_{\kappa+1})^{-1} \leqslant \cdots \leqslant a^{t-\kappa} u_\ell(r_{\kappa+t-\kappa})^{-1} = a^{t-\kappa} u_\ell(r_t)^{-1}.$$

With the latter we may estimate

$$\begin{aligned}
\sum_{\substack{\kappa \in N_\ell \\ \kappa \leqslant t+1}} 2A^2 a_\ell^{-1} u_\ell(r_\kappa)^{-1} &\leqslant \sum_{\kappa \leqslant t+1} 2A^2 a_\ell^{-1} u_\ell(r_\kappa)^{-1} \\
&= 2A^2 a_\ell^{-1} \Big(\sum_{\kappa=0}^{t} u_\ell(r_\kappa)^{-1} + u_\ell(r_{t+1})^{-1} \Big) \\
&\leqslant 2A^2 a_\ell^{-1} \Big(\sum_{\kappa=0}^{t} a^{t-\kappa} u_\ell(r_t)^{-1} + A^2 u_\ell(r_t)^{-1} \Big) \\
&= 2A^2 a_\ell^{-1} u_\ell(r_t)^{-1} \Big(\sum_{\sigma=0}^{t} a^\sigma + A^2 \Big) \\
&\leqslant 2A^2 a_\ell^{-1} u_\ell(r_t)^{-1} \Big(\sum_{\sigma=0}^{\infty} a^\sigma + A^2 \Big) \\
&\leqslant 2A^2 (B + A^2) a_\ell^{-1} u_\ell(z)^{-1}
\end{aligned}$$

where we used that $B > \sum_{\kappa \in \mathbb{N}} a^\kappa$, that u_ℓ is radial and decreasing for $r \nearrow 1$ and $|z| \geqslant r_t$, whence $u_\ell(r_t)^{-1} \leqslant u_\ell(z)^{-1}$. Thus we have

$$G_\ell(z) \leqslant 2A^2 (B + A^2) a_\ell^{-1} u_\ell(z)^{-1}.$$

(ii) Consider $H_\ell(z)$. Then all the occuring κ satisfy $\kappa > t+1$ and $\kappa \in N_\ell$. By the estimates in 2. we obtain

$$\begin{aligned}
H_\ell(z) \stackrel{\text{dfn}}{=} \sum_{\substack{\kappa \in N_\ell \\ \kappa > t+1}} |g_\kappa(z)| &= \delta_{i(t+2),\ell}\, |g_{t+2}| + \sum_{\substack{\kappa \in N_\ell \\ \kappa > t+2}} |g_\kappa(z)| \\
&\leqslant 4a_\ell^{-1} u_\ell(z)^{-1} A^2 + \sum_{\substack{\kappa \in N_\ell \\ \kappa > t+2}} 4 \cdot 2^{-\kappa} B a_\ell^{-1} u_\ell(z)^{-1} \\
&\leqslant \big(4A^2 + 4B \sum_{\kappa=0}^{\infty} 2^{-\kappa} \big) a_\ell^{-1} u_\ell(z)^{-1} \\
&= 4(A^2 + 2B) a_\ell^{-1} u_\ell(z)^{-1},
\end{aligned}$$

where δ denotes the Kronecker symbol.

Combining the estimates in (i) and (ii) we obtain

$$|h_\ell(z)| = G_\ell(z) + H_\ell(z) \leqslant (2A^2(B+A^2)+4(A^2+2B))a_\ell^{-1}u_\ell(z)^{-1} = Ta_\ell^{-1}u_\ell(z)^{-1}$$

that is

$$u_\ell(z)|h_\ell(z)| \leqslant Ta_\ell^{-1}$$

for each $z \in \mathbb{D}$ and $\ell = 0$, 1. By the definition of u_ℓ and a_ℓ this means

$$a_{N,n}(z)|h_0(z)| = u_0(z)|h_0(z)| \leqslant T\varepsilon' \leqslant \tfrac{\varepsilon}{2}$$

and

$$a_{K,k}(z)|h_1(z)| = u_1(z)|h_1(z)| \leqslant TS' \leqslant \tfrac{S}{2}$$

for each $z \in \mathbb{D}$. Hence $h_0 \in \frac{\varepsilon}{2}B_{N,n}$ by the first estimate and $h_1 \in \frac{S}{2}B_{K,k}$ by the second estimate. This yields

$$\begin{aligned} f_{r_{\nu+1}} &= g_0 + h_0 + h_1 \\ &\in \tfrac{\varepsilon}{2}B_{N,n} + \tfrac{S}{2}B_{K,k} + \tfrac{\varepsilon}{2}B_{N,n} + \tfrac{S}{2}B_{K,k} \\ &\subseteq \varepsilon B_{N,n} + SB_{K,k}. \end{aligned}$$

Since $\varepsilon B_{N,n} + SB_{K,k}$ is co-compact, $f \in \varepsilon B_{N,n} + SB_{K,k}$ follows from 8.1 and we are done. ■

Let us extend the remarks we made previous to 8.1.

Remark 8.3. (a) If in the situation of 8.1 $g \in Hv_0(\mathbb{D})$ holds, then $g_{r_n} \to g$ holds also in $Hv_0(\mathbb{D})$ and thus in each step $(\mathcal{A}_N)_0H(\mathbb{D})$ of the projective limit $(AH)_0(\mathbb{D})$. This is also a well known fact; we include a proof for the sake of completeness.

(b) [32, 4.5] implies that $(A_N)_0(\mathbb{D}) \subseteq H(\overline{V}_N)_0(\mathbb{D})$ is a topological subspace and hence by [27, 1.3.(a)] also $(\mathcal{A}_N)_0(\mathbb{D}) \subseteq (\mathcal{A}_N)_0C(\mathbb{D})$ is a topological subspace for each N which finally yields that $(AH)_0(\mathbb{D})$ is a topological subspace of the corresponding (PLB)-space $(AC)_0(\mathbb{D})$ of continuous functions, if we assume that $\mathcal{A}$ satisfies (LOG).

(c) It seems not to be possible to apply the method studied in this section to the o-growth case although it yields an approximation within the weighted inductive limits, see (a). As in the preceeding sections the lack of compactness of the unit balls (cf. the remarks at the beginning of section 6.2) anticipates the latter.

Proof. (a) We showed already the co-convergence. Let now $\varepsilon > 0$ be given. Since $g \in Hv_0(\mathbb{D})$ there exists $0 < R_0 < 1$ such that $v(z)|g(z)| \leqslant \frac{\varepsilon}{3}$ for each $|z| \geqslant R_0$. We select $0 < R_0 < R_1 < 1$. Then in particular $\sup_{|z|\geqslant R_1} v(z)|g(z)| \leqslant \frac{\varepsilon}{3}$. By the above we may select N such that $\sup_{|z|\leqslant R_1} v(z)|g(z) - g(r_nz)| \leqslant \frac{\varepsilon}{3}$ for $n \geqslant N$. By increasing N we may assume that $r_nR_1 \geqslant R_0$ for $n \geqslant N$. Now we get

$$\begin{aligned} \sup_{z\in\mathbb{D}} v(z)|g(z) - g(r_nz)| &\leqslant \sup_{|z|\leqslant R_1} v(z)|g(z) - g(r_nz)| + \sup_{|z|\geqslant R_1} v(z)|g(z) - g(r_nz)| \\ &\leqslant \tfrac{\varepsilon}{3} + \sup_{|z|\geqslant R_1} v(z)|g(z)| + \sup_{|z|\geqslant R_1} v(z)|g(r_nz)| \end{aligned}$$

$$
\begin{aligned}
&\leqslant \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \sup_{|\xi| \geqslant r_n R_1} v(\xi)|g(\xi)| \\
&\leqslant \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \sup_{|\xi| \geqslant R_0} v(\xi)|g(\xi)| \\
&\leqslant \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} + \tfrac{\varepsilon}{3} = \varepsilon
\end{aligned}
$$

for $n \geqslant N$. ■

8.2 Summary of results

Remark 8.4. Let us summarize the results of sections 5 and 8 in the following scheme.

$$
\begin{tikzcd}
(\mathrm{Q})^{\sim}_{\mathrm{in}} \arrow[dr, Rightarrow, "\text{i.g.}"] & & & & & & & \\
& (\mathrm{Q})^{\sim}_{\mathrm{out}} \arrow[r, Rightarrow, "G=\mathbb{D}", "\substack{\mathcal{A}\text{ sat.}\\(\mathrm{LOG})}"'] & \operatorname{Proj}{}^1\mathcal{A}H=0 \arrow[r, Rightarrow, "\text{i.g.}"] & \substack{\mathcal{A}H(G)\\ \text{ultra-}\\ \text{bornological}} \arrow[r, Rightarrow, "\text{i.g.}"] & \substack{\mathcal{A}H(G)\\ \text{barrelled}} \arrow[dr, Rightarrow, "\substack{\text{balanced}\\ \text{setting}}"'] & & (\mathrm{wQ})^{\sim}_{\mathrm{in}} \arrow[r, Rightarrow, "\text{i.g.}"] & (\mathrm{wQ})^{\sim}_{\mathrm{out}} \\
(\mathrm{Q}) \arrow[ur, Rightarrow, "\text{i.g.}"'] & & & & & \substack{(\mathcal{A}H)_0(G)\\ \text{barrelled}} \arrow[ur, Rightarrow, "\substack{\text{balanced}\\ \text{setting}}"'] & &
\end{tikzcd}
$$

Figure 5: Condition (LOG): Scheme of implications.

As in section 7.3 we state the following result for the case of essential weights.

Corollary 8.5. Assume that $\mathcal{A}$ satisfies (LOG) and that all weights in $\mathcal{A}$ are essential. Then we have (i)⇒(ii)⇒(iii)⇒(iv)⇒(v), where

(i) $\mathcal{A}$ satisfies condition (Q),
(ii) $\operatorname{Proj}^1 \mathcal{A}_0 H = 0$,
(iii) $\mathcal{A}_0 H(\mathbb{D})$ is ultrabornological,
(iv) $\mathcal{A}_0 H(\mathbb{D})$ is barrelled,
(v) $\mathcal{A}$ satisfies condition (wQ),

9 Projective limits of (DFN)-spaces of entire functions

For the following definitions we refer to Meise [56], Berenstein, Taylor [5, 6, 7] and the book of Berenstein, Gay [4]. In the references just mentioned the following definitions and the results we will quote deal with the space $\mathbb{C}^d$ for $d \geqslant 1$. Since we finally will only investigate weighted (PLB)-spaces of holomorphic functions over the plane, we will restrict ourselves to $d = 1$ right from the beginning. $p\colon \mathbb{C} \to \mathbb{R}_{\geqslant 0}$ is said to be a *weight function*, if it has the following properties.

(DFN 1) p is continuous and (pluri)subharmonic.

(DFN 2) $\log(1 + |z|^2) = O(p(z))$ for $|z| \to \infty$.

(DFN 3) $\exists\, C \geqslant 1\ \forall\, w \in \mathbb{C}$: $\sup_{|z-w|\leqslant 1} p(z) \leqslant C \inf_{|z-w|\leqslant 1} p(z) + C$.

We consider a double sequence $\mathcal{P} = ((p_{N,n})_{N\in\mathbb{N}})_{n\in\mathbb{N}}$ of weight functions on $\mathbb{C}$ with the following properties.

(DFN 4) $\forall\, N,\, n$: $p_{N+1,n} \leqslant p_{N,n} \leqslant p_{N,n+1}$.

(DFN 5) $\forall\, N,\, n\ \exists\, l,\ L \geqslant 0$: $2p_{N,n} \leqslant p_{N,l} + L$.

Now we put $\mathcal{A} = \exp(-\mathcal{P})$, i.e. $a_{N,n}(z) = \exp(-p_{N,n}(z))$ for $z \in \mathbb{C}$; as usual we call the members of $\mathcal{A}$ *weights*. Following Meise [56, 2.1 – 2.3], the above means that each sequence $\mathcal{P}_N = (p_{N,n})_{n\in\mathbb{N}}$ is a *weight system* in his notation and hence, the steps $\mathcal{A}_N H(\mathbb{C})$ are all (LB)-spaces of the type $A_{\mathcal{P}_N}(\mathbb{C})$, considered by Meise. Moreover, the special case $p_{N,n} = np_N$ with any decreasing sequence $(p_N)_{N\in\mathbb{N}}$ of weight functions yields as steps exactly the spaces $A_{p_N}(\mathbb{C})$ considered e.g. in Berenstein, Gay [4, Chapter 2]. Clearly, $\mathcal{A}$ consists of radial weights if and only if the same is true for $\mathcal{P}$ – and we will assume this for the whole section.

By Meise [56, 2.4] we know that the steps $\mathcal{A}_N H(\mathbb{C})$ are (DFN)-spaces, i.e. strong duals of nuclear Fréchet spaces. In particular, the $\mathcal{A}_N H(\mathbb{C})$ are nuclear, complete and reflexive. In addition, condition (Σ) which we introduced in 5.4 is always satisfied as we show in the following lemma.

Lemma 9.1. Let $\mathcal{A} = \exp(-\mathcal{P})$ and $\mathcal{P}$ be as above. Then $\mathcal{A}$ satisfies condition (Σ).

Proof. For given N we select $K := N$. For given k we select n and $L \geqslant 0$ as in (DFN 5), i.e. such that $2p_{N,k} \leqslant p_{N,n} + L$ that is $p_{N,n} \geqslant 2p_{N,k} - L$. For $r \geqslant 0$ we have

$$\tfrac{a_{N,n}(r)}{a_{K,k}(r)} = \tfrac{e^{-p_{N,n}(r)}}{e^{-p_{N,k}(r)}} = e^{p_{N,k}(r)-p_{N,n}(r)} \leqslant e^{p_{N,k}(r)-(2p_{N,k}(r)-L)} = e^{L-p_{N,k}(r)}.$$

Since $p_{N,k}(r) \to \infty$ for $r \to \infty$ by (DFN 2), $\frac{a_{N,n}(r)}{a_{K,k}(r)} \to 0$ holds for $r \to \infty$. That is, $\frac{a_{N,n}}{a_{K,k}}$ vanishes at ∞ on $\mathbb{C}$. ■

At the beginning of section 5 we described the assumptions of the balanced setting: The domain G has to be balanced, the weights have to be radial, the Banach space topologies have to be stronger than co and the polynomials have to be contained in all the considered spaces. We mentioned in section 5 that in this case the assumption concerning the polynomials means exactly that each weight $a_{N,n}$ is rapidly decreasing at ∞ (cf. the remark in [20, previous to 1.2]). Hence we get the following.

Lemma 9.2. Assume that $\log(1 + r^2) = o(p(r))$ for $r \to \infty$ holds for each $p \in \mathcal{P}$. Then the assumptions of the balanced setting apply to the space $AH(\mathbb{C})$.

Proof. Let $p \in \mathcal{P}$. It is enough to check that $a(z) := \exp(-p(z))$ is rapidly decreasing at ∞. Let $j \in \mathbb{N}$ be given. Since p is radial we have to show that $r^j a(r) \to 0$ for $r \to \infty$. We choose $\varepsilon > 0$ such that $\varepsilon \leqslant \frac{1}{j}$. Then there exists $R > 0$ such that

$\log(1+r^2) \leqslant \varepsilon p(r) \leqslant \frac{1}{j}p(r)$ and hence $-p(r) \leqslant -j\log(1+r^2) = \log((1+r^2)^{-j})$ holds for $r \geqslant R$. We get

$$r^j a(r) = r^j \exp(-p(r)) \leqslant r^j \exp(\log((1+r^2)^{-j})) = \tfrac{r^j}{(1+r^2)^j}$$

for $r \geqslant R$ and hence $r^j a(r) \to 0$ for $r \to \infty$ as desired. ∎

Now, the results of section 5.4 imply immediately that the spectra $\mathcal{A}H$ and $\mathcal{A}_0H$ are equivalent, $\operatorname{Proj}^1 \mathcal{A}H = 0$ if and only if $\operatorname{Proj}^1 \mathcal{A}_0H = 0$ and that the spaces $(AH)_0(\mathbb{C})$ and $AH(\mathbb{C})$ are equal algebraically and topologically, if we assume that $\log(1+|z|^2) = o(p_{N,1}(|z|))$ for $|z| \to \infty$ holds for each N. Moreover, for $\mathcal{A}$, (Q) and (wQ) are equivalent in this case.

In the sequel we will use three different (but related) methods to obtain sufficient conditions for the vanishing of $\operatorname{Proj}^1 \mathcal{A}H$. All methods will yield a sequence space representation of the space $AH(\mathbb{C})$, i.e. we represent $AH(\mathbb{C})$ as $\operatorname{proj}_N \operatorname{ind}_n \ell^\infty(b_{N,n})$ where the $b_{N,n}$ are the following.

1. In the first approach we use the representation arising from Meise [56, Proposition 2.8]; $b_{N,n}$ will be defined by some integral.
2. In the second approach we use methods of Domański, Vogt [41] to get $b_{N,n}$ which are defined as the weighted sup-norms (w.r.t. the original weights $a_{N,n}$) of the monomials. This approach uses the theory of (equicontinuous) bases.
3. Finally, we reformulate the latter in terms of the Young conjugates $\varphi^\star$.

9.1 The space $A_{\mathbb{P}}(\mathbb{C})$ – Summary of known and some supplementary results

Before we start with the "program" sketched above we collect certain facts on the steps of the (PLB)-spaces under consideration; for the sake of simplicity we stick to the notation of Meise (cf. [56, Definition 2.3]), which is the following.

For a weight function p on $\mathbb{C}$ we consider the following spaces.

$$H_p^\infty(\mathbb{C}) := \big\{\, f \in H(\mathbb{C})\,;\ \|f\|_{p,\infty} := \sup_{z\in\mathbb{C}} |f(z)|e^{-p(z)} < \infty \,\big\}$$

$$H_p^2(\mathbb{C}) := \big\{\, f \in H(\mathbb{C})\,;\ \|f\|_{p,2} := \Big(\int_{\mathbb{C}} |f(z)|^2 e^{-2p(z)}\, dm(z)\Big)^{1/2} < \infty \,\big\}$$

where m denotes the Lebesque measure on $\mathbb{C} = \mathbb{R}^2$. Let $\mathbb{P} = (p_n)_{n\in\mathbb{N}}$ be a sequence of weight functions (that is (DFN 1)-(DFN 3) holds) such that (cf. (DFN 4) and (DFN 5)) $p_n \leqslant p_{n+1}$ holds for all n and such that for each n there exists l and $L > 0$ such that $2p_n \leqslant p_l + L$. Then we put

$$A_{\mathbb{P}}(\mathbb{C}) := \operatorname{ind}_n H_{p_n}^\infty(\mathbb{C}).$$

Meise [56, Proposition 2.4.(c)] stated (without proof) the following. For the sake of completeness we will give a proof.

Lemma 9.3. In the situation of this section we have $A_{\mathbb{P}}(\mathbb{C}) = \operatorname{ind}_n H^2_{p_n}(\mathbb{C})$. In particular, the associated inductive spectra are equivalent.

Proof. We show

(a) $\forall\, k\ \exists\, m > k\colon H^2_{p_k}(\mathbb{C}) \subseteq H^\infty_{p_m}(\mathbb{C})$ with continuous inclusion and
(b) $\forall\, k\ \exists\, m > k\colon H^\infty_{p_k}(\mathbb{C}) \subseteq H^2_{p_m}(\mathbb{C})$ with continuous inclusion.

In order to show (a) we need the following well-known trick: Let $r > 0$, $0 < \rho \leqslant r$, $w \in \mathbb{C}$ and $g \in A_{\mathbb{P}}(\mathbb{C})$ be given. We have

$$g(w) = \tfrac{1}{2\pi i}\int_{\partial\overline{B}_\rho(w)} \tfrac{g(z)}{z-w}dz = \tfrac{1}{2\pi}\int_0^{2\pi} \tfrac{g(w+\rho e^{it})}{\rho e^{it}}\rho e^{it}dt = \tfrac{1}{2\pi}\int_0^{2\pi} g(w+\rho e^{it})dt$$

and hence

$$\begin{aligned} g(w)\tfrac{r^2}{2} &= \int_0^r g(w)\rho d\rho \\ &= \tfrac{1}{2\pi}\int_0^r\int_0^{2\pi} g(w+\rho(\cos t + i\sin t))\rho dt d\rho \\ &= \tfrac{1}{2\pi}\int_{\overline{B}_r(w)} g(z)dm(z) \end{aligned}$$

which finally implies

$$(\star)\qquad g(w) = \tfrac{1}{\pi r^2}\int_{\overline{B}_r(w)} g(z)dm(z).$$

(a) Let $k \in \mathbb{N}$ be given. We select $l > k$ and $L > 0$ such that $2p_k \leqslant p_l + L$ that is $p_k \leqslant \frac{1}{2}(p_l + L)$. Let $f \in H^\infty_{p_k}(\mathbb{C})$ and $w \in \mathbb{C}$ be given. We put $g := f^2$ and $r := \exp(-p_l(w))$ in the formula $(\star)$ and may thus compute

$$\begin{aligned} |f(w)|^2 = |f(w)^2| &= \big|\tfrac{e^{2p_l(w)}}{\pi}\int_{|w-z|\leqslant e^{-p_l(w)}} f(z)^2 dm(z)\big| \\ &\leqslant \tfrac{e^{2p_l(w)}}{\pi}\int_{|w-z|\leqslant e^{-p_l(w)}} |f(z)|^2 dm(z) \\ &= \tfrac{e^{2p_l(w)}}{\pi}\int_{|w-z|\leqslant e^{-p_l(w)}} |f(z)|^2 e^{-2p_k(z)}e^{2p_k(z)} dm(z) \\ &\leqslant \tfrac{e^{2p_l(w)}}{\pi}\sup_{|w-z|\leqslant e^{-p_l(w)}} e^{2p_k(z)}\int_{|w-z|\leqslant e^{-p_l(w)}} |f(z)|^2 e^{-2p_k(z)} dm(z) \\ &\leqslant \tfrac{e^{2p_l(w)}}{\pi}\sup_{|w-z|\leqslant 1} e^{2p_k(z)}\int_{\mathbb{C}} |f(z)|^2 e^{-2p_k(z)} dm(z) \\ &\overset{(\circ)}{\leqslant} \tfrac{e^{2p_l(w)}}{\pi} e^{Cp_k(w)+C}\|f\|^2_{p_k,2} \end{aligned}$$

$$\leqslant \tfrac{1}{\pi}e^{2p_l(w)+\frac{C}{2}(p_l(w)+L)+C}\|f\|^2_{p_k,2}$$
$$= \tfrac{1}{\pi}e^{C(\frac{L}{2}+1)}e^{(2+\frac{C}{2})p_l(w)}\|f\|^2_{p_k,2}$$

where we used in ($\circ$) that by (DFN 3) there exists a constant $C \geqslant 1$ – which is independent of w – such that $\sup_{|w-z|\leqslant 1} p_k(z) \leqslant C\inf_{|w-z|\leqslant 1} p_k(z) + C \leqslant Cp_k(w) + C$. Now we put $D := \frac{1}{\pi}e^{C(\frac{L}{2}+1)}$ and obtain $|f(w)|^2e^{-(2+\frac{C}{2})p_l(w)} \leqslant D\|f\|^2_{p_k,2}$ and hence $|f(w)|e^{-(1+\frac{C}{4})p_l(w)} \leqslant \sqrt{D}\|f\|_{p_k,2}$.

We select $N \in \mathbb{N}$ such that $(1+\frac{C}{4}) \leqslant 2^N$. Then there exist (by iterating the condition we used already) m and $M > 0$ such that $2^Np_l \leqslant p_m + M$ and we obtain $(1+\frac{C}{4})p_l(w) \leqslant 2^Np_l(w) \leqslant p_m(w) + M$ that is $-(1+\frac{C}{4})p_l(w) \geqslant -(p_m(w)+M)$ and thus $\exp(-(1+\frac{C}{4})p_l(w)) \geqslant \exp(-(p_m(w)+M)) = \exp(-p_m(w))\exp(-M)$ which implies $\exp(-p_m(w)) \leqslant \exp(M)\exp(-(1+\frac{C}{4})p_l(w))$.

Combining the two estimates we get

$$e^{-p_m(w)}|f(w)| \leqslant e^Me^{-(1+\frac{C}{4})p_l(w)}|f(w)| \leqslant e^M\sqrt{D}\|f\|_{p_k,2}$$

and thus (for arbitrary w) $\|f\|_{p_m,\infty} = \sup_{w\in\mathbb{C}} e^{-p_m(w)}|f(w)| \leqslant e^M\sqrt{D}\|f\|_{p_k,2}$ which shows the desired inclusion and its continuity.

(b) Let $k \in \mathbb{N}$ and $f \in H^2_{p_k}(\mathbb{C})$ be given. We select m and $M > 0$ such that $2p_k \leqslant p_m + M$ that is $p_m \geqslant 2p_k - M$ and hence $-p_m \leqslant -2p_k + M$. We compute

$$\begin{aligned}
\|f\|^2_{p_m,2} &= \int_{\mathbb{C}} |f(z)|^2e^{-2p_m(z)}dm(z)\\
&= \int_{\mathbb{C}} |f(z)|^2e^{-2p_k(z)}e^{2(p_k(z)-p_m(z))}dm(z)\\
&\leqslant \|f\|^2_{p_k,\infty}\int_{\mathbb{C}} e^{2(p_k(z)-p_m(z))}dm(z)\\
&\leqslant \|f\|^2_{p_k,\infty}\int_{\mathbb{C}} e^{2(p_k(z)-2p_k(z)+M)}dm(z)\\
&= e^{2M}\|f\|^2_{p_k,\infty}\int_{\mathbb{C}} e^{-2p_k(z)}dm(z) =: (\circ).
\end{aligned}$$

By (DFN 2) there exists $D \geqslant 1$ and $R \geqslant 0$ such that $\log(1+|z|^2) \leqslant Dp_m(z) + R$ and hence $1+|z|^2 \leqslant \exp(R)\exp(Dp_m(z))$ which implies $\exp(-Dp_m(z)) \leqslant \frac{\exp(R)}{1+|z|^2}$ for each $z \in \mathbb{C}$. Hence we get

$$\begin{aligned}
(\circ) &\leqslant e^{2M}\|f\|^2_{p_k,\infty}\int_{\mathbb{C}} \big(e^{-Dp_k(z)}\big)^2dm(z)\\
&\leqslant e^{2M}\|f\|^2_{p_k,\infty}\int_{\mathbb{C}} \big(\tfrac{e^R}{1+|z|^2}\big)^2dm(z)\\
&= e^{2(M+R)}\int_{\mathbb{C}} \tfrac{1}{(1+|z|^2)^2}dm(z)\ \|f\|^2_{p_k,\infty}.
\end{aligned}$$

Since the integral is finite the estimate yields the desired continuous inclusion. ■

The result 9.5 is also due to Meise [56, Proposition 2.8]. Since in the next section we want to form projective limits of spaces of type $A_{\mathbb{P}}(\mathbb{C})$ we need to know if the isomorphism in Meise's result will yield an isomorphism of these projective limits. Hence we precise the formulation of [56, Proposition 2.8] in this direction; the isomorphism is given in Meise's proof.

We use the following well-known notation. Let $B = (b_{j;n})_{j,n\in\mathbb{N}}$ be a Köthe matrix. We identify its entries $b_{j;n}$ with the maps $b_n(j) = b_{j;n}$ and consider the Köthe coechelon spaces of order 2 and ∞ that is

$$k^2(B) = \operatorname{ind}_n \ell^2(b_n) \;\text{ and }\; k^\infty(B) = \operatorname{ind}_n \ell^\infty(b_n)$$

where for $b = (b_j)_{j\in\mathbb{N}}$

$$\ell^2(b) := \big\{\, z = (z_j)_{j\in\mathbb{N}}\,;\, |z|_{b,2} := \big(\sum_{j=0}^{\infty}(|z_j|b_j^{-1})^2\big)^{1/2} < \infty \,\big\} \quad \text{and}$$

$$\ell^\infty(b) := \big\{\, z = (z_j)_{j\in\mathbb{N}}\,;\, |z|_{b,\infty} := \sup_{j\in\mathbb{N}} |z_j|b_j^{-1} < \infty \,\big\}.$$

For the proof of 9.5 we need the following lemma.

Lemma 9.4. Let p be a weight function and assume that $\log(1+|z|^2) = o(p(z))$ holds. Let

$$b_j := \big(2\pi \int_0^\infty r^{2j+1} e^{-2p(r)} dr\big)^{-1/2}.$$

Then the map $T\colon H_p^2(\mathbb{C}) \to \ell^2(b)$, $T(f) := \big(\frac{f^{(j)}(0)}{j!}\big)_{j\in\mathbb{N}}$ is an isometrical isomorphism.

Proof. It is well-known that the space $H_p^2(\mathbb{C})$ is Hilbert w.r.t. the scalar product $\langle f, g\rangle_{H_p^2} = \int_{\mathbb{C}} f(z)\overline{g(z)}e^{-2p(z)}dm(z)$. Since $\log(1+|z|^2) = o(p(z))$, for given j there exists $d > 0$ such that $\log(1+r^2) \leqslant \frac{2}{j+2}p(r) + d$, hence $\log((1+r^2)^{j+2}) \leqslant 2p(r) + d(j+2)$ and therefore $(1+r^2)^{j+2} \leqslant \exp(2p(r))\exp(d(j+2))$ which yields $\exp(-2p(r)) \leqslant \frac{\exp(d(j+2))}{(1+r^2)^q}$. Hence

$$\begin{aligned}\int_0^\infty r^{2j+1}e^{-2p(r)}dr &\leqslant \int_0^\infty r^{2j+1}\tfrac{e^{d(j+2)}}{(1+r^2)^{j+2}}dr\\ &\leqslant e^{d(j+2)}\int_0^\infty \tfrac{r^{2j+1}}{r^{2j+4}}dr\\ &= e^{d(j+2)}\int_0^\infty \tfrac{1}{r^3}dr\end{aligned}$$

and we have shown that the integral $\int_0^\infty r^{2j+1}\exp(-2p(r))dr$ is finite (which means in particular that the b_j are well-defined).

We define the sequence $(f_j)_{j\in\mathbb{N}} \subseteq H_p^2(\mathbb{C})$ where $f_j\colon \mathbb{C}\to\mathbb{C}$, $f_j(z) := b_j z^j$ and claim that $(f_j)_{j\in\mathbb{N}}$ is a complete orthonormal system in the Hilbert space $H_p^2(\mathbb{C})$. In fact

$$\langle f_j, f_l\rangle_{H_p^2} = \int_{\mathbb{C}} f_j(z)\overline{f_l(z)}e^{-2p(z)}dm(z)$$

$$\begin{aligned}
&= \int_0^\infty \int_0^{2\pi} f_j(re^{it})\overline{f_l(re^{it})}e^{-2p(r)}rdtdr \\
&= \int_0^\infty \int_0^{2\pi} b_j r^j e^{itj}\overline{b_l} r^i e^{-itl}e^{-2p(r)}rdtdr \\
&= \frac{\int_0^\infty r^{j+l+1}e^{-2p(r)}dr}{\big(\int_0^\infty r^{2j+1}e^{-2p(r)}dr\big)^{1/2}\big(\int_0^\infty r^{2l+1}e^{-2p(r)}dr\big)^{1/2}} \frac{1}{2\pi}\int_0^{2\pi} e^{i(j-l)t}dt \\
&= \begin{cases} 1 & j = l \\ 0 & j \neq l. \end{cases}
\end{aligned}$$

For $f \in H_p^2(\mathbb{C})$ we have (in $H(\mathbb{C})$) $f = \sum_{j=0}^\infty a_j(f)z^j$ where

$$a_j(f) = \tfrac{f^{(j)}(0)}{j!} = \tfrac{1}{2\pi i}\int_{\partial U_r^+(0)} \tfrac{f(z)}{z^{j+1}}dz = \tfrac{1}{2\pi}\int_0^{2\pi} \tfrac{f(re^{it})}{r^k e^{ijt}}dt$$

for arbitrary $r > 0$. We compute

$$\begin{aligned}
\langle f, \tfrac{1}{b_j}f_j\rangle_{H_p^2} &= \int_{\mathbb{C}} f(z)\tfrac{1}{b_j}\overline{f_j(z)}e^{-2p(z)}dz \\
&= \int_{\mathbb{C}} f(z)\overline{z^j}e^{-2p(z)}dz \\
&= \int_0^\infty \int_0^{2\pi} f(re^{it})r^j e^{-ijt}e^{-2p(r)}rdrdt \\
&= \int_0^\infty r^{2j+1}e^{-2p(r)}\left[\int_0^{2\pi} f(re^{it})r^{-j}e^{-ijt}dt\right]dr \\
&= 2\pi a_j(f)\int_0^\infty r^{2j+1}e^{-2p(r)}dr \\
&= \tfrac{a_j(f)}{b_j^2}.
\end{aligned}$$

Let $f \in H_p^2(\mathbb{C})$ satisfy $\langle f, f_j\rangle_{H_p^2} = 0$ for all $j \in \mathbb{N}$. Then the above shows that $a_j(f) = 0$ for each j, hence $f = 0$ and e.g. Meise, Vogt [60, Remark after 12.4] yields the claim.

Next we claim that $(b_je_j)_{j\in\mathbb{N}}$, where e_j denotes the j-th unit vector, forms a complete orthonormal system in the space $\ell^2(b)$ which is Hilbert (cf. [46, § 95 on page 51]) under the scalar product $\langle x, y\rangle_{\ell^2(b)} = \sum_{j=0}^\infty x_j\overline{y_j}b_j^{-2}$. We have

$$\langle b_je_j, b_le_l\rangle_{\ell^2(b)} = \langle(0,\dots,\underset{j\text{-th entry}}{b_j},\dots),(0,\dots,\underset{l\text{-th entry}}{b_l},\dots)\rangle_{\ell^2(b)} = \begin{cases} 1 & j = l \\ 0 & j \neq l. \end{cases}$$

If $\langle x, b_je_j\rangle_{\ell^2(b)} = 0$ for all j, the computation

$$\langle x, b_je_j\rangle_{\ell^2(b)} = \langle(x_0, x_1,\dots),(0,\dots,\underset{j\text{-th entry}}{b_j},\dots)\rangle_{\ell^2(b)} = x_jb_jb_j^{-2} = \tfrac{x_j}{b_j}$$

shows that all x_j have to be zero and hence $x = 0$ holds. Again, the claim follows e.g. by [60, Remark after 12.4].

The above computations show in particular that $a_j(f) = \langle f, f_j\rangle_{H^2_p} b_j$ for each $j \in \mathbb{N}$. Hence we have

$$T(f) = (a_j(f))_{j\in\mathbb{N}} = \sum_{j=0}^{\infty} a_j e_j = \sum_{j=0}^{\infty} \langle f, f_j\rangle_{H^2_p} b_j e_j$$

and thus

$$\begin{aligned}
\langle T(f), T(g)\rangle_{\ell^2(b)} &= \langle \sum_{j=0}^{\infty} \langle f, f_j\rangle_{H^2_p} b_j e_j, \sum_{i=0}^{\infty} \langle g, f_i\rangle_{H^2_p} b_i e_i\rangle_{\ell^2(b)} \\
&= \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \langle f, f_j\rangle_{H^2_p} \langle g, f_i\rangle_{H^2_p} \langle b_j e_j, b_i e_i\rangle_{\ell^2(b)} \\
&= \sum_{j=0}^{\infty} \langle f, f_j\rangle_{H^2_p} \langle g, f_j\rangle_{H^2_p} \\
&= \langle f, g\rangle_{H^2_p}
\end{aligned}$$

since the f_j form a complete orthonormal system (cf. [60, 12.5]). Thus, T is an isometry. In particular, for given j

$$T(f_j) = \sum_{l=0}^{\infty} \langle f_j, f_l\rangle_{H^2_p} b_l e_l = b_j e_j$$

holds that is each basis element of $\ell^2(b)$ is hit by the map T. Hence T is surjective and we are done. ■

Proposition 9.5. (Meise [56, Proposition 2.8]) Let $\mathbb{P}$ be as in 9.3. Assume in addition that $\log(1+|z|^2) = o(p_1(z))$. Then $\operatorname{ind}_n H^2_{p_n}(\mathbb{C}) \cong \operatorname{ind}_n \ell^2(b_n) = k^2(B)$, where $k^2(B)$ is the Köthe coechelon space of order 2 w.r.t. the Köthe matrix $B = (b_{j;n})_{j,n\in\mathbb{N}}$ with

$$b_{j;n} := \big(2\pi \int_0^{\infty} r^{2j+1} e^{-2p_n(r)} dr\big)^{-1/2}.$$

An isomorphism is given by $T\colon \operatorname{ind}_n H^2_{p_n}(\mathbb{C}) \to \operatorname{ind}_n \ell^2(b_n)$, $f \mapsto \big(\frac{f^{(j)}(0)}{j!}\big)_{j\in\mathbb{N}}$.

Proof. We put $T_n\colon H^2_{p_n}(\mathbb{C}) \to \ell^2(b_n)$, $T_n(f) := \big(\frac{f^{(j)}(0)}{j!}\big)_{j\in\mathbb{N}}$ – By 9.4, T_n is an isometrical isomorphism for each n. Denote by $i_{n+1,n}\colon \ell^2(b_n) \to \ell^2(b_{n+1})$ and by $j_{n+1,n}\colon H^2_{p_n}(\mathbb{C}) \to H^2_{p_{n+1}}(\mathbb{C})$ the inclusion maps. Since by definition $T_{n+1}|_{H^2_{p_n}(\mathbb{C})} = T_n$ holds we have $T_{n+1}\circ j_{n+1,n} = i_{n+1,n}\circ T_n$ for each n. We define the following sequences of maps: $\alpha_n\colon H^2_{p_n}(\mathbb{C}) \to \ell^2(b_n)$, $\alpha_n := T_n$ and $\beta_n\colon \ell^2(b_n) \to H^2_{p_{n+1}}(\mathbb{C})$, $\beta_n := j_{n+1,n}\circ T_n^{-1}$, that is

$$\begin{aligned}
\beta_n \circ \alpha_n &= j_{n+1,n}\circ T_n^{-1}\circ T_n = j_{n+1,n} \\
\alpha_{n+1}\circ\beta_n &= T_{n+1}\circ j_{n+1,n}\circ T_n^{-1} = i_{n+1,n}\circ T_n \circ T_n^{-1} = i_{n+1,n}
\end{aligned}$$

i.e. the diagram

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[dr] & \ell^2(b_n) \arrow[r, "i_{n+1,n}"] \arrow[dr, "\beta_n"] & \ell^2(b_{n+1}) \arrow[r, "i_{n+2,n+1}"] \arrow[dr, "\beta_{n+1}"] & \ell^2(b_{n+2}) \arrow[r] \arrow[dr] & \cdots \\
\cdots \arrow[r] & H^2_{p_n}(\mathbb{C}) \arrow[r, "j_{n+1,n}"'] \arrow[u, "\alpha_n"'] & H^2_{p_{n+1}}(\mathbb{C}) \arrow[r, "j_{n+2,n+1}"'] \arrow[u, "\alpha_{n+1}"'] & H^2_{p_{n+2}}(\mathbb{C}) \arrow[r] \arrow[u, "\alpha_{n+2}"'] & \cdots
\end{tikzcd}$$

is commutative and hence the inductive spectra $(H^2_{p_n}(\mathbb{C}))_{n\in\mathbb{N}}$ and $(\ell^2(b_n))_{n\in\mathbb{N}}$ of Banach spaces are equivalent and hence their inductive limits are isomorphic.

Let $j_n\colon H^2_{p_n}(\mathbb{C}) \to \operatorname{ind}_k H^2_{p_k}(\mathbb{C})$ and $i_n\colon \ell^2(b_n) \to \operatorname{ind}_k \ell^2(b_k)$ be the canonical (inclusion) maps. Then by the universal property of the inductive limit the maps $i_n \circ T_n\colon H^2_{p_n}(\mathbb{C}) \to \operatorname{ind}_k \ell^2(b_k)$ induce a map $S\colon \operatorname{ind}_k H^2_{p_k}(\mathbb{C}) \to \operatorname{ind}_k \ell^2(b_k)$ such that $S \circ j_n = i_n \circ T_n$ for all n which is by the above an isomorphism. Let $f \in \operatorname{ind}_k H^2_{p_k}(\mathbb{C})$ be given. Choose n such that $f \in H^2_{p_n}(\mathbb{C})$. Then $S(f) = S \circ j_n(f) = i_n \circ T_n(f) = T_n(f) = T(f)$ since the j_n and i_n are just inclusions of subspaces. That is $S = T$ is an isomorphism. ■

Remark 9.6. With the notation and assumptions of 9.5 we have $k^2(B) = k^\infty(B)$: By Meise [56, 2.4] the space $\operatorname{ind}_n H^2_{p_n}(\mathbb{C})$ is a (DFN)-space, hence by 9.5 the same is true for $k^2(B)$. Thus the Grothendieck-Pietsch condition (cf. [64, 6.1.2] and [26, Remark after 5.4]) yields that even all the spaces $k^p(B)$ for arbitrary orders $1 \leqslant p \leqslant \infty$ or $p = 0$ coincide (see [11, 2.15 and the subsequent remark]).

Consequence 9.7. Under the assumptions of the 9.5 we have

$$A_{\mathbb{P}}(\mathbb{C}) \overset{\text{dfn}}{=} \operatorname{ind}_n H^\infty_{p_n}(\mathbb{C}) \;=\; \operatorname{ind}_n H^2_{p_n}(\mathbb{C}) \overset{T}{\cong} \operatorname{ind}_n \ell^2(b_n) \;=\; \operatorname{ind}_n \ell^\infty(b_n) \overset{\text{dfn}}{=} k^\infty(B).$$

9.5 and an inspection of the proof of 9.4 yields the following result which will be important in section 9.3.

Scholium 9.8. (of 9.4) Under the assumptions of 9.5 the monomials $q_j\colon \mathbb{C} \to \mathbb{C}$, $q_j(z) = z^j$ constitute an equicontinuous basis in the space $A_{\mathbb{P}}(\mathbb{C})$ where the coefficient functionals are the Taylor coefficients.

Proof. First of all, by 9.2 the q_j are contained in each of the Banach spaces $H^\infty_{p_n}(\mathbb{C})$. Let $f \in A_{\mathbb{P}}(\mathbb{C})$. Hence there exists n such that $f \in H^2_{p_n}(\mathbb{C})$. The proof of 9.4 shows that we have

$$f = \sum_{j=0}^{\infty} \langle f, f_j \rangle_{H^2_{p_n}} f_j = \sum_{j=0}^{\infty} \tfrac{a_j(f)}{b_{j;n}} b_{j;n} z^j = \sum_{j=0}^{\infty} a_j(f) z^j = \sum_{j=0}^{\infty} \tfrac{f^{(j)}(0)}{j!} z^j$$

holds in $H^2_{p_n}(\mathbb{C})$ and hence in $A_{\mathbb{P}}(\mathbb{C})$.

Assume that $(a'_j)_{j\in\mathbb{N}}$ is a sequence such that $f = \sum_{j=0}^\infty a'_j z^j$ holds in $A_{\mathbb{P}}(\mathbb{C})$. Then the latter is also true in $(H(\mathbb{C}), \mathrm{co})$ and hence $a'_j = a_j(f)$ for each j since in $(H(\mathbb{C}), \mathrm{co})$ the power series representation is unique.

In order to show that $(p_j)_{j\in\mathbb{N}}$ is a Schauder basis we have to check that $a_j\colon A_{\mathbb{P}}(\mathbb{C}) \to \mathbb{C}$ is continuous for each $j \in \mathbb{N}$. We fix j and n. Let $f \in H^2_{p_n}(\mathbb{C})$. We estimate

$$\begin{aligned}
|a_j(f)| &= \big|\tfrac{1}{2\pi i}\int_{|z|=1} \tfrac{f(z)}{z^{j+1}}dz\big| \\
&\leqslant \tfrac{1}{2\pi}\int_{|z|=1} \tfrac{|f(z)e^{-p_n(z)}e^{p_n(z)}|}{|z|^{j+1}}|dz| \\
&\leqslant \sup_{|z|=1} e^{p_n(z)}\tfrac{1}{2\pi}\int_{|z|=1} |f(z)e^{-p_n(z)}||dz| \\
&\leqslant \tfrac{e^{p_n(1)}}{2\pi} 2\pi \sup_{z\in\mathbb{C}} |f(z)|e^{-p_n(z)} \\
&= e^{p_n(1)}\|f\|_{p_n}.
\end{aligned}$$

Hence the restriction $a_j\colon H^2_{p_n}(\mathbb{C}) \to \mathbb{C}$ is continuous for each n. By the universal property of the inductive limit a_j has to be continuous on $A_{\mathbb{P}}(\mathbb{C})$ for arbitrary j. Since the space $A_{\mathbb{P}}(\mathbb{C})$ is barrelled, the basis has to be equicontinuous (cf. e.g. Jarchow [50, 14.3.3]). ■

After these preparations we start with the "program" sketched at the beginning of this section.

9.2 Integral representation

We use the notation established at the beginning of section 9. Assume that $\log(1+|z|^2) = o(p_{N,1}(z))$ holds for each $N \in \mathbb{N}$. From 9.7 we get immediately

$$\begin{aligned}
\mathcal{A}_N H(\mathbb{C}) \overset{\text{dfn}}{=} \operatorname{ind}_n H^\infty_{p_{N,n}}(\mathbb{C}) &= \operatorname{ind}_n H^2_{p_{N,n}}(\mathbb{C}) \\
&\overset{T_N}{\cong} \operatorname{ind}_n \ell^2(b_{N,n}) = \operatorname{ind}_n \ell^\infty(b_{N,n}) \overset{\text{dfn}}{=} k^\infty(B_N).
\end{aligned}$$

for each $N \in \mathbb{N}$ where $B_N = (b_{j;N,n})_{j,N,n\in\mathbb{N}}$ with

$$b_{j;N,n} := \big(2\pi \int_0^\infty r^{2j+1}e^{-2p_{N,n}(r)}dr\big)^{-1/2}$$

and $T_N\colon \operatorname{ind}_n H^2_{p_{N,n}}(\mathbb{C}) \to \operatorname{ind}_n \ell^2(b_{N,n})$ is defined by $T_N(f) = \big(\frac{f^{(j)}(0)}{j!}\big)_{j\in\mathbb{N}}$.

Lemma 9.9. Assume that $\log(1+|z|^2) = o(p_{N,1}(z))$ holds for each $N \in \mathbb{N}$. Then the map

$$T\colon \operatorname{proj}_N \operatorname{ind}_n H^2_{p_{N,n}}(\mathbb{C}) \to \operatorname{proj}_N \operatorname{ind}_n \ell^2(b_{N,n}),\ f \mapsto \big(\tfrac{f^{(j)}(0)}{j!}\big)_{j\in\mathbb{N}}$$

is an isomorphism. In particular, the associated projective spectra are equivalent.

Proof. We show that the spectra $(\operatorname{ind}_n H^2_{p_{N,n}}(\mathbb{C}))_{N\in\mathbb{N}}$ and $(\operatorname{ind}_n \ell^2(b_{N,n}))_{N\in\mathbb{N}}$ are equivalent. We define the maps $\alpha_N\colon \operatorname{ind}_n H^2_{p_{N,n}}(\mathbb{C}) \to \operatorname{ind}_n \ell^2(b_{N,n})$, $\alpha_N := T_N$ and $\beta_N\colon \operatorname{ind}_n \ell^2(b_{N,n}) \to \operatorname{ind}_n H^2_{p_{N-1,n}}(\mathbb{C})$, $\beta_N := j_{N,N-1}\circ T_N^{-1}$. Then

$$\begin{aligned}\beta_N\circ\alpha_N &= j_{N,N-1}\circ T_N^{-1}\circ T_N = j_{N,N-1}\\ \alpha_N\circ\beta_{N+1} &= T_N\circ j_{N+1,N}\circ T_{N+1}^{-1} = i_{N+1,N}\circ T_{N+1}\circ T_{N+1}^{-1} = i_{N+1,N}\end{aligned}$$

where we used $T_N\circ j_{N+1,N} = i_{N+1,N}\circ T_{N+1}$ which holds since $T_N|_{\operatorname{ind}_n H^2_{p_{N+1,n}}(\mathbb{C})} = T_{N+1}$. Hence the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \operatorname{ind}_n H^2_{p_{N+1,n}}(\mathbb{C}) \arrow[r, "j_{N+1,N}"] \arrow[d, "\alpha_{N+1}"] & \operatorname{ind}_n H^2_{p_{N,n}}(\mathbb{C}) \arrow[r, "j_{N,N-1}"] \arrow[d, "\alpha_N"] & \operatorname{ind}_n H^2_{p_{N-1,n}}(\mathbb{C}) \arrow[r] \arrow[d, "\alpha_{N-1}"] & \cdots \\
\cdots \arrow[r] & \operatorname{ind}_n \ell^2(b_{N+1,n}) \arrow[r, "i_{N+1,N}"'] \arrow[ur, "\beta_{N+1}"] & \operatorname{ind}_n \ell^2(b_{N,n}) \arrow[r, "i_{N,N-1}"'] \arrow[ur, "\beta_N"] & \operatorname{ind}_n \ell^2(b_{N-1,n}) \arrow[r] & \cdots
\end{tikzcd}$$

commutes, the spectra are equivalent and their projective limits are isomorphic. Let us denote by j_N and i_N the canonical (inclusion) maps $\operatorname{proj}_K \operatorname{ind}_n H^2_{p_{K,n}}(\mathbb{C}) \to \operatorname{ind}_n H^2_{p_{K,n}}(\mathbb{C})$ and $\operatorname{proj}_K \operatorname{ind}_n \ell^2(b_{K,n}) \to \operatorname{ind}_n \ell^2(b_{N,n})$. Then the maps $T_N\circ j_N\colon \operatorname{proj}_K \operatorname{ind}_n H^2_{p_{K,n}}(\mathbb{C}) \to \operatorname{ind}_n \ell^2(b_{N,n})$ induce a mapping

$$S\colon \operatorname{proj}_N \operatorname{ind}_n H^2_{p_{N,n}}(\mathbb{C}) \to \operatorname{proj}_N \operatorname{ind}_n \ell^2(b_{N,n})$$

such that $i_N\circ S = T_N\circ j_N$ for all N which is by the above an isomorphism. Now let $f\in \operatorname{proj}_N \operatorname{ind}_n H^2_{p_{N,n}}(\mathbb{C})$ and $N\in\mathbb{N}$ be arbitrary. Then $S(f) = i_N\circ S(f) = T_N\circ j_N(f) = T(f)$ since i_N and j_N are just inclusion maps. Hence $T = S$ is an isomorphism. ■

In view of the equation previous to 9.9 we can regard the T_N also as maps $\mathcal{A}_N H(\mathbb{C}) \to k^\infty(B_N)$. Hence, 9.9 yields that

$$AH(\mathbb{C}) \overset{T}{\cong} \operatorname{proj}_N k^\infty(B_N) = \operatorname{proj}_N \operatorname{ind}_n \ell^\infty(b_{N,n}) = \operatorname{proj}_N \operatorname{ind}_n Cb^{-1}_{N,n}(\mathbb{N}) = BC(\mathbb{N})$$

where we use the the double sequence $\mathcal{B} = ((b^{-1}_{N,n})_{N\in\mathbb{N}})_{n\in\mathbb{N}}$ formed by the weights $b^{-1}_{N,n}(j) := b^{-1}_{j;N,n}$. Thus, 3.1.B immediately yields necessary and sufficient conditions for the vanishing of $\operatorname{Proj}^1 \mathcal{B}C$, ultrabornologicity and barrelledness of $BC(\mathbb{N})$ and since the spectra $\mathcal{A}H$ and $\mathcal{B}H$ are equivalent and the corresponding spaces are isomorphic, we thus get a result on $\operatorname{Proj}^1 \mathcal{A}H$ and $AH(\mathbb{C})$. By the following remark we even get a characterization of the forementioned properties.

Remark 9.10. In 9.6 we noted that the space $\operatorname{ind}_n H^2_{p_n}(\mathbb{C}) = k^\infty(B)$ is a (DFN)-space under the assumptions of 9.5 and that therefore the sequence $B = (b_{j;n})_{j,\,n\in\mathbb{N}}$ satisfies the Grothendieck-Pietsch condition

$$\text{(N)} \qquad \forall\, k\ \exists\, n>k\colon \tfrac{b_{j;k}}{b_{j;n}} \to 0 \text{ for } j\to\infty$$

in the notation of Bierstedt, Meise, Summers [26, Remark subsequent to 5.4].

Concerning the sequence $\mathcal{B} = ((b_{N,n}^{-1})_{N\in\mathbb{N}})_{n\in\mathbb{N}}$ the above implies that each step $\mathcal{B}_N C(\mathbb{N})$ of the projective spectrum satisfies condition (S) of Bierstedt, Meise, Summers [27], i.e. for fixed N we have

$$\forall\, k\ \exists\, n \geqslant k\colon \frac{b_{j;N,n}^{-1}}{b_{j;N,k}^{-1}} \to 0 \text{ for } j \to \infty$$

and therefore $\mathcal{B}$ satisfies in particular condition (Σ)

$$\forall\, N\ \exists\, K \geqslant N\ \forall\, k\ \exists\, n \geqslant k\colon \frac{b_{N,n}^{-1}}{b_{K,k}^{-1}} \text{ vanishes at } \infty \text{ on } \mathbb{N}.$$

Now we can state the desired characterization.

Theorem 9.11. Assume that $\log(1+|z|^2) = o(p_{N,1}(z))$ holds for each N. Then the following are equivalent.

(i) $\mathcal{B}$ satisfies condition (wQ).

(ii) $\operatorname{Proj}^1 \mathcal{A}H = \operatorname{Proj}^1 \mathcal{B}C = 0$.

(iii) $AH(\mathbb{C}) \cong BC(\mathbb{N})$ is ultrabornological.

(iv) $AH(\mathbb{C}) \cong BC(\mathbb{N})$ is barrelled.

9.3 Basis method

In this section we deduce a sequence space representation using methods of Domański, Vogt [41]. In order to do this, we need the following abstract result.

Proposition 9.12. Let $E = \operatorname{proj}_N E_N$ with locally convex spaces E_N such that $E_{N+1} \subseteq E_N$ and inclusions as linking maps. We denote by $\pi_N\colon E \to E_N$ the canonical (inclusion) map and consider $(e_j)_{j\in\mathbb{N}} \subseteq E$.

(1) Assume that $(e_j)_{j\in\mathbb{N}} \subseteq E_N$ is an equicontinuous basis for each N.

(2) For each N let $\xi_j^N\colon E_N \to \mathbb{C}$, $\sum_{i=0}^{\infty} \xi_i^N(x)e_i = x \mapsto \xi_j^N(x)$ be the j-th coefficient functionals. Assume that $\xi_j^N|_{E_{N+1}} = \xi_j^{N+1}$ holds for each N.

Then $(e_j)_{j\in\mathbb{N}}$ is an equicontinuous basis in E.

Proof. First we have to show that $(e_j)_{j\in\mathbb{N}}$ is a basis in E. Let $x \in E$ be given. We put $\xi_j(x) := \xi_j^1(x)$ $(= \xi_j^N(x)$ for each $N \in \mathbb{N})$ and claim $\sum_{j=1}^{\infty} \xi_j(x)e_j = x$. Let $p \in \operatorname{cs}(E)$. Since $E = \operatorname{proj}_N E_N$ we may assume $p = \max_{N\in\mathcal{M}} p_N$ with $\mathcal{M} \subseteq \mathbb{N}$ finite, $p_N \in \operatorname{cs}(E_N)$ for $N \in \mathcal{M}$, cf. Meise, Vogt [60, Definition after 24.4]. We compute

$$p\Big(\sum_{j=0}^{K} \xi_j(x)e_j - f\Big) = \max_{N\in\mathcal{M}} p_N\Big(\sum_{j=0}^{K} \xi_j^N(x)e_j - f\Big) \to 0 \text{ for } K \to \infty.$$

Now we assume that $(\zeta_j)_{j\in\mathbb{N}}$, $\zeta_j\colon E \to \mathbb{C}$ also satisfies $\sum_{j=0}^{\infty} \zeta_j(x)e_j = x$ in E. Then this equality holds also in E_1 and since $(e_j)_{j\in\mathbb{N}}$ is a basis in E_1 with coefficient functionals $(\xi_j^1)_{j\in\mathbb{N}}$, $\zeta_j(x) = \xi_j^1(x) = \xi_j(x)$ and hence $\zeta_j = \xi_j$ for each j.

Since $\xi_j = \xi_j^1 \circ \pi_1$ where ξ_j^1 and π_1 are continuous, the same is true for ξ_j. Hence,

$(e_j)_{j\in\mathbb{N}}$ is a Schauder basis in E.

It remains to check that $(e_j)_{j\in\mathbb{N}}$ is equicontinuous. By (1) we have

$$(\star)\quad \forall\, p\in \operatorname{cs}(E_N)\ \exists\, q\in \operatorname{cs}(E_N),\ C_N>0\ \forall\, x\in E_N,\ j\in\mathbb{N}\colon |\xi_j^N(x)|p(e_j)\leqslant C_N q(x)$$

for each N. Let $p\in \operatorname{cs}(E)$. As above we may assume $p=\max_{N\in\mathcal{M}} p_N$. For each $N\in\mathcal{M}$ we have $p_N\in \operatorname{cs}(E_N)$ and we thus may select $q_N\in \operatorname{cs}(E_N)$ and $C_N>0$ according to $(\star)$. Now we put $q:=\max_{N\in\mathcal{M}} q_N$ and $C:=\max_{N\in\mathcal{M}} C_N$. For given $x\in E$, $j\in\mathbb{N}$ we compute

$$\begin{aligned}|\xi_j(x)|p(e_j) &= |\xi_j(x)|\max_{N\in\mathcal{M}} p_N(e_j)=\max_{N\in\mathcal{M}}|\xi_j(x)|p_N(e_j)\\ &\overset{(\star)}{\leqslant} \max_{N\in\mathcal{M}} C_N q_N(x)\leqslant \max_{N\in\mathcal{M}} C q_N(x)=C\max_{N\in\mathcal{M}} q_N(x)=Cq(x)\end{aligned}$$

which finishes the proof. ■

Consequence 9.13. Assume that $\log(1+|z|^2)=o(p_{N,1}(z))$ holds for each N. The monomials $q_j\colon\mathbb{C}\to\mathbb{C}$, $p_j(z)=z^j$ constitute an equicontinuous basis in the space $AH(\mathbb{C})$.

Proof. This follows directly from 9.8 and 9.12. ■

Now, we may apply Domański, Vogt [41, Theorem 2.1] to get a sequence space representation of the (PLN)-space $AH(\mathbb{C})$. We define the Köthe (PLB)-matrix

$$B:=(b_{j;N,n})_{N,n\in\mathbb{N}},\ \ b_{j;N,n}:=p_{b_{N,n}}(p_j)=\|p_j\|_{N,n}=\sup_{z\in\mathbb{C}} e^{-p_{N,n}(z)}|z^j|$$

and obtain by [41, Theorem 2.1] that (in the notation of Domański, Vogt)

$$AH(\mathbb{C})\cong E_\infty(B)=\operatorname{proj}_N \operatorname{ind}_n E_\infty^{N,n}(B)$$

where $E_\infty^{N,n}(B)=\big\{\,x=(x_j)_{j\in I_N}\ ;\ \|x\|_{N,n}^{(\infty)}<\infty\,\big\}$, $I_N:=\{\,j\ ;\ \forall\, n\colon b_{j;N,n}>0\,\}$ and $\|x\|_{N,n}^{(\infty)}:=\sup_{j\in I_N}|x_j|b_{j;N,n}$. In view of the appearence of the $b_{j;N,n}$ we have $I_N=\mathbb{N}$ for each N which simplifies the above into

$$AH(\mathbb{C})\cong E_\infty(B)=\operatorname{proj}_N \operatorname{ind}_n\big\{\,x=(x_j)_{j\in\mathbb{N}}\ ;\ \sup_{j\in\mathbb{N}} b_{j;N,n}|x_j|<\infty\,\big\}.$$

If we identify the elements of the Köthe (PLB)-matrix with the maps $b_{N,n}\colon\mathbb{N}\to\mathbb{R}$, $b_{N,n}(j)=b_{j;N,n}$ the spaces $E_\infty^{N,n}(B)$ are exactly the weighted Banach spaces of continuous functions $Cb_{N,n}(\mathbb{N})$ investigated by Agethen, Bierstedt, Bonet [2]. Hence, the space $E_\infty(B)$ coincides with the weighted (PLB)-space of continuous functions $BC(\mathbb{N})$ for the double sequence $\mathcal{B}=((b_{N,n})_{N\in\mathbb{N}})_{n\in\mathbb{N}}$. In the proof of [41, Theorem 2.1] Domański, Vogt showed even that the projective spectra of (LB)-spaces are equivalent – in our situation this means that $\mathcal{B}C$ and $\mathcal{A}H$ are equivalent and hence that $\operatorname{Proj}^1\mathcal{B}C=0$ if and only if $\operatorname{Proj}^1\mathcal{A}H=0$. By our observations previous to 9.14 the space $BC(\mathbb{N})$ is a (PLN)-space and thus the steps $\mathcal{B}_N C(\mathbb{N})=\operatorname{ind}_n Cb_{N,n}(\mathbb{N})$, which can be identified with the coechelon spaces

$k^\infty(B_N)$ w.r.t. the Köthe matrix $B_N = (b_{j;N,n}^{-1})_{j,\,n\in\mathbb{N}}$ for fixed N, are (DFN)-spaces. Thus, we get in complete analogy to 9.10 (but with "inverted" entries of the Köthe matrices) that the sequence $\mathcal{B} = ((b_{N,n})_{N\in\mathbb{N}})_{n\in\mathbb{N}}$ satisfies condition (Σ). From the latter observations and 3.1.B we get immediately the following.

Theorem 9.14. Assume that $\log(1+|z|^2) = o(p_{N,1}(z))$ holds for each N. Then the following are equivalent.

(i) $\mathcal{B}$ satisfies condition (wQ).

(ii) $\operatorname{Proj}^1 \mathcal{B}C = \operatorname{Proj}^1 \mathcal{A}H = 0$.

(iii) $BC(\mathbb{N}) \cong AH(\mathbb{C})$ is ultrabornological.

(iv) $BC(\mathbb{N}) \cong AH(\mathbb{C})$ is barrelled.

9.4 Young conjugates

In this section we present a reformulation of the results of section 9.3 in terms of the Young conjugate. We stick to the assumptions made in the last section. In the notation established after 9.13 we considered the Köthe (PLB)-matrix $B := (b_{j;N,n})_{j,N,n\in\mathbb{N}}$ with $b_{j;N,n} = \|p_j\|_{N,n}$. We have

$$\|p_j\|_{N,n} = \sup_{z\in\mathbb{C}} e^{-p_{N,n}(z)}|z^j| = \sup_{r\geqslant 0} e^{j\log r - p_{N,n}(r)} = \exp\big(\sup_{r\geqslant 0}(j\log r - p_{N,n}(r))\big).$$

If we put $\varphi_{N,n}\colon \mathbb{C}\to\mathbb{R}$, $\varphi_{N,n}(z) := p_{N,n}(\exp(z))$, the last expression in the above computation is exactly $\exp\big(\varphi^\star_{N,n}(j)\big)$ where $\varphi^\star_{N,n}\colon \mathbb{R}_{\geqslant 0}\to\mathbb{R}$ is the Young conjugate of $\varphi_{N,n}$ defined by $\varphi^\star_{N,n}(y) = \sup_{x\in\mathbb{R}_{\geqslant 0}}(x\cdot y - \varphi_{N,n}(x))$. Hence we have

$$b_{j;N,n} = \exp\big(\varphi^\star_{N,n}(j)\big)$$

and thus 9.14 can be restated for the sequence $\mathcal{B}$ defined by $\mathcal{B} = ((b_{N,n})_{N\in\mathbb{N}})_{n\in\mathbb{N}}$ with $b_{N,n}(j) = \exp(\varphi^\star_{N,n}(j))$.

Theorem 9.15. Assume that $\log(1+|z|^2) = o(p_{N,1}(z))$ holds for each N. Then the following are equivalent.

(i) $\mathcal{B}$ satisfies condition (wQ),

(ii) $\operatorname{Proj}^1 \mathcal{A}H = \operatorname{Proj}^1 \mathcal{B}C = 0$,

(iii) $BC(\mathbb{N}) \cong AH(\mathbb{C})$ is ultrabornological,

(iv) $BC(\mathbb{N}) \cong AH(\mathbb{C})$ is barrelled.

Let us add, that condition (i) of 9.15 can be expressed as follows in terms of the Young conjugates.

Remark 9.16. The sequence $\mathcal{B}$ satisfies condition (wQ) if and only if

$$\forall\, N\ \exists\, M\geqslant N,\, n\ \forall\, K\geqslant M,\, m\ \exists\, k,\, S>0:$$

$$\exp(-\varphi^\star_{M,m}) \leqslant S\max\big(\exp(-\varphi^\star_{N,n}),\, \exp(-\varphi^\star_{K,k})\big).$$

9.5 A condition on $\mathcal{P}$

In this section we present a condition on the sequence $\mathcal{P}$, which is sufficient for all the sequences $\mathcal{B}$ of the sequence spaces representations established in the last three sections to satisfy condition (wQ). Since in this section the defining sequence of the weighted (PLB)-spaces is in fact $\mathcal{P}$ (since $\mathcal{A}$ is just $\exp(-\mathcal{P})$, the latter is a clearly a desirable result.

Definition 9.17. The sequence $\mathcal{P} = ((p_{N,n})_{N\in\mathbb{N}})_{n\in\mathbb{N}}$ is said to satisfy condition $(\mathrm{wQ})^{\mathrm{P}}$ if

$$\forall\, N\; \exists\, M \geqslant N,\, n\; \forall\, K \geqslant M,\, m\; \exists\, k,\, S > 0\; \forall\, r \geqslant 0\colon$$
$$p_{M,m}(r) \leqslant S + \max\big(p_{N,n}(r),\, p_{K,k}(r)\big).$$

Lemma 9.18. If $\mathcal{P} = ((p_{N,n})_{N\in\mathbb{N}})_{n\in\mathbb{N}}$ satisfies $(\mathrm{wQ})^{\mathrm{P}}$ then $\mathcal{B} = ((b_{N,n})_{N\in\mathbb{N}})_{n\in\mathbb{N}}$ with

$$b_{j;N,n} = \sup_{z\in\mathbb{C}} e^{-p_{N,n}(z)}|z^j|$$

satisfies (wQ).

Proof. For given N we select M and n as in $(\mathrm{wQ})^{\mathrm{P}}$. For given K and m we select k and $S > 1$ as in $(\mathrm{wQ})^{\mathrm{P}}$. We put $S' := \exp(S) > 0$. i.e. $S = \log S'$. Then for arbitrary j and $r \geqslant 0$ we have $p_{M,m}(r) \leqslant S + \max\big(p_{N,n}(r),\, p_{K,k}(r)\big) = \max\big(\log S' + p_{N,n}(r),\, \log S' + p_{K,k}(r)\big)$ and hence by multiplication with r^{-1}, taking $\exp(\cdot)$ and then the infimum on both sides we get

$$\begin{aligned}
\inf_{r\geqslant 0} e^{p_{M,m}(r)} r^{-j} &\leqslant \inf_{r\geqslant 0} e^{\max\big(\log S' + p_{N,n}(r),\, \log S' + p_{K,k}(r)\big)} r^{-j}\\
&= \inf_{r\geqslant 0} \Big(\max\big(e^{\log S' + p_{N,n}(r)}, e^{\log S' + p_{K,k}(r)}\big) r^{-j}\Big)\\
&= \inf_{r\geqslant 0} \max\big(e^{\log S' + p_{N,n}(r)} r^{-j}, e^{\log S' + p_{K,k}(r)} r^{-j}\big)\\
&= \inf_{r\geqslant 0} \max\big(S' e^{p_{N,n}(r)} r^{-j}, S' e^{p_{K,k}(r)} r^{-j}\big)\\
&= S' \max\big(\inf_{r\geqslant 0} e^{p_{N,n}(r)} r^{-j}, \inf_{r\geqslant 0} e^{p_{K,k}(r)} r^{-j}\big)
\end{aligned}$$

that is

$$\inf_{z\in\mathbb{C}} e^{p_{M,m}(z)}|z^{-j}| \leqslant S' \max\big(\inf_{z\in\mathbb{C}} e^{p_{N,n}(z)}|z^{-j}|, \inf_{z\in\mathbb{C}} e^{p_{K,k}(z)}|z^{-j}|\big).$$

By the definition of the $b_{N,n}$ this is exactly $\frac{1}{b_{M,m}(j)} \leqslant \max S'\big(\frac{1}{b_{N,n}(j)}, \frac{1}{b_{K,k}(j)}\big)$, which is the estimate in (wQ). ∎

Corollary 9.19. If $\mathcal{P} = ((p_{N,n})_{N\in\mathbb{N}})_{n\in\mathbb{N}}$ satisfies $(\mathrm{wQ})^{\mathrm{P}}$ then also sequences obtained in 9.2 and 9.4 satisfy (wQ).

Proof. This is clear in view of the equivalences stated in 9.11, 9.14 and 9.15. ∎

9.6 Summary of results

In the following theorem we summarize the three different sequence space representations, their consequences aswell as the necessary conditions we obtain from section 5.

Theorem 9.20. Let $\mathcal{P} = ((p_{N,n})_{N\in\mathbb{N}})_{n\in\mathbb{N}}$ a sequence of radial weight functions which is decreasing in n and increasing in N such that for each N and each n there exists l and $L \geqslant 0$ with $2p_{N,n} \leqslant p_{N,l} + L$. Assume that $\log(1+|z|^2) = o(p_{N,1})$ for each N and put $\mathcal{A} = \exp(-\mathcal{P})$. Let $\mathcal{B} = ((b_{N,n})_{N\in\mathbb{N}}$, $\mathcal{C} = ((c_{N,n})_{N\in\mathbb{N}})_{n\in\mathbb{N}}$ and $\mathcal{D} = ((d_{N,n})_{N\in\mathbb{N}})_{n\in\mathbb{N}}$ be defined via

$$\begin{aligned} b_{N,n}(j) &= \Big(\int_0^\infty r^{2j+1}e^{-2p_{N,n}(r)}dr\Big)^{1/2},\\ c_{N,n}(j) &= \sup_{r\geqslant 0} e^{-p_{N,n}(r)}r^j,\\ d_{N,n}(j) &= \exp(\exp^\star(p_{N,n}(j))). \end{aligned}$$

Then $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ satisfy condition (Σ), (ii) – (v) are equivalent, (i) implies the latter and the latter implies (vi), where

(i) $\mathcal{P}$ satisfies $(\mathrm{wQ})^{\mathrm{P}}$,

(ii) $\mathcal{B}$ satisfies (wQ), (ii′) $\mathcal{C}$ satisfies (wQ), (ii″) $\mathcal{D}$ satisfies (wQ),

(iii) $\operatorname{Proj}^1 \mathcal{A}H = 0$,

(iv) $AH(\mathbb{C})$ is ultrabornological,

(v) $AH(\mathbb{C})$ is barrelled,

(vi) $\mathcal{A}$ satisfies $(\mathrm{wQ})^\sim_{\mathrm{in}}$.

Proof. The equivalence of (ii), (iii), (iv) and (v) follows from 9.11. From 9.14 it follows that (ii′), (iii), (iv) and (v) are equivalent. The equivalence of (ii″), (iii), (iv) and (v) holds, since $\mathcal{C} = \mathcal{D}$ holds by our considerations in section 9.4. The implication "(i)⇒(ii′)" is exactly 9.18 and "(v)⇒(vi)" we get from 5.8. ■

Let us now assume that all the weights in $\mathcal{A}$ are essential in the sence that $\big(\frac{1}{a}\big)^\sim = \frac{1}{a}$ for all $a \in \mathcal{A}$. This is equivalent to the assumption that

$$p(z) = \sup_{\substack{f\in H(\mathbb{C})\\ |f|\leqslant e^p}} \log|f(z)|$$

holds for each $z \in \mathbb{C}$ where $a(z) = e^{-p(z)}$. Since in the current setting all the information contained in $\mathcal{A}$ is already contained in the sequence $\mathcal{P}$ the latter formulation of essentialness is in some sense more natural (and accessible) than the very definition.

Corollary 9.21. Assume in the situation of 9.20 that all the weights in $\mathcal{A}$ are essential. Then all the conditions in 9.20 are equivalent.

Proof. It is enough to check that (vi) implies (i). If the weights in $\mathcal{A}$ are essential, (vi) implies that $\mathcal{A}$ satisfies (wQ), see 5.9. We claim that $\mathcal{P}$ satisfies $(\mathrm{wQ})^{\mathrm{P}}$. For given N we select M and n as in (wQ). For given $K \geqslant M$ and m we select k as in (wQ) and denote by S the constant in (wQ), where we may assume that $S \geqslant 1$ holds. We put $S' := \log S > 0$. Let $r \geqslant 0$ be fixed. Then the estimate in (wQ) yields $e^{p_{M,m}(r)} \leqslant S\max(e^{p_{N,n}(r)}, e^{p_{K,k}(r)})$. Applying log, this yields $p_{M,m}(r) \leqslant S' + \max(p_{N,n}(r), p_{K,k}(r))$ which is the estimate in $(\mathrm{wQ})^{\mathrm{P}}$. ■

10 A non-radial setting for the complex plane

10.1 Meise, Taylor's decomposition Lemma

Meise, Taylor [57] proved a decomposition lemma for entire functions. Among other applications they showed that the space $A_{\mathbb{P}}(\mathbb{C})'_b$ for certain sequences $\mathbb{P}$ satisfies condition (DN). Since we want to use their decomposition method to investigate again a class of (PLB)-spaces arising from sequences of weights of a special type we will review the definitions and results of Meise, Taylor first. Most of their results are stated for $\mathbb{C}^d$, $d \geqslant 1$. Since we will finally have to restrict ourselves to the case $d = 1$, we will right from the beginning state everything for this case.

Let $\omega\colon [0,\infty[\to [0,\infty[$ be an increasing continuous function which satisfies $\omega(0) = 0$. According to [57, Definition 1.1] let us call ω a *weight function*, if it has the following properties.

(WF 1) $\exists\, C > 0\ \forall\, y \geqslant 0\colon \int_1^\infty \frac{\omega(yt)}{t^2}dt \leqslant C\omega(y) + C$.

(WF 2) The function $\varphi\colon \mathbb{R} \to [0,\infty[$, $t \mapsto \omega(e^t)$ is convex.

(WF 3) $\lim_{t\to\infty} \frac{\log t}{\omega(t)} = 0$.

For later use let us state the following remarks of Meise, Taylor.

Remark 10.1. (Meise, Taylor [57, Remark 1.2]) For each weight function ω there exists a concave weight function χ and $C > 0$ such that $\omega(y) \leqslant \chi(y) \leqslant C\omega(y) + C$ holds for each $y \geqslant 0$; χ is given by $\chi(y) = \int_1^\infty \frac{\omega(yt)}{t^2}dt = y\int_y^\infty \frac{\omega(s)}{s^2}ds$.

The above means that we w.l.o.g. may assume that ω is concave and hence subadditive. We will use this frequently in the sequel.

Let $u\colon \mathbb{R} \to \mathbb{R}$ be continuous such that $\int_{-\infty}^\infty \frac{|u(t)|}{1+t^2}dt < \infty$. By $P_u\colon \mathbb{C} \to \mathbb{R}$ we denote the *harmonic extension* of u which is defined by

$$P_u(x+iy) := \begin{cases} \frac{|y|}{\pi}\int_{-\infty}^\infty \frac{u(t)}{(t-x)^2+y^2}dt & \text{if } |y| > 0 \\ u(x) & \text{if } y = 0. \end{cases}$$

For a weight function ω we will understand P_ω as the harmonic extension of $\omega|_{\mathbb{R}}$ where we (as usual) regard ω as a radial function on $\mathbb{C}$.

Remark 10.2. (Meise, Taylor [57, Remark 1.4])

(a) For u as above, P_u is continuous on $\mathbb{C}$ and harmonic in the open upper half plane.

(b) For every weigt function ω there exists $D > 0$ such that for $\omega(z) \leqslant P_u(z) \leqslant D\omega(z) + D$ holds for all $z \in \mathbb{C}$

For the proof of 10.12 we need the following property of the harmonic extension, which is also used by Meise, Taylor [57]. For the sake of completeness we will give a proof.

Lemma 10.3. Let $u\colon \mathbb{R} \to \mathbb{R}$ be continuous such that $\int_{-\infty}^{\infty} \frac{|u(t)|}{1+t^2} dt < \infty$ and let ν, μ and $A \in \mathbb{R}$. Then

$$P_{\min(\nu u + A, \mu u - A)}(z) \leqslant \min(\nu P_u(z) + A, \mu P_u(z) - A)$$

holds for each $z \in \mathbb{C}$.

Proof. Let $z = x + iy$ be given. In view of the definition of P the case $y = 0$ is trivial. If $|y| > 0$, we have

$$\begin{aligned}
P_{\min(\nu u + A, \mu u - A)}(x+iy) &= \tfrac{|y|}{\pi} \int_{-\infty}^{\infty} \tfrac{\min(\nu u(t)+A, \mu u(t) - A)}{(t-x)^2+y^2} dt \\
&\leqslant \tfrac{|y|}{\pi} \min\left(\int_{-\infty}^{\infty} \tfrac{\nu u(t)+A}{(t-x)^2+y^2} dt, \int_{-\infty}^{\infty} \tfrac{\mu u(t) - A}{(t-x)^2+y^2} dt \right) \\
&= \tfrac{|y|}{\pi} \min\left(\nu \int_{-\infty}^{\infty} \tfrac{u(t)}{(t-x)^2+y^2} dt + A \int_{-\infty}^{\infty} \tfrac{1}{(t-x)^2+y^2} dt, \right. \\
&\qquad \left. \mu \int_{-\infty}^{\infty} \tfrac{u(t)}{(t-x)^2+y^2} dt - A \int_{-\infty}^{\infty} \tfrac{1}{(t-x)^2+y^2} dt \right) \\
&= \min\big(\mu P_u(x+iy) + A, \nu P_u(x+iy) - A\big),
\end{aligned}$$

where we used $\int_{-\infty}^{\infty} \frac{1}{(t-x)^2+y^2} dt = \frac{\pi}{|y|}$ for the last equality. ■

Let us now state the decomposition lemma.

Lemma 10.4. (Meise, Taylor [57, Lemma 2.1]) Let p_1, p_2 and $u\colon \mathbb{C} \to \mathbb{R}$ be continuous functions and let u be subharmonic. Let $f \in H(\mathbb{C})$, let $\Omega \subseteq \mathbb{C}$ be a region and assume that (1)-(4) holds, where

(1) $\forall\, z \in \overline{\Omega}\colon |f(z)| \leqslant e^{p_1(z)}$,
(2) $\forall\, z \notin \Omega\colon |f(z)| \leqslant e^{p_2(z)}$,
(3) $\forall\, z \in \mathbb{C}\colon u(z) \leqslant \min(p_1(z), p_2(z))$,
(4) $\forall\, z \in \partial\Omega$, $w \in \mathbb{C}$ with $|w| \leqslant 1\colon |f(z+w)| \leqslant e^{u(z+w)}$.

Then there exists C – independent of f – and f_1, $f_2 \in H(\mathbb{C})$ such that $f = f_1 + f_2$ and (5) and (6) are satisfied, where

(5) $\int_{\mathbb{C}} |f_j(z)|^2 e^{-2p_j(z) - 4\log(1+|z|^2)} d\lambda(z) \leqslant C^2$ for $j = 1$, 2,
(6) $\int_{\mathbb{C}} |f_j(z)|^2 e^{-2\max(u(z),\, \log|f(z)| - 4\log(1+|z|^2)} d\lambda(z) \leqslant C^2$ for $j = 1$, 2

and λ denoted the Lebesgue measure on $\mathbb{C}$.

The next result is only a slight modification of [57, Lemma 2.2].

Lemma 10.5. Let p, q and $r\colon \mathbb{C} \to \mathbb{R}$ be continuous. Put

$$\begin{aligned} U &:= \big\{f \in H(\mathbb{C})\,;\, \sup_{z\in\mathbb{C}}|f(z)|e^{-p(z)} \leqslant 1\big\},\\ V &:= \big\{f \in H(\mathbb{C})\,;\, \textstyle\int_{\mathbb{C}}|f(z)|^2 e^{-2q(z)-4\log(1+|z|^2)}d\lambda(z) \leqslant 1\big\},\\ W &:= \big\{f \in H(\mathbb{C})\,;\, \textstyle\int_{\mathbb{C}}|f(z)|^2 e^{-2r(z)-4\log(1+|z|^2)}d\lambda(z) \leqslant 1\big\}. \end{aligned}$$

Assume that there exist D and $A_0 > 0$, such that for each $A \geqslant A_0$ there is a subharmonic function $u_A\colon \mathbb{C} \to \mathbb{R}$ and a region $\Omega_A \subseteq \mathbb{C}$ such that

(1) $\forall\, z \in \overline{\Omega}_A\colon p(z) \leqslant q(z) + A$,
(2) $\forall\, z \notin \Omega_A\colon p(z) \leqslant r(z) - A$,
(3) $\forall\, z \in \mathbb{C}\colon u_A(z) \leqslant \min(q(z) + A, r(z) - A)$,
(4) $\forall\, z \in \partial\Omega_A$, $w \in \mathbb{C}$ with $|w| \leqslant 1\colon p(z) - D \leqslant u_A(z + w)$.

Then for all $\varepsilon > 0$ there is $S > 0$ such that $U \subseteq SV + \varepsilon W$.

Proof. Let C denote the constant of 10.4. Let $\varepsilon > 0$ be given and assume w.l.o.g. that $\varepsilon \leqslant Ce^{D-A_0}$. We put $\varepsilon' := C^{-1}e^{-D}\varepsilon$ and $S := Ce^D\varepsilon'^{-1}$. We select $A := \log\frac{1}{\varepsilon'}$, i.e. $\varepsilon' = e^{-A}$. By our assumption on ε we have $\varepsilon' = C^{-1}e^{-D}\varepsilon \leqslant e^{-A_0}$ hence $\frac{1}{\varepsilon'} \geqslant e^{A_0}$ and thus $A = \log\frac{1}{\varepsilon'} \geqslant A_0$.
Now we put $p_1(z) := q(z)+A$, $p_2(z) := r(z)-A$. If f is in U, we have $|f(z)| \leqslant e^{p(z)}$ for all $z \in \mathbb{C}$, hence

(1) $|e^{-D}f(z)| \leqslant e^{-D}e^{p(z)} \leqslant e^{q(z)+A-D} \leqslant e^{p_1}(z)$ for $z \in \overline{\Omega}_A$,
(2) $|e^{-D}f(z)| \leqslant e^{-D}e^{p(z)} \leqslant e^{r(z)-A-D} \leqslant e^{p_2}(z)$ for $z \notin \Omega_A$,
(2) $u_A(z) \leqslant \min(q(z) + A, r(z) - A) = \min(p_1(z), p_2(z))$ for $z \in \mathbb{C}$,
(4) $e^{-D}f(z + w)| \leqslant e^{-D}e^{p(z)} = e^{p(z)-D} \leqslant e^{u_A(z+w)}$ for $z \in \partial\Omega_A$ and $|w| \leqslant 1$.

Therefore, 10.4 yields f_1 and f_2 with $e^{-D}f = f_1 + f_2$, i.e. $f = e^D(f_1 + f_2)$ such that the estimates in 10.4.(5) are satisfied, i.e.

$$\begin{aligned} \textstyle\int_{\mathbb{C}} |f_1(z)|^2 e^{-2p_1(z)-4\log(1+|z|^2)}d\lambda(z) &= \textstyle\int_{\mathbb{C}} |f_1(z)|^2 e^{-2(q(z)+A)-4\log(1+|z|^2)}d\lambda(z)\\ &= e^{-2A}\textstyle\int_{\mathbb{C}} |f_1(z)|^2 e^{-2q(z)-4\log(1+|z|^2)}d\lambda(z)\\ &\leqslant C^2 \end{aligned}$$

and thus $f_1 \in Ce^AV$ and analogously $f_2 \in Ce^{-A}W$ for each $A \geqslant A_0$. Thus

$$f = e^D(f_1 + f_2) \in Ce^De^AV + Ce^De^{-A}W = Ce^D\varepsilon'^{-1}V + Ce^D\varepsilon'W = SV + \varepsilon W$$

and we are done. ■

A modification of the above proof yields immediately the following.

Scholium 10.6. Let p, q and r, U, V, W be as in 10.5. Assume that there exist D and $A > 0$, a subharmonic function $u\colon \mathbb{C} \to \mathbb{R}$ and a region $\Omega \subseteq \mathbb{C}$ such that

(1) $\forall\, z \in \overline{\Omega}\colon p(z) \leqslant q(z) + A$,
(2) $\forall\, z \notin \Omega\colon p(z) \leqslant r(z) - A$,
(3) $\forall\, z \in \mathbb{C}\colon u(z) \leqslant \min(q(z) + A, r(z) - A)$,
(4) $\forall\, z \in \partial\Omega,\ w \in \mathbb{C}$ with $|w| \leqslant 1\colon p(z) - D \leqslant u(z+w)$.

Then there is $S > 0$ such that $U \subseteq S(V + W)$.

For later use we state the following special case of [57, 2.3].

Lemma 10.7. (Meise, Taylor [57, Lemma 2.3 for $\alpha = \beta = 1$]) Let $\omega\colon \mathbb{R} \to [0, \infty[$ be even and subadditive with $\int_{-\infty}^{\infty} \frac{\omega(t)}{1+t^2} dt < \infty$. For $a,\ b \in \mathbb{R}$ with $a > b$ and for $A > 0$ define $\psi\colon \mathbb{R} \to \mathbb{R}$ by $\psi(t) := \min(a\omega(t) - \alpha A, b\omega(t) + \beta A)$. Let h denote the harmonic extension of ψ. Then the following statements are true.

(1) $\sup_{z\in\mathbb{C}} \sup_{|w|\leqslant 1} |h(z+w) - h(z)| \leqslant \max(|a|, |b|)\big(\omega(1) + 2\max_{y\in[0,1]} P_\omega(iy)\big)$.
(2) $\left|\frac{\partial h}{\partial y}\right| \leqslant \max(|a|, |b|)\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\omega(t)}{t^2+y^2} dt$ for all $x \in \mathbb{R}$ and $y > 0$.

Let us from now on assume that ω is a subadditive weight function. The following definitions are slight modifications of those stated in [57, Notation 3.1]. For $\mu > \nu > 0$ and $A > 0$ we define

$$\psi(\cdot, \nu, \mu, A)\colon \mathbb{R} \to \mathbb{R},\ \ \psi(t, \nu, \mu, A) := \min(\nu\omega(t) + A, \mu\omega(t) - A).$$

By $H(\cdot, \nu, \mu, A)$ we denote the harmonic extension P_ψ of $\psi(\cdot, \nu, \mu, A)$ which is continuous on $\mathbb{C}$ and harmonic in the open upper and lower half plane by 10.2.(a), since ψ is continuous by definition and $\int_{-\infty}^{\infty} \frac{\psi(t)}{1+t^2} dt < \infty$ since $\int_{-\infty}^{\infty} \frac{\omega(t)}{1+t^2} < \infty$ holds by (WF 1).

Since $\omega(0) = 0$ and $\lim_{t\to\infty} \omega(t) = \infty$ holds by (WF 3), we have $\omega([0, \infty[) = [0, \infty[$. Therefore, the equation $\omega(t) = \frac{2A}{\mu-\nu}$ has at least one positive solution t_1. Note that $t = 0$ cannot be a solution. Let $N := \{t \in [0, \infty[\,;\ \omega(t) = \frac{2A}{\mu-\nu}\}$ be the set of all solutions. Since ω is continuous and increasing with $\lim_{t\to\infty} \omega(t) = \infty$, N has to be of the form $[a', b']$ with $0 < a' \leqslant b'$. With φ as in (WF 2) we get that there is an intervall $[a, b] \subseteq \mathbb{R}$ such that $\varphi(t) < \frac{2A}{\mu-\nu}$ for $t < a$ and $\varphi(t) = \frac{2A}{\mu-\nu}$ for $a \leqslant t \leqslant b$. If N has more than one element we obtain $a < b$ and the latter statement contradicts the convexity of φ. Therefore there is one and only one solution of $\omega(t) = \frac{2A}{\mu-\nu}$ and we may denote this solution by $\xi(\nu, \mu, A)$. With this notation we have

$$\psi(t, \nu, \mu, A) = \begin{cases} \mu\omega(t) - A & \text{for } |t| \leqslant \xi(\nu, \mu, A), \\ \nu\omega(t) + A & \text{for } |t| \geqslant \xi(\nu, \mu, A). \end{cases}$$

By $R(\nu, \mu, A)$ we denote the set

$$R(\nu, \mu, A) := \big\{z \in \mathbb{C}\,;\ \max(|\operatorname{Re} z|, |\operatorname{Im} z|) < \xi(\nu, \mu, A)\big\},$$

and put $\overline{R}(\nu, \mu, A) := \overline{R(\nu, \mu, A)}$.

By the above, the function $H(\cdot,\nu,\mu,A)$ can be written as follows

$$H(x+iy,\nu,\mu,A) = \begin{cases} \frac{|y|}{\pi}\int_{-\infty}^{\infty}\frac{\psi(t,\nu,\mu,A)}{(t-x)^2+y^2}dt & \text{if } |y|>0, \\ \psi(x,\nu,\mu,A) & \text{if } |y|=0. \end{cases}$$

Therefore, we get immediately that $H(x+iy,\nu,\mu,A) = H(x-iy,\nu,\mu,A)$ holds for all x and $y\in\mathbb{R}$. Moreover, we have for $|y|>0$ and $\xi := \xi(\nu,\mu,A)$ by the substitution $s := -t$

$$\begin{aligned} H(-x+iy,\nu,\mu,A) &= \frac{|y|}{\pi}\int_{-\infty}^{\infty}\frac{\psi(t,\nu,\mu,A)}{(t+x)^2+y^2}dt \\ &= \frac{|y|}{\pi}\left[\int_{-\infty}^{-\xi}\frac{\nu\omega(t)+A}{(t+x)^2+y^2}dt + \int_{-\xi}^{\xi}\frac{\mu\omega(t)-A}{(t+x)^2+y^2}dt + \int_{\xi}^{\infty}\frac{\nu\omega(t)+A}{(t+x)^2+y^2}dt\right] \\ &= \frac{|y|}{\pi}\left[\int_{\infty}^{\xi}\frac{\nu\omega(-s)+A}{(-s+x)^2+y^2}(-1)ds + \int_{-\xi}^{\xi}\frac{\mu\omega(-s)-A}{(-s+x)^2+y^2}(-1)ds\right. \\ &\qquad\left. + \int_{-\xi}^{-\infty}\frac{\nu\omega(-s)+A}{(-s+x)^2+y^2}(-1)ds\right] \\ &= \frac{|y|}{\pi}\left[\int_{\xi}^{\infty}\frac{\nu\omega(s)+A}{(s-x)^2+y^2}ds + \int_{\xi}^{-\xi}\frac{\mu\omega(s)-A}{(s-x)^2+y^2}ds\right. \\ &\qquad\left. + \int_{-\infty}^{-\xi}\frac{\nu\omega(s)+A}{(s-x)^2+y^2}ds\right] \\ &= \frac{|y|}{\pi}\int_{-\infty}^{\infty}\frac{\psi(s,\nu,\mu,A)}{(s-x)^2+y^2}ds \\ &= H(x+iy,\nu,\mu,A). \end{aligned}$$

Finally, for $|y|=0$ we have $H(-x+iy,\nu,\mu,A) = \psi(-x,\nu,\mu,A) = \psi(x,\nu,\mu,A) = H(x+iy,\nu,\mu,A)$, hence $H(-x+iy,\nu,\mu,A) = H(x+iy,\nu,\mu,A)$ for all x and $y\in\mathbb{R}$.

The following lemma was stated by Meise, Taylor [57, Lemma 3.2] for the special case $\nu = 1$ and $\mu\in\mathbb{N}$, $\mu\geqslant 2$.

Lemma 10.8. For ω, ν, μ and A as above we have

$$\inf_{z\in\partial R(\nu,\mu,A)} H(z,\nu,\mu,A) \geqslant \delta A \quad \text{with} \quad 0<\delta := \tfrac{1}{\pi}(\arctan(2)-\arctan(\tfrac{3}{2})) < 1.$$

Proof. To simplify notation let us put $H := H(\cdot,\nu,\mu,A)$, $\psi := \psi(\cdot,\nu,\mu,A)$ and $\xi := \xi(\nu,\mu,A)$. By the symmetry properties of H we stated above it is enough to show the following

(1) $H(\xi+iy)\geqslant\delta A$ for $y\in[0,\xi]$,
(2) $H(x+i\xi)\geqslant\delta A$ for $x\in[0,\xi]$.

The case $y=0$ is easy since $H(\xi)=\psi(\xi)=\nu\omega(\xi)+A\geqslant A\geqslant\delta A$.
(1) We have

$$\frac{y}{\pi}\int_{\xi}^{\infty}\frac{1}{(t-\xi)^2+y^2}dt \overset{s:=t-\xi}{=} \frac{y}{\pi}\int_0^{\infty}\frac{1}{s^2+y^2}ds = \frac{1}{2},$$

$$\frac{y}{\pi}\int_{-\xi}^{\xi}\frac{1}{(t-\xi)^2+y^2}\,dt \overset{s:=t-\xi}{=} \frac{y}{\pi}\int_{-2\xi}^{0}\frac{1}{s^2+y^2}\,ds = \frac{1}{\pi}\arctan(\tfrac{2\xi}{y}) \leqslant \frac{1}{\pi}\frac{\pi}{2} = \frac{1}{2},$$

$$\frac{y}{\pi}\int_{-\infty}^{-\xi}\frac{1}{(t-\xi)^2+y^2}\,dt \overset{s:=t-\xi}{=} \frac{y}{\pi}\int_{-\infty}^{-2\xi}\frac{1}{s^2+y^2}\,ds \overset{s:=-t}{=} \frac{y}{\pi}\int_{\infty}^{2\xi}\frac{1}{t^2+y^2}(-1)\,dt = \frac{y}{\pi}\int_{2\xi}^{\infty}\frac{1}{t^2+y^2}\,dt.$$

Thus

$$\begin{aligned}
H(\xi+iy) &= \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\psi(t)}{(t-\xi)^2+y^2}\,dt\\
&= \frac{y}{\pi}\int_{|t|\geqslant\xi}\frac{\nu\omega(t)+A}{(t-\xi)^2+y^2}\,dt + \frac{y}{\pi}\int_{|t|\leqslant\xi}\frac{\mu\omega(t)-A}{(t-\xi)^2+y^2}\,dt\\
&= \frac{y}{\pi}\int_{|t|\geqslant\xi}\frac{\nu\omega(t)}{(t-\xi)^2+y^2}\,dt + \frac{y}{\pi}\int_{|t|\geqslant\xi}\frac{A}{(t-\xi)^2+y^2}\,dt\\
&\quad + \frac{y}{\pi}\int_{|t|\leqslant\xi}\frac{\mu\omega(t)}{(t-\xi)^2+y^2}\,dt - \frac{y}{\pi}\int_{|t|\leqslant\xi}\frac{A}{(t-\xi)^2+y^2}\,dt\\
&\geqslant \frac{y}{\pi}\int_{|t|\geqslant\xi}\frac{\nu\omega(t)}{(t-\xi)^2+y^2}\,dt + \frac{A}{2} + \frac{y}{\pi}\int_{|t|\leqslant\xi}\frac{\mu\omega(t)}{(t-\xi)^2+y^2}\,dt - \frac{A}{2}\\
&\geqslant \frac{y}{\pi}\int_{-\xi}^{\xi}\frac{\mu\omega(t)}{(t-\xi)^2+y^2}\,dt\\
&\overset{(\alpha)}{=} \frac{y}{\pi}\int_{-2\xi}^{0}\frac{\mu\omega(s+\xi)}{s^2+y^2}\,ds\\
&\geqslant \frac{y}{\pi}\int_{-\xi/2}^{0}\frac{\mu\omega(s+\xi)}{s^2+y^2}\,ds\\
&\overset{(\beta)}{\geqslant} \frac{y\mu\omega(\xi/2)}{\pi}\int_{-\xi/2}^{0}\frac{1}{s^2+y^2}\,ds\\
&= \frac{\mu\omega(\xi/2)}{\pi}\int_{-\xi/2}^{0}\frac{y}{s^2+y^2}\,ds\\
&= \frac{\mu\omega(\xi/2)}{\pi}\arctan(\tfrac{\xi}{2y})\\
&\overset{(\gamma)}{\geqslant} \frac{\mu\omega(\xi/2)}{\pi}\arctan(\tfrac{1}{2})\\
&\overset{(\delta)}{\geqslant} \frac{\mu}{2\pi}\omega(\xi)\arctan(\tfrac{1}{2})\\
&= \frac{\mu}{2\pi}\frac{2A}{\mu-\nu}\arctan(\tfrac{1}{2})\\
&\geqslant \frac{A}{\pi}\arctan(\tfrac{1}{2})\\
&\geqslant \delta A
\end{aligned}$$

where we get (α) by substituting $s := t-\xi$. (β) follows since $s \in [-\xi/2, 0]$ implies $s+\xi \in [\xi/2, 3\xi/2]$ and ω is increasing. (γ) we get since arctan is increasing and $y \leqslant \xi$ implies $\frac{1}{2} \leqslant \frac{\xi}{2y}$. Finally, (δ) follows from the fact that ω is subadditive, i.e. $(\omega(\xi) = \omega(\xi/2+\xi/2) \leqslant \omega(\xi/2)+\omega(\xi/2) = 2\omega(\xi/2)$ which yields $\omega(\xi)/2 \leqslant \omega(\xi/2)$.

(2) We define
$$u\colon \mathbb{R}\to\mathbb{R},\ u(t):=\begin{cases} 1 & \text{if } |t|\geqslant \xi,\\ -1 & \text{if } |t|<\xi.\end{cases}$$
By Meise, Taylor [57, p. 54] we have

$$(\circ)\quad \min_{x\in\mathbb{R}} P_u(x+i\xi)=P_u(i\xi)=\tfrac{\xi}{\pi}\int_{-\infty}^{\infty}\tfrac{\xi}{t^2+\xi^2}dt=\tfrac{2}{\pi}\int_{\xi}^{\infty}\tfrac{\xi}{t^2+\xi^2}dt-\tfrac{2}{\pi}\int_{0}^{\xi}\tfrac{\xi}{t^2+\xi^2}dt=0.$$

For $x\in[0,\xi]$ and $t\in[-\xi,-\xi/2]$ we have $t-\xi\leqslant t-x\leqslant -\xi/2$ and thus
$$(\star)\quad (t-\xi)^2\geqslant (t-x)^2.$$
Now we estimate

$$\begin{aligned}
H(x+i\xi)&=\tfrac{\xi}{\pi}\int_{-\infty}^{\infty}\tfrac{\psi(t)}{(t-x)^2+\xi^2}dt\\
&=\tfrac{\xi}{\pi}\int_{|t|\geqslant\xi}\tfrac{\nu\omega(t)+A}{(t-x)^2+\xi^2}dt+\tfrac{\xi}{\pi}\int_{|t|\leqslant\xi}\tfrac{\mu\omega(t)-A}{(t-x)^2+\xi^2}dt\\
&\geqslant\tfrac{\xi}{\pi}\int_{|t|\geqslant\xi}\tfrac{A}{(t-x)^2+\xi^2}dt+\tfrac{\xi}{\pi}\int_{|t|\leqslant\xi}\tfrac{-A}{(t-x)^2+\xi^2}dt+\tfrac{\xi}{\pi}\int_{|t|\leqslant\xi}\tfrac{\mu\omega(t)}{(t-x)^2+\xi^2}dt\\
&=A\tfrac{\xi}{\pi}\int_{-\infty}^{\infty}\tfrac{u(t)}{(t-x)^2+\xi^2}dt+\tfrac{\xi}{\pi}\int_{|t|\leqslant\xi}\tfrac{\mu\omega(t)}{(t-x)^2+\xi^2}dt\\
&\overset{\text{dfn}}{=}AP_u(x+i\xi)+\tfrac{\xi}{\pi}\int_{-\xi}^{\xi}\tfrac{\mu\omega(t)}{(t-x)^2+\xi^2}dt\\
&\overset{(\circ)}{\geqslant}A\cdot 0+\tfrac{\xi}{\pi}\int_{-\xi}^{-\frac{\xi}{2}}\tfrac{\mu\omega(t)}{(t-x)^2+\xi^2}dt\\
&\overset{(\alpha)}{\geqslant}\tfrac{\xi}{\pi}\mu\omega(\tfrac{\xi}{2})\int_{-\xi}^{-\frac{\xi}{2}}\tfrac{1}{(t-x)^2+\xi^2}dt\\
&\overset{(\star)}{\geqslant}\tfrac{\mu}{\pi}\omega(\tfrac{\xi}{2})\int_{-\xi}^{-\frac{\xi}{2}}\tfrac{\xi}{(t-\xi)^2+\xi^2}dt\\
&\overset{(\beta)}{=}\tfrac{\mu}{\pi}\omega(\tfrac{\xi}{2})\int_{-2\xi}^{-\frac{3}{2}\xi}\tfrac{\xi}{s^2+\xi^2}ds\\
&=\tfrac{\mu}{\pi}\omega(\tfrac{\xi}{2})[\arctan(2)-\arctan(\tfrac{3}{2})]\\
&\geqslant\tfrac{A}{\pi}[\arctan(2)-\arctan(\tfrac{3}{2})]\\
&=\delta A
\end{aligned}$$

where (α) holds since $\min_{t\in[-\xi,-\frac{\xi}{2}]}\omega(t)=\min_{t\in[\frac{\xi}{2},\xi]}\omega(t)=\omega(\frac{\xi}{2})$ as ω is increasing. (β) follows by substituting $s:=t-\xi$. ∎

Now we have collected all the preliminary results and thus in the next section we can start our investigation of the (PLB)-spaces.

10.2 A sufficient condition for the vanishing of $\mathrm{Proj}^1\mathcal{A}\mathrm{H}$

As we indicated already, in [57, Proposition 3.4] Meise, Taylor showed that the strong dual of the space $A_{\mathbb{P}}(\mathbb{C}) = \mathrm{ind}_n\, Ha_n(\mathbb{C})$ with $a_n(z) = \exp(-(|\operatorname{Im} z| + n\omega(z)))$, i.e. $\mathbb{P} = (|\operatorname{Im} z| + n\omega(z))_{n\in\mathbb{N}}$ in their notation, enjoys the property (DN) for each weight function ω.

In the sequel we study (PLB)-spaces $AH(\mathbb{C})$ for a sequence $\mathcal{A} = ((a_{N,n})_{n\in\mathbb{N}})_{N\in\mathbb{N}}$ of the following form. As in the previous section, let ω be a subadditive weight function. Let $\mathcal{U} := ((u_{N,n})_{n\in\mathbb{N}})_{N\in\mathbb{N}} \subseteq]0,\infty[$ be a double sequence which satisfies $u_{N+1,n} \leqslant u_{N,n} \leqslant u_{N,n+1}$. Then we put $p_{N,n}(z) := |\operatorname{Im} z| + u_{N,n}\omega(z)$, $\mathcal{P} := ((p_{N,n})_{n\in\mathbb{N}})_{N\in\mathbb{N}}$ and finally $\mathcal{A} := \exp(-\mathcal{P})$.

As Meise, Taylor [57, proof of 3.4] we need the following representation of the steps of the projective spectrum. For the sake of simplicitly in the next result we ommit the N from our notation. Note the similarities to the proof of 9.3.

Lemma 10.9. Let ω be as above and $(u_n)_{n\in\mathbb{N}} \subseteq]0,\infty[$ be increasing. Assume that for each n there is $k > n$ such that $u_k > u_n$. Put $p_n(z) := |\operatorname{Im} z| + u_n\omega(z)$. Then

$$\mathrm{ind}_n\, H^\infty_{p_n}(\mathbb{C}) \cong \mathrm{ind}_n\, H^2_{p_n}(\mathbb{C})$$

where

$$H^\infty_{p_n}(\mathbb{C}) := \big\{ f \in H(\mathbb{C})\,;\, \|f\|_{p_n,\infty} := \sup_{z\in\mathbb{C}} |f(z)| e^{-p_n(z)} < \infty \big\},$$
$$H^2_{p_n}(\mathbb{C}) := \big\{ f \in H(\mathbb{C})\,;\, \|f\|_{p_n,2} := \big[\int_{\mathbb{C}} |f(z)|^2 e^{-2p_n(z)-4\log(1+|z|^2)} d\lambda(z)\big]^{1/2} < \infty \big\}$$

are endowed with the natural Banach space topologies. The proof will show that the associated inductive spectra are equivalent.

Proof. We claim

(a) $\forall\, k\, \exists\, m > k\colon H^2_{p_k}(\mathbb{C}) \subseteq H^\infty_{p_m}(\mathbb{C})$ with continuous inclusion,
(b) $\forall\, k\, \exists\, m > k\colon H^\infty_{p_k}(\mathbb{C}) \subseteq H^2_{p_m}(\mathbb{C})$ with continuous inclusion.

(a) Let k be given. We select m such that $u_m > u_k$. Let $f \in H^2_{p_k}(\mathbb{C})$ and $w \in \mathbb{C}$ be given. We use the trick $(\star)$ of 9.3 with $g := f^2$ and $r := 1$, thus

$$\begin{aligned}
|f(w)|^2 &= |f(w)^2| \\
&\leqslant \tfrac{1}{\pi} \int_{|w-z|\leqslant 1} |f(z)|^2 dm(z) \\
&= \tfrac{1}{\pi} \int_{|w-z|\leqslant 1} |f(z)|^2 e^{-2p_k(z)-4\log(1+|z|^2)} e^{2p_k(z)+4\log(1+|z|^2)} dm(z) \\
&\leqslant \tfrac{1}{\pi} \sup_{|w-z|\leqslant 1} e^{2p_k(z)+4\log(1+|z|^2)} \int_{\mathbb{C}} |f(z)|^2 e^{-2p_k(z)-4\log(1+|z|^2)} dm(z) \\
&= \tfrac{1}{\pi} \sup_{|w-z|\leqslant 1} e^{2|\operatorname{Im} z|+2[u_k\omega(z)+2\log(1+|z|^2)]} \|f\|^2_{p_k,2} \;=: (\circ).
\end{aligned}$$

Since we have $\frac{2}{u_m-u_k}\log(1+t^2) \leqslant \frac{2}{u_m-u_k}\log(t^3) = \frac{6}{u_m-u_k}\log(t)$ for big t, that is $\frac{2\log(1+t^2)}{(u_m-u_k)\omega(t)} \leqslant \frac{6}{u_m-u_k}\frac{\log(t)}{\omega(t)}$ and thus by (WF 3) $\lim_{t\to\infty}\frac{2\log(1+t^2)}{(u_m-u_k)\omega(t)} = 0$, there exists $T \geqslant 0$ such that $\frac{2\log(1+t^2)}{(u_m-u_k)\omega(t)} \leqslant 1$, i.e. $2\log(1+t^2) \leqslant (u_m-u_k)\omega(t)$ for $t \in [T,\infty[$. With $C := \max_{t\in[0,T]} 2\log(1+t^2)$ we thus get $2\log(1+t^2) \leqslant (u_m - u_k)\omega(t)+C$ for all $t \in [0,\infty[$ and therefore $2\log(1+|z|^2) \leqslant (u_m-u_k)\omega(z)+C$ for all $z \in \mathbb{C}$. From the latter we obtain

$$u_k\omega(z) + 2\log(1+|z|^2) \leqslant u_m\omega(z) + C$$

for all $z \in \mathbb{C}$. Now we continue the estimate of $(\circ)$ by

$$\begin{aligned}
(\circ) &= \tfrac{1}{\pi}\sup_{|w-z|\leqslant 1} e^{2|\operatorname{Im} z|+2[u_k\omega(z)+2\log(1+|z|^2)]}\|f\|^2_{p_k,2}\\
&\leqslant \tfrac{1}{\pi}\sup_{|w-z|\leqslant 1} e^{2|\operatorname{Im} z|+2[u_m\omega(z)+C]}\|f\|^2_{p_k,2}\\
&\leqslant \tfrac{1}{\pi}\sup_{|w-z|\leqslant 1} e^{2[1+|\operatorname{Im} w|]+2[u_m(\omega(w)+\omega(1))+C]}\|f\|^2_{p_k,2}\\
&= \tfrac{1}{\pi}e^{2+2u_m\omega(1)+2C}e^{2(|\operatorname{Im} w|+u_m\omega(w))}\|f\|^2_{p_k,2}\\
&= \tfrac{1}{\pi}e^{2+2u_m\omega(1)+2C}e^{2p_m(z)}\|f\|^2_{p_k,2}.
\end{aligned}$$

Here we used $|\operatorname{Im} z| \leqslant 1+|\operatorname{Im} w|$ for $|z-w| \leqslant 1$: Let $z = a+ib$ and $w = x+iy$. Then $1 \geqslant |w-z| = |a-x|+|b-y|$, in particular $|b-y| \leqslant 1$. Hence $|\operatorname{Im} z| = |b| = |b-y+y| \leqslant |b-y|+|y| \leqslant 1+|y| = 1+|\operatorname{Im} w|$. Moreover, we used $\omega(z) \leqslant \omega(w)+\omega(1)$ for $|z-w| \leqslant 1$: We have $\omega(z) = \omega(z-w+w) = \omega(|z-w+w|) \leqslant \omega(|z-w|+|w|) \leqslant \omega(1+|w|) \leqslant \omega(1)+\omega(|w|) = \omega(1)+\omega(w)$, since ω is radial, increasing and subadditive.

Now we put $D := \exp(2+2u_m\omega(1)+2C)$ and get $e^{-2p_m(z)}|f(w)|^2 \leqslant D\|f\|^2_{p_k,2}$, i.e. $e^{-p_m(z)}|f(w)| \leqslant \sqrt{D}\|f\|_{p_k,2}$. Since w was arbitrary, this implies $\|f\|_{p_m,\infty} = \sup_{w\in\mathbb{C}} e^{-p_m(z)}|f(w)| \leqslant \sqrt{D}\|f\|_{p_k,2}$ which shows the desired inclusion and its continuity.

(b) Let $k \in \mathbb{N}$ and $f \in H^2_{p_k}(\mathbb{C})$ be given. We select $m > k$ such that $u_m > u_k$ and compute

$$\begin{aligned}
\|f\|^2_{p_m,2} &= \int_{\mathbb{C}} |f(z)|^2 e^{-2p_m(z)-4\log(1+|z|^2)}d\lambda(z)\\
&= \int_{\mathbb{C}} |f(z)|^2 e^{-2p_k(z)}e^{2p_k(z)-2p_m(z)-4\log(1+|z|^2)}d\lambda(z)\\
&\leqslant \sup_{z\in\mathbb{C}}|f(z)|^2 e^{-2p_k(z)}\int_{\mathbb{C}} e^{2|\operatorname{Im} z|+2u_k\omega(z)-2|\operatorname{Im} z|-2u_m\omega(z)-4\log(1+|z|^2)}d\lambda(z)\\
&= \|f\|^2_{p_k,\infty}\int_{\mathbb{C}} e^{2(u_k-u_m)-4\log(1+|z|^2)}d\lambda(z).
\end{aligned}$$

As in (a) we select $T \geqslant 0$ such that $2\log(1+|z|^2) \leqslant (u_m-u_k)\omega(z)$, i.e. $2(u_k - u_m)\omega(z) \leqslant -4\log(1+|z|^2)$ holds for $|z| \geqslant T$. Thus, we have

$$\int_{\mathbb{C}} e^{2(u_k-u_m)-4\log(1+|z|^2)}d\lambda(z) \leqslant \int_{\mathbb{C}} e^{-8\log(1+|z|^2)}d\lambda(z) = \int_{\mathbb{C}} \tfrac{1}{(1+|z|^2)^8}d\lambda(z) < \infty$$

which finishes the proof. ■

Under the rather natural condition from the beginning of 10.9 for each step (that is $\forall\, N, n\ \exists\, k > n\colon u_{N,k} > u_{N,n}$) it follows from the above that for each N the sequence $(B_{N,n})_{n\in\mathbb{N}}$ and also $(C_{N,n})_{n\in\mathbb{N}}$, where

$$B_{N,n} = \{f \in H(\mathbb{C})\,;\, \|f\|_{N,n} \leqslant 1\} \;\text{ and }\; C_{N,n} = \{f \in H(\mathbb{C})\,;\, |f|_{N,n} \leqslant 1\},$$

both form fundamental sequences of bounded sets in the inductive limit $\mathcal{A}_N(\mathbb{C})$.

Remark 10.10. Assume that

$$\forall\, N\ \exists\, M \geqslant N\ \forall\, K \geqslant M\ \exists\, n\ \forall\, m,\, \varepsilon > 0\ \exists\, k,\, S > 0\colon B_{M,m} \subseteq \varepsilon C_{N,n} + SC_{K,k}$$

holds. Then

$$\forall\, N\ \exists\, M \geqslant N\ \forall\, K \geqslant M\ \exists\, B \in \mathcal{BD}_N\ \forall\, A \in \mathcal{BD}_M\ \exists\, C \in \mathcal{BD}_K\colon A \subseteq B + C$$

holds, where $\mathcal{BD}_N$ denotes the set of Banach discs in the space $\mathcal{A}_N H(\mathbb{C})$. The latter condition is those of [84, 3.2.14], i.e. it implies that $\operatorname{Proj}^1 \mathcal{A}H = 0$.

Proof. For given N we select M as in the first condition. For given $K \geqslant M$ we select n as in the first condition and put $B := C_{N,n}$, which is clearly a Banach disc. Let A be given. A is a Banach disc in the space $\mathcal{A}_M H(\mathbb{C})$ and thus there exists m such that $A \subseteq Ha_{M,m}(\mathbb{C})$ and A is a Banach disc in this space, i.e. in particular A is bounded hence there exists $S' > 0$ such that $A \subseteq S' B_{M,m}$. We put $\varepsilon := 1/S'$ and the forementioned m into the first condition and obtain k and $S > 0$. We put $C := SS'C_{K,k}$ and obtain

$$A \subseteq S'B_{M,m} \subseteq S'(\varepsilon C_{N,n} + SC_{K,k}) \subseteq C_{N,n} + SS'C_{K,k} = B + C$$

which finishes the proof. ■

Definition 10.11. Let ω be as above. By 10.2.(b) there exists $C(\omega) \geqslant 1$ such that $P_\omega(z) \leqslant C(\omega)\omega(z) + C(\omega)$ for $z \in \mathbb{C}$. We put

$$S(\omega) := 4\big(\max(C(\omega), (\tfrac{1}{\pi}(\arctan(2) - \arctan(\tfrac{3}{2})))^{-1})\big)^2$$

and say that the sequence $\mathcal{U} = ((u_{N,n})_{n\in\mathbb{N}})_{N\in\mathbb{N}}$ satisfies condition $(\overline{\mathrm{Q}})_\omega$, if

$$\forall\, N\ \exists\, M \geqslant N\ \forall\, K \geqslant M\ \exists\, n\ \forall\, m\ \exists\, k\colon u_{K,k} \geqslant S(\omega)u_{M,m} + u_{N,n}.$$

Proposition 10.12. Let $\mathcal{U}$ satisfy $(\overline{\mathrm{Q}})_\omega$. Then

$\exists\, C \geqslant 1\ \forall\, N\ \exists\, M \geqslant N\ \forall\, K \geqslant M\ \exists\, n\ \forall\, m\ \exists\, k,\, D > 0\ \forall\, A \geqslant 1\ \exists\, u_A$ subharmonic:

(1) $p_{M,m}(z) \leqslant p_{N,n}(z) + A$ for $z \in \overline{R}(u_{N,n}, u_{K,k}, CA)$,
(2) $p_{M,m}(z) \leqslant p_{K,k}(z) - A$ for $z \notin R(u_{N,n}, u_{K,k}, CA)$,
(3) $u_A(z) \leqslant \min(p_{N,n}(z) + A, p_{K,k} - A)$ for $z \in \mathbb{C}$,
(4) $p_{M,m}(z + w) - D \leqslant u_A(z + w)$ for $z \in \partial R(u_{N,n}, u_{K,k}, CA)$ and $|w| \leqslant 1$.

Proof. We put $C := \max(C(\omega), (\frac{1}{\pi}(\arctan(2) - \arctan(\frac{3}{2})))^{-1})$, i.e. $C \geqslant C(\omega)$, hence

$$(5) \qquad P_\omega(z) \leqslant C\omega(z) + C$$

holds for $z \in \mathbb{C}$. Moreover, we put $\delta := \min(C(\omega)^{-1}, \frac{1}{\pi}(\arctan(2) - \arctan(\frac{3}{2})))$ that is the statement of 10.8 is true for this δ (and arbitrary ν, μ and A). In addition we have $S(\omega) = \frac{4C}{\delta}$. Now let N be given. We select $M \geqslant N$ according to $(\overline{\mathrm{Q}})_\omega$. Given $K \geqslant M$, we select n according to $(\overline{\mathrm{Q}})_\omega$. Given m we select k according to $(\overline{\mathrm{Q}})_\omega$, i.e. $u_{K,k} \geqslant \frac{4C}{\delta} u_{M,m} + u_{N,n}$. Now we select $b > 0$ such that

$$(6) \qquad \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\omega(t)}{t^2+b^2} dt \leqslant \frac{C}{u_{K,k}}$$

holds. This is possible since $f_n \stackrel{\mathrm{co}}{\to} 0$ where $f_n(t) := \frac{\omega(t)}{n^2+t^2}$, $g(t) := \frac{\omega(t)}{1+t^2}$ satisfies $|f_n| \leqslant g$ on $\mathbb{R}$ and $\int_{-\infty}^{\infty} g(t)dt < \infty$ by our assumptions on ω. Now we define $u(\cdot, A) \colon \mathbb{C} \to \mathbb{R}$ by

$$u(z, A) := \begin{cases} |\operatorname{Im} z| + \frac{1}{C} H(z + ib, u_{N,n}, u_{K,k}, CA) & \text{for } \operatorname{Im} z \geqslant 0, \\ |\operatorname{Im} z| + \frac{1}{C} H(\overline{z} + ib, u_{N,n}, u_{K,k}, CA) & \text{for } \operatorname{Im} z < 0. \end{cases}$$

We claim that $u(\cdot, A)$ is subharmonic. $u(\cdot, A)$ is continuous on $\mathbb{C}$ and harmonic in the open upper and lower half plane. From (6) and 10.7.(2) we get that $\big|H(x + iy, u_{N,n}, u_{K,k}, CA)\big| \leqslant C$ for each $x \in \mathbb{R}$. Hence we have for each $v \in C_c(\mathbb{C})$ with $v \leqslant 0$

$$\int_{\mathbb{C}} u(z, A)\delta v(z) d\lambda(z) = 2 \int_{-\infty}^{\infty} \big(1 - \tfrac{1}{C} \tfrac{\partial}{\partial y} H(x + iy, u_{N,n}, u_{K,k}, CA)\big) v(x) dx \geqslant 0,$$

which shows the claim.

To prove (1), note that we have

$$(7) \qquad \omega(z) \leqslant \omega(2\xi(u_{N,n}, u_{K,k}, CA)) \leqslant 2\omega(\xi(u_{N,n}, u_{K,k}, CA)) = \frac{4CA}{u_{K,k} - u_{N,n}}$$

for $z \in \overline{R}(u_{N,n}, u_{K,k}, CA)$, since ω is increasing, $|z| \leqslant 2\xi(u_{N,n}, u_{K,k}, CA)$ and ω is subadditive. The last equality in (7) is just the definition of $\xi(u_{N,n}, u_{K,k}, CA)$. Since we have $u_{K,k} \geqslant \frac{4C}{\delta} u_{M,m} + u_{N,n}$ we get $u_{K,k} - u_{N,n} \geqslant \frac{4C}{\delta} u_{M,m}$ and hence $\delta \geqslant \frac{4C}{u_{K,k} - u_{N,n}} u_{M,m}$. Thus

$$u_{M,m}\omega(z) \stackrel{(7)}{\leqslant} u_{M,m} \frac{4C}{u_{K,k} - u_{N,n}} A \leqslant \delta A \stackrel{\delta<1}{\leqslant} A \leqslant u_{N,n}\omega(z) + A$$

for each $z \in \overline{R}(u_{N,n}, u_{K,k}, CA)$, which shows (1).

To prove (2), we note that $z \notin R(u_{N,n}, u_{K,k}, CA)$ implies $|z| \geqslant \xi(u_{N,n}, u_{K,k}, CA)$ which implies $\omega(z) \geqslant \omega(\xi(u_{n,n}, u_{K,k}, CA) = \frac{2CA}{u_{K,k} - u_{N,n}}$ and thus $\omega(z) \frac{u_{K,k} - u_{N,n}}{2C} \leqslant A$, since ω is increasing. This yields

$$u_{M,m}\omega(z) + A \leqslant \omega(z)\big(u_{M,m} + \tfrac{u_{K,k} - u_{N,n}}{2C}\big) \stackrel{(\star)}{\leqslant} u_{K,k}\omega(z)$$

for $z \notin R(u_{N,n}, u_{K,k}, CA)$. The estimate $(\star)$ can be seen as follows. Since we have

$(4C)(2C-1) \geqslant 2 = 2C\delta$ it follows $\frac{4C}{\delta} \geqslant \frac{2C}{2C-1}$ and hence $u_{K,k} \geqslant \frac{4C}{\delta}u_{M,m} + u_{N,n} \geqslant \frac{2C}{2C-1}u_{M,m} - \frac{1}{2C-1}u_{N,n} = \frac{1}{2C-1}(2Cu_{M,m} - u_{N,n})$ which yields $2Cu_{K,k} - u_{K,k} = (2C-1)u_{K,k} \geqslant 2Cu_{M,m} - u_{N,n}$, therefore $2Cu_{K,k} \geqslant 2Cu_{M,m} + u_{K,k} - u_{N,n}$ and finally $u_{K,k} \geqslant u_{M,m} + \frac{u_{K,k}-u_{N,n}}{2C}$. This finishes the proof of (2).

In order to show (3) we note that by 10.7.(1) there exists $C_1 = C_1(u_{N,n}, u_{K,k}, \omega, b)$ such that

$$(\circ) \qquad \sup_{z\in\mathbb{C}} \big|H(z+ib, u_{N,n}, u_{K,k}, CA) - H(z, u_{N,n}, u_{K,k}, CA)\big| \leqslant C_1$$

holds, since $u_{K,k} \geqslant \frac{4C}{\delta}u_{M,m} + u_{N,n} > u_{N,n}$. Now we have

$$\begin{aligned} u(z,A) &\overset{(\circ)}{\leqslant} |\operatorname{Im} z| + \tfrac{1}{C}\big(H(z, u_{N,n}, u_{K,k}, CA) + C_1\big) \\ &\overset{10.3}{\leqslant} |\operatorname{Im} z| + \tfrac{1}{C}\big(\min(u_{N,n}P_\omega(z) + CA, u_{K,k}P_\omega(z) - CA)\big) + \tfrac{C_1}{C} \\ &\overset{(5)}{\leqslant} |\operatorname{Im} z| + \min\big(u_{N,n}\omega(z) + A, u_{K,k}\omega(z) - A\big) + u_{N,n} + u_{K,k} + \tfrac{C_1}{C}. \end{aligned}$$

Hence, (3) holds for $u_A(z) := u(z,A) - u_{N,n} - u_{K,k} - \frac{C_1}{C}$.

It remains to check (4). By (7) and since $u_{K,k} - u_{N,n} \geqslant \frac{4C}{\delta}u_{M,m}$ we have

$$A \geqslant \tfrac{u_{K,k}-u_{N,n}}{4C}\omega(z) \geqslant \tfrac{4Cu_{M,m}}{4C\delta}\omega(z) = \tfrac{u_{M,m}}{\delta}\omega(z)$$

and thus

$$(8) \qquad \delta A \geqslant u_{M,m}\omega(z)$$

for $z \in \partial R(u_{N,n}, u_{K,k}, CA)$. By 10.7.(1) there exists $C_2 = C_2(u_{N,n}, u_{K,k}, \omega)$ with

$$\big|H(z+w, u_{N,n}, u_{K,k}, CA) - H(z, u_{N,n}, u_{K,k}, CA)\big| \leqslant C_2$$

for all $z \in \mathbb{C}$ and $|w| \leqslant 1$. Therefore, 10.8 and (8) imply

$$\begin{aligned} u(z+w,A) &\geqslant |\operatorname{Im}(z+w)| + \tfrac{1}{C}\big(H(z+w, u_{N,n}, u_{K,k}, CA) - C_1\big) \\ &\geqslant |\operatorname{Im}(z+w)| + \tfrac{1}{C}H(z, u_{N,n}, u_{K,k}, CA) - \tfrac{C_1+C_2}{C} \\ &\geqslant |\operatorname{Im}(z+w)| + \tfrac{\delta CA}{C} - \tfrac{C_1+C_2}{C} \\ &\geqslant |\operatorname{Im}(z+w)| + u_{M,m}\omega(z) - \tfrac{C_1+C_2}{C} \\ &\geqslant |\operatorname{Im}(z+w)| + u_{M,m}(\omega(z+w) - \omega(1)) - \tfrac{C_1+C_2}{C} \\ &= |\operatorname{Im}(z+w)| + u_{M,m}\omega(z+w) - u_{M,m}\omega(1) - \tfrac{C_1+C_2}{C} \end{aligned}$$

for $z \in \partial R(u_{N,n}, u_{K,k}, CA)$ and $|w| \leqslant 1$. By our choice of u_A we have for $z \in \partial R(u_{N,n}, u_{K,k}, CA)$ and $|w| \leqslant 1$

$$\begin{aligned} u_A(z+w) &= u(z+w,A) - u_{N,n} - u_{K,k} - \tfrac{C_1}{C} \\ &\geqslant |\operatorname{Im}(z+w)| + u_{M,m}\omega(z+w) - u_{M,m}\omega(1) - \tfrac{C_1+C_2}{C} - u_{N,n} - u_{K,k} - \tfrac{C_1}{C} \\ &= p_{M,m}(z+w) - \big(u_{M,m}\omega(1) + \tfrac{C_1+C_2}{C} + u_{N,n} + u_{K,k} + \tfrac{C_1}{C}\big) \end{aligned}$$

that is (4) holds with $D := u_{M,m}\omega(1) + \frac{C_1+C_2}{C} + u_{N,n} + u_{K,k} + \frac{C_1}{C}$ (which is independent of A). ∎

Corollary 10.13. If $\mathcal{U}$ satisfies condition $(\overline{\mathrm{Q}})_\omega$, then

$$\forall\, N\; \exists\, M \geqslant N\; \forall\, K \geqslant M\; \exists\, n\; \forall\, m,\, \varepsilon > 0\; \exists\, k,\, S > 0\colon B_{M,m} \subseteq \varepsilon C_{N,n} + SC_{K,k}$$

holds.

Proof. 10.12 shows that the assumptions of 10.5 are satisfied. ∎

At this point we could already state a result on the vanishing of Proj^1 just by combining the assumptions of 10.13 and those we made previous to 10.10. But the next observation shows that it is possible to drop the second assumption.

Observation 10.14. (1) If $\mathcal{U}$ satisfies condition $(\overline{\mathrm{Q}})_\omega$, then there exists an integer J such that

$$(\star) \qquad \forall\, N \geqslant J,\, n\; \exists\, k > n\colon u_{N,k} > u_{N,n}$$

holds.

(2) Let J be as in (1). Define the spectrum $\mathcal{A}^{>J}H := (\mathcal{A}_{J+N}H(\mathbb{C}))_{N\in\mathbb{N}}$ by cutting off the first $J+1$ spaces. Then clearly $\mathcal{A}^{>J}H \sim \mathcal{A}H$ and $\operatorname{Proj}^1 \mathcal{A}^{>J}H = 0$ if and only if $\operatorname{Proj}^1 \mathcal{A}H = 0$. Moreover, if the original spectrum $\mathcal{A}H$ satisfies

$$\forall\, N\; \exists\, M \geqslant N\; \forall\, K \geqslant M\; \exists\, n\; \forall\, m,\, \varepsilon > 0\; \exists\, k,\, S > 0\colon B_{M,m} \subseteq \varepsilon C_{N,n} + SC_{K,k}$$

the same is true for the truncated spactrum $\mathcal{A}^{>J}H$. In particular, $(\star)$ implies that the sequence underlying the truncated spectrum satisfies

$$\forall\, N,\, n\; \exists\, k > n\colon u_{N,k} > u_{N,n}.$$

Proof. It is enough to show (1). Let $(\overline{\mathrm{Q}})_\omega$ be satisfied. We select $N = 1$. Then there is $M \geqslant 1$ such that

$$\forall\, K \geqslant M\; \exists\, n\; \forall\, m\; \exists\, k\colon u_{K,k} \geqslant S(\omega)u_{M,m} + u_{N,n}.$$

Since $S(\omega) \geqslant 1$, we have $S(\omega)u_{M,m} + u_{N,n} > u_{M,m}$. Moreover, $u_{M,m} \geqslant u_{K,m}$ holds. Thus we get

$$\forall\, K \geqslant M,\, m\; \exists\, k > m\colon u_{K,k} \geqslant u_{K,m},$$

where we used that $u_{K,k+1} \geqslant u_{K,k}$. Thus we get the desired statement by selecting $J := M(1)$. ∎

Theorem 10.15. If $\mathcal{U}$ satisfies condition $(\overline{\mathrm{Q}})_\omega$, then $\operatorname{Proj}^1 \mathcal{A}H = 0$.

Proof. First we apply 10.13. Then we use 10.14.(2) to get that the spectrum $\mathcal{A}^{>J}H$ satisfies the assumptions previous to 10.10. Then we use 10.10 to get the condition in [84, 3.2.14] which yields that $\operatorname{Proj}^1 \mathcal{A}H = 0$. ∎

Remark 10.16. (a) Let us first note that every sequence $\mathcal{U}$ with

$$\forall\, N\; \exists\, M \geqslant N\; \forall\, K \geqslant M\colon u_{K,k} \overset{k\to\infty}{\longrightarrow} \infty$$

satifies $(\overline{\mathrm{Q}})_\omega$. In particular the latter is the case for sequences which satisfy $\lim_{k\to\infty} u_{K,k} = \infty$ for each K.

(b) If $\mathcal{U}$ satisfies condition $(\overline{\mathrm{Q}})_\omega$, then the sequence $\mathcal{A}$ satisfies condition (Q) – Thus, $(\overline{\mathrm{Q}})_\omega$ is a rather strong condition.

(c) Taylor [73] obtained results on the density of the polynomials in weighted Banach spaces $H(a)_0(\mathbb{C})$ for weights of the form $a(z) = \exp(-p(z))$ with $p(z) = u(\mathrm{Re}(z)) + v(|z|)$ under technical conditions on v similar to the conditions (WF 1)-(WF 3). It might be possible to modify his proofs in order to get a similar result for $p(z) = u(\mathrm{Im}(z)) + v(|z|)$. With this it might then be possible to get information on reducedness for the spectra considered in this section what clearly is desireable (cf. (d)). But since Taylor showed that the polynomials are dense in $H(v)_0(\mathbb{C})$ if and only if $\int_0^\infty \frac{v(yt)}{1+t^2}dt = \infty$ for each $y \geqslant 0$ (compare with (WF 1)), that is for quasianalytic weights, it looks as if this method will yield only negative results concerning reducedness in our (non-quasianalytic) setting.

(d) Many concrete sequences $\mathcal{U}$ will yield that the steps of the projective spectrum under investigation are (LS)-spaces. If in addition the spectrum is reduced (cf. (c)) it would be possible to apply 10.6 in conjuction with [84, 3.2.18] to get results on the vanishing of $\mathrm{Proj}^{\,1}\mathcal{A}H = 0$. However, it is questionable, if this approach would yield an improvement of 10.15, since seems not to be possible to replace in this way $(\overline{\mathrm{Q}})_\omega$ by any weaker condition.

(e) In the article [58] Meise and Taylor modified the methods of [57] to prove that the strong dual of $A_{\mathbb{P}}(\mathbb{C}) = \mathrm{ind}_n\, a_n H(\mathbb{C})$ with $a_n(z) = \exp(-(|\operatorname{Im} z| - \frac{1}{n}\omega(z)))$ has property $(\underline{\mathrm{DN}})$ for each weight function ω. A modification of their proofs might yield results similar to those of this section for another class of weights, namely for $p_{N,n}(z) = |\operatorname{Im} z| - \frac{1}{u_{N,n}}\omega(z)$ with $u_{N,n}$ as in 10.2.

11 Condition (B1) revisited

11.1 The (PLB)-case

In this section we extend the remarks on condition (B1), which we made in section 4.4. We used the techniques of section 4.3 (in particular 4.10) in the settings of class $\mathcal{W}$ and $(\mathrm{E})_{\mathrm{C,c}}$. These assumptions had been necessary, since to check condition (B2) we had to decompose holomorphic functions. However, condition (B1) can also be studied in the less-restrictive balanced setting.

Remark 11.1. Assume that we are in the balanced setting. Then we have the following.

(a) $\mathcal{A}_0 H$ satisfies condition (B1).

(b) If $\mathcal{A}H$ satisfies condition (B1), then $\mathcal{A}$ satisfies $(\underline{\mathrm{Q}})^{\sim}_{\mathrm{out}}$ that is

$$\forall\, N \;\exists\, M \;\forall\, m \;\exists\, n \;\forall\, K,\, \varepsilon > 0 \;\exists\, k,\, S > 0 : \big(\tfrac{1}{a_{M,m}}\big)^{\sim} \leqslant \big(\max\big(\tfrac{\varepsilon}{a_{N,n}}, \tfrac{S}{a_{K,k}}\big)\big)^{\sim}.$$

Proof. (a) For fixed N, $n \in \mathbb{N}$ we claim

$$B^{\circ}_{N,n} \subseteq \overline{B^{\circ}_{N,n} \cap (AH)_0(G)}^{H(a_{N,n})_0(G)}.$$

"$\supseteq$" Trivial.

"$\subseteq$" Let $f \in B^{\circ}_{N,n}$ that is $a_{N,n}|f|$ vanishes at infinity and $a_{N,n}|f| \leqslant 1$ on G. Since we are in the balanced setting, the polynomials are contained in $H(a_{N,n})_0(G)$ hence $S_t f \in H(a_{N,n})_0(G)$ and $a_{N,n}|S_t f| \leqslant a_{N,n}|f| \leqslant 1$, i.e. $s_t f \in B^{\circ}_{N,n} \cap (AH)_0(G)$ and by [20, Proposition 1.2.(e)] $S_t f \to f$ w.r.t. $\|\cdot\|_{N,n}$, that is $f \in \overline{B^{\circ}_{N,n} \cap (AH)_0(G)}^{H(a_{N,n})_0(G)}$, which yields the claim.

To check condition (B1), let N be given. We put $M := N$. For given m we put $n := m$. Then we have

$$B^{\circ}_{N,n} \subseteq \overline{B^{\circ}_{N,n} \cap (AH)_0(G)}^{H(a_{N,n})_0(G)} = \bigcap_{k \in \mathbb{N}} B^{\circ}_{N,n} \cap (AH)_0(G) + \tfrac{1}{k} B^{\circ}_{N,n}.$$

(b) As in 4.13.(b), condition (B1) implies

$$B_{M,m} \subseteq \bigcap_{\varepsilon > 0} \big(AH(G) + \varepsilon B_{N,n}\big).$$

We fix $\varepsilon > 0$. Since $(\frac{1}{a_{M,m}})^{\sim} \in B_{M,m}$ the above yields $(\frac{1}{a_{M,m}})^{\sim} \in AH(G) + \frac{\varepsilon}{2} B_{N,n}$. Thus, there exist f and g such that $(\frac{1}{a_{M,m}})^{\sim} = f + \frac{\varepsilon}{2} g$ with $f \in AH(G)$ and $g \in B_{N,n}$. That is, for each K there exists k and $\lambda > 0$ with $|f| \leqslant \frac{\lambda}{a_{K,k}}$ and $|g| \leqslant \frac{1}{a_{M,m}}$ and we get similar to the proof of 4.13.(b) that $(\frac{1}{a_{M,m}})^{\sim} \leqslant \max(\frac{\varepsilon}{a_{N,n}}, \frac{2\lambda}{a_{K,k}})$ holds. Now we apply [21, Observation 1.2.(vii) and (v)] to obtain $(\underline{\mathrm{Q}})^{\sim}_{\mathrm{out}}$ by setting $S := 2\lambda$. ■

Concerning O-growth conditions we have the following informations on condition (B1) in the setting of the classes $\mathcal{W}$ and $(\mathrm{E})_{\mathrm{C,c}}$, under the condition (LOG) and also in the setting of (DFN)-spaces.

Remark 11.2. (a) Assume that $\mathcal{A} \subseteq W$. If $\mathcal{A}$ satisfies $(\mathrm{Q})^{\sim}_{\mathrm{in}}$ then (B1) holds.

(b) Assume that $\mathcal{A} \subseteq E$. If $\mathcal{A}$ satisfies $(\mathrm{Q})^{\sim}_{\mathrm{in}}$ then (B1) holds.

(c) Assume that $\mathcal{A}$ satisfies condition (LOG). If $\mathcal{A}$ satisfies $(\mathrm{Q})^{\sim}_{\mathrm{in}}$ then (B1) holds.

(d) Assume that $\mathcal{A} = \exp(-\mathcal{P})$ such that $\mathcal{P}$ satisfies the assumptions made in 9.20. If one of the conditions 9.20.(i)–(ii$''$) is satisfied, then condition (B1) holds.

Proof. (a) If $\mathcal{A} \subseteq W$ satisfies $(\mathrm{Q})^{\sim}_{\mathrm{in}}$ we have $\mathrm{Proj}^{\,1} \mathcal{A}H = 0$ by 6.1 and hence by 4.11.(c) condition (B1) holds.

(b) We may conclude as in (a), but the vanishing of $\mathrm{Proj}^{\,1} \mathcal{A}H$ follows from 7.1.

(c) We may conclude as in (a), but the vanishing of $\mathrm{Proj}^{\,1} \mathcal{A}H$ follows from 8.2.

(d) As above, $\operatorname{Proj}^1 \mathcal{A}H = 0$ holds by 9.20 and therefore 4.11.(c) yields condition (B1). ■

Remark 11.3. As we remarked already at several points, in the case of o-growth conditions we have no result on sufficient conditions for $\operatorname{Proj}^1 \mathcal{A}_0 H = 0$ in the settings of classes $\mathcal{W}$ and $(\mathrm{E})_{\mathrm{C,c}}$ and in the situation of condition (LOG). The abstract results in section 4.3, in particluar 4.10 we proved exactly to get at least a sufficient condition for barrelledness of $(AH)_0(G)$. In view of 3.1.A (where $\operatorname{Proj}^1 \mathcal{A}_0 C = 0$ is characterized via the weight condition (wQ)) for the continuous situation the results of section 4.3 cannot yield anything new, cf. 4.15. However the methods we used for $(AH)_0(G)$ can be applied in a similar way, that is by replacing the space of polynomials by the space of continuous functions with compact support. In this way it is possible to show the implication "(wQ) $\Rightarrow$ $(AC)_0(X)$ barrelled" without using the machinery of Proj^1: In order to show this implication, we proceed as in the holomorphic case. In what follows Claim A. follows from [14, Lemma 5.1]; for the sake of completeness we give a proof.

Claim A. Let $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ be decreasing. Then $C_c(X) \subseteq \mathcal{V}_0 C(X)$ is a limit subspace.

We define $\mathcal{V}_c C(X) := \operatorname{ind}_n(C_c(X), \|\cdot\|_n)$ and denote by $C_c(X)$ this space endowed with the topology induced by $\mathcal{V}_0 C(X)$. Since the identity $\mathcal{V}_c C(X) \to \mathcal{V}C(X)$ is continuous it is enough to show

$$\forall\, U \in \mathcal{U}_0(\mathcal{V}_c C(X))\; \exists\, V \in \mathcal{U}_0(C_c(X))\colon V \subseteq U.$$

By Bierstedt, Meise, Summers [27, 1.3.(a)] $\mathcal{V}_0 C(X) \subseteq C\overline{V}_0(X)$ is a topological subspace and hence the topology of $C_c(X)$ is given by $\|\cdot\|_{\overline{v}}$, $\overline{v} \in \overline{V}$. Let now U be a 0-neighborhood in $\mathcal{V}_c C(X)$. Then we may assume $U = \Gamma(\cup_{n=1}^{\infty} \varepsilon_n C_n)$ where $\varepsilon_n > 0$ is decreasing and $C_n := B_n^{\circ} \cap C_c(X) = B_n \cap C_c(X)$. We claim that

$$V := \big\{ f \in C_c(X)\, ;\, \|f\|_{\overline{v}} < 1 \big\} \subseteq U$$

holds for $\overline{v} := \inf_{n\in\mathbb{N}} 2^n \varepsilon_n^{-1} v_n$. The following is very similar to the proof of [27, Lemma 1.1]. Let $f \in V$ that is $\|f\|_{\overline{v}} < 1$. For $n \in \mathbb{N}$

$$F_n := \big\{ x \in X\, ;\, s^2 \varepsilon_n^{-1} v_n(x)|f(x)| \geqslant 1 \big\}$$

is a closed subspace of $\operatorname{supp} f$. If $x \in \cap_{n\in\mathbb{N}} F_n$, then $2^n \varepsilon_n^{-1} v_n(x)|f(x)| \geqslant 1$ holds for each n and hence $\overline{v}|f(x)| \geqslant 1$, which contradicts $\|f\|_{\overline{v}} < 1$. Thus, $\cap_{n\in\mathbb{N}} = \emptyset$. Now put $U_n := X \backslash F_n$ for $n \in \mathbb{N}$. By the above, $\cup_{n\in\mathbb{N}} U_n = X$ and since $\operatorname{supp} f$ is compact, there exists m such that $\operatorname{supp} f \subseteq \cup_{n=1}^{m} U_n$. Let $(\varphi_n)_{n=1,\dots,m}$ be a finite continuous partition of unity on $\operatorname{supp} f$ which is subordinate to the covering $(U_n)_{n=1,\dots,m}$ and set $g_n := 2^n \varphi_n f$ for $n = 1, \dots, m$. Then $g_n \in C_c(X)$ and $g_n(x) = 0$ if $x \in X \backslash U_n$. For $x \in U_n$ we have $v_n(x)|g_n(x)| = \varphi_n(x) 2^n v_n(x)|f(x)| < \varepsilon_n$ that is $g_n \in \varepsilon_n C_n$ for $n = 1, \dots, m$. Therefore

$$f = \sum_{n=1}^{m} \varphi_n f = \sum_{n=1}^{m} 2^{-n} g_n \in \Gamma\big(\cup_{n\in\mathbb{N}} \varepsilon_n C_n \big)$$

and we have shown $V \subseteq U$ which establishes claim A.

Up to now we have (analogously to the holomorphic setting) that

$$\mathcal{V}_c C(X) \subseteq \mathcal{V}_0 C(X) \subseteq \mathcal{V}C(X)$$

are all topological subspaces. Moreover, $\mathcal{V}C(X)$ is regular (see section 3.1). Hence the C_n which we defined above form a fundamental system of bounded sets in $\mathcal{V}_c C(X)$ by 6.3. For a given double sequence $\mathcal{A}$ we consider $C(a_{N,n})_c(X) := (C_c(X), \|\cdot\|_{N,n})$ and $(AC)_c(X) := \operatorname{proj}_N \operatorname{ind}_n C(a_{N,n})_c(X)$.

Claim B. Let $\mathcal{A}$ satisfy condition (wQ). Then $(AC)_c(X)$ is bornological.

We proceed as in 6.6: By Bierstedt, Bonet [18], condition (wQ) implies condition $(\mathrm{wQ})^\star$ that is

$$\exists\, (n(\sigma))_{\sigma\in\mathbb{N}} \subseteq \mathbb{N} \text{ increasing } \forall\, N\ \exists\, M\ \forall\, K,\ m\ \exists\, S>0,\ k\colon$$

$$\tfrac{1}{a_{M,m}} \leqslant S \max\big(\tfrac{1}{a_{K,k}}, \min_{\sigma=1,\dots,N} \tfrac{1}{a_{\sigma,n(\sigma)}}\big).$$

We fix an absolutely convex and bornivorous set $T \subseteq (AC)_c(X)$. Since $(AC)_c(X) = C(a_{N,n})_c(X)$ algebraically for all N, n we may consider T as a subset of the latter space and claim that there exists N such that for each n the ball $C_{N,n} = B^{(\circ)} \cap C_c(X)$ is absorbed by T. We proceed by contradiction and hence assume

$$(\star) \qquad \forall\, M\ \exists\, m(M)\colon C_{M,m(M)} \text{ is not absorbed by } T.$$

By 6.5, there exists N such that $\cap_{\sigma=1}^N C_{\sigma,m(\sigma)}$ is absorbed by T. For the sequence $(n(\sigma))_{\sigma\in\mathbb{N}}$ and this N we choose M as in $(\mathrm{wQ})^\star$. By $(\star)$ there exists $m(M)$ such that for each K there exist $S_K>0$ and $k(K)$ such that $\frac{1}{a_{M,m(M)}} \leqslant S_K \max\big(\frac{1}{a_{K,k(K)}}, \min_{\sigma=1,\dots,N} \frac{1}{a_{\sigma,n(\sigma)}}\big)$ holds and get as in the proof of 6.6

$$(\circ) \qquad \forall\, K\colon \tfrac{1}{a_{M,m(M)}} \leqslant S'_K \max\big(\min_{\mu=1,\dots,K} \tfrac{1}{a_{\mu,k(\mu)}}, \min_{\sigma=1,\dots,N} \tfrac{1}{a_{\sigma,n(\sigma)}}\big)$$

with $S'_K := \max_{\mu=1,\dots,K} S_\mu$.

Now we use the decomposition lemma [2, Lemma 3.1] – which clearly is also the main ingredient to show "(wQ) $\Rightarrow \operatorname{Proj}^1 \mathcal{A}_0 C = 0$" – to show

$$\forall\, K\ \exists\, \tau_K>0\colon C_{M,m(M)} \subseteq \tau_K \big[\bigcap_{\sigma=1}^{N} C_{\sigma,n(\sigma)} + \bigcap_{\mu=1}^{K} C_{\mu,k(\mu)}\big].$$

We fix $K\in\mathbb{N}$. Let $f\in C_{M,m(M)}$, i.e. $a_{M,m(M)}|f| \leqslant 1$ hence $|f| \leqslant \frac{1}{a_{M,m(M)}}$. By $(\circ)$ we get the estimate $|f| \leqslant \max\big(\min_{\sigma=1,\dots,N} \frac{S'_K}{a_{\sigma,n(\sigma)}}, \min_{\mu=1,\dots,K} \frac{S'_K}{a_{\mu,k(\mu)}}\big)$. We define $\frac{1}{u} := \min_{\sigma=1,\dots,N} \frac{S'_K}{a_{\sigma,n(\sigma)}}$, $\frac{1}{v} := \min_{\mu=1,\dots,K} \frac{S'_K}{a_{\mu,k(\mu)}}$ and $\alpha_1 := \alpha_2 := \frac{1}{S'_K}$ to obtain $u = \max_{\sigma=1,\dots,N} \frac{a_{\sigma,n(\sigma)}}{S'_K}$, $\quad v = \max_{\mu=1,\dots,K} \frac{a_{\mu,k(\mu)}}{S'_K}$. By [2, Lemma 3.1] there exist $f_1, f_2 \in C(X)$ – and since f has compact support we can even find $f_1, f_2 \in C_c(X)$ – such that $|f_1| \leqslant \frac{2}{u}$, $|f_2| \leqslant \frac{2}{v}$ and $f = f_1 + f_2$, i.e. $\max_{\sigma=1,\dots,N} a_{\sigma,n(\sigma)}|f_1| \leqslant 2S'_K$ and $\max_{\mu=1,\dots,K} a_{\mu,k(\mu)}|f_2| \leqslant 2S'_K$ and therefore

$$f = f_1 + f_2 \in 2S'_K \bigcap_{\sigma=1}^{N} C_{\sigma,n(\sigma)} + 2S'_K \bigcap_{\mu=1}^{K} C_{\mu,k(\mu)}.$$

Hence we get the desired inclusion by setting $\tau_K := 2S'_K$. Now, claim B follows as in 6.6, since for the spectrum under consideration the statement (B1) of 4.9 is clearly trivial and our claim was exactly condition (B2) of 4.9, i.e. 4.10 implies that $(AC)_c(X)$ is bornological.

Analogously to the holomorphic setting, 6.7 together with the above statements and 6.4 implies that $(AC)_0(X)$ is quasi-barrelled and since the spectrum $\mathcal{A}_0C$ is reduced, barrelledness follows immediately.

11.2 The Fréchet case

Also in the case of holomorphic functions we can consider the Fréchet cases of $AH(G)$ and $(AH)_0(G)$. As we have seen in 4.4 in this special case there is a connection between our conditions and quasinormability of the considered spaces.

Since Wolf [86, 87] investigated quasinormability of the Fréchet spaces $AH(G)$ and $(AH)_0(G)$, we will summarize her results first and then draw the line to the conditions investigated in 4.4.

Remark 11.4. (Wolf [86, section IV] and [87, Theorem 4]) Consider the following two conditions introduced by Wolf.

(i) $\mathcal{A}$ satisfies condition (W1) if

$$\forall\, N\; \exists\, M > N\; \forall\, \varepsilon > 0\; \exists\, \overline{a} \in \overline{A}\colon \big(\tfrac{1}{a_M}\big)^{\sim} \leqslant \overline{a} + \tfrac{\varepsilon}{a_N}.$$

(ii) $\mathcal{A}$ satisfies condition (W2) if

$$\forall\, N\; \exists\, M > N\; \forall\, K > N,\, \varepsilon > 0\; \exists\, S > 0\colon \big(\tfrac{1}{a_M}\big)^{\sim} \leqslant \tfrac{\varepsilon}{a_N} + \tfrac{S}{a_K}.$$

Here, $\overline{A}$ is defined analogously to the continuous case, see the remarks after 4.13. Wolf showed the following.

(a) "(i) $\Rightarrow$ (ii)" holds in general.

(b) If in the balanced setting $AH(G)$ is quasinormable, then (i) holds.

(c) If $G = \mathbb{D}$ and $\mathcal{A} \subseteq W$ then "(i) $\Leftrightarrow$ (ii)" and these conditions are equivalent to quasinormability.

(d) If $G = \mathbb{C}$ and $\mathcal{W} \subseteq E$ then "(i) $\Leftrightarrow$ (ii)" and these conditions are equivalent to quasinormability.

Note that in the Fréchet case, condition $(\underline{\mathrm{Q}})^{\sim}_{\mathrm{out}}$ reduces to

$$\forall\, N\; \exists\, M\; \forall\, K,\, \varepsilon > 0\; \exists\, S > 0\colon \big(\tfrac{1}{a_M}\big)^{\sim} \leqslant \big(\max(\tfrac{\varepsilon}{a_N}, \tfrac{S}{a_K})\big)^{\sim}.$$

Now we can – under the assumption of class $\mathcal{W}$ or $(\mathrm{E})_{\mathrm{C,c}}$ – state a theorem analogously to 4.23. Several implications of the next proposition are true in more general settings as we have seen already. At the end of this section we will – within a comparison of continuous and holomorphic Fréchet case – subdivide the situation once more.

Proposition 11.5. Let $AH(G) = \mathrm{proj}_N\, Ha_N(G)$ be a Fréchet space. Let $\mathcal{A} \subseteq W$ and $G = \mathbb{D}$ or $\mathcal{A} \subseteq E$ and $G = \mathbb{C}$. Then the following are equivalent.

(i)	$AH(G)$ is quasinormable.	(iv)	$\mathcal{A}$ satisfies $(\underline{\mathrm{Q}})^{\sim}_{\mathrm{out}}$
(ii)	$\mathcal{A}H$ is reduced.	(v)	$\mathcal{A}$ satisfies (W1).
(iii)	$\mathcal{A}H$ satisfies (B1).	(vi)	$\mathcal{A}$ satisfies (W2).

Moreover, condition $(\mathrm{Q})^{\sim}_{\mathrm{in}}$ as well as condition $(\overline{\mathrm{B1}})$ implies (i)-(vi).

Proof. "(i)⇒(ii)" This is 4.20.

"(ii)⇒(iii)" This is 4.20.

"(iii)⇒(iv)" This is 11.1.(b).

"(iv)⇒(vi)" We estimate $\big(\max(\frac{\varepsilon}{a_N}, \frac{S}{a_K})\big)^{\sim} \leqslant \max(\frac{\varepsilon}{a_N}, \frac{S}{a_K}) \leqslant \frac{\varepsilon}{a_N} + \frac{S}{a_K}$.

"(vi)⇒(i)" This is 11.4.

"(i)⇒(v)" This is 11.4.

"(v)⇒(vi)" This is 11.4.

$(\mathrm{Q})^{\sim}_{\mathrm{in}} \Rightarrow (\underline{\mathrm{Q}})^{\sim}_{\mathrm{out}}$ holds by definition and $(\overline{\mathrm{B1}})$ implies quasinormability by definition. ■

Remark 11.6. Note that if $W = \mathcal{W}(\varepsilon_0, k_0)$ by 5.9 condition $(\mathrm{Q})^{\sim}_{\mathrm{in}}$ is equivalent to (i)–(vi) of the latter Proposition. In this case, the statements (i)–(vi) above are even equivalent to (Q). The latter is also true if $\mathcal{A} \subseteq E$ consists of essential weights.

Let us now – in analogy to 4.24 – collect the results on o-growth conditions in the Fréchet case.

Proposition 11.7. Let $(AH)_0(G) = \mathrm{proj}_N\, H(a_N)_0(G)$ be a Fréchet space.

(1) In the balanced setting $\mathcal{A}_0H$ is reduced.

(2) (wQ) is always satisfied.

(3) In the balanced setting $\mathcal{A}_0H$ satisfies condition (B1).

(4) Assume $\mathcal{A} \subseteq W$ and $G = \mathbb{D}$ or $\mathcal{A} \subseteq E$ and $G = \mathbb{C}$. Then we have (i)⇒(ii)⇔(iii)⇔(iv), where

(i)	$\mathcal{A}_0H$ satisfies $(\overline{\mathrm{B1}})$,	(iii)	$\mathcal{A}$ satisfies (W1),
(ii)	$(AH)_0(G)$ is quasinormable,	(iv)	$\mathcal{A}$ satisfies (W2).

In particular, $AH(G)$ being quasinormable is also equivalent to (ii)–(iv).

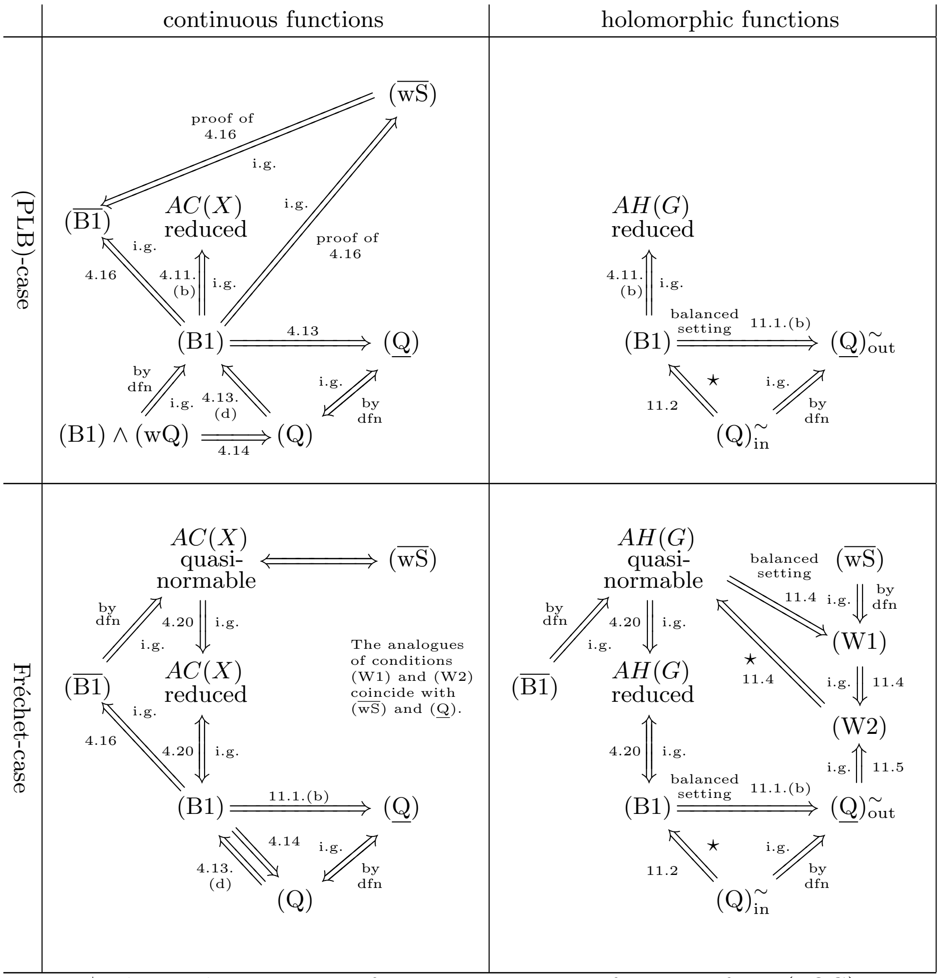
Proof. (1) This follows from 5.1 and 4.12.

(2) See the proof "(iii)⇒(ii)" in 4.23.

(3) By 4.20, condition (B1) is equivalent to the reducedness of $\mathcal{A}_0H$. Now, (a) implies the assertion.

(4) In 4.18 we noted that in the case of a projective spectrum of Banach spaces with inclusions as linking maps $(\overline{\mathrm{B1}})$ implies quasinormability. The equivalences follow from Wolf [86, Theorem 21] resp. [87, Theorem 4]. ∎

Remark 11.8. Let us compare the results within the following two tables.



⋆ This implication is true for $\mathcal{A} \subseteq W$, $\mathcal{A} \subseteq E$ or for $\mathcal{A}$ satisfying (LOG)

Table 1: Comparison of the conditions under investigation for continuous and holomorphic functions, Fréchet and (PLB)-spaces – O-growth conditions.

	continuous functions	holomorphic functions
(PLB)-case	(B1) is always satisfied $\mathcal{A}_0C$ is always reduced	(B1) is satisfied in the balanced setting $\mathcal{A}_0H$ is reduced in the balanced setting
Fréchet-case	(B1) is always satisfied $\mathcal{A}_0C$ is always reduced (wQ) is always satisfied	(B1) is satisfied in the balanced setting $\mathcal{A}_0H$ is reduced in the balanced setting (wQ) is always satisfied

Table 2: Comparison of the conditions under investigation for continuous and holomorphic functions, Fréchet and (PLB)-spaces – o-growth conditions.

It is open if in the (PLB)-case of $AH(G)$ some implications concerning $(\overline{\mathrm{wS}})$ and $(\overline{\mathrm{B1}})$ are valid.

12 Interchangeability of projective and inductive limit

12.1 The algebraic equality

In this section we will need (as it is needed for the corresponding results on continuous functions 3.3) at several points (12.4 and 12.5.(2)) that the steps $(\mathcal{A}_N)_0H(G)$ of the (PLB)-space $(AH)_0(G)$ are complete. Unfortunately, there is (in contrast to the continuous case, cf. 3.18) no characterization of completeness of the (LB)-space $\mathcal{V}_0H(G)$ for a decreasing sequence $\mathcal{V}$ of weights. However, under the assumption that $\mathcal{V}_0H(G) \subseteq \mathcal{V}_0C(G)$ is a topological subspace, $\mathcal{V}_0H(G)$ is complete if $\mathcal{V}$ is regular decreasing, cf. Bierstedt [12, Corollary C]. In the setting of class $\mathcal{W}$, the setting of condition (LOG) and also in the setting of class $(\mathrm{E})_{\mathrm{C,c}}$ the latter is satisfied as we noted in the corresponding sections and hence in these cases one might replace the completeness assumption in 12.4 and 12.5.(2) by the (a priori stronger but in some sense more accessible) requirement that $\mathcal{A}_N = (a_{N,n})_{n\in\mathbb{N}}$ is regularly decreasing for each N.

Lemma 12.1. $\mathcal{V}H(G) \hookrightarrow AH(G)$ and $\mathcal{V}_0H(G) \hookrightarrow (AH)_0(G)$ holds with continuous inclusions.

Proof. We fix $f \in \mathcal{V}H(G) = \operatorname{ind}_n \operatorname{proj}_N Ha_{N,n}(G)$, i.e. there exists n such that $a_{N,n}|f| < \infty$ holds for each N. Thus, for each N there exists n such that $a_{N,n}|f| < \infty$, that is, $f \in \mathcal{A}_NH(G)$ holds for each N and hence $f \in AH(G)$. We have shown $\mathcal{V}H(G) \subseteq AH(G)$ and we obtain similarly $\mathcal{V}_0H(G) \subseteq (AH)_0(G)$. By definition the projective limit $H(V_N)_{(0)}(G)$ is included continuously in the steps $H(V_N)_{(0)}(G)$

which are all included continously in the inductive limit $(\mathcal{A}_N)_{(0)}H(G)$. Now, applying the universal properties of inductive and projective limit yields the desired continuity of the mappings $\mathcal{V}H(G) \hookrightarrow AH(G)$ and $\mathcal{V}_0H(G) \hookrightarrow (AH)_0(G)$. ■

In order to characterize the equality we need the following lemma.

Lemma 12.2. Let $F \subseteq H(G)$ be a linear subspace which contains the polynomials and let v and w be two radial weights on G, where G is assumed to be balanced. If there exists $c > 0$ such that

$$(\circ) \qquad \sup_{z\in G} v(z)|f(z)| \leqslant c \sup_{z\in G} w(z)|f(z)|$$

holds for each $f \in F$, then we have $\tilde{v} \leqslant c\tilde{w}$ on G.

Proof. Let $g \in H(G)$ with $|g| \leqslant \frac{1}{w}$ on G. Then we have $|S_t g| \leqslant \frac{1}{w}$ on G. Since $S_t g \in \mathbb{P} \subseteq F$, we may apply $(\circ)$ and obtain $\frac{1}{c}\sup_{z\in G} v(z)|S_t g(z)| \leqslant \sup_{z\in G} w(z)|S_t(z)| \leqslant 1$, i.e. $|S_t g| \leqslant \frac{c}{v}$ on G and since $S_t g \to g$ converges pointwise on G for $t \to \infty$, we obtain $|g| \leqslant \frac{c}{v}$ on G, hence $|\frac{g}{c}| \leqslant \frac{1}{v}$ and therefore, since $\frac{g}{c} \in H(G)$, $|\frac{g}{c}| \leqslant \frac{1}{\tilde{v}}$ holds on G. Finally we have

$$\frac{1}{\tilde{w}(z)} = \sup_{\substack{g\in H(G)\\ |g|\leqslant \frac{1}{w} \text{ on } G}} |g(z)| \leqslant \sup_{\substack{g\in H(G)\\ |g|\leqslant \frac{c}{\tilde{v}} \text{ on } G}} |g(z)| \leqslant \frac{c}{\tilde{v}(z)}$$

for arbitrary $z \in G$. ■

Proposition 12.3. Assume that we are in the balanced setting. Then $AH(G) = \mathcal{V}H(G)$ holds algebraically if and only if $\mathcal{A}$ satisfies condition $(\mathrm{B})^{\sim}$.

Proof. "$\Rightarrow$" For a given sequence $(n(N))_{N\in\mathbb{N}}$ we consider the space

$$F := \bigcap_{N\in\mathbb{N}} Ha_{N,n(N)}(G),$$

endowed with the topology given by the system

$$(p_L)_{L\in\mathbb{N}}, \;\; p_L(f) := \max_{N=1,\dots,L} \sup_{z\in G} a_{N,n(N)}(z)|f(z)|$$

of seminorms. Then we have $F \hookrightarrow AH(G)$ with continuous inclusion, which is complete and has a topology finer that co. Therefore F is a Fréchet space. $AH(G) = \mathcal{V}H(G)$ implies that F is included in the (LF)-space $\mathcal{V}H(G)$. Hence, 4.1 implies that the mapping $F \hookrightarrow \mathcal{V}H(G)$ has closed graph and with deWilde's closed graph theorem (e.g. [60, 24.31]) we get that it is even continuous. Now we may apply Grothendieck's factorization (e.g. [60, 24.33]) theorem to obtain m such that $F \subseteq HV_m(G)$ holds with continuous inclusion. Hence for given M there exists L and $c > 0$ such that for each $f \in F$ the estimate

$$\sup_{z\in G} a_{M,m}(z)|f(z)| \;\leqslant\; c \max_{N=1,\dots,L} \sup_{z\in G} a_{N,n(N)}(z)|f(z)|$$

$$\leqslant\ c\ \sup_{z\in G}\big(\max_{N=1,\dots,L} a_{N,n(N)}\big)(z)|f(z)|$$

holds. Since we are in the balanced setting, $\mathbb{P}\subseteq F$ holds and we may apply 12.2, to obtain $\tilde{a}_{M,m}\leqslant c\big(\max_{N=1,\dots,L}a_{N,n(N)}\big)^{\sim}$ on G, which is exactly the estimate in (B)$^{\sim}$.

"$\Leftarrow$" Let $f\in AH(G)$. By definition, for each N there exists $n(N)$ and $b_n>0$ such that $a_{N,n(N)}|f|\leqslant b_n$. Now we select m according to (B)$^{\sim}$ w.r.t. the sequence $(n(N))_{N\in\mathbb{N}}$ and claim that $f\in Ha_{M,m}(G)$ holds for each $M\in\mathbb{N}$. Given M we select L and $c>0$ as in (B)$^{\sim}$ and put $b:=\max(b_1,\dots,b_L)$. Then we have $a_{N,n(N)}|\frac{f}{b}|\leqslant 1$ on G for $N=1,\dots,L$ and therefore $\max_{N=1,\dots,L}a_{N,n(N)}|\frac{f}{b}|\leqslant 1$ on G. We put $w_N:=\max_{N=1,\dots,L}a_{N,n(N)}$ to obtain $|\frac{f}{b}|\leqslant\frac{1}{w_N}$ and since $\frac{f}{b}\in H(G)$, we get $|\frac{f}{b}|\leqslant\frac{1}{\tilde{w}_N}$ on G. Now, (B)$^{\sim}$ implies $\frac{1}{\tilde{w}_N}\leqslant\frac{c}{\tilde{a}_{M,m}}$ and hence $|\frac{f}{b}|\leqslant\frac{c}{\tilde{a}_{M,m}}$ holds on G. This finally yields $\tilde{a}_{M,m}|f|\leqslant cb$, i.e. $f\in H\tilde{a}_{M,m}(G)=Ha_{M,m}(G)$ and therefore we established the claim. But now $f\in HV_m(G)\subseteq\mathcal{V}H(G)$ holds and we are done. ■

Proposition 12.4. Assume that we are in the balanced setting. If $\mathcal{A}$ satisfies condition (B)$^{\sim}$ then $(AH)_0(G)=\mathcal{V}_0H(G)$ holds algebraically. If all $(\mathcal{A}_N)_0H(G)$ are complete, the converse is also true.

Proof. "$\Rightarrow$" Let $\mathcal{A}$ satisfy (B)$^{\sim}$ and $f\in(AH)_0(G)$. By definition, for each N there exists $n(N)$ such that for each $\varepsilon>0$ there exists $K_{N,\varepsilon}\subseteq G$ compact with $a_{N,n(N)}(z)|f(z)|\leqslant\varepsilon$ for all $z\in G\backslash K_{N,\varepsilon}$. Now we select m according to (B)$^{\sim}$ w.r.t. the sequence $(n(N))_{N\in\mathbb{N}}$ and claim that $f\in H(a_{M,m})_0(G)$ for each $M\in\mathbb{N}$. Given M and $\varepsilon>0$ we select L and $c>0$ as in (B)$^{\sim}$ and put $K_\varepsilon:=K_{1,\frac{\varepsilon}{c}}\cup\dots\cup K_{L,\frac{\varepsilon}{c}}$, where the $K_{1,\frac{\varepsilon}{c}},\dots,K_{L,\frac{\varepsilon}{c}}$ are chosen as above. Hence we have $a_{N,n(N)}|f|\leqslant\frac{\varepsilon}{c}$ on $G\backslash K_\varepsilon$ for $N=1,\dots,L$ and therefore $\max_{N=1,\dots,L}a_{N,n(N)}|f|\leqslant\frac{\varepsilon}{c}$ on $G\backslash K_\varepsilon$. We put $w_N:=\max_{N=1,\dots,L}a_{N,n(N)}$ and have $|\frac{cf}{\varepsilon}|\leqslant\frac{1}{w_N}$ on $G\backslash K_\varepsilon$ and since $\frac{cf}{\varepsilon}\in H(G)$, we get $|\frac{cf}{\varepsilon}|\leqslant\frac{1}{\tilde{w}_N}$ on $G\backslash K_\varepsilon$. Now, (B)$^{\sim}$ implies $\frac{1}{\tilde{w}_N}\leqslant\frac{c}{\tilde{a}_{M,m}}$ and hence we have $|\frac{f}{\varepsilon}|\leqslant\frac{1}{\tilde{a}_{M,m}}$ on $G\backslash K_\varepsilon$ and therefore $|\frac{f}{\varepsilon}|\leqslant\frac{1}{a_{M,m}}$ on $G\backslash K_\varepsilon$, i.e. $a_{M,m}|f|\leqslant\varepsilon$ on $G\backslash K_\varepsilon$. This establishes the claim. But now $f\in H(V_m)_0(G)\subseteq\mathcal{V}_0H(G)$ holds and we are done.

"$\Leftarrow$" For a given sequence $(n(N))_{N\in\mathbb{N}}$ we consider the space

$$F_0:=\bigcap_{N\in\mathbb{N}}H(a_{N,n(N)})_0(G),$$

endowed with the topology given by the system

$$(p_L)_{L\in\mathbb{N}},\ \ p_L(f):=\max_{N=1,\dots,L}\sup_{z\in G}a_{N,n(N)}(z)|f(z)|$$

of seminorms. Then we have $F_0\hookrightarrow(AH)_0(G)$ with continuous inclusion, which is – by our additional assumption – complete and has a topology finer than co. Therefore F_0 is a Fréchet space. $(AH)_0(G)=\mathcal{V}_0H(G)$ implies that F_0 is included in the (LF)-space $\mathcal{V}_0H(G)$. Again, 4.1 implies that the mapping $F_0\hookrightarrow\mathcal{V}_0H(G)$ has closed graph and with de Wilde's closed graph theorem (e.g. [60, 24.31]) we get that it is even continuous. Now we apply Grothendieck's factorization theorem (e.g. [60, 24.33]) to obtain m such that $F_0 \subseteq H(V_m)_0(G)$ holds with continuous inclusion. Hence for given M there exists L and $c > 0$ such that for each $f \in F_0$ the estimate

$$\begin{aligned}\sup_{z\in G} a_{M,m}(z)|f(z)| &\leqslant c \max_{N=1,\dots,L} \sup_{z\in G} a_{N,n(N)}(z)|f(z)| \\ &\leqslant c \sup_{z\in G} \big(\max_{N=1,\dots,L} a_{N,n(N)}\big)(z)|f(z)|\end{aligned}$$

holds. Since we are in the balanced setting, we have $\mathbb{P} \subseteq F_0$ and thus can apply 12.2 to obtain $\tilde{a}_{M,m} \leqslant c\big(\max_{N=1,\dots,L} a_{N,n(N)}\big)^{\sim}$ on G, which is exactly the estimate in $(\mathrm{B})^{\sim}$. ■

12.2 Necessary conditions for the topological equality

Theorem 12.5. (1) Assume that we are in the balanced seting. If $AH(G) = \mathcal{V}H(G)$ holds algebraically and topologically then $\mathcal{A}$ satisfies the conditions $(\mathrm{B})^{\sim}$ and $(\mathrm{wQ})^{\sim}_{\mathrm{in}}$.

(2) Assume that we are in the balanced setting and all $(\mathcal{A}_N)_0H(G)$ are complete. If $(AH)_0(G) = \mathcal{V}_0H(G)$ holds algebraically and topologically then $\mathcal{A}$ satisfies the conditions $(\mathrm{B})^{\sim}$ and $(\mathrm{wQ})^{\sim}_{\mathrm{in}}$.

Proof. (1) Condition $(\mathrm{B})^{\sim}$ follows with 12.3 from the algebraical equality. The topological equality implies that $AH(G)$ is ultrabornological as it is isomorphic to an (LF)-space. With Proposition 6.1 it follows that $\mathcal{A}$ satisfies condition $(\mathrm{wQ})^{\sim}_{\mathrm{in}}$.
(2) Condition $(\mathrm{B})^{\sim}$ follows with 12.4 from the algebraical equality. Moreover, the topological equality $(AH)_0(G) = \mathcal{V}_0H(G)$ implies that $(AH)_0(G)$ is ultrabornological and therefore it follows from 5.4 that $\mathcal{A}$ satisfies $(\mathrm{wQ})^{\sim}_{\mathrm{in}}$. ■

12.3 Sufficient conditions for the topological equality

Theorem 12.6. (1) Let $\mathcal{A} \subseteq W$. If $\mathcal{A}$ satisfies the conditions $(\mathrm{B})^{\sim}$ and $(\mathrm{Q})^{\sim}_{\mathrm{out}}$ then $AH(\mathbb{D}) = \mathcal{V}H(\mathbb{D})$ holds algebraically and topologically.

(2) Let $\mathcal{A} \subseteq E$. If $\mathcal{A}$ satisfies the conditions $(\mathrm{B})^{\sim}$ and $(\mathrm{Q})^{\sim}_{\mathrm{out}}$ then $AH(\mathbb{C}) = \mathcal{V}H(\mathbb{C})$ holds algebraically and topologically.

(3) Let $\mathcal{A}$ satisfy condition (LOG). If $\mathcal{A}$ satisfies the conditions $(\mathrm{B})^{\sim}$ and $(\mathrm{Q})^{\sim}_{\mathrm{out}}$ then $AH(\mathbb{C}) = \mathcal{V}H(\mathbb{C})$ holds algebraically and topologically.

(4) Let $\mathcal{A} = \exp(-\mathcal{P})$ where $\mathcal{P}$ is a sequence of weight functions which satisfies the assumptions we made in section 9 (cf. in particular 9.20) and let $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ be defined as in 9.20. If $\mathcal{P}$ satisfies $(\mathrm{wQ})^{\mathrm{P}}$ or $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ satisfy (wQ) and $\mathcal{A}$ satisfies $(\mathrm{B})^{\sim}$ then $AH(\mathbb{C}) = \mathcal{V}H(\mathbb{C})$ holds algebraically and topologically.

As we noted already in section 2, in the statements above we may replace $(\mathrm{B})^{\sim}$ with (B) and $(\mathrm{Q})^{\sim}_{\mathrm{out}}$ with (Q).

Proof. (1) Let $(\mathrm{B})^{\sim}$ and $(\mathrm{Q})^{\sim}_{\mathrm{out}}$ be satisfied. By 12.3 the identity $\mathrm{id}\colon AH(\mathbb{D}) \to \mathcal{V}H(\mathbb{D})$ is one-to-one and by Lemma 12.1 it is continuous. Since $AH(\mathbb{D})$ is ultrabornological by Proposition 6.1 and $\mathcal{V}H(\mathbb{D})$ is webbed, we can apply the open mapping theorem (cf. Meise, Vogt [60, 24.30]) and obtain that id^{-1} is continuous and hence we have a topological isomorphism $\mathrm{id}\colon AH(\mathbb{D}) \to \mathcal{V}H(\mathbb{D})$.

(2), (3) and (4) We may copy the above proof verbatim except for the fact, that ultrabornologicity of $AH(\mathbb{C})$ now follows from 7.1, 8.2, resp. 9.20. ∎

Remark 12.7. The above proof uses the open mapping theorem of de Wilde and hence it depends on the ultrabornologicity of the (PLB)-space. In the settings of the classes $\mathcal{W}$ and $(\mathrm{E})_{\mathrm{C,c}}$ and in the situation of condition (LOG) we had not been able to show that a certain weight condition is sufficient for the latter property in the o-growth case. Therefore we have also no result concerning the topological equality $(AH)_0(G) = \mathcal{V}_0H(G)$.

12.4 Summary of results

Let us summarize the results of the last two sections in the following corollaries. We include also the special cases of essential weights. Unfortunately it is not clear, if essentialness of all weights in $\mathcal{A}$ yields that the conditions (B) and $(\mathrm{B})^{\sim}$ are equivalent. At least for the case $\mathcal{A} \subseteq \mathcal{W}(\varepsilon_0, k_0)$ this is indeed true since all weights in $\mathcal{W}(\varepsilon_0, k_0)$ are essential and $\mathcal{W}(\varepsilon_0, k_0)$ is closed under finite minima.

Corollary 12.8. Let $\mathcal{A} \subseteq W$ and $G = \mathbb{D}$, $\mathcal{A} \subseteq E$ and $G = \mathbb{C}$ or $\mathcal{A}$ satisfy (LOG) and $G = \mathbb{D}$. Then the implications (i)⇒(ii)⇒(iii) hold, where

(i) $\mathcal{A}$ satisfies $(\mathrm{B})^{\sim}$ and $(\mathrm{Q})^{\sim}_{\mathrm{out}}$,
(ii) $AH(G) = \mathcal{V}H(G)$ holds algebraically and topologically,
(iii) $\mathcal{A}$ satisfies $(\mathrm{B})^{\sim}$ and $(\mathrm{wQ})^{\sim}_{\mathrm{out}}$.

In statement (i) we may replace $(\mathrm{B})^{\sim}$ and $(\mathrm{Q})^{\sim}_{\mathrm{out}}$ by (B) and (Q). If moreover all weights in $\mathcal{A}$ are essential, we may also in statement (iii) replace $(\mathrm{wQ})^{\sim}_{\mathrm{out}}$ by (wQ).

If $W = \mathcal{W}(\varepsilon_0, k_0)$, we have (i′)⇒(ii′)⇒(iii′), where

(i′) $\mathcal{A}$ satisfies (B) and (Q),
(ii′) $AH(G) = \mathcal{V}H(G)$ holds algebraically and topologically,
(iii′) $\mathcal{A}$ satisfies (B) and (wQ).

Proof. It is enough to check that $(\mathrm{B})^{\sim}$ implies (B) if $\mathcal{A} \subseteq \mathcal{W}(\varepsilon_0, k_0)$. In order to show this, let $(\mathrm{B})^{\sim}$ be satisfied. To check (B) let a sequence $(n(N))_{N\in\mathbb{N}} \subseteq \mathbb{N}$ be given. We choose m as in $(\mathrm{B})^{\sim}$. For given M, we choose L and $c' > 0$ as in $(\mathrm{B})^{\sim}$. Since $\mathcal{W}(\varepsilon_0, k_0)$ is closed under finite maxima, by the remarks we made in section 6.4 there exists $C > 0$ such that $\big(\max_{N=1,\dots,L} a_{N,n(N)}\big)^{\sim} \leqslant C \cdot \max_{N=1,\dots,L} a_{N,n(N)}$. We put $c := c'C$. Then

$$a_{M,m} \leqslant \tilde{a}_{M,m} \leqslant c'\big(\max_{N=1,\dots,L} a_{N,n(N)}\big)^{\sim} \leqslant c \max_{N=1,\dots,L} a_{N,n(N)}$$

and we are done. ∎

Corollary 12.9. Let $\mathcal{A} = \exp(-\mathcal{P})$ where $\mathcal{P}$ is a sequence of weight functions which satisfies the assumptions we made in section 9 (cf. in particular 9.20) and let $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ be defined as in 9.20. Then (i)$\Rightarrow$(ii)$\Leftrightarrow$(iii)$\Rightarrow$(iv), where

(i) $\mathcal{A}$ satisfies $(\mathrm{B})^{\sim}$ and $\mathcal{P}$ satisfies $(\mathrm{wQ})^{\mathrm{P}}$,
(ii) $AH(\mathbb{C}) = \mathcal{V}H(\mathbb{C})$ holds algebraically and topologically,
(iii) $\mathcal{A}$ satisfies $(\mathrm{B})^{\sim}$ and $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ satisfy (wQ),
(iv) $\mathcal{A}$ satisfies $(\mathrm{B})^{\sim}$ and $(\mathrm{wQ})^{\sim}_{\mathrm{out}}$.

In statement (i) we may replace $(\mathrm{B})^{\sim}$ by (B). If moreover all the weights in $\mathcal{A}$ are essential, the following are equivalent.

(i′) $\mathcal{A}$ satisfies $(\mathrm{B})^{\sim}$ and $\mathcal{P}$ satisfies $(\mathrm{wQ})^{\mathrm{P}}$.
(ii′) $AH(\mathbb{C}) = \mathcal{V}H(\mathbb{C})$ holds algebraically and topologically.
(iii′) $\mathcal{A}$ satisfies $(\mathrm{B})^{\sim}$ and $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ satisfy (wQ).
(iv′) $\mathcal{A}$ satisfies $(\mathrm{B})^{\sim}$ and (wQ).

13 Condition (Σ): $\mathcal{W}$, $(\mathrm{E})_{\mathrm{C,c}}$ and (LOG) revisited

In section 9 we discovered that the sequences $\mathcal{A}$ in the setting of that section always satisfy (Σ). For the other settings, namely class $\mathcal{W}$, class $(\mathrm{E})_{\mathrm{C,c}}$ and condition (LOG) this might be not the case. However, in many examples the latter is indeed true. Therefore, in the sequel we willassume that condition (Σ) (or $(\Sigma)^{\sim}$) is satisfies and present the corresponding corollaries of the previous results.

13.1 Class $\mathcal{W}$

As suggested earlier, condition $(\Sigma)^{(\sim)}$ makes it possible to get a sufficient condition for the vanishing of Proj^1 in the o-growth case. Clearly this is no real analog to the result of section 6.1 since the spaces $(AH)_0(G)$ and $AH(G)$ coincide if $\mathcal{A}$ satisfies $(\Sigma)^{(\sim)}$. However, in view of examples, $(\Sigma)^{(\sim)}$ is – as we will see – a quite natural condition and hence the following statements are useful for the investigation of concrete spaces.

Proposition 13.1. Let $\mathcal{A} \subseteq W$ satisfy condition $(\Sigma)^{\sim}$. Then the spaces $(AH)_0(\mathbb{D})$ and $AH(\mathbb{D})$ coincide algebraically and topologically and $\operatorname{Proj}^1 \mathcal{A}_0 H = \operatorname{Proj}^1 \mathcal{A}H$. Moreover the following are equivalent.

(i) $\mathcal{A}$ satisfies condition $(\mathrm{Q})^{\sim}_{\mathrm{in}}$.
(ii) $\mathcal{A}$ satisfies condition $(\mathrm{Q})^{\sim}_{\mathrm{out}}$.
(iii) $\operatorname{Proj}^1 \mathcal{A}_{(0)} H = 0$.
(iv) $(AH)_{(0)}(\mathbb{D})$ is ultrabornological.
(v) $(AH)_{(0)}(\mathbb{D})$ is barrelled.
(vi) $\mathcal{A}$ satisfies condition $(\mathrm{wQ})^{\sim}_{\mathrm{in}}$.
(vii) $\mathcal{A}$ satisfies condition $(\mathrm{wQ})^{\sim}_{\mathrm{out}}$.

Proof. “(i)$\Rightarrow$(ii)” This is true in general.

“(ii)$\Rightarrow$(iii)” This is 6.1.

“(iii)$\Rightarrow$(iv)$\Rightarrow$(v)” This is true in general.

"(v)⇒(vi)" This is 5.4.

"(vi)⇒(vii)" This is true in general.

"(vii)⇒(ii)" and "(vi)⇒(i)" This is 5.15. ■

Corollary 13.2. Let $\mathcal{A}$ be in the set $\mathcal{W}(\varepsilon_0, k_0)$ and assume that $\mathcal{A}$ satisfies condition (Σ). Then the spaces $(AH)_0(\mathbb{D})$ and $AH(\mathbb{D})$ coincide algebraically and topologically and $\operatorname{Proj}^1 \mathcal{A}_0 H = \operatorname{Proj}^1 \mathcal{A}H$. Moreover the following are equivalent.

(i) $\mathcal{A}$ satisfies condition (Q).
(ii) $\operatorname{Proj}^1 \mathcal{A}_{(0)} H = 0$.
(iii) $(AH)_{(0)}(\mathbb{D})$ is ultrabornological.
(iv) $(AH)_{(0)}(\mathbb{D})$ is barrelled.
(v) $\mathcal{A}$ satisfies condition (wQ).

If we assume that the sequence $\mathcal{A}$ satisfies $(\Sigma)^\sim$, 13.1 yields the following.

Proposition 13.3. Let $\mathcal{A} \subseteq W$ satisfy condition $(\Sigma)^\sim$. Then the spaces $(AH)_0(\mathbb{D})$ and $AH(\mathbb{D})$ coincide algebraically and topologically. Moreover, $AH(\mathbb{D}) = \mathcal{V}H(\mathbb{D})$ holds algebraically and topologically if and only if $\mathcal{A}$ satisfies the conditions $(\mathrm{wQ})^\sim_{\mathrm{in}}$ and $(\mathrm{B})^\sim$.

Corollary 13.4. Let $\mathcal{A} \subseteq \mathcal{W}(\varepsilon_0, k_0)$ and assume that $\mathcal{A}$ satisfies condition (Σ). Then the spaces $(AH)_0(\mathbb{D})$ and $AH(\mathbb{D})$ coincide algebraically and topologically. Moreover, $AH(\mathbb{D}) = \mathcal{V}H(\mathbb{D})$ holds algebraically and topologically if and only if $\mathcal{A}$ satisfies the conditions (wQ) and (B).

13.2 Class $(\mathrm{E})_{\mathrm{C,c}}$

Again we combine the previous results under the assumption that $\mathcal{A}$ satisfies condition $(\Sigma)^\sim$ and obtain a result completely analog to 13.1.

Proposition 13.5. Let $\mathcal{A} \subseteq E$ satisfy condition $(\Sigma)^\sim$. Then the spaces $(AH)_0(\mathbb{C})$ and $AH(\mathbb{C})$ coincide algebraically and topologically and $\operatorname{Proj}^1 \mathcal{A}H = \operatorname{Proj}^1 \mathcal{A}_0 H$. Moreover the following are equivalent.

(i) $\mathcal{A}$ satisfies condition $(\mathrm{Q})^\sim_{\mathrm{in}}$.
(ii) $\mathcal{A}$ satisfies condition $(\mathrm{Q})^\sim_{\mathrm{out}}$.
(iii) $\operatorname{Proj}^1 \mathcal{A}_{(0)} H = 0$.
(iv) $(AH)_{(0)}(\mathbb{C})$ is ultrabornological.
(v) $(AH)_{(0)}(\mathbb{C})$ is barrelled.
(vi) $\mathcal{A}$ satisfies condition $(\mathrm{wQ})^\sim_{\mathrm{in}}$.
(vii) $\mathcal{A}$ satisfies condition $(\mathrm{wQ})^\sim_{\mathrm{out}}$.

Proof. We may copy the proof of 13.1 except that "(ii)⇒(iii)" now follows from 7.1. ■

Corollary 13.6. Let $\mathcal{A} \subseteq E$ consist of essential weights and satisfy condition (Σ). Then the spaces $(AH)_0(\mathbb{C})$ and $AH(\mathbb{C})$ coincide algebraically and topologically and $\operatorname{Proj}^1 \mathcal{A}H = \operatorname{Proj}^1 \mathcal{A}_0 H$. Moreover the following are equivalent.

(i) $\mathcal{A}$ satisfies condition (Q).
(ii) $\operatorname{Proj}^1 \mathcal{A}_{(0)} H = 0$.
(iii) $\mathcal{A}_{(0)} H(\mathbb{C})$ is ultrabornological.
(iv) $\mathcal{A}_{(0)} H(\mathbb{C})$ is barrelled.
(v) $\mathcal{A}$ satisfies condition (wQ).

Finally let us state the analog of 13.3 for the complex plane.

Proposition 13.7. Let $\mathcal{A} \subseteq E$ satisfy condition $(\Sigma)^\sim$. Then the spaces $(\mathcal{A}H)_0(\mathbb{C})$ and $\mathcal{A}H(\mathbb{C})$ coincide algebraically and topologically. Moreover, $\mathcal{A}H(\mathbb{C}) = \mathcal{V}H(\mathbb{C})$ holds algebraically and topologically if and only if $\mathcal{A}$ satisfies the conditions $(\mathrm{wQ})^\sim_{\mathrm{in}}$ and $(\mathrm{B})^\sim$.

Corollary 13.8. Let $\mathcal{A} \subseteq E$ consist of essential weights and satisfy condition (Σ). Then the spaces $(\mathcal{A}H)_0(\mathbb{C})$ and $\mathcal{A}H(\mathbb{C})$ coincide algebraically and topologically. Moreover, $\mathcal{A}H(\mathbb{C}) = \mathcal{V}H(\mathbb{C})$ holds algebraically and topologically if and only if $\mathcal{A}$ satisfies the conditions (wQ) and $(\mathrm{B})^\sim$.

13.3 Condition (LOG)

Proposition 13.9. Let $\mathcal{A}$ satisfy condition (LOG) and condition $(\Sigma)^\sim$. Then the spaces $(\mathcal{A}H)_0(\mathbb{D})$ and $\mathcal{A}H(\mathbb{D})$ coincide algebraically and topologically and $\operatorname{Proj}^1 \mathcal{A}H = \operatorname{Proj}^1 \mathcal{A}_0 H$. Moreover the following are equivalent.

(i) $\mathcal{A}$ satisfies condition $(\mathrm{Q})^\sim_{\mathrm{in}}$.
(ii) $\mathcal{A}$ satisfies condition $(\mathrm{Q})^\sim_{\mathrm{out}}$.
(iii) $\operatorname{Proj}^1 \mathcal{A}_{(0)} H = 0$.
(iv) $(\mathcal{A}H)_{(0)}(\mathbb{D})$ is ultrabornological.
(v) $(\mathcal{A}H)_{(0)}(\mathbb{D})$ is barrelled.
(vi) $\mathcal{A}$ satisfies condition $(\mathrm{wQ})^\sim_{\mathrm{in}}$.
(vii) $\mathcal{A}$ satisfies condition $(\mathrm{wQ})^\sim_{\mathrm{out}}$.

Proof. Again, we may copy the proof of 13.1 except that “(ii)⇒(iii)” now follows from 8.2. ■

Corollary 13.10. Let $\mathcal{A}$ consist of essential weights and satisfy the conditions (LOG) and (Σ). Then the spaces $(\mathcal{A}H)_0(\mathbb{D})$ and $\mathcal{A}H(\mathbb{D})$ coincide algebraically and topologically and $\operatorname{Proj}^1 \mathcal{A}H = \operatorname{Proj}^1 \mathcal{A}_0 H$. Moreover the following are equivalent.

(i) $\mathcal{A}$ satisfies condition (Q).
(ii) $\operatorname{Proj}^1 \mathcal{A}_{(0)} H = 0$.
(iii) $\mathcal{A}_{(0)} H(\mathbb{D})$ is ultrabornological.
(iv) $\mathcal{A}_{(0)} H(\mathbb{D})$ is barrelled.
(v) $\mathcal{A}$ satisfies condition (wQ).

Proposition 13.11. Let $\mathcal{A}$ satisfy the conditions (LOG) and $(\Sigma)^\sim$. Then the spaces $(\mathcal{A}H)_0(\mathbb{D})$ and $\mathcal{A}H(\mathbb{D})$ coincide algebraically and topologically. Moreover, $\mathcal{A}H(\mathbb{D}) = \mathcal{V}H(\mathbb{D})$ holds algebraically and topologically if and only if $\mathcal{A}$ satisfies the conditions $(\mathrm{wQ})^\sim_{\mathrm{in}}$ and $(\mathrm{B})^\sim$.

Corollary 13.12. Let $\mathcal{A} \subseteq E$ consist of essential weights and satisfy condition (Σ). Then the spaces $(\mathcal{A}H)_0(\mathbb{D})$ and $\mathcal{A}H(\mathbb{D})$ coincide algebraically and topologically. Moreover, $\mathcal{A}H(\mathbb{D}) = \mathcal{V}H(\mathbb{D})$ holds algebraically and topologically if and only if $\mathcal{A}$ satisfies the conditions (wQ) and $(\mathrm{B})^\sim$.

14 Examples

14.1 Examples for sequences of weights in $\mathcal{W}(\boldsymbol{\varepsilon_0},\mathbf{k_0})$

Example 14.1. Based on an example of Mattila, Saksman, Taskinen [55, 3.8] we put $a_{N,n}(z) := N(1-|z|)^{\alpha\frac{n}{n+1}}$ for some $\alpha > 0$. Since $a_{N,n}$ is clearly radial, non-increasing on $[0,1[$ and satisfies $\lim_{|z|\nearrow 1} a_{N,n}(z) = 0$ we have to check that the conditions (L1) and (L2) are satified.

(L1) Let N, n and $k \in \mathbb{N}$ be fixed. We compute

$$\frac{a_{N,n}(r_{k+1})}{a_{N,n}(r_k)} = \frac{N(1-(1-2^{-(k+1)}))^{\alpha\frac{n}{n+1}}}{N(1-(1-2^{-k}))^{\alpha\frac{n}{n+1}}} = (2^{-k-1+k})^{\alpha\frac{n}{n+1}} = \frac{1}{2^{\alpha\frac{n}{n+1}}} \geqslant \frac{1}{2^{\alpha}}$$

where we used $\frac{n}{n+1} \leqslant 1$. Hence we may put $\varepsilon_0 := \frac{1}{2^{\alpha}}$.

(L2) Let N, n and $k \in \mathbb{N}$ be fixed. For arbitrary k_0 we compute

$$\frac{a_{N,n}(r_{k+k_0})}{a_{N,n}(r_k)} = \frac{1}{2^{\alpha k_0\frac{n}{n+1}}} \leqslant \frac{1}{2^{\frac{\alpha}{2}k_0}}$$

where we used $\frac{n}{n+1} \geqslant \frac{1}{2}$. Since the right hand side tends to zero for $k_0 \to \infty$ there exists k_0 such that $\frac{1}{2^{\frac{\alpha}{2}k_0}} < 1 - \varepsilon_0$.

After we have shown that $\mathcal{A} \subseteq \mathcal{W}(\varepsilon_0, k_0)$, we claim that the above sequence satisfies (Σ). For given N we select $K := N$. For given k we select $n := k+1$. For $r \geqslant 0$ we have

$$\frac{a_{N,n}(r)}{a_{K,k}(r)} = \frac{N(1-r)^{\frac{\alpha n}{n+1}}}{N(1-r)^{\frac{\alpha k}{k+1}}} = (1-r)^{\alpha(\frac{n}{n+1}-\frac{k}{k+1})} = (1-r)^{\alpha\frac{1}{(k+2)(k+1)}} \stackrel{r\to 1}{\longrightarrow} 0$$

that is $\frac{a_{N,n}}{a_{K,k}}$ vanishes at ∞ on $\mathbb{D}$.

Now we claim that $\mathcal{A}$ satisfies condition (wQ). For given N we select $M := N$ and put $n := 1$. For given $K \geqslant M(=N)$ and m we select $k := m$ and $S := K$. Then we have to check that

$$\forall\, r \in]0,1]\colon r^{-\frac{\alpha m}{m+1}} M^{-1} \leqslant S \max\big(r^{-\frac{\alpha n}{n+1}} N^{-1}, r^{-\frac{\alpha k}{k+1}} K^{-1}\big)$$

that is

$$\forall\, r \in]0,1]\colon \big(\tfrac{1}{r}\big)^{\frac{\alpha m}{m+1}} N^{-1} \leqslant K \max\Big(\big(\tfrac{1}{r}\big)^{\frac{\alpha 1}{1+1}} N^{-1}, \big(\tfrac{1}{r}\big)^{\frac{\alpha m}{m+1}} K^{-1}\Big)$$

holds. The latter is equivalent to

$$\forall\, r \geqslant 1\colon r^{\frac{\alpha m}{m+1}} N^{-1} \leqslant K \max\big(r^{\frac{\alpha}{2}} N^{-1}, r^{\frac{\alpha m}{m+1}} K^{-1}\big).$$

Thus, let $r \geqslant 1$ be arbitrary. Then

$$r^{\frac{\alpha m}{m+1}} N^{-1} \leqslant r^{\frac{\alpha m}{m+1}} = K r^{\frac{\alpha k}{k+1}} K^{-1} \leqslant K \max\big(r^{\frac{\alpha}{2}} N^{-1}, r^{\frac{\alpha m}{m+1}} K^{-1}\big)$$

and we are done.

Finally let us show that $\mathcal{A}$ satisfies condition (B). Let $(n(N))_{N\in\mathbb{N}}$ be given. We put $m := n(1)$. For given M we select $L := 1$ and $c := M$. For given $r \in [0,1[$ we have

$$\begin{aligned} a_{M,m}(r) &= a_{M,n(1)}(r) \\ &= M(1-r)^{\alpha\frac{n(1)}{n(1)+1}} \end{aligned}$$

$$= c \cdot 1(1-r)^{\alpha \frac{n(1)}{n(1)+1}}$$
$$= c \cdot \max_{N=1,\dots,L} \big(N(1-r)^{\alpha \frac{n(N)}{n(N)+1}}\big)$$
$$= c \cdot \max_{N=1,\dots,L} (a_{N,n(N)})$$

and thus are done.

Note that the weights in the above sequence are all essential by [21, 1.7.(c)].

By 13.2, the spaces $(AH)_0(\mathbb{D})$ and $AH(\mathbb{D})$ coincide algebraically and topologically. They are ultrabornological, $\operatorname{Proj}^1 \mathcal{A}_0 H = \operatorname{Proj}^1 \mathcal{A}H = 0$ and by 13.4 the equality $(AH)_{(0)}(\mathbb{D}) = \mathcal{V}H(\mathbb{D})$ holds algebraically and topologically.

14.1 was in some sense the easiest way to construct a double sequence which gives rise to a (PLB)-space using the example of [55, 3.8] – we simply multiplied each weight $v_n(z) = (1-|z|)^{\alpha \frac{n}{n+1}}$ of the sequence $\mathcal{V} = (v_n)_{n\in\mathbb{N}}$ (which was studied by Mattila, Saksman, Taskinen in the context of (LB)-spaces) with N. The next example is of the same type that is we again put $a_{N,n} := u_N v_n$ but now u_N will be more complicated than above – Due to this "complication" of u_N we simplify v_n by selecting $\alpha = 1$.

Example 14.2. We put $a_{N,n}(z) := (1-|z|)^{\frac{N+1}{N}}(1-|z|)^{\frac{n}{n+1}}$. Again, $a_{N,n}$ is clearly radial, non-decreasing on $[0,1[$ and satisfies $\lim_{|z|\nearrow 1} a_{N,n}(z) = 0$. Thus, we have to check that the condition (L1) and (L2) are satified.

(L1) Let N, n and $k \in \mathbb{N}$ be fixed. We compute

$$\frac{a_{N,n}(r_{k+1})}{a_{N,n}(r_k)} = 2^{-\frac{N+1}{N}} 2^{-\frac{n}{n+1}} = \frac{1}{2^{\frac{N+1}{N}}} \frac{1}{2^{\frac{\alpha n}{n+1}}} \geqslant \frac{1}{2^2}\frac{1}{2} = 2^{-3},$$

where we used $\frac{N+1}{N} \leqslant 2$ and $\frac{n}{n+1} \leqslant 1$. Hence we may put $\varepsilon_0 := 2^{-3}$.

(L2) Let N, n and $k \in \mathbb{N}$ be fixed. For arbitrary k_0 we compute

$$\frac{a_{N,n}(r_{k+k_0})}{a_{N,n}(r_k)} = \frac{1}{2^{\frac{k_0(N+1)}{N}}} \frac{1}{2^{\frac{k_0 n}{n+1}}} \leqslant \frac{1}{2^{k_0}} \frac{1}{2^{k_0/2}} = 2^{-\frac{3}{2}k_0}$$

where we used $\frac{N+1}{N} \geqslant 1$ and $\frac{n}{n+1} \geqslant \frac{1}{2}$. Since, the right hand side tends to zero for $k_0 \to \infty$ there exists k_0 such that $2^{-\frac{3}{2}k_0} < 1 - \varepsilon_0$.

As in the previous example we claim that the above sequence satisfies (Σ). For given N we select $K := N$. For given k we select $n := k+1$. For $r \geqslant 0$ we have

$$\frac{a_{N,n}(r)}{a_{K,k}(r)} = \frac{(1-r)^{\frac{N+1}{N}}(1-r)^{\frac{n}{n+1}}}{(1-r)^{\frac{N+1}{N}}(1-r)^{\frac{k}{k+1}}} = (1-r)^{\frac{n}{n+1}-\frac{k}{k+1}} = (1-r)^{\frac{1}{(k+2)(k+1)}} \xrightarrow{r\to 1} 0$$

that is $\frac{a_{N,n}}{a_{K,k}}$ vanishes at ∞ on $\mathbb{D}$.

Let us now show that $\mathcal{A}$ satisfies (wQ). Let N be given. We select $M > N$ arbitrary. Then $\frac{M+1}{M} < \frac{N+1}{N}$ that is $\frac{N+1}{N} - \frac{M+1}{M} > 0$ and $\frac{N+1}{N} - \frac{M+1}{M} \leqslant 2 - \frac{M+1}{M} < 2 - 1 = 1$ and hence $\frac{N+1}{N} - \frac{M+1}{M} =: \alpha \in]0,1[$. We choose n such that $1 - \frac{n}{n+1} < \alpha$ which is possible since $1 - \frac{n}{n+1} \searrow 0$ for $n \to \infty$. Let $K \geqslant M$ and m be given. We put $\beta := \frac{m}{m+1} - \frac{n}{n+1}$.

CASE 1. $m \leqslant n$: In this case $\frac{m}{m+1} \leqslant \frac{n}{n+1}$ implies $\beta = \frac{m}{m+1} - \frac{n}{n+1} \leqslant 0$ and thus $(1-r)^\beta \geqslant 1$ for each $r \in [0,1[$. On the other hand we have $(1-r)^\alpha \leqslant 1$ for each $r \in [0,1[$ since $\alpha \in]0,1[$, thus $(1-r)^\alpha(1-r)^\beta$ for all $r \in [0,1[$.

CASE 2. $m > n$: Now, $\frac{m}{m+1} > \frac{n}{n+1}$ implies $\beta = \frac{m}{m+1} - \frac{n}{n+1} > 0$. Moreover, $\frac{m}{m+1} - \frac{n}{n+1} \leqslant 1 - \frac{n}{n+1} < \alpha$ by the choice of n. Thus, $0 < \beta < \alpha < 1$ and hence $(1-r)^\alpha \leqslant (1-r)^\beta$ for each $r \in [0,1[$.

To end the proof of (wQ) we select k arbitrarily and put $S := 1$. Now let $r \in [0,1[$ be fixed. By the above we have $(1-r)^\alpha \leqslant (1-r)^\beta$ that is by the definition of α and β just $(1-r)^{\frac{N+1}{N} - \frac{M+1}{M}} \leqslant (1-r)^{\frac{m}{m+1} - \frac{n}{n+1}}$ that is $(1-r)^{-\frac{M+1}{M} - \frac{m}{m+1}} \leqslant (1-r)^{-\frac{N+1}{N} - \frac{n}{n+1}}$ which yields

$$\begin{aligned}(a_{M,m}(r))^{-1} &= (1-r)^{-\frac{m}{m+1}}(1-r)^{-\frac{M+1}{M}}\\ &\leqslant (1-r)^{-\frac{N+1}{N}}(1-r)^{-\frac{n}{n+1}}\\ &= S(a_{N,n}(r))^{-1}\\ &\leqslant S\max\big((a_{N,n}(r))^{-1},(a_{K,k}(r))^{-1}\big)\end{aligned}$$

and thus finishes the proof of (wQ).

Finally let us show that condition (B) is not satisfied. That is we have to show $\neg$(B), i.e.

$$\exists\,(n(N))_{N\in\mathbb{N}}\,\forall\, m\,\exists\, M\,\forall\, L,\, c>0\,\exists\, r\in\ [0,1[\,: a_{M,m}(r) > c\cdot\max_{N=1,\dots,L} a_{N,n(N)}(r).$$

Put $n(N) := N$ and let m be given. Since $\frac{m}{m+1} < 1$ we have $2 - \frac{m}{m+1} > 1$. Therefore we may select M such that $\frac{M+1}{M} \leqslant 2 - \frac{m}{m+1}$ that is $\frac{M+1}{M} + \frac{m}{m+1} \leqslant 2$, since $\frac{M+1}{M} \to 1$ for $M \to \infty$. Let L be given. Then we have $\min_{N=1,\dots,L} \frac{N+1}{N} + \frac{N}{N+1} > 2$ since $\frac{N+1}{N} + \frac{N}{N+1} > 2$ holds for each N. Thus we have $\alpha := \frac{M+1}{M} + \frac{m}{m+1} \leqslant 2 < \min_{N=1,\dots,L} \frac{N+1}{N} + \frac{N}{N+1} =: \beta$. Let $c > 0$ be given. Since $\alpha - \beta < 0$ we have $\lim_{r\nearrow 1}(1-r)^{\alpha-\beta} = \infty$ and thus may choose $r \in\ [0,1[$ such that $(1-r)^{\alpha-\beta} > c$, i.e. $(1-r)^\alpha > c(1-r)^\beta$. By definition of α and β the latter yields

$$\begin{aligned}a_{M,m}(r) &= (1-r)^{\frac{M+1}{M}+\frac{m}{m+1}}\\ &> c\cdot(1-r)^{\min_{N=1,\dots,L}\frac{N+1}{N}+\frac{N}{N+1}}\\ &= c\cdot\max_{N=1,\dots,L}(1-r)^{\frac{N+1}{N}+\frac{N}{N+1}}\\ &= c\cdot\max_{N=1,\dots,L} a_{N,n(N)}(r).\end{aligned}$$

Note that the weights in the above sequence are all essential by [21, 1.7.(c)].

By 13.2, the spaces $(AH)_0(\mathbb{D})$ and $AH(\mathbb{D})$ coincide algebraically and topologically. They are ultrabornological, $\operatorname{Proj}^1 \mathcal{A}_0 H = \operatorname{Proj}^1 \mathcal{A}H = 0$ and 13.4 implies $(AH)_{(0)}(\mathbb{D}) \neq \mathcal{V}H(\mathbb{D})$.

The next example is very natural, since it involves the space $A^{-\infty}$ which is a space of Bergman type (see e.g. [47, section 4.1]). Unfortunately, there is no possibility to construct from the latter inductive limit a (PLB)-space which fits into the setting of class $\mathcal{W}(\varepsilon_0, k_0)$.

Example 14.3. For $v_n \colon \mathbb{D} \to \mathbb{R}$, $v_n(z) = (1-|z|)^n$ (or equivalently $v_n(z) = (1-|z|^2)^n$) we have $s' \cong A^{-\infty} = \operatorname{ind}_n Hv_n(\mathbb{D})$. A natural candidate for a double sequence $\mathcal{A} = ((a_{N,n})_{n\in\mathbb{N}})_{N\in\mathbb{N}}$ defining a (PLB)-space would be

$$a_{N,n}(z) := u_N(z)(1-|z|)^n$$

for some sequence $(u_N)_{N\in\mathbb{N}}$ with $u_N \colon \mathbb{D} \to \mathbb{R}$ and $u_N \leqslant u_{N+1}$.

The double sequence $\mathcal{A}$ always satisfies (Σ): For given N we select $K := N$ and for given k we select $n := k+1$. Then

$$\tfrac{a_{N,n}(z)}{a_{K,k}(z)} = \tfrac{u_N(z)(1-|z|)^{k+1}}{u_N(z)(1-|z|)^k} = (1-|z|)^{k+1-k} = (1-|z|) \overset{|z|\nearrow 1}{\longrightarrow} 0$$

that is $\frac{a_{N,n}(z)}{a_{K,k}(z)}$ vanishes at ∞ on $\mathbb{D}$.

Unfortunately, no sequence of this type satisfies (L1): We compute

$$\tfrac{a_{N,n}(1-2^{-(k+1)})}{a_{N,n}(1-2^{-k})} = \tfrac{u_N(1-2^{-(k+1)})}{u_N(1-2^{-k})} = \tfrac{u_N(1-2^{-(k+1)})(2^{-(k+1)})^n}{u_N(1-2^{-k})(2^{-k})^n} = \tfrac{u_N(r_{k+1})}{u_N(r_k)}2^{-n}.$$

Since $\big(\frac{u_N(r_{k+1})}{u_N(r_k)}\big)_{k\in\mathbb{N}} \subseteq]0,\infty[$, $\inf_{k\in\mathbb{N}} \frac{u_N(r_{k+1})}{u_N(r_k)} =: c \geqslant 0$ exists and

$$\inf_{k\in\mathbb{N}} \tfrac{a_{N,n}(r_{k+1})}{a_{N,n}(r_k)} = c2^{-n}.$$

But there cannot exist $\varepsilon_0 > 0$ such that the latter is bounded from below by $\varepsilon_0 > 0$ for all n.

Therefore the results in the setting of $\mathcal{W}(\varepsilon_0, k_0)$ are not applicable for double sequences $\mathcal{A}$ of this type.

Remark 14.4. The simplest modification of 14.3 might be to change n into $-1/n$ that is to put $a_{N,n}(z) := u_N(z)(1-|z|)^{-1/n}$ for $(u_N)_{N\in\mathbb{N}}$ as in 14.3. Again, (Σ) is satisfied: For given N we select $K := N$ and for given k we select $n := k+1$. Then

$$\tfrac{a_{N,n}(r)}{a_{K,k}(r)} = \tfrac{u_N(r)(1-r)^{-1/(k+1)}}{u_N(r)(1-r)^{-1/k}} = (1-r)^{\frac{1}{k}-\frac{1}{k+1}} \overset{r\nearrow 1}{\longrightarrow} 0,$$

hence $\frac{a_{N,n}}{a_{K,k}}$ vanishes at infinity on $\mathbb{D}$.

To get that the assumptions of the balanced setting are satisfied u_N has to be selected such that $\lim_{r\nearrow 1} u_N(r)(1-r)^{-1/n} = 0$ and for the assumptions of $\mathcal{W}(\varepsilon_0, k_0)$ we need in addition that $u_N(r)(1-r)^{-1/n}$ is non-increasing for $r \in [0,1[$.

Concerning the condition (L1) we compute

$$\begin{aligned} \tfrac{a_{N,n}(1-2^{-(k+1)})}{a_{N,n}(1-2^{-k})} &= \tfrac{u_N(1-2^{-(k+1)})(2^{-(k+1)})^{-1/n}}{u_N(1-2^{-k})(2^{-k})^{-1/n}} \\ &= \tfrac{u_N(1-2^{-(k+1)})}{u_N(1-2^{-k})}\big(2^{-1}\big)^{-1/n} = \tfrac{u_N(r_{k+1})}{u_N(r_k)}2^{1/n}. \end{aligned}$$

Since $2^{1/n} \geqslant 1$, condition (L1) is satisfied if and only if there is $\varepsilon_0 > 0$ such that

$\inf_k \frac{u_N(r_{k+1})}{u_N(r_k)} \geqslant \varepsilon_0$ holds for each N. We compute

$$\frac{a_{N,n}(1-2^{-(k+k_0)})}{a_{N,n}(1-2^{-k})} = \frac{u_N(1-2^{-(k+k_0)})(2^{-(k+k_0)})^{-1/n}}{u_N(1-2^{-k})(2^{-k})^{-1/n}} = \frac{u_N(r_{k+k_0})}{u_N(r_k)} 2^{k_0/n}.$$

Since $\max_n 2^{k_0/n} = 2^{k_0}$, condition (L2) is satisfied if and only if there is k_0 such that $2^{k_0} \limsup_k \frac{u_N(r_{k+k_0})}{u_N(r_k)} < (1-\varepsilon_0)$ with ε_0 as in (L1).

Example 14.5. In the situation above we put $u_N(z) := (1-|z|)^{\alpha_N}$ for some sequence $(\alpha_N)_{N\in\mathbb{N}}$ with $\alpha_N \searrow \alpha$ for $\alpha > 1$ and $\alpha_N \neq \alpha$ for each N. Then $a_{N,n}(r) = (1-r)^{\alpha_N - 1/n}$ with $\alpha_N - 1/n > 0$ is non-increasing for $r \in [0,1[$ and $\lim_{r\nearrow 1} a_{N,n}(r) = 0$ is satisfied for all N, $n \in \mathbb{N}$. By 14.4, the sequence satisfies (Σ).

We have $\frac{u_N(r_{k+1})}{u_N(r_k)} = 2^{-\alpha_N} \geqslant 2^{-\alpha_1} > 0$ and thus may put $\varepsilon_0 := 2^{-\alpha_1}$. Moreover,

$$2^{k_0} \frac{u_N(r_{k+k_0})}{u_N(r_k)} = 2^{k_0} 2^{-\alpha_N k_0} = 2^{-k_0(\alpha_N - 1)} < 2^{-k_0(\alpha-1)} \overset{k_0\to\infty}{\longrightarrow} 0,$$

since $\alpha - 1 > 0$. Hence it is possible to find k_0 such that $2^{-k_0(\alpha-1)} < 1-\varepsilon_0$. Therefore we have shown that $\mathcal{A} \subseteq \mathcal{W}(\varepsilon_0, k_0)$ holds.

Let us show that condition (wQ) is satisfied. For given N we select $M \geqslant N$ such that $\alpha_M < \alpha_N$ (this is possible since $(\alpha_N)_{N\in\mathbb{N}}$ is decreasing with limit α and all α_N are distinct from α). Therefore $\alpha' := \alpha_N - \alpha_M > 0$. We select n such that $\frac{1}{n} < \alpha'$. Let $K \geqslant M$ and m be given. We put $\beta' := \frac{1}{n} - \frac{1}{m}$.

CASE 1. $m \leqslant n$: In this case, $\beta' = \frac{1}{n} - \frac{1}{m} \leqslant 0$ that is $(1-r)^{\beta'} \geqslant 1$ and (since $\alpha' > 0$) $(1-r)^{\alpha'} \leqslant 1$, hence $(1-r)^{\alpha'} \leqslant (1-r)^{\beta'}$ for each $r \in [0,1[$.

CASE 2. $m > n$: Now, $\beta' = \frac{1}{n} - \frac{1}{m} > 0$ and $\beta' = \frac{1}{n} - \frac{1}{m} \leqslant \frac{1}{n} < \alpha'$. Thus $0 < \beta' < \alpha'$ and hence $(1-r)^{\alpha'} \leqslant (1-r)^{\beta'}$ for each $r \in [0,1[$.

To finish the proof of (wQ) we select k arbitrarily, put $S := 1$ and fix $r \in [0,1[$. By the above we have $(1-r)^{\alpha'} \leqslant (1-r)^{\beta'}$ that is by the definition of α' and β' just $(1-r)^{\alpha_N - \alpha_M} \leqslant (1-r)^{1/n - 1/m}$ that is $(1-r)^{1/m - \alpha_M} \leqslant (1-r)^{1/n - \alpha_N}$ which yields

$$\begin{aligned}
(a_{M,m}(r))^{-1} &= (1-r)^{1/m - \alpha_M} \\
&\leqslant (1-r)^{1/n - \alpha_N} \\
&= S(a_{M,m}(r))^{-1} \\
&\leqslant S \max\big(a_{M,m}(r))^{-1}, a_{K,k}(r))^{-1}\big).
\end{aligned}$$

Thus, the proof of (wQ) is complete.

Note that the weights in the above sequence are all essential by [21, 1.7.(c)].

By 13.2, the spaces $(AH)_0(\mathbb{D})$ and $AH(\mathbb{D})$ coincide algebraically and topologically. They are ultrabornological and $\operatorname{Proj}^1 \mathcal{A}_0 H = \operatorname{Proj}^1 \mathcal{A}H = 0$ holds.

14.2 Examples for sequences of weights in $(\mathsf{E})_{\mathsf{C,c}}$

Example 14.6. Based on the first example given by Bierstedt, Bonet, Taskinen in [22, Examples after 2.1], we define a double sequence $\mathcal{A} = ((a_{N,n})_{n\in\mathbb{N}})_{N\in\mathbb{N}}$ by $a_{N,n}(|z|) := (1+|z|)^{\alpha_{N,n}}e^{-|z|}$, where $((\alpha_{N,n})_{n\in\mathbb{N}})_{N\in\mathbb{N}} \subseteq [0,\infty[$ is a bounded double sequence which satisfies $\alpha_{N,n+1} \leqslant \alpha_{N,n} \leqslant \alpha_{N+1,n}$. By [22, Examples after 2.1], all $a_{N,n}$ belong to $(E)_{A,1}$ with $A := \sup_{N,n\in\mathbb{N}} \alpha_{N,n} < \infty$.

Remark 14.7. The sequence $\mathcal{A}$ in 14.6 satisfies condition (Σ) if and only if (Σ_α)

$$\forall\, N\; \exists\, K \geqslant N\; \forall\, k\; \exists\, n \geqslant k \colon \alpha_{N,n} < \alpha_{K,k}$$

holds.

Proof. "⇒" Let (Σ) be satisfied. For given N we select $K \geqslant N$ as in (Σ). For given k we select $n \geqslant k$ as in (Σ). Then

$$\tfrac{a_{N,n}(r)}{a_{K,k}(r)} = \tfrac{(1+r)^{\alpha_{N,n}}}{(1+r)^{\alpha_{K,k}}} = (1+r)^{\alpha_{N,n}-\alpha_{K,k}} \stackrel{r\to\infty}{\longrightarrow} 0$$

holds, since $\frac{a_{N,n}}{a_{K,k}}$ vanishes at ∞ on $\mathbb{C}$. But this clearly implies $\alpha_{N,n} - \alpha_{K,k} < 0$ that is $\alpha_{N,n} < \alpha_{K,k}$.

"⇐" Let (Σ_α) be satisfied. For given N we select $K \geqslant N$ as in (Σ_α). For given k we select $n \geqslant k$ as in (Σ_α). Then $\alpha_{N,n} < \alpha_{K,k}$ that is $\alpha_{N,n} - \alpha_{K,k} < 0$ and the computation above shows that $\frac{a_{N,n}(r)}{a_{K,k}(r)}$ vanishes at ∞. ■

Roughly speaking, the latter means that "enough" of the estimates $\alpha_{N,n+1} \leqslant \alpha_{N,n} \leqslant \alpha_{N+1,n}$ have to be strict, which is clearly a quite natural assumption.

Example 14.8. In the situation of 14.6 we put $\alpha_{N,n} := \frac{N}{N+1} + \frac{n+1}{n}$ that is we consider the double sequence $\mathcal{A} = ((a_{N,n})_{n\in\mathbb{N}})_{N\in\mathbb{N}}$ with $a_{N,n}(|z|) = (1+|z|)^{\frac{N}{N+1}+\frac{n+1}{n}}e^{-|z|}$. 14.6 implies that $\mathcal{A} \subseteq (E)_{A,a}$ holds for suitable constants a and A. Moreover, 14.7 yields that $\mathcal{A}$ satisfies condition (Σ). We claim that condition (wQ) is satisfied. For given N we select $M > N$ arbitrary. Then $\frac{M}{M+1} > \frac{N}{N+1}$ that is $\alpha := \frac{M}{M+1} - \frac{N}{N+1} > 0$. We choose n such that $\frac{1}{n} < \alpha$. Let $K \geqslant M$ and m be given.

CASE 1. $m \leqslant n$: In this case $\frac{m+1}{m} \geqslant \frac{n+1}{n}$ and therefore $\frac{m+1}{m} - \frac{n+1}{n} \geqslant 0$. Hence

$$(1+r)^{\frac{N}{N+1}-\frac{M}{M+1}} = (1+r)^{-\alpha} \leqslant 1 \leqslant (1+r)^{\frac{m+1}{m}-\frac{n+1}{n}}$$

holds for each $r \geqslant 0$ since $1+r \geqslant 1$ and $\alpha > 0$.

CASE 2. $m > n$: In this case $\frac{m+1}{m} < \frac{n+1}{n}$ that is $\beta := \frac{n+1}{n} - \frac{m+1}{m} > 0$ and $\beta := \frac{n+1}{n} - \frac{m+1}{m} < \frac{n+1}{n} - 1 = \frac{1}{n} < \alpha$. Thus, we have $0 < \beta < \alpha$ and hence $(1+r)^\beta \leqslant (1+r)^\alpha$, thus $(1+r)^{-\beta} \geqslant (1+r)^{-\alpha}$ and hence $(1+r)^{\frac{N}{N+1}-\frac{M}{M+1}} = (1+r)^{-\alpha} \leqslant (1+r)^{-\beta} = (1+r)^{\frac{m+1}{m}-\frac{n+1}{n}}$ for each $r \geqslant 0$.

Both cases together yield $(1+r)^{-\frac{M}{M+1}-\frac{m+1}{m}} \leqslant (1+r)^{-\frac{N}{N+1}-\frac{n+1}{n}}$ for arbitrary $r \geqslant 0$.

Now we select k arbitrarily and put $S := 1$. Then

$$\begin{aligned}
(a_{M,m}(r))^{-1} &= (1+r)^{-\frac{M}{M+1}-\frac{m+1}{m}} \\
&\leqslant (1+r)^{-\frac{N}{N+1}-\frac{n+1}{n}} \\
&= S(a_{N,n}(r))^{-1} \\
&\leqslant S \max\big((a_{N,n}(r))^{-1}, (a_{N,n}(r))^{-1}\big),
\end{aligned}$$

which is the desired estimate in (wQ).
The weights in the above sequence are all essential by [21, 1.7.(c)].
By 13.6, the spaces $AH(\mathbb{C})$ and $(AH)_0(\mathbb{C})$ coincide algebraically and topologically, $\operatorname{Proj}^1 \mathcal{A}_{(0)}H$ vanishes and the spaces are ultrabornological and barrelled.

14.3 Examples for sequences of weights satisfying (LOG)

Example 14.9. Based on the example [32, Example 5] we define the double sequence $\mathcal{A} = ((a_{N,n})_{n\in\mathbb{N}})_{N\in\mathbb{N}}$ by

$$a_{N,n}(z) := \begin{cases} 1 & \text{if } |z| < r', \\ (-1/\log(1-|z|))^{\alpha_n\beta_N} & \text{otherwise,} \end{cases}$$

where $r' := 1 - 1/e$ and $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_N)_{N\in\mathbb{N}}$ are sequences with $0 < \alpha_n \nearrow \alpha$ and $\beta_N \searrow \beta > 0$ and $\alpha_n \neq \alpha$, $\beta_N \neq \beta$ for all n, $N \in \mathbb{N}$. All $a_{N,n}$ are radial and approach monotonically 0 as $r \nearrow 1$. We put $A := 2^{\alpha\beta_1}$ and $a := (2\log 2)^{-\alpha_1\beta}$. Then $0 < a < 1 < A$ holds. Remember that we used the abbreviation $r_k = 1 - 2^{2^k}$ for $k \geqslant 1$ and $r_0 = 0$ within the setting of condition (LOG). For $k \geqslant 1$ we have $r_k \geqslant r_1 = 1 - 2^{-2^1} = 3/4 \geqslant 1 - 1/e = r'$. Thus, $a_{N,n}(r_0) = 1$ and $a_{N,n}(r_k) = (-1/(\log(1-(1-2^{-k})))^{\alpha_n\beta_N} = (1/(2^k\log 2))^{\alpha_n\beta_N}$ for $k \geqslant 1$. Let us check the two estimates in condition (LOG).

(LOG 1) For $k \geqslant 1$ we have

$$\begin{aligned}
Aa_{N,n}(r_{k+1}) &= 2^{\alpha\beta_1}(1/(2^{k+1}\log 2))^{\alpha_n\beta_N} \\
&= 2^{\alpha\beta_1-(k+1)\alpha_n\beta_N}(\log 2)^{-\alpha_n\beta_N} \\
&\geqslant 2^{\alpha_n\beta_N-(k+1)\alpha_n\beta_N}(\log 2)^{-\alpha_n\beta_N} \\
&= 2^{\alpha_n\beta_N(1-(k+1))}(\log 2)^{-\alpha_n\beta_N} \\
&= (1/(2^k\log 2))^{\alpha_n\beta_N} \\
&= a_{N,n}(r_k).
\end{aligned}$$

If on the other hand $k = 0$ we have

$$\begin{aligned}
Aa_{N,n}(r_{k+1}) &= 2^{\alpha\beta_1}(1/(2\log 2))^{\alpha_n\beta_N} \\
&= 2^{\alpha\beta_1}(2\log 2)^{-\alpha_n\beta_N} \\
&\geqslant 2^{\alpha\beta_1}(2\log 2)^{-\alpha\beta_1}
\end{aligned}$$

$$\begin{aligned}&= (\log 2)^{-\alpha\beta_1}\\ &\geqslant a_{N,n}(r_0).\end{aligned}$$

since $\log 2 \leqslant 1$ and $a_{N,n}(r_0) = 1$.

(LOG 2) For $k \geqslant 1$ we have

$$\begin{aligned}a_{N,n}(r_{k+1}) &= (1/(2^{k+1}\log 2))^{\alpha_n\beta_N}\\ &= 2^{-\alpha_n\beta_N - k\alpha_n\beta_N}(\log 2)^{-\alpha_n\beta_N}\\ &\leqslant 2^{-\alpha_1\beta - k\alpha_n\beta_N}(\log 2)^{-\alpha_n\beta_N}\\ &= 2^{-\alpha_1\beta}(1/(2^k\log 2))^{\alpha_n\beta_N}\\ &\leqslant (2\log 2)^{-\alpha_1\beta}(1/(2^k\log 2))^{\alpha_n\beta_N}\\ &= a a_{N,n}(r_k).\end{aligned}$$

If on the other hand $k = 0$ we have

$$\begin{aligned}a_{N,n}(r_{k+1}) &= (1/(2\log 2))^{\alpha_n\beta_N}\\ &= (2\log 2)^{-\alpha_n\beta_N}\\ &\leqslant (2\log 2)^{-\alpha_1\beta}\\ &= a a_{N,n}(r_0)\end{aligned}$$

since $a_{N,n}(r_0) = 1$.

Next, we claim that the sequence $\mathcal{A}$ satisfies condition (Σ). For given N we select $K > N$ such that $\beta_K < \beta_N$ which is possible by our assumptions on the sequence $(\beta_N)_{N\in\mathbb{N}}$. For given k we select $n := k$. For $r \geqslant r'$ we may compute

$$\tfrac{a_{N,n}(r)}{a_{K,k}(r)} = \big(\tfrac{-1}{\log(1-r)}\big)^{\alpha_k\beta_N - \alpha_k\beta_K} = \big(\tfrac{-1}{\log(1-r)}\big)^{\alpha_k(\beta_N-\beta_K)} \overset{r\nearrow 1}{\longrightarrow} 0,$$

since $\alpha_k(\beta_N - \beta_K) > 0$ holds by our choice of K.

Let us now investigate condition (wQ) for a concrete example of a sequence of the above type, that is we select concrete sequences $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_N)_{N\in\mathbb{N}}$.

Example 14.10. In the notation of 14.9 we put $\alpha_n := \frac{n}{n+1}$ and $\beta_N := \frac{N+1}{N}$ that is we consider $\mathcal{A} = ((a_{N,n})_{n\in\mathbb{N}})_{N\in\mathbb{N}}$ with

$$a_{N,n}(z) := \begin{cases} 1 & \text{if } |z| < r', \\ (-1/\log(1-|z|))^{\frac{n}{n+1}\frac{N+1}{N}} & \text{otherwise.}\end{cases}$$

For given N we have $\frac{N+1}{N} > 1$. Since $\frac{n}{n+1} \nearrow 1$ it is possible to select n such hat $\frac{n}{n+1}\frac{N+1}{N} > 1$ holds. On the other hand $\frac{M+1}{M} \searrow 1$ and hence we may select M such that $\frac{M+1}{M} < \frac{n}{n+1}\frac{N+1}{N}$ holds. Let $K \geqslant M$ and m be given. Then $\frac{m}{m+1}\frac{M+1}{M} < \frac{n}{n+1}\frac{N+1}{N}$ holds since $\frac{m}{m+1} < 1$. We select k arbitrarily and put $S := 1$. Let now $r \in [0,1[$ be given. If $r \leqslant r'$ the estimate in (wQ) is clearly true. If $r \in]r',1[$ we have $-1/\log(1-r) \in]0,1[$. Thus, the estimate $\frac{m}{m+1}\frac{M+1}{M} < \frac{n}{n+1}\frac{N+1}{N}$ implies

$(-1/\log(1-r))^{\frac{m}{m+1}\frac{M+1}{M}} > (-1/\log(1-r))^{\frac{n}{n+1}\frac{N+1}{N}}$ and hence

$$\begin{aligned}(a_{M,m}(r))^{-1} &= (-1/\log(1-r))^{-\frac{m}{m+1}\frac{M+1}{M}} \\ &< (-1/\log(1-r))^{-\frac{n}{n+1}\frac{N+1}{N}} \\ &= S(a_{N,n}(r))^{-1}\end{aligned}$$

which shows that (wQ) holds.

By [18, 5.2] the above provides that (Q) is satisfied and (Q) implies $(\mathrm{Q})^{\sim}_{\mathrm{out}}$. Thus 8.2 implies that $\operatorname{Proj}^1 \mathcal{A}H = 0$ holds and therefore $AH(\mathbb{D})$ is ultrabornological and barrelled.

14.4 Examples for projective limits of (DFN)-algebras

Remark 14.11. Let p be a weight function, $(p_n)_{n\in\mathbb{N}}$ be an increasing radial weight system in the notation of Meise [56, 2.2] and $(p_N)_{N\in\mathbb{N}}$ be an arbitrary decreasing sequence of weight functions. Assume that $((u_{N,n})_{N\in\mathbb{N}})_{n\in\mathbb{N}} \subseteq \mathbb{R}_{>0}$, $(u_N^{(1)})_{N\in\mathbb{N}} \subseteq \mathbb{R}_{>0}$ and $(u_n^{(2)})_{n\in\mathbb{N}} \subseteq \mathbb{R}_{>0}$ be sequences such that $u_{N+1} \leqslant u_{N,n} \leqslant u_{N,n+1}$, $u_{N+1}^{(1)} \leqslant u_N^{(1)}$ and $u_n^{(2)} \leqslant u_{n+1}^{(2)}$ holds for all N and $n \in \mathbb{N}$. Moreover we assume that $\limsup_{n\to\infty} u_{N,n} = \limsup_{n\to\infty} u_n^{(2)} = \infty$ (for each N). We consider a double sequence $\mathcal{P}$ which is defined by

(i) $p_{N,n}(z) = u_{N,n}p(z)$ or

(ii) $p_{N,n}(z) = u_N^{(1)}p_n(z)$ or

(iii) $p_{N,n}(z) = u_n^{(2)}p_N(z)$.

Then, $\mathcal{P}$ satisfies the (DFN 1)-(DFN 5).

Proof. (i) Let N, n be fixed. Since p is (pluri)subharmonic the same is true for $u_{N,n}p$. Further, $\log(1+|z|^2) = O(p(z))$ implies that $\log(1+|z|^2) = O(u_{N,n}p(z))$ holds. As (DFN 3) holds for p, that is there is $C \geqslant 1$ such that for each $w \in \mathbb{C}$, we may compute

$$\begin{aligned}\sup_{|z-w|\leqslant 1} p_{N,n}(z) &= \sup_{|z-w|\leqslant 1} u_{N,n}p(z) \\ &= u_{N,n} \sup_{|z-w|\leqslant 1} p(z) \\ &\leqslant u_{N,n}\big(C \inf_{|z-w|\leqslant 1} p(z) + C\big) \\ &= C \inf_{|z-w|\leqslant 1} u_{N,n}p(z) + u_{N,n}C \\ &\leqslant C' \inf_{|z-w|\leqslant 1} p_{N,n}(z) + C'.\end{aligned}$$

where $C' := \max(C, u_{N,n}C) \geqslant 1$. Thus, we have shown that each $p_{N,n}$ is a weight function. The estimate (DFN 4) follows immediately from the assumptions on the sequence $((u_{N,n})_{N\in\mathbb{N}})_{n\in\mathbb{N}}$. To check (DFN 5) let N and n be given. Then by our assumptions on $((u_{N,n})_{N\in\mathbb{N}})_{n\in\mathbb{N}}$ we may put $L := 0$ and select l such that $2p_{N,n} = 2u_{N,n}p \leqslant u_{N,l}p = p_{N,l} + L$.

(ii) The proof of the properties (DFN 1)-(DFN 3) can be performed in complete analogy to (i). Since $(u_N^{(1)})_{N\in\mathbb{N}}$ is decreasing (and $(p_n)_{n\in\mathbb{N}}$ is increasing) by definition, (DFN 4) follows immediately. In order to check (DFN 5), let N and n be given. Then by [56, 2.2.(2)] there exist l and $L' \geqslant 0$ such that $2p_n \leqslant p_l + L'$ and hence $2p_{N,n} = 2u_N^{(1)}p_n \leqslant u_N^{(1)}(p_n + L') = u_N^{(1)}p_n + u_N^{(1)}L' \leqslant p_{N,n} + u_1^{(1)}L'$ since $(u_N^{(1)})_{N\in\mathbb{N}}$ is decreasing. Thus we my put $L := u_1^{(1)}L'$ and have shown (DFN 5).

(iii) Again, (DFN 1)-(DFN 3) can be proved as in (i) and (DFN 4) follows from the assumptions on $(u_n^{(2)})_{n\in\mathbb{N}}$ and $(p_N)_{N\in\mathbb{N}}$. For (DFN 5) let again N and n be given. By our assumptions on $(u_n^{(2)})_{n\in\mathbb{N}}$ we may put $L := 0$ and select l such that $2p_{N,n} = 2u_n^{(2)}p_N \leqslant u_l^{(2)}p_N = p_{N,l} + L$ as desired. ∎

In the sequel we investigate concrete examples of sequences of the above types. Let us start with a very natural and in some sense simple one of type (i): We put $p(z) := |z|$ and $u(N,n) := n/N$ that is $p_{N,n}(z) = \frac{n}{N}|z|$. The example is very natural, since the weights $z \mapsto \exp(-np(z))$ for a weight function p yield (LB)-spaces of the type investigated by Berenstein, Gay and on the other hand weights $z \mapsto \exp(-\frac{1}{N}p(z))$ give rise to well-known Fréchet spaces (cf. e.g. [56, p. 60]). Finally, $p(z) = |z|$ appears to be a “quite simple” example for a weight function.

Example 14.12. Let $\mathcal{P}$ be defined via $p_{N,n} = \frac{n}{N}|z|$. In view of 14.11 it remains to check that $\log(1+|z|^2) = o(p)$ holds in order to apply 9.20. But the latter is clear since $\log(1+r^2)/r \to 0$ for $r \to \infty$. Let us now check if condition $(\mathrm{wQ})^{\mathrm{P}}$ is satisfied. For given N we select $M \geqslant N$ and n arbitrarily. Given $K \geqslant N$ and m we put $k := \lceil \frac{mK}{M} \rceil$ and $S := 1$. Then for $r \geqslant 0$ we have

$$\begin{aligned} p_{M,m}(r) &= \tfrac{m}{M}r \\ &\leqslant \log S + \tfrac{k}{K}r \\ &\leqslant \log S + \max\left(\tfrac{n}{N}r,\ \tfrac{k}{K}r\right) \\ &= \log S + \max(p_{N,n}(r),\ p_{K,k}(r)) \end{aligned}$$

that is condition $(\mathrm{wQ})^{\mathrm{P}}$ holds. In view of 9.20 the above provides already that $\operatorname{Proj}^1 \mathcal{A}H = 0$, $AH(\mathbb{C})$ is ultrabornological and barrelled. However, we will give an explicit description of the matrices $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ in the notation of 9.20. We compute

$$\begin{aligned} b_{N,n}(j)^2 &= \int_0^\infty r^{2j+1}e^{-2p_{N,n}(r)}dr \\ &= \int_0^\infty r^{2j+1}e^{-2\frac{n}{N}r}dr \\ &= \int_0^\infty \left(\tfrac{r}{2n/N}\right)^{2j+1}e^{-r}\tfrac{1}{2n/N}dr \\ &= (2n/N)^{-2(j+1)}\int_0^\infty r^{2j+2-1}e^{-r}dr \end{aligned}$$

$$= (2n/N)^{-2(j+1)}\,\Gamma(2j+2)$$
$$= (2n/N)^{-2(j+1)}(2j+1)!$$

that is

$$b_{N,n}(j) = (2n/N)^{-(j+1)}\sqrt{(2j+1)!}.$$

Moreover, the only zero of $\frac{\partial}{\partial r}e^{-\frac{n}{N}r}r^j = N^{-1}e^{-nr/N}(r^{j-1}jN - nr^j)$ is $r = \frac{N}{n}j$, whence $\sup_{r\geqslant 0} e^{-p_{N,n}(r)}r^j = \max_{r\geqslant 0} e^{-\frac{n}{N}r}r^j = \frac{N}{n}j$ since $r \mapsto e^{-\frac{n}{N}r}r^j$ is positive on $[0,\infty[$ and has value 0 for $r=0$. That is

$$d_{N,n}(j) = c_{N,n}(j) = \tfrac{N}{n}j.$$

Compared with $\mathcal{B}$ the sequence $\mathcal{C}$ (and $\mathcal{D}$) is first simpler and second very similar to the original sequence $\mathcal{P}$. Indeed, condition (wQ) for $\mathcal{C}$ is even easier to verify than (wQ)^{P} for $\mathcal{P}$: The estimate to check reduces to $\frac{m}{M} \leqslant S\max\big(\frac{n}{N}, \frac{k}{K}\big)$ since the j's can be cancelled. The selection of indices and constants according to the quantifiers in (wQ) resp. (wQ)^{P} can be performed similarly in both cases, but because of the logarithms occuring in (wQ)^{P}, the approach using $\mathcal{C}$ turns finally out to be the easiest one concerning this example.

Example 14.13. Arguments similar to those used in 14.12 yields for the sequence $\mathcal{P}$ with $p_{N,n} = \frac{n}{N}|z|^q$ for $q>0$ satisfies $\log(1+|z|^2) = o(p_{N,1}(z))$ for $|z|\to\infty$ and arbitrary N. It can be verified similarly to the above that $\mathcal{P}$ satisfies condition (wQ)^{P}. Finally it is easy to see that

$$b_{N,n}(j) = q^{-1/2}(2n/N)^{-\frac{j+1}{q}}\sqrt{\Gamma\big(\tfrac{2j+1}{q}\big)}$$

and

$$d_{N,n}(j) = c_{N,n}(j) = \exp\big(q^{-1}\log\big(\tfrac{N}{n}\tfrac{j}{q}\big)\big).$$

Hence, in this example the easiest approach is checking that $\mathcal{P}$ satisfies (wQ)^{P}.

Example 14.14. In addition to the examples 14.12 and 14.13 let us give three more examples of type (i), namely three more weight functions p, which satisfy $\log(1+|z|^2) = o(p(z))$ for $|z|\to\infty$ that is

(1) $p(z) = \big((\log(1+|z|^2)\big)^\alpha$ for $\alpha > 1$,
(2) $p(z) = \exp\big(\log(1+|z|^2)^\beta\big)$ for $0<\beta<1$,
(3) $p(z) = \exp(|z|^\gamma)$, for $\gamma > 0$.

In all three cases a concrete computation of $\mathcal{B}$, $\mathcal{C}$ or $\mathcal{D}$ seems to be rather difficult. However, if e.g. $u(N,n) = n/N$ one can proceed as in 14.12 to show that that $\mathcal{P} = ((u(N,n)p)_{n\in\mathbb{N}})_{N\in\mathbb{N}}$ satisfies condition (wQ)^{P}.

Let us now investigate two more concrete examples, where the first one is of type (ii) and the second one is of type (iii).

Example 14.15. We put $p_{N,n} = \frac{1}{N} p_n$ where

$$p_n(z) = \begin{cases} 1 & \text{if } z \in \mathbb{D}, \\ |z|^n & \text{otherwise.} \end{cases}$$

Since $p_n(z) = \max(1, |z|^n)$, p_n is subharmonic for each $n \in \mathbb{N}$ and we clearly have $p_n \leqslant p_{n+1}$. Moreover, $\log(1 + |z|^2) = O(p_n(z))$ for $|z| \to \infty$ holds for each $n \in \mathbb{N}$. Since $z \mapsto |z|^n$ is a weight function there exists $C \geqslant 1$ such that $\sup_{|z-w|\leqslant 1} |z|^n \leqslant C \inf_{|z-w|\leqslant 1} |z|^n + C$ for each $w \in \mathbb{C}$. Hence

$$\begin{aligned}
\sup_{|z-w|\leqslant 1} p_n(z) &= \sup_{|z-w|\leqslant 1} \max(1,\, |z|^n) \\
&= \max(1,\, \sup_{|z-w|\leqslant 1} |z|^n) \\
&= \max(1,\, C \inf_{|z-w|\leqslant 1} |z|^n + C) \\
&\leqslant \max(C,\, C \inf_{|z-w|\leqslant 1} |z|^n + C) \\
&\leqslant C \max(1,\, \inf_{|z-w|\leqslant 1} |z|^n) + C \\
&= C \inf_{|z-w|\leqslant 1} \max(1, |z|^n) + C \\
&= C \inf_{|z-w|\leqslant 1} p_n(z) + C
\end{aligned}$$

holds for each $n \in \mathbb{N}$. It remains to check [56, 2.2.(2)]: For given k we put $m := k + 1$. Then $\lim_{r\to\infty} 2r^k - r^{k+1} = -\infty$ that is there exists $L \geqslant 1$ such that $2r^k - r^{k+1} \leqslant L$, i.e. $2r^k \leqslant r^{k+1} + L$ for each $r \geqslant 0$. Hence

$$2p_k(z) = \max(2, 2|z|^k) \leqslant \max(1 + L, |z|^m + L) = \max(1, |z|^m) + L = p_m(z) + L.$$

for arbitrary $z \in \mathbb{C}$. Before we investigate if $\mathcal{P}$ satisfies condition $(\mathrm{wQ})^{\mathrm{P}}$, we note that clearly $\log(1 + |z|^2) = o(p_{N,n}(z))$ holds for $|z| \to \infty$.

In order to check $(\mathrm{wQ})^{\mathrm{P}}$, let n be given. We select $M \geqslant N$ and n arbitrarily. Let $K \geqslant M$ and m be given. We select $k := m + 1$. Then $\lim_{r\to\infty} \frac{1}{M} r^m - \frac{1}{K} r^k = -\infty$ and hence $S := \max\big(\exp(\frac{1}{M} - \frac{1}{K}), \exp(\max_{r\geqslant 1} \frac{1}{M} r^m - \frac{1}{K} r^k)\big) < \infty$. Now let $r \geqslant 0$ be arbitrary.

CASE 1. $r < 1$: Since $\log S \geqslant \frac{1}{M} - \frac{1}{K}$, we may compute

$$\log S + p_{K,k}(r) = \log S + \tfrac{1}{K} \geqslant \tfrac{1}{M} - \tfrac{1}{K} + \tfrac{1}{K} = \tfrac{1}{M} = p_{M,m}(r).$$

CASE 2. $r \geqslant 1$: Since we have $\log S \geqslant \max_{r'\geqslant 1} \frac{1}{M} r'^m - \frac{1}{K} r'^{m+1} \geqslant \frac{1}{M} r^m - \frac{1}{K} r^{m+1}$ for each $r \geqslant 1$, we have

$$\log S + p_{K,k}(r) \geqslant \tfrac{1}{M} r^m - \tfrac{1}{K} r^{m+1} + \tfrac{1}{K} r^k = \tfrac{1}{M} r^m = p_{M,m}(r)$$

for each $r \geqslant 1$.

Finally, the above yields $p_{M,m}(r) \leqslant \log S + p_{K,k}(r) \leqslant \log S + \max(p_{N,n}(r),\, p_{K,k}(r))$

for each $r \geqslant 0$ that is (wQ)^{P} holds.

Example 14.16. We put $p_{N,n} = np_N$ where

$$p_N(z) = \begin{cases} 1 & \text{if } z \in \mathbb{D}, \\ |z|^{1/(N+1)} & \text{otherwise.} \end{cases}$$

$(p_N)_{N\in\mathbb{N}}$ is a decreasing sequence. It can be seen as in 14.15 that each p_N is a weight function. Moreover, we have $\log(1+|z|^2) = o(p_{N,n}(z))$ for $|z| \to \infty$.

We claim, that the sequence above satisfies (wQ)^{P}. Given N we select $M > N$ and n arbitrarily. Let $K \geqslant M$ and m be given. We select $k = m$. By our choice of $M > N$ we have $\frac{1}{M+1} < \frac{1}{N+1}$ and hence $\lim_{r\to\infty} mr^{1/(M+1)} - nr^{1/(N+1)} = -\infty$. Thus there exists $r_0 \geqslant 1$ such that $mr^{1/(M+1)} - nr^{1/(N+1)} \leqslant 0$ for each $r \geqslant r_0$. We put $S := \exp(\max_{r\in[1,r_0]}(mr^{1/(M+1)}))$, which is clearly finite and greater or equal to one. Now let $r \geqslant 0$ be given.

CASE 1. $r < 1$: In this case we have $p_{M,m}(r) = m \leqslant \log S + m = \log S + k = \log S + p_{K,k}(r)$, since $\log S \geqslant 0$ by the choice of S.

CASE 2. $1 \leqslant r \leqslant r_0$: We have $p_{M,m}(r) = mr^{1/(M+1)} \leqslant \max_{r\in[1,r_0]} mr^{1/(M+1)} = \log S \leqslant \log S + kr^{1/(K+1)} = \log S + p_{K,k}(r)$ by our choice of S.

CASE 3. By the definition of $r > r_0$ we have $mr^{1/(M+1)} - nr^{1/(N+1)} \leqslant 0 \leqslant \log S$ for $r \geqslant r_0$, whence $p_{M,m}(r) = mr^{1/(M+1)} \leqslant \log S + nr^{1/(N+1)} = \log S + p_{N,n}$ holds.

Combining the three cases, we get exactly the the estimate in (wQ)^{P} for each $r \geqslant 0$.

14.5 Examples for the non-radial setting

Example 14.17. In view of the second condition in 10.16.(a) it is easy to see that $\mathcal{U} = ((u_{N,n})_{N\in\mathbb{N}})_{n\in\mathbb{N}}$ with $u_{N,n} = \frac{n}{N}$ satisfies $(\overline{\text{Q}})_\omega$. For this example we may even adjust the proof of 10.12 such that we only need the original results of Meise, Tayler and not the modifications presented in section 10.1. However, for $\mathcal{A}$ defined via $\mathcal{U}$ as in section 10, $\operatorname{Proj}^1 \mathcal{A}H = 0$ and $AH(\mathbb{C})$ is ultrabornological and barrelled.

Example 14.18. From the second condition in 10.16.(a) it is easy to get many examples of sequences $\mathcal{U} = ((u_{N,n})_{n\in\mathbb{N}})_{N\in\mathbb{N}}$ such that $(\overline{\text{Q}})_\omega$ is satisfied, e.g.

(i) $u_{N,n} = \frac{n^\alpha}{N^\beta}$ for α, $\beta > 0$,

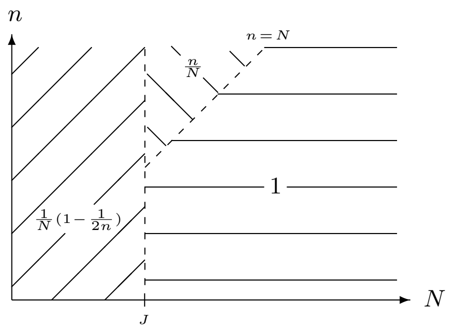
(ii) $u_{N,n} = \frac{\log(n)}{\exp(N)}$.

Example 14.19. Finally let us give one example for a sequence $((u_{N,n})_{n\in\mathbb{N}})_{N\in\mathbb{N}}$ which does not satisfy the second condition but the first one in 10.16.(a). Let

$J \in \mathbb{N}$ be arbitrary. Put

$$u_{N,n} = \begin{cases} \frac{1}{N}(1-\frac{1}{2n}) & \text{for } N \leqslant J, \\ \begin{cases} 1 & \text{for } n < N, \\ \frac{n}{N} & \text{otherwise,} \end{cases} & \text{otherwise.} \end{cases}$$

The following picture



illustrates the definition of the $u_{N,n}$ and shows that $u_{N+1,n} \leqslant u_{N,n} \leqslant u_{N,n+1}$ holds for all N and $n \in \mathbb{N}$. To show that the first condition in 10.16.(a) holds, let N be given. We select $M = \min(J,N)$. Let $K \geqslant M$ be given. By the selection of M we have $K \geqslant J$ that is for $k \geqslant K$ we have $u_{K,k} = \frac{k}{K}$ and therefore $u_{K,k} \to \infty$ for $k \to \infty$. As above, $(\overline{\mathrm{Q}})_\omega$ yields that $\operatorname{Proj}^1 \mathcal{A}H = 0$ and that $AH(\mathbb{C})$ is ultrabornological and barrelled.

15 Appendix: Mixed spaces of ultradistributions

The notion of a weight function, which we will use in this section goes back to the seminal article [37] of Braun, Meise, Vogt. For the sake of simplicitly we will use the definition stated by Bonet, Meise [33, 34].

Let us call a function $\omega\colon \mathbb{R} \to [0,\infty[$ a *weight function* if it is continuous, even, increasing on $[0,\infty[$ and if it satisfies $\omega(0) = 0$ and also the following conditions.

(α) There exists $K \geqslant 1$ such that $\omega(2t) \leqslant K\omega(t) + K$.

(β) $\omega(t) = o(t)$ for $t \to \infty$.

(γ) $\log(t) = o(\omega(t))$ for $t \to \infty$.

(δ) $\varphi\colon [0,\infty[\to \mathbb{R}$, $\varphi(t) = \omega(\exp(t))$ is convex.

If a weight function ω satisfies

$$\int_1^\infty \tfrac{\omega(t)}{t^2} dt = \infty$$

then it is called a *quasianalytic weight*. Otherwise it is called *non-quasianalytic*. A weight function ω is called a *strong weight function* if there exists $C > 0$ such that $\int_1^\infty \frac{\omega(yt)}{t^2}dt \leqslant C\omega(y) + C$. ω is called a *(DN)-weight function* if for each $C > 1$ there exists $R_0 > 0$ and $0 < \delta < 1$ such that for each $R \geqslant R_0$ the inequality $\omega^{-1}(CR)\omega^{-1}(\delta R) \leqslant (\omega^{-1}(R))^2$ is satisfied.

Following Bonet, Meise [33, 2.3] we define the following spaces. Let K be a compact subset of $\mathbb{R}$. By $C^\infty(K)$ we denote the space of all C^∞-Whitney jets on K. For $n \in \mathbb{N}$ we put

$$\mathcal{E}^n_{(\omega)}(K) := \{f \in C^\infty(K)\,;\, p_{K,n}(f) := \sup_{x\in K}\sup_{\alpha\in\mathbb{N}_0} |f^{(\alpha)}(x)| \exp(-n\varphi^\star(\tfrac{|\alpha|}{n})) < \infty\}$$

and

$$\mathcal{E}_{(\omega)}(K) := \operatorname{proj}_n \mathcal{E}^n_{(\omega)}(K)$$

which is a Fréchet space if we endow it with the topology of the seminorms $p_{K,n}$. For $N \in \mathbb{N}$ we put

$$\mathcal{E}^N_{\{\omega\}}(K) := \{f \in C^\infty(K)\,;\, q_{K,N}(f) := \sup_{x\in K}\sup_{\alpha\in\mathbb{N}_0} |f^{(\alpha)}(x)| \exp(-\tfrac{1}{N}\varphi^\star(N|\alpha|)) < \infty\}$$

and

$$\mathcal{E}_{\{\omega\}}(K) := \operatorname{ind}_N \mathcal{E}^N_{\{\omega\}}(K)$$

which is an (LB)-space if we endow it with the natural inductive topology.

The latter spaces are the building blocks of the *ω-ultradifferentiable functions of Beurling (resp. Roumieu) type*, cf. [33, 2.3] and [34, 2.3].

According to [33, 2.4] and [34, 2.4] we put

$$A(K,\lambda) := \{f \in H(\mathbb{C})\,;\, \|f\|_{K,\lambda} := \sup_{z\in\mathbb{C}} |f(z)| \exp(-h_K(\operatorname{Im} z) - \lambda\omega(|z|)) < \infty\},$$

for $\lambda > 0$ where $h_K\colon \mathbb{R} \to \mathbb{R}$, $h_K(x) = \sup_{y\in K} x\cdot y$. Moreover we put

$$A_{(\omega,R)} := \operatorname{ind}_n A([-R,R],n) \;\text{ and }\; A_{\{\omega,R\}} := \operatorname{proj}_N A([-R,R],\tfrac{1}{N})$$

which is the notation of [33, 4.4] and [34, 4.2]. Note that in the case $K = [-R,R]$

$$h_K(x) = \sup_{y\in K} x\cdot y = \sup_{y\in[-R,R]} x\cdot y = R|x|$$

holds. We put $w_n(z) := \exp(-R|\operatorname{Im} z| - n\omega(|z|))$ and $v_N(z) := \exp(-R|\operatorname{Im} z| - \frac{1}{N}\omega(|z|))$ and thus get

$$\|f\|_{K,n} = \sup_{z\in\mathbb{C}} w_n(z)|f(z)| \;\text{ and }\; \|f\|_{K,\frac{1}{N}} = \sup_{z\in\mathbb{C}} w_N(z)|f(z)|.$$

With the notation above we have the following result.

Theorem F. ([33, 4.5] and [34, 4.3])

(1) If ω is a (DN)-weight function or a strong weight function then the Fourier-Laplace transform $\mathcal{F}\colon \mathcal{E}'_{(\omega)}[-R,R] \to A_{(\omega,R)}$ is a linear topological isomorphism.

(2) The Fourier-Laplace transform $\mathcal{F}\colon \mathcal{E}'_{\{\omega\}}[-R,R] \to A_{\{\omega,R\}}$ is a linear topological isomorphism.

Moreover, in the situation of (1), $A_{(\omega,R)}$ is a (DFN)-space and $A'_{(\omega,R)}$ satisfies the conditions (DN) and (Ω).

In the sequel we consider tensor products of the spaces defined above. The resulting spaces are connected with the so-called mixed spaces of ultradistibutions considered recently by Schmets, Valdivia [68, 69, 70].

We need the following preparatory statements.

Lemma 15.1. Let $\mathcal{V} := (v_N)_{N\in\mathbb{N}}$ be an increasing sequence of weights on an arbitrary open subset of $\mathbb{C}^d$. Assume that $\mathcal{V}$ satisfies condition

$$(\mathrm{S}') \quad \forall\, N \,\exists\, M > N\colon \tfrac{v_N}{v_M} \text{ vanishes at } \infty \text{ on } G.$$

Then the two Fréchet spaces

$$HV(G) := \mathrm{proj}_N\, Hv_N(G)$$
$$HV_0(G) := \mathrm{proj}_N\, H(v_N)_0(G)$$

coincide algebraically and topologically.

Proof. $HV_0(G) \subseteq HV(G)$ is a topological subspace in general. Therefore it is enough to check that (S′) implies that $HV(G) \subseteq HV_0(G)$ holds algebraically. In order to show this let $f \in HV(G)$ and $N \in \mathbb{N}$ be given. We select M according to (S′). Let $\varepsilon > 0$ be given. Since $f \in HV(G)$ there is $C > 0$ such that $v_M|f| \leqslant C$ on G. By (S′) there exists $K \subseteq G$ compact such that $\frac{v_N}{v_M} < \frac{\varepsilon}{C}$ on $G\backslash K$. Hence $v_N|f| = \frac{v_N}{v_M} v_M|f| \leqslant \frac{\varepsilon}{C}C = \varepsilon$ on $G\backslash K$. ■

Remark 15.2. Bierstedt, Meise, Summers [27, p. 108] introduced the condition

$$(\mathrm{S}) \quad \forall\, n \,\exists\, m > n\colon \tfrac{w_m}{w_n} \text{ vanishes at } \infty \text{ on } G$$

(or (V) in their notation) for a decreasing sequence $\mathcal{W} := (w_n)_{n\in\mathbb{N}}$ of weights on an arbitrary open subset G of $\mathbb{C}^d$. We mentioned this condition already several times. Bierstedt, Meise, Summers [27, 0.4] showed that the (LB)-spaces $\mathcal{W}H(G) := \mathrm{ind}_N\, Hw_n(G)$ and $\mathcal{W}_0H(G) := \mathrm{ind}_n\, H(w_n)_0(G)$ coincide algebraically and topologically if $\mathcal{W}$ satisfies (S).

Lemma 15.3. Let ω be a weight function.

(1) The sequence $(v_N)_{N\in\mathbb{N}}$, $v_N(z) := \exp(-R|\operatorname{Im} z| - \frac{1}{N}\omega(|z|))$ for $z \in \mathbb{C}$ safisfies condition (S′).

(2) The sequence $(w_n)_{n\in\mathbb{N}}$, $w_n(z) := \exp(-R|\operatorname{Im} z| - n\omega(|z|))$ for $z \in \mathbb{C}$ safisfies condition (S).

Proof. (1) Let $N \in \mathbb{N}$ be given. We select $M := N+1$. For $z \in \mathbb{C}$ we have

$$\frac{v_N(z)}{v_M(z)} = \frac{e^{-|\operatorname{Im} z| - \frac{1}{N}\omega(z)}}{e^{-|\operatorname{Im} z| - \frac{1}{M}\omega(z)}} = e^{(\frac{1}{M} - \frac{1}{N})\omega(z)}.$$

The above implies in particular that $\frac{v_N}{v_M}$ is radial and it is enough to show that $\lim_{r\to\infty} \frac{v_N(r)}{v_M(r)} = 0$ holds. By condition (γ) there is $r_0 \geqslant 1$ such that $\log r \leqslant \omega(r)$ for $r \geqslant r_0$. Thus for $r \geqslant r_0$ we may compute

$$\frac{v_N(r)}{v_M(r)} = e^{(\frac{1}{M}-\frac{1}{N})\omega(r)} \leqslant e^{(\frac{1}{M}-\frac{1}{N})\log r} = r^{\frac{1}{M}-\frac{1}{N}} \overset{r\to\infty}{\longrightarrow} 0$$

since $\frac{1}{M} - \frac{1}{N} < 0$ by the choice of M and $\log r \geqslant 0$ for $r \geqslant r_0$.

(2) For given $n \in \mathbb{N}$ we select $m := n+1$. For $z \in \mathbb{C}$ we have

$$\frac{w_m(z)}{w_n(z)} = \frac{e^{-|\operatorname{Im} z|-m\omega(z)}}{e^{-|\operatorname{Im} z|-n\omega(z)}} = e^{(n-m)\omega(z)} = e^{-\omega(z)}$$

by the choice of m. Again $\frac{w_m}{w_n}$ is radial and we have to show that $\lim_{r\to\infty} \frac{w_m(r)}{w_n(r)} = 0$ holds. As above condition (γ) yields $r_0 \geqslant 1$ such that $\log r \leqslant \omega(r)$ for $r \geqslant r_0$. Thus for $r \geqslant r_0$ we may compute

$$\frac{w_m(r)}{w_n(r)} = e^{-\omega(r)} \leqslant e^{-\log r} = r^{-1} \overset{r\to\infty}{\longrightarrow} 0$$

where we again used that $\log r \geqslant 0$ for $r \geqslant r_0$. ■

In the sequel we investigate the space

$$\begin{aligned}\mathcal{E}'_{\{\omega\}}[-R,R] \,\check{\otimes}_\varepsilon\, \mathcal{E}'_{(\omega)}[-R,R] &\overset{(1)}{=} A_{\{\omega,R\}} \,\check{\otimes}_\varepsilon\, A_{(\omega,R)} \\ &\overset{\text{dfn}}{=} \operatorname{proj}_N Hv_N(\mathbb{C}) \,\check{\otimes}_\varepsilon\, \operatorname{ind}_n Hw_n(\mathbb{C}) \\ &\overset{(2)}{=} \operatorname{proj}_N H(v_N)_0(\mathbb{C}) \,\check{\otimes}_\varepsilon\, \operatorname{ind}_n H(w_n)_0(\mathbb{C}),\end{aligned}$$

where the isomorphy (1) can be seen as follows: By Jarchow [50, 16.2.2.(b)] the map $\mathcal{F} \otimes_\varepsilon \mathcal{F}\colon \mathcal{E}'_{\{\omega\}}[-R,R] \otimes_\varepsilon \mathcal{E}'_{(\omega)}[-R,R] \to A_{\{\omega,R\}} \otimes_\varepsilon A_{(\omega,R)}$ is injective and open since this is true for $\mathcal{F}$ by 15.F. If we consider the above spaces just as linear spaces it is clear that the map is also bijective. Hence, $\mathcal{F} \otimes_\varepsilon \mathcal{F}$ is a linear topological isomorphism. Now we take the completions and therefore there has to be an isomorphism $\mathcal{E}'_{\{\omega\}}[-R,R] \,\check{\otimes}_\varepsilon\, \mathcal{E}'_{(\omega)}[-R,R] \cong A_{\{\omega,R\}} \,\check{\otimes}_\varepsilon\, A_{(\omega,R)}$. The equality (2) follows from 15.3 in combination with 15.1 and 15.2, respectively.

Our aim is to show that this space is a weighted (PLB)-space of holomorphic functions over $\mathbb{C}\times\mathbb{C}$. For this purpose we define the double sequence of weights $\mathcal{A} = ((a_{N,n})_{N\in\mathbb{N}})_{n\in\mathbb{N}}$ by $a_{N,n} := v_N \otimes w_n\colon \mathbb{C}\times\mathbb{C} \to \mathbb{R}$, $a_{N,n}(z_1,z_2) := v_N(z_1)\cdot w_n(z_2)$ where v_N and w_n are the weights defined above.

Proposition 15.4. Let ω be a (DN)-weight function or a strong weight function and $R > 0$. Define the double sequence of weights $\mathcal{A} = ((a_{N,n})_{N\in\mathbb{N}})_{n\in\mathbb{N}}$ by $a_{N,n} := v_N \otimes w_N\colon \mathbb{C}\times\mathbb{C} \to \mathbb{R}$, $a_{N,n}(z_1,z_2) := v_N(z_1)\cdot v_n(z_2)$. Then there exists a linear topological isomorphism

$$\mathcal{E}'_{\{\omega\}}[-R,R] \,\check{\otimes}_\varepsilon\, \mathcal{E}'_{(\omega)}[-R,R] \cong (AH)_0(\mathbb{C}\times\mathbb{C}).$$

Proof. We have

$$\mathcal{E}'_{\{\omega\}}[-R,R] \,\check{\otimes}_\varepsilon\, \mathcal{E}'_{(\omega)}[-R,R] \overset{(1)}{=} \operatorname{proj}_N H(v_N)_0(\mathbb{C}) \,\check{\otimes}_\varepsilon\, \operatorname{ind}_n H(w_n)_0(\mathbb{C})$$

$$
\begin{aligned}
&\overset{(2)}{=} \operatorname{proj}_N \big[H(v_N)_0(\mathbb{C}) \check{\otimes}_\varepsilon \operatorname{ind}_n H(w_n)_0(\mathbb{C})\big]\\
&\overset{(3)}{=} \operatorname{proj}_N \big[H(v_N)_0(\mathbb{C})\, \varepsilon \operatorname{ind}_n H(w_n)_0(\mathbb{C})\big]\\
&\overset{(4)}{=} \operatorname{proj}_N \operatorname{ind}_n \big[H(v_N)_0(\mathbb{C})\, \varepsilon\, H(w_n)_0(\mathbb{C})\big]\\
&\overset{(5)}{=} \operatorname{proj}_N \operatorname{ind}_n \big[H(v_N \otimes w_n)(\mathbb{C}\times\mathbb{C})\big]\\
&\overset{\text{dfn}}{=} (AH)_0(\mathbb{C}\times\mathbb{C}).
\end{aligned}
$$

(1) we have already shown above.

(2) is true in general, see Jarchow [50, 16.3.2].

(3) follows from Bierstedt, Meise [23, remarks previous to 3.11], since

a. $\operatorname{ind}_n H(w_n)_0(\mathbb{C}) = \operatorname{ind}_n Hw_n(\mathbb{C})$ is complete as we noted already in section 2,
b. $\operatorname{ind}_n H(w_n)_0(\mathbb{C})$ is nuclear as it is a (DFN)-space (see e.g. Bierstedt [11, 2.4.(c)]),
c. $H(v_N)_0(\mathbb{C})$ is a complete bornological (DF)-space as it is a Banach space

in combination with 3.14, since for each $N \in \mathbb{N}$ the isomorphisms

$$H(v_N)_0(\mathbb{C}) \check{\otimes}_\varepsilon \operatorname{ind}_n H(w_n)_0(\mathbb{C}) \to H(v_N)_0(\mathbb{C})\, \varepsilon \operatorname{ind}_n H(w_n)_0(\mathbb{C})$$

are just the canonical maps, that is the condition in 3.15 holds.

(4) follows from Bierstedt, Meise [23, 3.13] since

a. $\operatorname{ind}_n H(w_n)_0(\mathbb{C}) = \operatorname{ind}_n Hw_n(\mathbb{C})$ is complete as we noted already in section 2,
b. $\operatorname{ind}_n H(w_n)_0(\mathbb{C})$ is a (DFS)-space, since $(w_n)_{n\in\mathbb{N}}$ satisfies (S) by 15.3, cf. Bierstedt, Meise, Summers [27, 0.4.(d)], and therefore it is compactly regular (by [11, Appendix, remarks after 1] contable compact inductive limits are boundedly retractive and by [11, Appendix, remark after 7] the latter yields compact regularity),
c. $H(v_N)_0(\mathbb{C})$ is a Banach space,
d. $\operatorname{ind}_n H(w_n)_0(\mathbb{C})$ is nuclear as it is a (DFN)-space (see e.g. Bierstedt [11, 2.4.(c)])

again in combination with 3.14, since the isomorphisms

$$\operatorname{ind}_n \big[H(v_N)_0(\mathbb{C})\, \varepsilon\, H(w_n)_0(\mathbb{C})\big] \to H(v_N)_0(\mathbb{C})\, \varepsilon \big[\operatorname{ind}_n H(w_n)_0(\mathbb{C})\big]$$

are again just the canonical maps (cf. [23, proof of 3.10]).

(5) follows from Bierstedt [10, Corollary 42]. He proved that $H(V_1 \otimes V_2)_0(X_1 \times X_2) \cong H(V_1)_0(X_1)\, \varepsilon\, H(V_2)_0(X_2)$ for $X_1 \subseteq \mathbb{C}^N$, $X_2 \subseteq \mathbb{C}^M$ $(N,\, M \geqslant 1)$ and V_i a Nachbin family on X_i such that that $W(X_i) \subseteq V_i$ for $i = 1, 2$ where $W(X) = \{\lambda\chi_K \,;\, \lambda \geqslant 0,\, K \subseteq X \text{ compact}\}$ and $W \subseteq V$ if and only if for each $w \in W$ there is $v \in V$ such that $w \leqslant v$ on X. In the case of a weighted Banach space $Hv_0(\mathbb{C})$ the corresponding Nachbin family is $V = \{\lambda v \,;\, \lambda \geqslant 0\}$ and it is easy to see that

$W(\mathbb{C}) \subseteq V$ holds, since $v > 0$. The isomorphisms

$$H(v_N \otimes w_n)_0(\mathbb{C} \times \mathbb{C}) \xrightarrow{\sim} H(v_N)_0(\mathbb{C}) \, \varepsilon \, H(w_n)_0(\mathbb{C})$$

in [10, Corollary 42] are obtained from the result [9, Satz 3.2] on slice products. From the proof of the latter result it follows that they are given by

$$f \mapsto \Big[\mu \mapsto \big(x_1 \mapsto \mu f(x_1, \cdot)\big)\Big].$$

Thus they satisfy the conditions in 3.13 and we may apply 3.14. ■

Proposition 15.5. The sequence $\mathcal{A}_N = (a_{N,n})_{n\in\mathbb{N}}$ satisfies condition (S) for each $N \in \mathbb{N}$.

Proof. Let $N \in \mathbb{N}$ be fixed and n be given. Select $m = n + 1$. Since

$$\begin{aligned} a_{N,n}(z_1, z_2) &= v_N(z_1) \cdot w_n(z_2) \\ &= \exp(-R|\operatorname{Im} z_1| - \tfrac{1}{N}\omega(|z_1|)) \exp(-R|\operatorname{Im} z_2| - n\omega(|z_2|)) \\ &= \exp(-R(|\operatorname{Im} z_1| + |\operatorname{Im} z_2|) - \tfrac{1}{N}\omega(|z_1|) - n\omega(|z_2|)) \end{aligned}$$

we have

$$\begin{aligned} \tfrac{a_{N,m}(z_1,z_2)}{a_{N,n}(z_1,z_2)} &= \tfrac{\exp(-R(|\operatorname{Im} z_1|+|\operatorname{Im} z_2|)-\frac{1}{N}\omega(|z_1|)-m\omega(|z_2|))}{\exp(-R(|\operatorname{Im} z_1|+|\operatorname{Im} z_2|)-\frac{1}{N}\omega(|z_1|)-n\omega(|z_2|))} \\ &= \exp(-R(|\operatorname{Im} z_1| + |\operatorname{Im} z_2|) - \tfrac{1}{N}\omega(|z_1|) - m\omega(|z_2|) \\ &\qquad +R(|\operatorname{Im} z_1| + |\operatorname{Im} z_2|) + \tfrac{1}{N}\omega(|z_1|) + n\omega(|z_2|)) \\ &= \exp((n-m)\omega(|z_2|) \end{aligned}$$

and may conclude as in 15.3.(2). ■

Remark 15.6. From 15.5 and 15.2 we get that the spaces $\operatorname{ind}_n H(a_{N,n})_0(\mathbb{C} \times \mathbb{C})$ and $\operatorname{ind}_n Ha_{N,n}(\mathbb{C} \times \mathbb{C})$ coincide for each N. Thus, we get from 15.4 even the isomorphism

$$\mathcal{E}'_{\{\omega\}}[-R, R] \, \check{\otimes}_\varepsilon \, \mathcal{E}'_{(\omega)}[-R, R] \cong AH(\mathbb{C} \times \mathbb{C})$$

where we stick to the notation and definitions of 15.4.

As we pointed out in section 1 the latter shows that several of the so-called mixed spaces of ultradistributions (introduced recently by Schmets, Valdivia [68, 69, 70]) can be regarded as weighted (PLB)-spaces of holomorphic functions. However, the results established in this thesis do not cover the situation above, since we obtained a weighted space over $\mathbb{C} \times \mathbb{C}$ and are dealing with non-radial weights.

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