A simultaneous canonical form of a pair of matrices and applications involving the weighted Moore-Penrose inverse

Néstor Thome
Instituto Universitario de Matemática Multidisciplinar
Universitat Politècnica de València, Valencia, Spain, njthome@mat.upv.es

Abstract

In this paper, a simultaneous canonical form of a pair of rectangular complex matrices is developed. Using this new tool we give a necessary and sufficient condition to assure that the reverse order law is valid for the weighted Moore-Penrose inverse. Additionally, we characterize matrices ordered by the weighted star partial order and adjacent matrices as applications.

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1 Introduction

For an $m \times n$ complex matrix $A \in \mathbb{C}_r^{m\times n}$ of rank $r > 0$, a singular value decomposition (SVD) of A [3, pp. 206]

$$A = U(\Sigma \oplus O)V^*$$

is a well-known factorization where $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary matrices and $\Sigma \in \mathbb{R}^{r\times r}$ is a diagonal matrix; the so called singular values $\sigma_1, \sigma_2, \dots, \sigma_r$ are on the diagonal of Σ ordered as $\sigma_1 > \sigma_2 > \cdots > \sigma_r > 0$.

A simultaneous diagonalization for rectangular matrices is also possible under a certain condition. That is, a pair of matrices $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{m\times n}$ has a simultaneous diagonalization [3, Ex. 15, pp. 208] such as

$$A = U\Sigma_A V^* \qquad \text{and} \qquad B = U\Sigma_B V^*,$$

with $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ unitary and Σ_A, Σ_B diagonal real matrices if and only if AB^* and B^*A are both hermitian matrices.

On the other hand, a Hartwig-Spindelböck decomposition of a square matrix $A \in \mathbb{C}^{n\times n}$ of rank $r > 0$ [6, 1] is given by

$$A = U \begin{bmatrix} \Sigma K & \Sigma L \\ O & O \end{bmatrix} U^*, \tag{1}$$

where $U \in \mathbb{C}^{n\times n}$ is unitary, $\Sigma \in \mathbb{C}^{r\times r}$ is a positive definite diagonal matrix and $K \in \mathbb{C}^{r\times r}$, $L \in \mathbb{C}^{r\times(n-r)}$ satisfy the condition $KK^* + LL^* = I_r$.

By keeping as far as possible the essential properties of all these factorizations, the main aim of this paper is to present a simultaneous decomposition of a pair of rectangular complex matrices without restrictions. Such a factorization is given in Section 2. In Section 3, we present some applications. First of all, we study the reverse order law for the weighted Moore-Penrose inverse. Secondly, we show the form of the matrices ordered by the weighted star partial order. And finally, we characterize the adjacent matrices related by the weighted star partial order.

2 A simultaneous canonical form of a pair of matrices

Theorem 1 *Let $A \in \mathbb{C}_r^{m\times n}$ and $B \in \mathbb{C}_s^{m\times n}$. Then there exist unitary matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ such that*

$$A = U \begin{bmatrix} \Sigma_A A_1 & \Sigma_A A_2 \\ O & O \end{bmatrix} V^* \qquad \text{and} \qquad B = U \begin{bmatrix} B_1^* \Sigma_B & O \\ B_2^* \Sigma_B & O \end{bmatrix} V^*,$$

where $\Sigma_A \in \mathbb{R}^{r\times r}$ and $\Sigma_B \in \mathbb{R}^{s\times s}$ are positive definite diagonal matrices (with non-increasing diagonal entries) and blocks $A_1 \in \mathbb{C}^{r\times s}$, $A_2 \in \mathbb{C}^{r\times(n-s)}$, $B_1 \in \mathbb{C}^{s\times r}$, and $B_2 \in \mathbb{C}^{s\times(m-r)}$ satisfy

$$A_1 A_1^* + A_2 A_2^* = I_r \qquad \text{and} \qquad B_1 B_1^* + B_2 B_2^* = I_s.$$

Proof. First, let us consider singular value decompositions of A and B^*:

$$A = U_A \begin{bmatrix} \Sigma_A & O \\ O & O \end{bmatrix} V_A^* \qquad \text{and} \qquad B^* = U_B \begin{bmatrix} \Sigma_B & O \\ O & O \end{bmatrix} V_B^*,$$

where $U_A, V_B \in \mathbb{C}^{m\times m}$ and $V_A, U_B \in \mathbb{C}^{n\times n}$ are unitary matrices and $\Sigma_A \in \mathbb{R}^{r\times r}$ and $\Sigma_B \in \mathbb{R}^{s\times s}$ are positive definite diagonal matrices (with non-increasing diagonal entries). It is clear that $V_A^* U_B$ and $V_B^* U_A$ are unitary as well. Now, according to the decompositions of A and B, we partition

$$V_A^* U_B = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \qquad \text{and} \qquad V_B^* U_A = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}.$$

Then, computing the $(1,1)$-block in $V_A^* U_B (V_A^* U_B)^* = I_n$ and $V_B^* U_A (V_B^* U_A)^* = I_m$ we obtain $A_1 A_1^* + A_2 A_2^* = I_r$ and $B_1 B_1^* + B_2 B_2^* = I_s$, respectively. Finally,

$$A = U_A \begin{bmatrix} \Sigma_A & O \\ O & O \end{bmatrix} V_A^* U_B U_B^* = U_A \begin{bmatrix} \Sigma_A & O \\ O & O \end{bmatrix} \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} U_B^* = U_A \begin{bmatrix} \Sigma_A A_1 & \Sigma_A A_2 \\ O & O \end{bmatrix} U_B^*$$

and

$$B^* = U_B \begin{bmatrix} \Sigma_B & O \\ O & O \end{bmatrix} V_B^* U_A U_A^* = U_B \begin{bmatrix} \Sigma_B & O \\ O & O \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} U_A^* = U_B \begin{bmatrix} \Sigma_B B_1 & \Sigma_B B_2 \\ O & O \end{bmatrix} U_A^*.$$

Defining $U = U_A$ and $V = U_B$ and computing the conjugate transpose of B^* we get the required form for A and B. ■

3 Applications

3.1 The reverse order law for the weighted Moore-Penrose inverse

Next result characterizes the reverse order law for Moore-Penrose inverses. For matrices A, B such that AB exists, the following conditions are equivalent [3, pp. 176]:

$$(AB)^\dagger = B^\dagger A^\dagger \Leftrightarrow \mathcal{R}(A^*AB) \subseteq \mathcal{R}(B), \mathcal{R}(BB^*A^*) \subseteq \mathcal{R}(A^*) \Leftrightarrow \mathcal{R}(A^*ABB^*) = \mathcal{R}(BB^*A^*A), \tag{2}$$

where $\mathcal{R}(.)$ denotes the range of the matrix $(.)$. For more properties and applications we refer the reader to [4, 17, 18].

Next, we need the following technical result.

Lemma 2 *Let $X \in \mathbb{C}^{s\times r}$, $Y \in \mathbb{C}^{s\times(k-r)}$, $Z \in \mathbb{C}^{p\times s}$ be matrices such that $XX^* + YY^* = I_s$ and*

$$M = \begin{bmatrix} ZX & ZY \\ O & O \end{bmatrix} \in \mathbb{C}^{\ell\times k}.$$

Then

$$M^\dagger = \begin{bmatrix} X^* Z^\dagger & O \\ Y^* Z^\dagger & O \end{bmatrix}.$$

Proof. If we define

$$E = \begin{bmatrix} X^*Z^\dagger & O \\ Y^*Z^\dagger & O \end{bmatrix},$$

it is easy to check the properties $MEM = M$, $EME = E$, $(ME)^* = ME$, and $(EM)^* = EM$. The uniqueness of the Moore-Penrose inverse gives $M^\dagger = E$. ■

Notice that this lemma is a slight extension of [2, Formula (1.13)] and [12, Lemma 3] to rectangular matrices, since both of them are valid for square matrices.

The equivalences in (2) give conditions on matrices A and B such that the reverse order law is valid. Related results can be found in [5, 14, 15]. Next theorem describes the form of both matrices A and B for which the Moore-Penrose inverse satisfies that property.

Theorem 3 *Let $A \in \mathbb{C}_r^{m\times n}$ and $B \in \mathbb{C}_s^{n\times m}$. Then $(AB)^\dagger = B^\dagger A^\dagger$ if and only if there exist unitary matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ such that*

$$A = U \begin{bmatrix} \Sigma_A A_1 & \Sigma_A A_2 \\ O & O \end{bmatrix} V^* \qquad \text{and} \qquad B = V \begin{bmatrix} \Sigma_B B_1 & \Sigma_B B_2 \\ O & O \end{bmatrix} U^*,$$

where $\Sigma_A \in \mathbb{R}^{r\times r}$ and $\Sigma_B \in \mathbb{R}^{s\times s}$ are positive definite diagonal matrices (with non-increasing diagonal entries), blocks $A_1 \in \mathbb{C}^{r\times s}$, $A_2 \in \mathbb{C}^{r\times(n-s)}$, $B_1 \in \mathbb{C}^{s\times r}$, and $B_2 \in \mathbb{C}^{s\times(m-r)}$ satisfy $A_1A_1^* + A_2A_2^* = I_r$, $B_1B_1^* + B_2B_2^* = I_s$, and

$$(\Sigma_A A_1 \Sigma_B)^\dagger = \Sigma_B^{-1} A_1^* \Sigma_A^{-1}.$$

Proof. Applying Theorem 1 to the pair of matrices A and B^* we can assure that there exist unitary matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ such that

$$A = U \begin{bmatrix} \Sigma_A A_1 & \Sigma_A A_2 \\ O & O \end{bmatrix} V^* \qquad \text{and} \qquad B^* = U \begin{bmatrix} B_1^*\Sigma_B & O \\ B_2^*\Sigma_B & O \end{bmatrix} V^*,$$

where $\Sigma_A \in \mathbb{R}^{r\times r}$ and $\Sigma_B \in \mathbb{R}^{s\times s}$ are positive definite diagonal matrices (with non-increasing diagonal entries) and blocks $A_1 \in \mathbb{C}^{r\times s}$, $A_2 \in \mathbb{C}^{r\times(n-s)}$, $B_1 \in \mathbb{C}^{s\times r}$, and $B_2 \in \mathbb{C}^{s\times(m-r)}$ satisfy

$$A_1A_1^* + A_2A_2^* = I_r, \qquad B_1B_1^* + B_2B_2^* = I_s.$$

Then,

$$U^*ABU = \begin{bmatrix} \Sigma_A A_1 & \Sigma_A A_2 \\ O & O \end{bmatrix} \begin{bmatrix} \Sigma_B B_1 & \Sigma_B B_2 \\ O & O \end{bmatrix} = \begin{bmatrix} (\Sigma_A A_1 \Sigma_B)B_1 & (\Sigma_A A_1 \Sigma_B)B_2 \\ O & O \end{bmatrix}$$

and applying Lemma 2 we get

$$U^*(AB)^\dagger U = \begin{bmatrix} B_1^*(\Sigma_A A_1 \Sigma_B)^\dagger & O \\ B_2^*(\Sigma_A A_1 \Sigma_B)^\dagger & O \end{bmatrix}.$$

Applying twice Lemma 2 we obtain

$$U^*B^\dagger A^\dagger U = \begin{bmatrix} B_1^*\Sigma_B^{-1} & O \\ B_2^*\Sigma_B^{-1} & O \end{bmatrix} \begin{bmatrix} A_1^*\Sigma_A^{-1} & O \\ A_2^*\Sigma_A^{-1} & O \end{bmatrix} = \begin{bmatrix} B_1^*\Sigma_B^{-1}A_1^*\Sigma_A^{-1} & O \\ B_2^*\Sigma_B^{-1}A_1^*\Sigma_A^{-1} & O \end{bmatrix}.$$

Hence, $(AB)^\dagger = B^\dagger A^\dagger$ if and only if $B_1^*(\Sigma_A A_1 \Sigma_B)^\dagger = B_1^*\Sigma_B^{-1}A_1^*\Sigma_A^{-1}$ and $B_2^*(\Sigma_A A_1 \Sigma_B)^\dagger = B_2^*\Sigma_B^{-1}A_1^*\Sigma_A^{-1}$. Pre-multiplying both equalities by B_1 and B_2, respectively, and using $B_1B_1^* + B_2B_2^* = I_s$ we arrive at $(\Sigma_A A_1 \Sigma_B)^\dagger = \Sigma_B^{-1}A_1^*\Sigma_A^{-1}$. ■

Now, if we consider three Hermitian positive definite matrices $M, R \in \mathbb{C}^{m\times m}$, and $N \in \mathbb{C}^{n\times n}$, we can apply Theorem 3 to the pair of matrices $\widetilde{A} := M^{1/2}AN^{-1/2}$ and $\widetilde{B} := N^{1/2}BR^{-1/2}$ to get a generalization of the reverse order law [9, 16] taking into account that the $\{M, N\}$-weighted Moore-Penrose inverse of $A \in \mathbb{C}^{m\times n}$ is given by

$$A^\dagger_{M,N} = N^{-1/2}(M^{1/2}AN^{-1/2})^\dagger M^{1/2}.$$

Corollary 4 *Let $A \in \mathbb{C}_r^{m\times n}$, $B \in \mathbb{C}_s^{n\times m}$ and consider three Hermitian positive definite matrices $M, R \in \mathbb{C}^{m\times m}$, and $N \in \mathbb{C}^{n\times n}$. Then $(AB)^\dagger_{M,R} = B^\dagger_{N,R} A^\dagger_{M,N}$ if and only if there exist unitary matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ such that*

$$A = M^{-1/2} U \begin{bmatrix} \Sigma_A A_1 & \Sigma_A A_2 \\ O & O \end{bmatrix} V^* N^{1/2} \qquad \text{and} \qquad B = N^{-1/2} V \begin{bmatrix} \Sigma_B B_1 & \Sigma_B B_2 \\ O & O \end{bmatrix} U^* R^{1/2},$$

where $\Sigma_A \in \mathbb{R}^{r\times r}$ and $\Sigma_B \in \mathbb{R}^{s\times s}$ are positive definite diagonal matrices (with non-increasing diagonal entries), blocks $A_1 \in \mathbb{C}^{r\times s}$, $A_2 \in \mathbb{C}^{r\times(n-s)}$, $B_1 \in \mathbb{C}^{s\times r}$, and $B_2 \in \mathbb{C}^{s\times(m-r)}$ satisfy $A_1A_1^* + A_2A_2^* = I_r$, $B_1B_1^* + B_2B_2^* = I_s$, and $(\Sigma_A A_1 \Sigma_B)^\dagger = \Sigma_B^{-1} A_1^* \Sigma_A^{-1}$.

3.2 (M, N)-Star partial order and adjacent matrices

We remind that a pair of matrices $A, B \in \mathbb{C}^{m\times n}$ are ordered under the star order $\leq^*$, and written $A \leq^* B$, if $AA^* = BA^*$ and $A^*A = A^*B$ [7, 8, 11, 13]. It is well-known that inequalities under $\leq^*$ are preserved under unitary equivalences, that is $A \leq^* B$ if and only if $SAT \leq^* SBT$ for all unitary matrices $S \in \mathbb{C}^{m\times m}$ and $T \in \mathbb{C}^{n\times n}$. We will denote by $\mathcal{N}(.)$ the null space of the matrix (.).

Theorem 5 *Let $A \in \mathbb{C}_r^{m\times n}$ and $B \in \mathbb{C}_s^{m\times n}$. Then $A \leq^* B$ if and only if there exist unitary matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ and a matrix $Z \in \mathbb{C}^{(m-r)\times s}$ such that*

$$A = U \begin{bmatrix} \Sigma_A A_1 & O \\ O & O \end{bmatrix} V^* \qquad \text{and} \qquad B = U \begin{bmatrix} \Sigma_A A_1 & O \\ Z(I_s - A_1^* A_1) & O \end{bmatrix} V^*,$$

where $\Sigma_A \in \mathbb{R}^{r\times r}$ is a positive definite diagonal matrix (with non-increasing diagonal entries), and block $A_1 \in \mathbb{C}^{r\times s}$ satisfies $A_1A_1^* = I_r$.

Proof. Applying Theorem 1 to the pair of matrices A and B we can assure that there exist unitary matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ such that

$$A = U \begin{bmatrix} \Sigma_A A_1 & \Sigma_A A_2 \\ O & O \end{bmatrix} V^* \qquad \text{and} \qquad B = U \begin{bmatrix} B_1^* \Sigma_B & O \\ B_2^* \Sigma_B & O \end{bmatrix} V^*,$$

where $\Sigma_A \in \mathbb{R}^{r\times r}$ and $\Sigma_B \in \mathbb{R}^{s\times s}$ are positive definite diagonal matrices (with non-increasing diagonal entries) and blocks $A_1 \in \mathbb{C}^{r\times s}$, $A_2 \in \mathbb{C}^{r\times(n-s)}$, $B_1 \in \mathbb{C}^{s\times r}$, and $B_2 \in \mathbb{C}^{s\times(m-r)}$ satisfy

$$\text{(A)}\ \ A_1A_1^* + A_2A_2^* = I_r, \qquad \text{(B)}\ \ B_1B_1^* + B_2B_2^* = I_s.$$

Then, $A \leq^* B$ if and only if $U^*AV \leq^* U^*BV$. Using the block forms of A and B and making some computations, the last inequality is equivalent to the matrix equation system given by:

$$\text{(a)}\ B_1^*\Sigma_B A_1^* = \Sigma_A,\ \ \text{(b)}\ B_2^*\Sigma_B A_1^* = O,\ \ \text{(c)}\ A_1^*\Sigma_A^2 A_1 = A_1^*\Sigma_A B_1^*\Sigma_B,\ \ \text{(d)}\ A_2^*\Sigma_A^2 A_2 = O.$$

From (d) we get $(\Sigma_A A_2)^*(\Sigma_A A_2) = O$, which yields $A_2 = O$. So, $A_1A_1^* = I_r$. Then, we have found the form of matrix A.

The remaining computations will give the form of matrix B. Indeed, pre-multiplying (a) by B_1, (b) by B_2 and adding them we obtain $\Sigma_B A_1^* = B_1 \Sigma_A$ after using (B). Thus, $B_1 = \Sigma_B A_1^* \Sigma_A^{-1}$. On the other hand, pre-multiplying (c) by A_1 and using the non-singularity of Σ_A we arrive at $\Sigma_A A_1 = B_1^* \Sigma_B$, or equivalently, $B_1 = \Sigma_B^{-1} A_1^* \Sigma_A$. Now, we obtain A_1 from both expressions of B_1 and using $A_1A_1^* = I_r$ we have $I_r = (\Sigma_A^{-1} B_1^* \Sigma_B)(\Sigma_B^{-1} B_1 \Sigma_A)$, that is, $B_1^*B_1 = I_r$. Hence, $(B_1B_1^*)^2 = B_1B_1^*$. In order to find an expression for B_2, we observe that $B_2^*B_1 = B_2^*(\Sigma_B A_1^* \Sigma_A^{-1}) = (B_2^*\Sigma_B A_1^*)\Sigma_A^{-1} = O$ by (b) and so $B_1B_1^*B_2 = O$ holds. So, $\mathcal{R}(B_2) \subseteq \mathcal{N}(B_1B_1^*) = \mathcal{R}(I_s - B_1B_1^*)$, from which $B_2 = (I_s - B_1B_1^*)\widetilde{Z}$ for some $\widetilde{Z}$. Now, $B_1^*\Sigma_B = (\Sigma_A A_1 \Sigma_B^{-1})\Sigma_B = \Sigma_A A_1$ and $B_2^*\Sigma_B = \widetilde{Z}^*(I_s - B_1B_1^*)\Sigma_B = \widetilde{Z}^*(I_s - \Sigma_B A_1^* \Sigma_A^{-1} \Sigma_A A_1 \Sigma_B^{-1})\Sigma_B = \widetilde{Z}^*\Sigma_B(I_s - A_1^*A_1) = Z(I_s - A_1^*A_1)$, for some Z. ■

Considering the (M, N)-star partial order [9] given by $A \leq^*_{M,N} B$ if and only if $A^\dagger_{M,N}A = A^\dagger_{M,N}B$ and $AA^\dagger_{M,N} = BA^\dagger_{M,N}$ for $A, B \in \mathbb{C}^{m\times n}$, we can extend Theorem 5 to the weighted case (using again the matrices $\widetilde{A}$ and $\widetilde{B}$ as in Subsection 3.1).

Corollary 6 *Let $A \in \mathbb{C}_r^{m\times n}$, $B \in \mathbb{C}_s^{m\times n}$ and let two Hermitian positive definite matrices $M \in \mathbb{C}^{m\times m}$ and $N \in \mathbb{C}^{n\times n}$. Then $A \leq^*_{M,N} B$ if and only if there exist unitary matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ and a matrix $Z \in \mathbb{C}^{(m-r)\times s}$ such that*

$$A = M^{1/2}U \begin{bmatrix} \Sigma_A A_1 & O \\ O & O \end{bmatrix} V^* N^{-1/2} \qquad \text{and} \qquad B = M^{1/2}U \begin{bmatrix} \Sigma_A A_1 & O \\ Z(I_s - A_1^* A_1) & O \end{bmatrix} V^* N^{-1/2},$$

where $\Sigma_A \in \mathbb{R}^{r\times r}$ is a positive definite diagonal matrix (with non-increasing diagonal entries), and block $A_1 \in \mathbb{C}^{r\times s}$ satisfies $A_1 A_1^* = I_r$.

In order to state the last application, we recall that two matrices $A \in \mathbb{C}_r^{m\times n}$ and $B \in \mathbb{C}_s^{m\times n}$ are called adjacent if $rank(B-A) = 1$ [10].

Theorem 7 *Let $A \in \mathbb{C}_r^{m\times n}$ and $B \in \mathbb{C}_s^{m\times n}$ be two matrices such that $A \leq^* B$. Then A and B are adjacent if and only if there exist unitary matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ and a matrix $Z \in \mathbb{C}^{(m-r)\times s}$ such that*

$$A = U \begin{bmatrix} \Sigma_A A_1 & O \\ O & O \end{bmatrix} V^* \qquad \text{and} \qquad B = U \begin{bmatrix} \Sigma_A A_1 & O \\ Z(I_s - A_1^* A_1) & O \end{bmatrix} V^*,$$

where $\Sigma_A \in \mathbb{R}^{r\times r}$ is a positive definite diagonal matrix (with non-increasing diagonal entries), $s = r+1$ and block $A_1 \in \mathbb{C}^{r\times s}$ satisfies $A_1 A_1^* = I_r$ and $\mathcal{N}(Z) \cap \mathcal{N}(A_1) = \{0\}$.

Proof. Applying Theorem 5 to the pair of matrices A and B we can assure that there exist unitary matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ and a matrix $Z \in \mathbb{C}^{(m-r)\times s}$ such that

$$A = U \begin{bmatrix} \Sigma_A A_1 & O \\ O & O \end{bmatrix} V^* \qquad \text{and} \qquad B = U \begin{bmatrix} \Sigma_A A_1 & O \\ Z(I_s - A_1^* A_1) & O \end{bmatrix} V^*,$$

where $\Sigma_A \in \mathbb{R}^{r\times r}$ is a positive definite diagonal matrix (with non-increasing diagonal entries) and block $A_1 \in \mathbb{C}^{r\times s}$ satisfies $A_1 A_1^* = I_r$. Since star order implies minus order, that is, $A \leq^* B$ implies $A \leq^- B$ (see [7, 13]), we notice that $rank(B-A) = rank(B) - rank(A) = s - r$ holds. From $(A_1^* A_1)^2 = A_1^* A_1 = (A_1^* A_1)^*$ and $rank(A_1^* A_1) = rank(A_1 A_1^*) = r$ we can assure that there exists a unitary matrix $S \in \mathbb{C}^{s\times s}$ such that $A_1^* A_1 = S(I_r \oplus O_{s-r})S^*$. Then $I_s - A_1^* A_1 = S(O_r \oplus I_{s-r})S^*$, that is $rank(I_s - A_1^* A_1) = s-r$. Hence, A and B are adjacent if and only if $rank(Z(I_s - A_1 A_1^*)) = 1$. In this case, $s = r+1$. Using the Sylvester formula $rank(Z(I_{r+1} - A_1^* A_1)) = rank(I_{r+1} - A_1^* A_1) - dim(\mathcal{N}(Z) \cap \mathcal{R}(I_{r+1} - A_1^* A_1))$ and the fact that $\mathcal{R}(I_{r+1} - A_1^* A_1) = \mathcal{N}(A_1)$ holds, we obtain that $rank(Z(I_s - A_1^* A_1)) = 1$ if and only if $\mathcal{N}(Z) \cap \mathcal{N}(A_1) = \{0\}$. ■

The weighted case is given in the following result.

Corollary 8 *Let $A \in \mathbb{C}_r^{m\times n}$, $B \in \mathbb{C}_s^{m\times n}$ and consider two Hermitian positive definite matrices $M \in \mathbb{C}^{m\times m}$ and $N \in \mathbb{C}^{n\times n}$ such that $A \leq^*_{M,N} B$. Then A and B are adjacent if and only if there exist unitary matrices $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ and a matrix $Z \in \mathbb{C}^{(m-r)\times s}$ such that*

$$A = M^{-1/2}U \begin{bmatrix} \Sigma_A A_1 & O \\ O & O \end{bmatrix} V^* N^{1/2} \qquad \text{and} \qquad B = M^{-1/2}U \begin{bmatrix} \Sigma_A A_1 & O \\ Z(I_s - A_1^* A_1) & O \end{bmatrix} V^* N^{1/2},$$

where $\Sigma_A \in \mathbb{R}^{r\times r}$ is a positive definite diagonal matrix (with non-increasing diagonal entries), $s = r+1$ and block $A_1 \in \mathbb{C}^{r\times s}$ satisfies $A_1 A_1^* = I_r$ and $\mathcal{N}(Z) \cap \mathcal{N}(A_1) = \{0\}$.

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