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Computational Cost Reduction for *N+2* Order Coupling Matrix Synthesis based on Desnanot-Jacobi Identity

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***Abstract*—Matrix inversion is routinely performed in computational engineering, with coupling matrix filter synthesis considered here as just one of many example applications. When calculating the elements of the inverse of a matrix, the determinants of the submatrices are evaluated.**

The recent mathematical proof of the Desnanot-Jacobi (also known as the 'Lewis Carol') identity shows how the determinant of an *N+2* order square matrix can be directly computed from the determinants of the *N+1* order principal submatrices and *N* order core submatrix. For the first time, this identity is applied directly to an electrical engineering problem; simplifying *N+2* order coupled matrix filter synthesis (general case, which includes lossy and asymmetrical filters). With general 2-port network theory, we prove the simplification using the Desnanot-Jacobi identity and show that the *N*+2 coupling matrix can be directly extracted from the zeros of the admittance parameters (given by *N+1* order determinants) and poles of the impedance parameters(given by the *N* order core matrix determinant). The results show that it is possible to decrease the computational complexity (by eliminating redundancy), reduce the associated cost function (by using less iterations) and under certain circumstances obtain different equivalent solutions. Nevertheless the method also proves its practical usefulness under constrained optimizations (when the user desires specific coupling matrix topologies and also constrained coefficients values (purely real/ imaginary/positive/negative) when it manages to lead to a direct coupling matrix constrained configuration when other similar methods fail (using the same optimization algorithms)

***Index Terms*—coupling matrix, determinant, filter synthesis**

I. Introduction

IN computational engineering, matrix inversion is routinely performed and this requires the calculation of its determinant. While generally considered a mature subject, there is still scope for new algorithms [1] and methods [2], which is critical for simplifying computational effort and ultimately speeding up simulation time.

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For an *N* order filter, *N* order coupling matrix filter synthesis requires *N* order matrix inversion [3-4]. The *N+2* coupling matrix, on the other hand, includes an extra pair of rows (top and bottom) and extra pair of columns (to the left and right) surrounding the *N* order core submatrix, to describe all the couplings between the source and load and the different nodes of the circuit [5-6]. The *N+2* order coupling matrix synthesis can start from the transversal coupling matrix [6], (lossless case) [7] (lossy case) which can be obtained directly from the poles and residues of the short-circuit admittance or *Y*-parameters. Since transversal coupling is not practical for physical implementations, the authors of [6-7] search for a new coupling matrix that shares the same target frequency response. Classical synthesis/reconfiguration techniques employ similarity transformations; based on either rotations [6, 8] or reflections [9] for reciprocal lossless filters (having symmetrical real coupling matrices), hyperbolic rotations [10-11] or hyperbolic reflections [12] for reciprocal lossy filters (having symmetrical complex coupling matrices). These transformations are reapplied until the coupling matrix is transformed into the desired filter topology. The drawbacks of these methodologies are represented by the fact that one has to find the sometimes complicated sequence of transformations which has to be applied in order to obtain the desired filter topology. Further one cannot impose an ideal user constrained reconfiguration of the coupling matrix (supposing one desires a coupling topology with coefficients values within a specific range).Once the proper sequence of transformations is found (which reconfigures an initial coupling matrix to a new one in a desired topology), one may still have to work on changing the signs of the coupling coefficients to adjust them to the practical ones. This can be analytically using the method of enclosures (proposed by Cameron) and /or using scaling matrices in order to work with more practical coupling coefficient values [7] Alternatively, since it can be cumbersome to find the appropriate sequence of transformations, it is possible to use optimization techniques to replace the transversal coupling matrix with one that can generate an equivalent network topology; for example the technique proposed and applied in [13] for a reciprocal symmetrical lossless filter.

Another synthesis procedure transforms the *Y*-parameters of a lossy filter directly into the desired complex coupling matrix, based on the computation of four determinants of three

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principal (sub)matrices [14]; one of order *N+2* and three of order *N+1*. However, by exploiting a recent mathematic proof (Desnanot-Jacobi identity) [2], it is shown that the synthesis in [14] can be simplified using lower order determinants for its (sub)matrices; three of order *N+1* and one of order *N*. With general 2-port network theory, we prove the simplification using the Desnanot Jacoby identity and propose a simplified hybrid coupling matrix extraction/reconfiguration method based on the zeros of the admittance parameters and on the poles of the impedance parameters (or vice versa if one works with admittance inverters coupling matrices models).

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II. Desnanot-Jacobi Identity with Network Theory

In 2012 it was shown that for a matrix $\boldsymbol{W}_{N+2}$ of order N+2 (with $N \geq 1$) its determinant $|\boldsymbol{W}_{N+2}|$ can be computed directly from the determinants of its *N+1* order principal submatrices $\boldsymbol{C}$, $\boldsymbol{D}$, $\boldsymbol{E}$ and $\boldsymbol{F}$ and N order core submatrix $\boldsymbol{W}_N$, with $|\boldsymbol{W}_N| \neq 0$ [2]. With reference to (1), $\boldsymbol{C}$ is obtained by deleting the last row and last column, $\boldsymbol{D}$ by deleting the last row and first column, $\boldsymbol{E}$ by deleting the first row and last column and $\boldsymbol{F}$ by deleting the first row and first column; while $\boldsymbol{W}_N$ (the core N*N submatrix of $\boldsymbol{W}_{N+2}$) is obtained by deleting the first and last rows and the first and last columns:

$$\boldsymbol{W}_{N+2} = \begin{pmatrix} w_{1,1} & w_{1,2} & w_{1,3} & \cdot\cdot & w_{1,N+2} \\ w_{2,1} & w_{2,2} & w_{2,3} & \cdot\cdot & w_{2,N+2} \\ w_{3,1} & w_{3,2} & w_{3,3} & \cdot\cdot & w_{3,N+2} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ w_{N+2,1} & w_{N+2,2} & w_{N+2,3} & \cdot\cdot & w_{N+2,N+2} \end{pmatrix} \quad (1a)$$

$$|\boldsymbol{W}_{N+2}| = \frac{1}{|\boldsymbol{W}_N|}\begin{Vmatrix} |\boldsymbol{C}| & |\boldsymbol{D}| \\ |\boldsymbol{E}| & |\boldsymbol{F}| \end{Vmatrix} = \frac{|\boldsymbol{C}||\boldsymbol{F}|-|\boldsymbol{D}||\boldsymbol{E}|}{|\boldsymbol{W}_N|} \quad (1b)$$

A. Y-parameters and N+2 coupling matrix

Using a low-pass filter prototype and impedance inverters [4], from [16], the *Y*-parameters for a 2-port network are related to the extended coupling matrix $\boldsymbol{M}_{N+2}$ [14]:

$$\boldsymbol{W}_{N+2} = (j\boldsymbol{M}_{N+2} + j\omega \boldsymbol{U}_{\boldsymbol{N+2}}) \quad (2a)$$

$$\boldsymbol{M}_{N+2} = \begin{pmatrix} 0 & M_{S,1} & M_{S,2} & \cdot\cdot & M_{S,N} & M_{S,L} \\ M_{1,S} & M_{1,1} & M_{1,2} & \cdot\cdot & M_{1,N} & M_{1,L} \\ M_{2,S} & M_{2,1} & M_{2,2} & \cdot\cdot & M_{2,N} & M_{2,L} \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ M_{N,S} & M_{N,1} & M_{N,2} & \cdot\cdot & M_{N,N} & M_{N,L} \\ M_{L,S} & M_{L,1} & M_{L,2} & \cdot\cdot & M_{L,N} & 0 \end{pmatrix} \quad (2b)$$

$$[\boldsymbol{Y}] = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} = \begin{pmatrix} \boldsymbol{W}^{-1}_{1,1} & \boldsymbol{W}^{-1}_{1,N+2} \\ \boldsymbol{W}^{-1}_{N+2,1} & \boldsymbol{W}^{-1}_{N+2,N+2} \end{pmatrix} =$$

$$-j\begin{pmatrix} (\boldsymbol{M}_{N+2} + \omega \boldsymbol{U}_{\boldsymbol{N+2}})^{-1}_{1,1} & (\boldsymbol{M}_{N+2} + \omega \boldsymbol{U}_{\boldsymbol{N+2}})^{-1}_{1,N+2} \\ (\boldsymbol{M}_{N+2} + \omega \boldsymbol{U}_{\boldsymbol{N+2}})^{-1}_{N+2,1} & (\boldsymbol{M}_{N+2} + \omega \boldsymbol{U}_{\boldsymbol{N+2}})^{-1}_{N+2,N+2} \end{pmatrix} \quad (3)$$

where $j = \sqrt{-1}$ is the complex operator, ω is angular frequency, $\boldsymbol{W}_{N+2}$ is the (N+2)*(N+2) impedance matrix [15], and $\boldsymbol{U}_{\boldsymbol{N+2}}$ is a diagonal matrix of order *N+2* [14] with elements equal to *1*, with the exception of $U_{11} = U_{N+2,N+2} = 0$. Since our computational reduction procedure is based on (1), where all the coefficients can be complex, if (2) includes an additional summing term (represented by a diagonal matrix that includes resonator losses [17]), the following analysis is unaffected.

Now, (3) can be further simplified as follows. Using determinants for its submatrices, in their (3), the authors of [14] interchange y_{22} with y_{11} by mistake; as can be seen in our (4), y_{22} is related to $|\boldsymbol{C}|$):

$$[\boldsymbol{Y}] = \frac{1}{|\boldsymbol{W}_{N+2}|}\begin{pmatrix} |\boldsymbol{F}| & (-1)^{N+3}|\boldsymbol{D}| \\ (-1)^{N+3}|\boldsymbol{E}| & |\boldsymbol{C}| \end{pmatrix} \quad (4)$$

From now, if we only address reciprocal networks having symmetrical coupling matrices, (4) is further simplified to:

$$[\boldsymbol{Y}] = \frac{1}{|\boldsymbol{W}_{N+2}|}\begin{pmatrix} |\boldsymbol{F}| & (-1)^{N+1}|\boldsymbol{D}| \\ (-1)^{N+1}|\boldsymbol{D}| & |\boldsymbol{C}| \end{pmatrix} \quad (5)$$

Finally, for the first time, we introduce (1b) to make a further simplification:

$$[\boldsymbol{Y}] = \frac{|\boldsymbol{W}_N|}{|\boldsymbol{C}||\boldsymbol{F}|-|\boldsymbol{D}|^2}\begin{pmatrix} |\boldsymbol{F}| & (-1)^{N+1}|\boldsymbol{D}| \\ (-1)^{N+1}|\boldsymbol{D}| & |\boldsymbol{C}| \end{pmatrix} \quad (6)$$

This result is important because it can now be seen that admittance parameters are completely determined by lower order determinants $|\boldsymbol{C}|$, $|\boldsymbol{D}|$, $|\boldsymbol{F}|$ and $|\boldsymbol{W}_N|$. In addition, by combining (6) and (1), with $|\boldsymbol{W}_{N+2}| \neq 0$, the determinant of $[\boldsymbol{Y}]$ can be calculated from the new and elegant relationship:

$$|\boldsymbol{Y}| = \left(\frac{|\boldsymbol{W}_N|}{|\boldsymbol{C}||\boldsymbol{F}|-|\boldsymbol{D}|^2}\right)^2 (|\boldsymbol{C}||\boldsymbol{F}| - |\boldsymbol{D}|^2) =$$

$$\frac{|\boldsymbol{W}_N|^2}{|\boldsymbol{C}||\boldsymbol{F}|-|\boldsymbol{D}|^2} = \frac{|\boldsymbol{W}_N|^2}{|\boldsymbol{W}_{N+2}|\,|\boldsymbol{W}_N|} = \frac{|\boldsymbol{W}_N|}{|\boldsymbol{W}_{N+2}|} \quad (7)$$

B. ABCD-, Y- and Z-parameters

With traditional normalized synthesis, for a low-pass filter prototype, the *ABCD*-parameters can be expressed as [13]:

$$[\boldsymbol{ABCD}(s)] = \frac{\varepsilon}{P(s)}\begin{pmatrix} A(s) & B(s) \\ C(s) & D(s) \end{pmatrix} \quad (8)$$

where complex frequency $s = j\omega$ (ignoring transient behavior), *P(s)* is a polynomial whose degree is given by the number of finite transmission zeros for the filter and ε is a normalization constant.

The admittance parameters for a reciprocal network are related to the *ABCD*-parameters as:

$$[\boldsymbol{Y}(s)] = \frac{[Y_{\boldsymbol{NU}}(s)]}{y_D(s)} = \frac{1}{B(s)}\begin{pmatrix} D(s) & -\frac{P(s)}{\epsilon} \\ -\frac{P(s)}{\epsilon} & A(s) \end{pmatrix} \quad (9)$$

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where $[\boldsymbol{Y_{NU}}(s)]$ is the numerator matrix of the admittance parameters and $y_D(s) = B(s)$ is the common denominator. Using a low-pass filter prototype and impedance inverters, the associated degrees are: *N* for $C(s)$; *N-1* for $A(s)$ and $D(s)$; and $B(s)$ would have degree *N-2*, except with the fully canonical case (i.e. source-load coupling occurs with $\boldsymbol{W}_{1,N+2} = \boldsymbol{W}_{N+2,1} \neq 0$) when it is also *N*. Note that, if admittance inverter coupling matrix models are used, $\boldsymbol{W}_{N+2}$ would be the (*N*+2)*(*N*+2) admittance matrix and the associated degrees would be *N* for $B(s)$; *N-1* for $A(s)$ and $D(s)$; and $C(s)$ would have degree *N-2*, except with the fully canonical case when it is also *N* [15]).

In a similar way, we can determine the relationship between the open-circuit impedance or *Z*-parameters for the 2-port network and *ABCD*-parameters:

$$[\boldsymbol{Z}(s)] = \frac{[\boldsymbol{Z_{NU}}(s)]}{z_D(s)} = \frac{1}{C(s)}\begin{pmatrix} A(s) & \frac{P(s)}{\epsilon} \\ \frac{P(s)}{\epsilon} & D(s) \end{pmatrix} \tag{10}$$

where $[\boldsymbol{Z_{NU}}(s)]$ is the numerator matrix of the impedance parameters while $z_D(s) = C(s)$ is the common denominator.

C. Desnanot-Jacobi simplification to the coupling matrix

By inverting (6) we obtain:

$$[\boldsymbol{Z}] = \frac{1}{|\boldsymbol{W_N}|}\begin{pmatrix} |\boldsymbol{C}| & (-1)^N|\boldsymbol{D}| \\ (-1)^N|\boldsymbol{D}| & |\boldsymbol{F}| \end{pmatrix} \tag{11}$$

This shows an important point in that the poles of the *Z*-parameters are given by the core *N***N* coupling matrix Eigenvalues (with $C(s) = |\boldsymbol{W_N}|$).

With direct synthesis (in a translated generalized Eigenvalue problem), the authors of [14] force the zeros of y_{11}, y_{21} and y_{22} to be equal to the zeros of $|\boldsymbol{C}| = 0$, $|\boldsymbol{D}| = 0$ and $|\boldsymbol{F}| = 0$; while the poles of the *Y*-parameters should be equal to the zeros of $|\boldsymbol{W}_{N+2}|$. Thus, the authors of [14] impose the optimized network to share the same values for $A(s)$, $B(s)$, $D(s)$ and $P(s)/\varepsilon$ in (9) as the target filter.

Translating this into a simplified Eigenvalue problem, for the condition that an *N+2* order coupling matrix generates the same admittance poles as the target network (with $B(s) = |\boldsymbol{W}_{N+2}|$), gives [14]:

$$(\boldsymbol{M}_{N+2} - \lambda_i \boldsymbol{U_{N+2}})\boldsymbol{x}_i = 0 \tag{12}$$

where $\boldsymbol{x}_i$ are the Eigenvectors and λ_i are the corresponding generalized Eigenvalues.

Using similar expressions for the zeros of the *Y*-parameters, based on *N+1* order determinants $|\boldsymbol{C}|, |\boldsymbol{D}|$ and $|\boldsymbol{F}|$, the associated cost function ΔC can be defined as [14]:

$$\Delta C = [p_y, z_y^{11}, z_y^{21}, z_y^{22}] - [\lambda_y, \lambda', \lambda'', \lambda'''] \tag{13}$$

where p_y are the poles of the prototype *Y*-parameters, z_y^{11}, z_y^{21} and z_y^{22} are the zeros of the target y_{11}, y_{21} and y_{22}, respectively. Also, λ_y are the generalized Eigenvalues of (12), corresponding to the poles of the prototype *Y*-parameters (while λ_y are the zeros of $B(s) = |\boldsymbol{W}_{N+2}|$ in either (5) or (9), and equal to the zeros of $y_D(s)$ in (9)); while λ', λ'' and λ''' are the set of generalized Eigenvalues for equations similar to (14), but now using the *N+1* order principal submatrices matrices $\boldsymbol{C}$, $\boldsymbol{D}$ and $\boldsymbol{F}$ [14]. During the optimization process $\lambda_y, \lambda', \lambda'', \lambda'''$ are calculated for each iteration, until the target values are reached.

Now, a classical Eigenvalue equation has the standard form $(\boldsymbol{A} - \lambda_i \boldsymbol{I})\boldsymbol{x_i} = 0$, where $\boldsymbol{A}$ denotes an arbitrary matrix and $\boldsymbol{I}$ is the identity matrix (with unity along the main diagonal and zeros elsewhere); both square matrices having the same order. With (12), while $\boldsymbol{U_{N+2}}$ is a diagonal matrix of order *N+2*, its diagonal elements are not all equal to *1*, since $U_{11} = U_{N+2,N+2} = 0$; this can be seen as providing additional and unnecessary redundancy. Therefore, the 'generalized Eigenvalues' [14] obtained from (12) cannot be the same as the classical Eigenvalues of $\boldsymbol{M}_{N+2}$ and, thus, (12) represents a non-standard solution.

Here, since $\boldsymbol{U}_N$ (the core *N***N* submatrix of $\boldsymbol{U_{N+2}}$) has only unity along its main diagonal, using (11) with (2), gives us the standard form of the Eigenvalue equation:

$$(\boldsymbol{M}_N - \lambda_i \boldsymbol{I})\boldsymbol{x}_i = 0 \tag{14}$$

where $\boldsymbol{M}_N$ is the core *N***N* submatrix of $\boldsymbol{M_{N+2}}$. The coupling matrix now has the same values for $A(s)$, $C(s)$, $D(s)$ and $P(s)/\varepsilon$ in (10) as the target filter. The resulting cost function now changes to $\Delta C'$:

$$\Delta C' = [p_z, z_y^{11}, z_y^{21}, z_y^{22}] - [\lambda_z, \lambda', \lambda'', \lambda'''] \tag{15}$$

where p_z are the poles of the prototype *Z*-parameters; while λ_z are the zeros of $C(s) = |\boldsymbol{W}_N|$ in either (10) or (11), and equal to the zeros of $z_D(s)$ in (10).

It should be noted that (14) generates *N* different solutions, corresponding to the zeros of $C(s)$ in (10). However, (12) generates either *N-2* solutions, corresponding to the zeros of $B(s)$ in (9), if there is no source-load coupling, or *N* solutions with the fully canonical case. This can also be seen from matrix theory. With reference to (2b), the extended coupling matrix $\boldsymbol{M}_{N+2}$ has the first and last elements on the main diagonal equal to zero [6, 8] (corresponding to no source-source and load-load self-coupling) and if the source (S) and load (L) are not coupled together (i.e. $M_{S,L} = M_{L,S} = 0$) then $\boldsymbol{W}_{1,1} = \boldsymbol{W}_{1,N+2} = \boldsymbol{W}_{N+2,1} = \boldsymbol{W}_{N+2,N+2} = 0$. Using a Laplace expansion, it can be seen that $B(s) = |\boldsymbol{W}_{N+2}|$ or (12) only generates *N-2* solutions. With the canonical case, $M_{S,L} = M_{L,S} \neq 0$ and, therefore, $\boldsymbol{W}_{1,N+2} = \boldsymbol{W}_{N+2,1} \neq 0$ and again, using Laplace expansion, it can be seen that $B(s) = |\boldsymbol{W}_{N+2}|$ or (12) now generates *N* solutions.

D. Computational cost advantage

It will be found that (14) has the following computational

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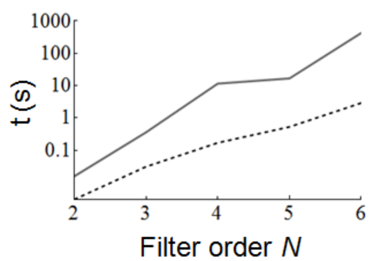
advantages over (12): (i) It uses an N order matrix, unlike the $N+2$ order matrix in (12); (ii) It is a classical Eigenvalue problem unlike a generalized Eigenvalue problem in (12); (iii) It always uses the core N order coupling matrix and, thus, avoids taking into account the elements corresponding to the couplings between the source-load and different resonators – using less variables in the optimization steps; and (iv) it is less affected when source-load coupling occurs, as (12) also generates N solutions.

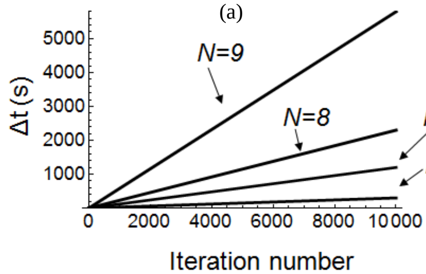
The resulting computational gain with our standard form is represented by the difference in time Δt needed for a processor to solve (12), when compared to (14), at each iteration step of the optimization process.

The computational gain for arbitrary 2nd to 6th order filters having symmetrical $N+2$ order coupling matrices can be significant, as shown in Fig. 1(a). Here, one needs more than 400 seconds using *Mathematica 9.0* for $N = 6$ (coupling matrix of order 8) to solve (12) and only 2.91 seconds for (14), using random symbolic (non-numerical) coupling matrix coefficients [18].

With a typical optimization process [19], depending on the initial numerical values, algorithm used and accuracy required, from ten to thousands of iterartions may be required to achieve the target Y-parameters; the associate cost advantage can be clearly seen in Fig. 1(b).



(a)



(b)

Fig. 1. Computation gain using *Mathematica 9.0* with an Intel i7 processor and 2 GB RAM: (a) Time taken t for solving (12) (solid lines) and (14) (dashed line) for arbitrary 2nd to 6th order filters with symmetrical $N+2$ order coupling matrix (using symbolic non-numerical coefficients); (b) Results against iteration number for different order coupling matrices (using numerical coefficients) (with the cost functions ΔC and $\Delta C'$ giving the same final results).

III. Filter Design Examples

Design examples will now be given to an arbitrarily chosen asymmetrical lossy filter [p. 60 in 20]; the target filter response is given by the scattering or S-parameters in Fig. 2, for a 4th order filter having both source and load couplings. With all design examples, the Nelder Mead, Simulated Annealing and Differential Evolution algorithms available in [18] for constrained optimizations (direct search optimization algorithms) are used and compared. Gradient based optimization algorithms [21] are available in Mathematica for unconstrained optimization problems. Thus we tested too the Fletcher (conjugate gradient algorithm) which gave the best results among the the latter ones available.(in the cases when we put no restrictions on the coupling matrix coefficients).

The initial solutions for the coupling matrix coefficients are considered always the ones Mathematica generates authomatically, mainly a set of points with random numbers in the interval [-1,1]. We consider this authomatical intial solution always unchanged since we aim to test our coupling matrix extraction/ reconfiguration method in a variety of cases (different topologies sharing the same frequency response but with different constraints on the coefficients too). Optimizations is implememnted using an Intel i5 3317u processor, with 4 GB of RAM, until the filter is optimized to match with the target given in Fig. 2. Having a pre-defined topology, a search starts for coupling matrix coefficient values that share those for the target filter response (as in Fig. 2); optimization is completed once this is achieved. The iteration number represents the number of times the search algorithm re-computes the values of the coupling matrix coefficients during the optimization process.

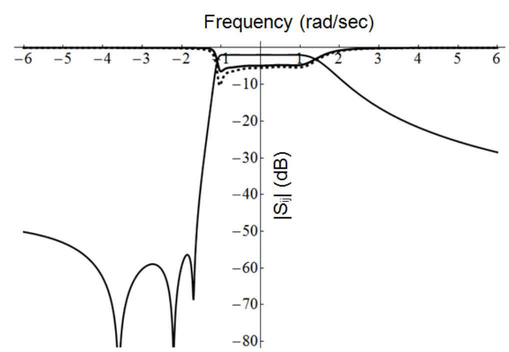


Fig. 2. Target S-parameters for an asymmetrical lossy filter (low-pass prototype with unity source and load resistances) [p. 60 in 20]: solid lines for $|S_{21}|$ and $|S_{11}|$; dashed line for $|S_{22}|$.

The general form for the 4th order extended coupling matrix for a reciprocal filter is given by:

$$\boldsymbol{M}_{4+2} = \begin{pmatrix} M_{S,S} & M_{S,1} & M_{S,2} & M_{S,3} & M_{S,4} & M_{S,L} \\ M_{S,1} & M_{1,1} & M_{1,2} & M_{1,3} & M_{1,4} & M_{1,L} \\ M_{S,2} & M_{1,2} & M_{2,2} & M_{2,3} & M_{2,4} & M_{2,L} \\ M_{S,3} & M_{1,3} & M_{2,3} & M_{3,3} & M_{3,4} & M_{3,L} \\ M_{S,4} & M_{1,4} & M_{2,4} & M_{3,4} & M_{4,4} & M_{4,L} \\ M_{S,L} & M_{1,L} & M_{2,L} & M_{3,L} & M_{4,L} & M_{L,L} \end{pmatrix} \quad (19)$$

To meet the target specification given by Fig. 2, if the elements of the coupling matrix are used as optimization variables (with the exceptions of $M_{S,S} = M_{L,L} = 0$ and $M_{S,L} = 0$), there will be 18(36) independent complex(real) variables; loss is considered here to be distributed evenly over all elements.

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A. Proposed general synthesis

Our optimization routine performs iterations until either $\Delta C<10^{-6}$ or $\Delta C'<10^{-6}$ or better is achieved; at which point the reconfigured topology shares the same response (which would be indistinguishable to see if we plotted this out). Using our synthesis procedure with (15), we obtain the extended coupling matrix given in Table I with the associated cost function against iteration number shown in Fig. 3. With 17 iterations and in 28 seconds (fastest using the Nelder Mead algorithm) we get $\Delta C' \leq 10^{-6}$). With Simulated Annealing and with Differential Evoultion we obtain the same results but in 33 and 40 seconds respectively. On the other hand the gradient based optimizer reaches too for this unconstrained coupling matrix solution in around 30 seconds in 300 iterations.

TABLE I
N+2 COUPLING MATRIX OBTAINED WITH (15)

0	0.1295-0.0549j	-0.0430+0.0067j	0.4643+0.0022j	-0.1604-0.0396j	0
0.1295-0.0549j	0.52264-0.0141j	-0.5781-0.0570j	-0.7806-0.0195j	0.20735-0.049j	-0.1146-0.0156j
-0.0430+0.0067j	-0.5781-0.0570j	0.0120+0.0518j	-0.9083-0.0267j	-0.9114+0.0462j	-0.1461-0.0098j
0.4643+0.0022j	-0.7806-0.0195j	-0.9083-0.0267j	-0.2344-0.1036j	-0.2518+0.1410j	-0.5176-0.0990j
-0.1604-0.0396j	0.20735-0.049j	-0.9114+0.0462j	-0.2518+0.1410j	0.4070+0.0059j	-1.2698+0.0429j
0	-0.1146-0.0156j	-0.1461-0.0098j	-0.5176-0.0990j	-1.2698+0.0429j	0

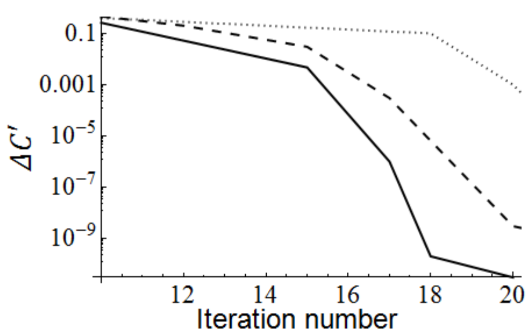

Fig. 3. Cost function associated with the extended coupling matrix from Table I, calculated using (15). (solid line- Nelder Mead, dashed- Simulated Annealing, dots-Differential Evolution)

B. Previous general synthesis

By comparison, using the previous synthesis procedure [14] with (13), we obtain the extended coupling matrix given in Table II with the associated cost function against iteration number shown in Fig. 4. With 27 iterations and in 102 seconds, we get $\Delta C \leq 10^{-6}$ (fastest using the Nelder Mead algorithm). With Simulated Annealing and with Differential Evoultion we obtain the same results but in 150 and 244 seconds respectively. On the other hand the gradient based optimizer Fletscher fails to obtain in 5000 iterations and 10 minutes any solution in this case.

TABLE II
N+2 COUPLING MATRIX OBTAINED WITH (13)

0	-0.253-0.0031j	0.2947-0.1285j	0.1877-0.0220j	0.3203+0.1286j	0
-0.253-0.0031j	-0.0327+0.0274j	-0.4058-0.17588j	-0.8475+0.0246j	-0.2752+0.1700j	-0.3891+0.3176j
0.2947-0.1285j	-0.4058-0.17588j	0.3567+0.0134j	-0.5005+0.2018j	-1.1917-0.0349j	-0.6164+0.1380j
0.1877-0.0220j	-0.8475+0.0246j	-0.5005 +0.2018j	-0.1806+0.1008j	-0.4826-0.2372j	-1.1248 -0.3357j
0.3203+0.1286j	-0.2752+0.1700j	-1.1917-0.0349j	-0.4826-0.2372j	0.5637-0.2017j	0.6393-0.2642j
0	-0.3891+0.3176j	-0.6164+0.1380j	-1.1248 -0.3357j	0.6393-0.2642j	0

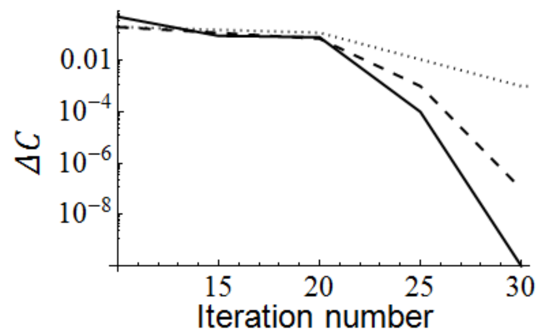

Fig. 4. Cost function associated with the extended coupling matrix from Table II, calculated using (16). (solid line- Nelder Mead, dashed- Simulated Annealing, dots-Differential Evolution)

Both matrices can be then uneasy manipulated using hyperbolic rotations [7, 10-11] or hyperbolic reflections [12], with the proper sequence, to generate a coupling matrix that is optimal for the implementation technology used.

C. Proposed direct synthesis

Now, we try to directly synthetize a more practical coupling matrix topology (as in many cases no solutions exists), having symmetry (being reciprocal), and assume complex coupling coefficients (representing lossy elements) on the main diagonal only (giving a total of 22 independent real variables), using our synthesis procedure with (15). We obtain the extended coupling matrix given in Table III, with the associated cost function against iteration number shown in Fig. 5. With 11 iterations and in 5.2 seconds, $\Delta C' \leq 10^{-6}$ we get the soultion with the Nelder Mead algorithm and in less than 6 seconds with the Simulated Annealing and Differential Evolution. On the other hand the gradient based optimizer we get the solution in 2 seconds in this particular case(190 iterations).

TABLE III
N+2 COUPLING MATRIX OBTAINED WITH (15) WITH 4 COMPLEX AND 14 REAL NON-ZERO COUPLING COEFFICIENTS

0	-0.1855	0.0190	0.1883	0.4302	0
-0.1855	0.5773-0.015j	0.9122	0.8296	0.0726	1.2139
0.0190	0.9122	-0.4613-0.015j	-0.9086	-0.2891	-0.4153
0.1883	0.8296	-0.9086	0.6373-0.015j	-0.0732	-0.1200
0.4302	0.0726	-0.2891	-0.0732	-0.0461-0.015j	0.4921
0	1.2139	-0.4153	-0.1200	0.4921	0

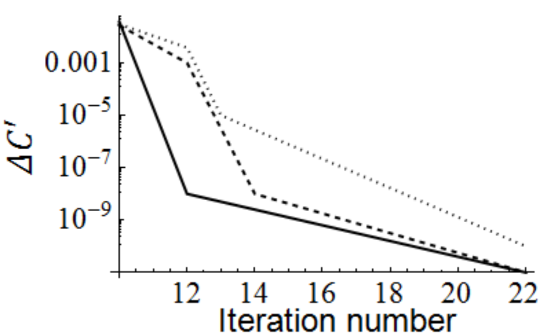

Fig. 5. Cost function associated with the extended coupling matrix from Table III, calculated using (15). (solid line- Nelder Mead, dashed- Simulated Annealing, dots-Differential Evolution)

D. Previous direct synthesis

By comparison, using the previous synthesis procedure [14] with (13), by imposing the same coupling configuration, we obtain the extended coupling matrix given in Table IV with the associated cost function against iteration number shown in Fig. 4. With 18 iterations and in 9.7 seconds, $\Delta C \leq 10^{-6}$. we get the soultion with the Nelder Mead algorithm and in less than 10 seconds with the Simulated Annealing and Differential Evolution . On the other hand the gradient based optimizer (Fletscher) fails to find in this case any solution.

TABLE IV
N+2 COUPLING MATRIX OBTAINED WITH (13) WITH 4 COMPLEX AND 14 REAL NON-ZERO COUPLING COEFFICIENTS

0	0.1551	0.1744	-0.4403	0.0826	0
0.1551	-0.1742-0.015j	0.9346	1.0709	-0.5499	-0.9147
0.1744	0.9346	-0.1045-0.015j	-0.2545	-0.2572	0.5724
-0.4403	1.0709	-0.2545	0.2069-0.015j	0.3439	0.1624
0.0826	-0.5499	-0.2572	0.3439	0.7790-0.015j	0.8438
0	-0.9147	0.5724	0.1624	0.8438	0

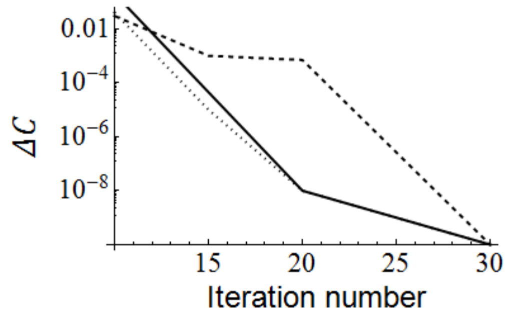

Fig. 6. Cost function associated with the extended coupling matrix from Table IV, calculated using (13).

E. Synthesis for conditions imposed by [20]

We now impose the coupling configuration given in [20], having lossy elements only along the main diagonal and with the only non-zero cross-coupling coefficients $M_{1,4}$, $M_{1,L}$ and $M_{2,4}$. We obtain the same coupling matrix as in [20], shown in Table V, in less than 3 seconds with both procedures (13) and (15) and with all optimization algorithms. The associated cost function against iteration number is shown in Fig.7. In this particular case the Fletcher algorithm is fastest finding the solution in 0.5 seconds for (15) and in 1.5 seconds for the previous procedure (13).

TABLE V
N+2 COUPLING MATRIX OBTAINED WITH (13) AND (15) WHILE SEARCHING IMPOSING THE COUPLING ELEMENTS GIVEN IN [20] WITH 4 COMPLEX AND 8 REAL NON-ZERO COUPLING COEFFICIENTS

0	-0.5053	0	0	0	0
-0.5053	-0.2562-0.0150j	-0.8464	0	-0.4143	0.0873
0	-0.864	0.0679-0.0150j	0.2310	-0.9313	0
0	0	0.2310	0.9841-0.0150j	0.7653	0
0	-0.4143	-0.9313	0.7653	-0.0886-0.0150j	1.3767
0	0.0873	0	0	1.3767	0

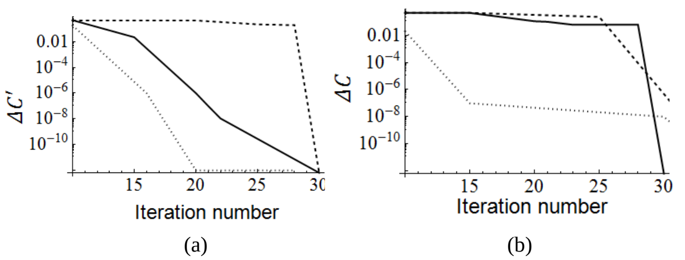

Fig. 7. Cost functions associated with the extended coupling matrix from Table V, calculated using: (a) (15); and (b) (13). (solid line- Nelder Mead, dashed- Simulated Annealing, dots-Differential Evolution)

F. Synthesis for arbitrary lossless resonator condition

Finally, we now arbitrarily impose that the second and third resonators be lossless. The extended coupling matrix using

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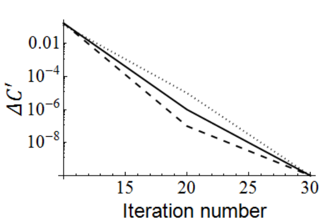
(15) is given in Table VI, while that using (13) is given in Table VII. The former now requires 151 seconds with Nelder Mead algorithm (which proves to be the faster in this case as the other algorithms fail to converge in 10 minutes), while the latter only requires 30 seconds using the Nelder Mead algorithm or around 40 seconds using Differential Evolution or Simulated Annealing. Clearly, for this example, there is a computation loss with our technique and also a flexibility in the choice of the optimization algorithm. Further, if we impose new constrains in the optimization search we may find that the proposed reconfiguration/ extraction procedure leads to the only direct solution.

TABLE VI
N+2 COUPLING MATRIX OBTAINED WITH (15) WITH 2 LOSSLESS CENTRAL RESONATORS (16 COMPLEX AND 2 REAL VARIABLES)

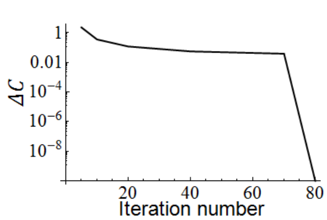
0	0.3050+ 0.0020j	0.1279- 0.0148j	0.0712+ 0.0026j	-0.3755- 0.0028j	0
0.3050+ 0.0020j	0.4114- 0.0154j	-0.5329- 0.0063j	-1.051- 0.0019j	0.0846+ 0.0194j	-0.9965- 0.0011j
0.1279- 0.0148j	-0.5329- 0.0063j	0.3237	-0.9911- 0.0100j	0.4332- 0.0136j	-0.3188- 0.0146j
0.0712+ 0.0026j	-1.051- 0.0019j	-0.9911- 0.0100j	-0.3105	-0.3557+ 0.0193j	-0.2801+ 0.0016j
-0.3755- 0.0028j	0.0846+ 0.0194j	0.4332- 0.0136j	-0.3557+ 0.0193j	-0.2825- 0.0044j	-0.8543+ 0.006 j
0	-0.9965- 0.0011j	-0.3188- 0.0146j	-0.2801+ 0.0016j	-0.8543+ 0.006 j	0

TABLE VII
N+2 COUPLING MATRIX OBTAINED WITH (13) WITH 2 LOSSLESS CENTRAL RESONATORS (16 COMPLEX AND 2 REAL VARIABLES)

0	0.0160- 0.0506j	-0.1851- 0.0348j	-0.1797+ 0.0058j	-0.4386+ 0.0104j	0
0.0160- 0.0506j	0.1795- 0.059j	0.3077- 0.1284j	-0.9165+ 0.0061j	0.3905+ 0.0321j	-1.2773- 0.0282j
-0.1851- 0.0348j	0.3077- 0.1284j	0.1665	0.6426- 0.0334j	-1.1372- 0.0144j	0.5008- 0.0398j
-0.1797+ 0.0058j	-0.9156 +0.0061j	0.6426- 0.0334j	0.2750	0.3005+ 0.1256j	-0.1564+ 0.0342j
-0.4386 +0.0104j	0.3905+ 0.0321j	-1.1372- 0.0144j	0.3005+ 0.1256j	0.0860- 0.0003j	-0.1028 +0.1047j
0	-1.2773- 0.0282j	0.5008- 0.0398j	-0.1564+ 0.0342j	-0.1028 +0.1047j	0



(a)



(b)

Fig. 8. Cost functions associated with the extended coupling matrix from: (a) Table VI; and (b) Table VII. (solid line- Nelder Mead, dashed- Simulated Annealing, dots-Differential Evolution)

G. Synthesis of a paralell coupled pair filter without resistive coupling coefficients

As an example, the parallel coupled pair filter is considered, having its routing schematic shown in Fig. 9 [7]. A search is made for a practical coupling matrix configuration with lossy resonators, without resistive couplings. Using our method (15) and the Nelder Mead algorithm we obtain the coupling matrix given in Table VIII in 72 seconds, as seen in Fig.10. The same solution is obtained with the Simulated Annealing and Differential Evolution Algorithm within between 72 and 104 seconds; while no solution could be found with the Fletcher-Powell algorithm. Using these four optimization algorithms with the previous methodology (13), no solution could be found for this coupling scheme in over 45 minutes of simulation time.

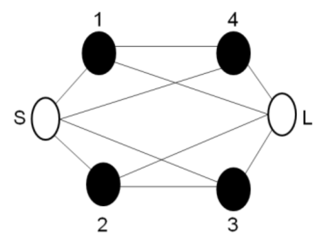


Fig. 9. Parallel coupled filter topology

TABLE VIII
N+2 COUPLING MATRIX OBTAINED WITH (15) WITH A PARALEL COUPLED TOPOLOGY WITHOUT ANY RESISTIVE COUPLINGS : $\boldsymbol{M_1}$

0	-0.3428	0.054	0.362	0.061	0
-0.3428	-0.064- 0.015j	0	0	1.291	0.820
0.054	0	1.391- 0.015j	-0.2565	0	-0.611
0.362	0	-0.2565	0.035- 0.015j	0	0.8193
0.061	1.291	0	0	-0.655- 0.015j	-0.4285
0	0.820	-0.611	0.8193	-0.4285	0

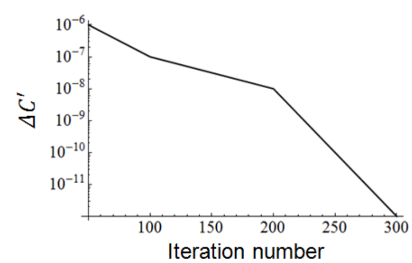


Fig. 10. Cost function associated with the extended coupling matrix from Table VIII, calculated using (15) using the Nelder Mead algorithm. Using (15) we reach a solution while using (13) we fail.

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IV. Further Discussion

The classical deterministic reconfiguration process of the lossless coupling matrices (for reciprocal networks) is based on similarity transformations involving rotations for the lossless cases [1-4], [6, 8] and hyperbolic rotations [7, 10, 11] for lossy cases which are applied to the coupling matrix via (20). The authors themselves introduced a new class of similarity transformations (for reciprocal networks) based on reflections (lossless cases) and hyperbolic reflections (lossy cases) [9, 12]. By this means the authors reconfigure a given matrix M_1 to a new one M_2 which will have the same frequency response as the first one while keeping the symmetry of it (and thus converting a reciprocal network into a new reciprocal one).

$$\boldsymbol{M_2} = \boldsymbol{T} * \boldsymbol{M_1} * \boldsymbol{T}^{-1} \quad (20)$$

Recently lossless non-reciprocal networks synthesis has gathered the attention of the microwave community [22], the authors proposing there a first technique to synthesize and reconfigure lossless nonreciprocal networks based on coupling matrices. The transformations used to reconfigure the coupling matrices are still based in [22] on a modified form of (20) and thus on complex similarity transformations.

Let us now consider the simple rotation matrix presented in Table IX (one can replace it with a complex rotation matrix, but for keeping results simple we will consider it a simple rotation matrix).

TABLE IX
Rotation matrix T

1	0	0	0	0	0
0	Cos(π/4)	0	Cos(π/4)	0	0
0	0	1	0	0	0
0	Cos(π/4)	0	Cos(π/4)	0	0
0	0	0	0	1	0
0	0	0	0	0	1

For the matrix given in Table VIII and let us consider the extended generalized impedance matrices:

$$\boldsymbol{W}_1 = (j\boldsymbol{M}_1 + j\omega \boldsymbol{U_6}) \quad (21)$$

$$\boldsymbol{W}_2 = (\boldsymbol{T} * M_1 * \boldsymbol{T}^{-1} + j\omega \boldsymbol{U_6}) \quad (22)$$

$$\boldsymbol{W}_3 = \boldsymbol{T} * \boldsymbol{W}_1 = (\boldsymbol{T} * j\boldsymbol{M}_1 + \boldsymbol{T} * j\omega \boldsymbol{U_6}) \quad (23)$$

Using the Desnanot Jacobi property it can be proved that the new proposed form (23) will generate the same admittance parameters as (21) and (22) if the rotation matrix $\boldsymbol{T}$ has no pivot on the first and last lines and rows (and thus has just ones and zeros there). Unlike (21) or (22) the matrix in (23) will not be any more a symmetrical matrix since (23) is not a similarity transformation, it is a rotation (please beware that we do not apply a simple rotation to the coupling matrix $\boldsymbol{M_1}$ (this would be completely wrong), we apply it to 2a) and thus to $\boldsymbol{W_1}$).

Indeed applying $\boldsymbol{T}$ to $\boldsymbol{W_1}$ via (23) we get the matrix in Table X which represents a frequency dependent extended non reciprocal impedance matrix sharing the same frequency response with $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$. Equation (23) shows that the direct application of rotation matrices (without any pivots on the first and last rows/columns) to the generalized extended impedance matrices leads to the same frequency response as the initial one. Even though the matrix in Table X has the inconveniency of frequency dependence, the result may be of a theoretical interest in the new topic proposed in [22], since it is obtained without a similarity transformation and it is valid for lossy networks too.

TABLE X
Frequency dependent W_3 extended impedance matrix

0	-0.3428	0.054	0.362	0.061	0
-0.4984	0.707 ω--0.045-0.01j	0.1814	-0.707 ω -0.025+0.01j	0.9128	0.0009
0.054	0	ω +1.3911-0.015j	-0.2565	0	-0.611
0.01357	0.707 ω--0.045-0.01j	-0.1814	0.707 ω -0.025-0.01j	0.9128	1.1596
0.061	1.291	0	0	ω -0.655-0.015j	-0.4285
0	0.820	-0.611	0.8193	-0.4285	0

V. Conclusion

It has been shown that, based on the recent mathematic proof of the Desnanot-Jacobi identity in (1b), the optimization process for coupling matrix filter extraction and reconfiguration can be simplified; decreasing computational complexity, by eliminating redundancy, and reducing the cost function. Until now, (1b) was previous used in the so-called 'Dodgson condensation procedure' [23]. However, by exploiting the properties of (1b), we derived simplifying expressions for the *Y*- and *Z*- parameters. These results prove that the poles of the *Z*-parameters are given by the Eigenvalues of $\boldsymbol{M}_N$ (unlike the poles of the *Y*-parameters, which are given by the zeros in $|\boldsymbol{W}_{N+2}|$). The technique proves especially suitable when the search is made for specific constrained coupling matrices configurations, in this case leading always to a solution, even though the previous technique irrespective of the proposed algorithm failed to do (using the same initial coupling matrix). The proposed method, which exploits our new equations (14) and (15), simplifies the associated cost function by computing different lower-order determinants; significantly speeding up the optimization procedure used in [14], based on impedance inverter coupling matrix models [4, 15, 17]. Similarly, it will be found that (1b) can also be used when working with admittance inverter coupling matrix models [6, 15].

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