Journal of Mathematics Research; Vol. 8, No. 5; October 2016
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

A Note on Relative (p, q) th Proximate Order of Entire Functions

Luis M. Sánchez Ruiz[1], Sanjib Kumar Datta[2], Tanmay Biswas[3] & Chinmay Ghosh[4]

[1] ETSID-Depto. de Matemática Aplicada & CITG, Universitat Politècnica de València, E-46022 Valencia, Spain.

[2] Department of Mathematics, University of Kalyani Kalyani, Dist-Nadia, PIN-741235, West Bengal, India.

[3] Rajbari, Rabindrapalli, R. N. Tagore Road P.O.- Krishnagar, Dist-Nadia, PIN- 741101, West Bengal, India.

[4] Guru Nanak Institute of Technology 157/F, Nilgunj Road, Panihati, Sodepur Kolkata-700114, West Bengal, India.

Correspondence: Luis M. Sánchez Ruiz, ETSID-Depto. de Matemática Aplicada & CITG, Universitat Politècnica de València, E-46022 Valencia, Spain. E-mail: LMSR@mat.upv.es

Received: June 27, 2016 Accepted: July 12, 2016 Online Published: September 14, 2016

doi:10.5539/jmr.v8n5p1 URL: http://dx.doi.org/10.5539/jmr.v8n5p1

Abstract

Relative order of functions measures specifically how different in growth two given functions are which helps to settle the exact physical state of a system. In this paper for any two positive integers p and q, we introduce the notion of relative (p, q) th proximate order of an entire function with respect to another entire function and prove its existence.

Keywords: entire function, index-pair, relative (p, q) th order, relative (p, q) th proximate order

1. Introduction

A single valued analytic function in the finite complex plane is called an entire (or integral) function. It is well known that for example exp, sin, cos are all entire functions. In 1926 Rolf Nevanlinna initiated the value distribution theory of entire functions which is a prominent branch of Complex Analysis and is the prime concern of this paper. In this line the value distribution theory studies how an entire function assumes some values and conversely, what is in some specific manner the influence on a function of taking certain values. It also deals with various aspects of the behaviour of entire functions one of which is the study of comparative growth properties of entire functions. For any entire function f, the so called maximum modulus function and denoted by M_f, is defined on each non-negative real value r by

$$M_f(r) = \max_{|z|=r} |f(z)|.$$

With the aim of estimating the growth of a nonconstant entire function f, Boas (Boas, 1954) introduced the concept of order as the value ρ_f which is generally used in computational purpose and is defined in terms of the growth of f respect to the $\exp z$ function as

$$\rho_f = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log\log M_{\exp}(r)} = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log(r)} \left(0 \le \rho_f \le \infty\right).$$

Given another entire function g, the ratio $\frac{M_f(r)}{M_g(r)}$ as $r \to \infty$ is called the growth of f with respect to g in terms of their maximum moduli. If this relative growth happens to be $k \in \mathbb{R}$, then

$$M_f(r) \propto k M_g(r) \text{ as } r \to \infty.$$

With the aim of knowing the relative growth of functions of the same nonzero finite order, the type of a given such funtion f was introduced as

$$\tau_f = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^{\rho_f}} \left(0 \le \tau_f \le \infty\right).$$

L. Bernal (Bernal, 1988) introduced the relative order between two entire functions to avoid comparing growth just with exp. Thus the growth of entire functions may be studied in terms of its relative orders. In fact, some works on relative order of entire functions and the growth estimates of composite entire functions on the basis of it have been explored in (Chakraborty & Roy, 2006; Datta, Biswas, 2009; Datta, Biswas, 2010; Datta, Biswas, Biswas, 2013; Datta, Biswas & Biswas, 2013; Datta, Biswas, & Pramanick, 2012; Lahiri & Banerjee, 2005). This has different applications related to entropy as this is the amount of additional information needed to specify the exact physical state of a system, and relative order of functions measures how different in growth two given functions are. Indeed very recently these ideas have been used by Alburquerque et al. (Albuquerque, Bernal-González, Pellegrino, & Seoane-Sepúlveda, 2014) who obtained new Peano type results by showing that the subset of continuous surjections from $\mathbb{R}^m$ to $\mathbb{C}^n$ such that each value a in $\mathbb{C}^n$ is assumed on an unbounded set of $\mathbb{R}^m$ is maximal strongly algebrable, i.e. there exists a $\mathfrak{c}$-generated free algebra contained in $CS(\mathbb{R}^m, \mathbb{C}^n) \cup \{0\}$, where $CS(\mathbb{R}^m, \mathbb{C}^n)$ denotes the set of all continuous surjective mappings $\mathbb{R}^m \to \mathbb{C}^n$.

On the other hand, Sánchez Ruiz et al. (Sánchez Ruiz, Datta, Biswas, & Mondal, 2014) have introduced a new type of relative (p, q)th order of entire functions where p, q are any two positive integers revisiting the ideas developed by a number of authors including Lahiri and Banerjee (Lahiri & D. Banerjee, 2005).

However, these concepts are not adequate for comparing the growth of entire functions with either zero or infinite order. For this reason Valiron (Valiron, 1949) introduced the concept of a positive continuous function $\rho_f(r)$ for an entire function f having finite order ρ_f with the following properties:

(i) $\rho_f(r)$ is non-negative and continuous for $r > r_0$, say,

(ii) $\rho_f(r)$ is differentiable for $r \geq r_0$ except possibly at isolated points at which $\rho_f'(r+0)$ and $\rho_f'(r-0)$ exist,

(iii) $\lim_{r\to\infty} \rho_f(r) = \rho_f$,

(iv) $\lim_{r\to\infty} r\rho_f'(r) \log r = 0$ and

(v) $\limsup_{r\to\infty} \dfrac{\log M_f(r)}{r^{\rho_f(r)}} = 1.$

Such a function is called a Lindelöf proximate order which makes unnecessary to consider functions of minimal or maximal type, its existence being established op. cit. It was simplified by Shah (Shah, 1946), and Nandan et al. (Nandan, Doherey, & Srivastava, 1980) extended this notion of proximate order for an entire function of one complex variable with index-pair (p, q) with positive integers $p \geq q$. Also Lahiri (Lahiri, 1989) generalised the idea of the proximate order for a meromorphic function with finite generalised order and proved its existence.

As a consequence of the above it seems reasonable for any two positive integers, p, q, to define the relative (p, q)th proximate order of an entire function with respect to another entire function. In this paper we do so and prove its existence.

2. Notation and Preliminary Remarks

Our notation is standard within the theory of Nevanlinna's value distribution of entire functions, For short, given a real function h and whenever the corresponding domain and range allow it we will use the notation

$$h^{[0]}(x) = x, \text{ and}$$
$$h^{[k]}(x) = h\left(h^{[k-1]}(x)\right) \text{ for } k = 1, 2, 3, ...$$

omitting the parenthesis when h happens to be the log or exp function. Taking this into account the *order* (resp. *lower order*) of an entire function f is given by

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r} \text{ (resp. } \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log r}).$$

Let us recall that Juneja, Kapoor and Bajpai (Juneja, Kapoor, Bajpai, 1976) defined the (p, q)-th order (resp. (p, q)-th lower order) of an entire function f as follows:

$$\rho_f(p, q) = \limsup_{r\to\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} \text{ (resp. } \lambda_f(p, q) = \liminf_{r\to\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r}),$$

where p, q are any two positive integers with $p \geq q$. These definitions extended the generalized order $\rho_f^{[l]}$ (resp. generalized lower order $\lambda_f^{[l]}$) of an entire function f considered in (Sato, 1963) for each integer $l \geq 2$ since these correspond to the particular case $\rho_f^{[l]} = \rho_f(l, 1)$ (resp. $\lambda_f^{[l]} = \lambda_f(l, 1)$). Clearly $\rho_f(2, 1) = \rho_f$ and $\lambda_f(2, 1) = \lambda_f$. Related to this, let us recall the following properties. If $0 < \rho_f(p, q) < \infty$, then

$$\begin{aligned} \rho_f(p-n, q) &= \infty \text{ for } n < p, \\ \rho_f(p, q-n) &= 0 \text{ for } n < q, \\ \rho_f(p+n, q+n) &= 1 \text{ for } n = 1, 2, ... \end{aligned}$$

Similarly for $0 < \lambda_f(p,q) < \infty$, one can easily verify that

$$\begin{aligned}\lambda_f(p-n,q) &= \infty \text{ for } n < p,\\ \lambda_f(p,q-n) &= 0 \text{ for } n < q,\\ \lambda_f(p+n,q+n) &= 1 \text{ for } n = 1,2,...\end{aligned}$$

Recalling that for any pair of integer numbers m,n the Kroenecker function is defined by $\delta_{m,n} = 1$ for $m = n$ and $\delta_{m,n} = 0$ for $m \neq n$, the aforementioned properties provide the following definition.

Definition 1. *(Juneja, Kapoor, Bajpai, 1976) An entire function f is said to have index-pair $(1,1)$ if $0 < \rho_f(1,1) < \infty$. Otherwise, f is said to have index-pair $(p,q) \neq (1,1)$, $p \geq q \geq 1$, if $\delta_{p-q,0} < \rho_f(p,q) < \infty$ and $\rho_f(p-1,q-1) \notin \mathbb{R}^+$.*

Definition 2. *(Juneja, Kapoor, Bajpai, 1976) An entire function f is said to have lower index-pair $(1,1)$ if $0 < \lambda_f(1,1) < \infty$. Otherwise, f has lower index-pair $(p,q) \neq (1,1)$, $p \geq q \geq 1$, if $\delta_{p-q,0} < \lambda_f(p,q) < \infty$ and $\lambda_f(p-1,q-1) \notin \mathbb{R}^+$.*

Given a non-constant entire function f defined in the open complex plane, its maximum modulus function M_f is strictly increasing and continuous. Hence there exists its inverse function $M_f^{-1} : (|f(0)|, \infty) \to (0,\infty)$ with $\lim_{s\to\infty} M_f^{-1}(s) = \infty$. Bernal (Bernal, 1988) introduced the definition of relative order of f with respect to g, denoted by $\rho_g(f)$, as follows:

$$\begin{aligned}\rho_g(f) &= \inf\left\{\mu > 0 : M_f(r) < M_g(r^\mu) \text{ for all } r > r_0(\mu) > 0\right\}\\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.\end{aligned}$$

This definition coincides with the classical one (Titchmarsh, 1968) if $g = \exp$. Analogously, the relative lower order of f with respect to g, denoted by $\lambda_g(f)$, is defined as

$$\lambda_g(f) = \liminf_{r\to\infty} \frac{\log M_g^{-1} M_f(r)}{\log r}.$$

Recently, Sánchez Ruiz et al. (Sánchez Ruiz, Datta, Biswas, & Mondal, 2014) have introduced a definition of relative (p,q)-th order $\rho_g^{(p,q)}(f)$ of an entire function f with respect to another entire function g, sharpenning an earlier definiton of relative (p,q)-th order of Lahiri and Banerjee (Lahiri & Banerjee, 2005), from which the more natural particular case $\rho_g^{(k,1)}(f) = \rho_g^k(f)$ arises. This is done as follows.

Definition 3. *Let f,g be two entire functions with index-pairs (m,q) and (m,p), respectively, where p,q,m are positive integers with $m \geq \max(p,q)$. Then the relative (p,q)-th order of f with respect to g is defined by*

$$\rho_g^{(p,q)}(f) = \limsup_{r\to\infty} \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r}.$$

And the relative (p,q)-th lower order of f with respect to g is defined by

$$\lambda_g^{(p,q)}(f) = \liminf_{r\to\infty} \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r}.$$

When $(m,1)$ and (m,k) are the index-pairs of f and g respectively, then Definition 3 reduces to definition of generalized relative order (Lahiri & Banerjee, 2002). If the entire functions f and g have the same index-pair $(p,1)$, we get the definition of relative order introduced by Bernal (Bernal, 1988) and if $g = \exp^{[m-1]}$, then $\rho_g(f) = \rho_f^{[m]}$ and $\rho_g^{(p,q)}(f) = \rho_f(m,q)$. Also Definition 3 becomes the classical one given in (Titchmarsh, 1968) if f is an entire function with index-pair $(2,1)$ and $g = \exp$.

In order to refine the above growth scale, now we intend to introduce the definition of an intermediate comparison function, called relative (p,q)th proximate order of entire function with respect to another entire function in the light of their index-pair which is as follows. Its consistency will be established in Section 3.

Definition 4. *Let f,g be two entire functions with index-pairs (m,q) and (m,p) respectively where p,q,m are positive integers with $m \geq \max(p,q)$. For a finite relative (p,q)-th order $\rho_g^{(p,q)}(f)$ of f with respect to g, then a function $\rho_g^{(p,q)}(f)(r) : \mathbb{R}^+ \to \mathbb{R}$ is said to be a relative (p,q)th proximate order of f with respect to g if there is some $r_0 > 0$ so that it satisfies:*

(i) $\rho_g^{(p,q)}(f)(r)$ is non-negative and continuous for $r > r_0$,

(ii) $\rho_g^{(p,q)}(f)(r)$ is differentiable for $r \geq r_0$ except possibly at isolated points where $\rho_g^{(p,q)\prime}(f)(r+0)$ and $\rho_g^{(p,q)\prime}(f)(r-0)$ exist,

(iii) $\lim_{r\to\infty} \rho_g^{(p,q)}(f)(r) = \rho_g^{(p,q)}(f)$,

(iv) $\lim_{r\to\infty} \rho_g^{(p,q)\prime}(r) \prod_{i=0}^{\max(p,q)} \log^{[i]} r = 0$,

(v) $\limsup_{r\to\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(r)}{\left[\log^{[q-1]} r\right]^{\rho_g^{(p,q)}(f)(r)}} = 1$.

When $(m, 1)$ and (m, k) are the index-pairs of f and g respectively, Definition 4 reduces to definition of generalized relative proximate order. If the entire functions f and g have the same index-pair $(p, 1)$, the above definition provides the relative proximate order $\rho_g(f)(r)$.

The relative (p, q)th lower proximate order of an entire function with respect to another entire function may analogously be defined, consistency being held by virtue of Section 3, too.

Definition 5. *Let f and g be any two entire functions with index-pairs (m, q) and (m, p) respectively where p, q, m are positive integers such that $m \geq \max(p, q)$. For a finite relative (p, q)-th lower order of f with respect to g, $\lambda_g^{(p,q)}(f)$, then a function $\lambda_g^{(p,q)}(f)(r) : \mathbb{R}^+ \to \mathbb{R}$ is said to be a relative (p, q)th lower proximate order of f with respect to g if there is some $r_0 > 0$ so that it satisfies:*

(i) $\lambda_g^{(p,q)}(f)(r)$ is non-negative and continuous for $r > r_0$,

(ii) $\lambda_g^{(p,q)}(f)(r)$ is differentiable for $r \geq r_0$ except possibly at isolated points at which $\lambda_g^{(p,q)\prime}(f)(r+0)$ and $\lambda_g^{(p,q)\prime}(f)(r-0)$ exist,

(iii) $\lim_{r\to\infty} \lambda_g^{(p,q)}(f)(r) = \lambda_g^{(p,q)}(f)$,

(iv) $\lim_{r\to\infty} \lambda_g^{(p,q)\prime}(r) \prod_{i=0}^{\max(p,q)} \log^{[j]} r = 0$,

(v) $\liminf_{r\to\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(r)}{\left[\log^{[q-1]} r\right]^{\lambda_g^{(p,q)}(f)(r)}} = 1$.

3. Main Results

In this section we state the main results of the paper. We include the proof of the first main Theorem 1 for the sake of completeness. The others are basically omitted since they are easily proved with the same techniques or with some easy reasonings.

Theorem 1. *Let f, g be any two entire functions with index-pairs (m, q) and (m, p) respectively where p, q, m are positive integers with $m \geq \max(p, q)$. If the relative (p, q)-th order $\rho_g^{(p,q)}(f)$ is finite, then the relative (p, q)th proximate order $\rho_g^{(p,q)}(f)(r)$ of f with respect to g exists.*

Proof. We distinguish the following two cases:
Case I. Assume $p \geq q$. Then we write

$$\sigma(r) = \frac{\log^{[p]} M_g^{-1} M_f(r)}{\log^{[q]} r}$$

and it can be easily proved that $\sigma(r)$ is continuous and

$$\limsup_{r\to\infty} \sigma(r) = \rho_g^{(p,q)}(f).$$

Now we consider the following three sub cases:
Sub Case A$_\mathbf{I}$. Let $\sigma(r) > \rho_g^{(p,q)}(f)$ for at least a sequence of values of r tending to infinity. Then we define the non increasing real function

$$\phi(r) = \max_{x\geq r}\{\sigma(x)\}.$$

Now let us take $R_1 > R$ with $R_1 > \exp^{[p+2]} 1$ and $\sigma(R) > \rho_g^{(p,q)}(f)$.
Then for any given $r \geq R_1$, we obtain that $\sigma(r) \leq \sigma(R)$. As $\sigma(r)$ is continuous, there exists $r_1 \in [R, R_1]$ such that

$$\sigma(r_1) = \max_{R \leq x \leq R_1} \{\sigma(x)\}.$$

Clearly $r_1 > \exp^{[p+2]} 1$ and $\phi(r_1) = \sigma(r_1)$, there being a sequence of such r_1 values tending to infinity.
Let us now consider that $\rho_g^{(p,q)}(f)(r_1) = \phi(r_1)$ and let t_1 be the smallest integer not smaller than $1 + r_1$ such that $\phi(r_1) > \phi(t_1)$. Also we define $\rho_g^{(p,q)}(f)(r) = \rho_g^{(p,q)}(f)(r_1)$ for $r_1 < r \leq t_1$. Now we observe that:

(i) $\phi(r)$ and $\rho_g^{(p,q)}(f)(r_1) - \log^{[p+2]} r + \log^{[p+2]} t_1$ are continuous functions,

(ii) $\rho_g^{(p,q)}(f)(r_1) - \log^{[p+2]} r + \log^{[p+2]} t_1 > \phi(t_1)$ for $r\,(> t_1)$ sufficiently close to t_1 and

(iii) $\phi(r)$ is non increasing.

Consequently we can define $u_1 > t_1$ as follows:

$$\rho_g^{(p,q)}(f)(r) = \rho_g^{(p,q)}(f)(r_1) - \log^{[p+2]} r + \log^{[p+2]} t_1 \text{ for } t_1 \leq r \leq u_1,$$

$$\rho_g^{(p,q)}(f)(r) = \phi(r) \text{ for } r = u_1 \text{ and}$$

$$\rho_g^{(p,q)}(f)(r) > \phi(r) \text{ for } t_1 \leq r < u_1.$$

Let now r_2 be the smallest value of r for which $r_2 \geq u_1$ and $\phi(r_2) = \sigma(r_2)$. If $r_2 > u_1$ then let $\rho_g^{(p,q)}(f)(r) = \phi(r)$ for $u_1 \leq r \leq r_2$. Then it can be easily shown that $\phi(r)$ and $\rho_g^{(p,q)}(f)(r)$ are both constant in $u_1 \leq r \leq r_2$. By repeating this process, we obtain that $\rho_g^{(p,q)}(f)(r)$ is differentiable in adjacent intervals.

Moreover $\rho_g^{(p,q)\prime}(r)$ coincides with 0 or $\left(\prod_{i=0}^{p+1} \log^{[i]} r\right)^{-1}$ and

$$\rho_g^{(p,q)}(f)(r) \geq \phi(r) \geq \sigma(r) \quad \text{for} \quad \text{all} \quad r \geq r_1.$$

Also $\rho_g^{(p,q)}(f)(r) = \sigma(r)$ for a sequence of values of r tending to infinity and $\rho_g^{(p,q)}(f)(r)$ is non increasing for $r \geq r_1$.

So

$$\begin{aligned} \rho_g^{(p,q)}(f) &= \limsup_{r\to\infty} \sigma(r) = \lim_{r\to\infty} \phi(r) \\ \text{i.e., } \limsup_{r\to\infty} \rho_g^{(p,q)}(f)(r) &= \liminf_{r\to\infty} \rho_g^{(p,q)}(f)(r) \\ &= \lim_{r\to\infty} \rho_g^{(p,q)}(f)(r) = \rho_g^{(p,q)}(f) \end{aligned}$$

and

$$\lim_{r\to\infty} \rho_g^{(p,q)\prime}(r) \prod_{i=0}^{p} \log^{[i]} r = 0.$$

Again we get that

$$\log^{[p-1]} M_g^{-1} M_f(r) = \left[\log^{[q-1]} r\right]^{\sigma(r)} = \left[\log^{[q-1]} r\right]^{\rho_g^{(p,q)}(f)(r)}$$

for a sequence of values of r tending to infinity and

$$\log^{[p-1]} M_g^{-1} M_f(r) < \left[\log^{[q-1]} r\right]^{\rho_g^{(p,q)}(f)(r)}$$

for the remaning r's. Hence

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(r)}{\left[\log^{[q-1]} r\right]^{\rho_g^{(p,q)}(f)(r)}} = 1.$$

The continuity of $\rho_g^{(p,q)}(f)(r)$ for $r \geq r_1$ follows by construction.
Sub Case $\mathbf{B_I}$. Let $\sigma(r) < \rho_g^{(p,q)}(f)$ for all sufficiently large values of r tending to infinity. Now we define the real function

$$\xi(r) = \max_{X \leq x \leq r} \{\sigma(x)\},$$

where $X > \exp^{[p+2]} 1$ is such that $\sigma(r) < \rho_g^{(p,q)}(f)$ whenever $x \geq X$.

Here we note that $\xi(r)$ is non decreasing and the roots of

$$\xi(x) = \rho_g^{(p,q)}(f) + \log^{[p+2]} x - \log^{[p+2]} r$$

are smaller than r for all sufficiently large values of $r \geq X$.

Now for a suitable large value $v_1 > X$, we define
$\rho_g^{(p,q)}(f)(v_1) = \rho_g^{(p,q)}(f)$,

$\rho_g^{(p,q)}(f)(r) = \rho_g^{(p,q)}(f) + \log^{[p+2]} r - \log^{[p+2]} v_1$ for $s_1 \leq r \leq v_1$ where $s_1 < v_1$ is such that $\xi(s_1) = \rho_g^{(p,q)}(f)(s_1)$.

In fact s_1 is given by the largest positive root of

$$\xi(x) = \rho_g^{(p,q)}(f) + \log^{[p+2]} x - \log^{[p+2]} v_1.$$

If $\xi(s_1) \neq \sigma(s_1)$ let ω_1 be an upper bound of the $\omega < s_1$ at which $\xi(\omega)$ is different from $\sigma(\omega)$. If we define $\rho_g^{(p,q)}(f)(r) = \xi(r)$ for $\omega_1 \leq r \leq s_1$, it is clear that $\xi(r)$ is constant in $[\omega_1, s_1]$, hence $\rho_g^{(p,q)}(f)(r)$ is constant in $[\omega_1, s_1]$, too.
If $\xi(s_1) = \sigma(s_1)$ we take $\omega_1 = s_1$. Now we choose $v_2 > v_1$ suitably large and let $\rho_g^{(p,q)}(f)(v_1) = \rho_g^{(p,q)}(f)$ and

$$\rho_g^{(p,q)}(f)(r) = \rho_g^{(p,q)}(f) + \log^{[p+2]} r - \log^{[p+2]} v_2$$

for $s_2 \leq r \leq v_2$ where $s_2 < v_2$ is such that $\xi(s_2) = \rho_g^{(p,q)}(f)(s_2)$.

If $\xi(s_2) \neq \rho_g^{(p,q)}(f)(s_2)$ then suppose that $\rho_g^{(p,q)}(f)(r) = \xi(r)$ for $\omega_2 \leq r \leq s_2$, with ω_2 mimicking the behavour of ω_1. Hence $\rho_g^{(p,q)}(f)(r)$ is constant in $[\omega_2, s_2]$.
If $\xi(s_2) = \sigma(s_2)$ we take $\omega_2 = s_2$.

Also suppose that $\rho_g^{(p,q)}(f)(r) = \rho_g^{(p,q)}(f)(\omega_2) - \log^{[p+2]} r + \log^{[p+2]} \omega_2$ for $q_1 \leq r \leq \omega_2$ where $q_1 < \omega_2$ is the point of intersection of $y = \rho_g^{(p,q)}(f)$ with $y = \rho_g^{(p,q)}(f)(\omega_2) - \log^{[p+2]} x + \log^{[p+2]} \omega_2$. Now it is also possible to choose v_2 so large that $v_1 < q_1$ and for the case under consideration, let us consider $\rho_g^{(p,q)}(f)(r) = \rho_g^{(p,q)}(f)$ for $v_1 \leq r \leq q_1$. Therefore if we repeat this process it can be shown that for all $r \geq v_1$, $\rho_g^{(p,q)}(f) \geq \rho_g^{(p,q)}(f)(r) \geq \xi(r) \geq \sigma(r)$ and $\rho_g^{(p,q)}(f)(r) = \sigma(r)$ for $r = \omega_1, \omega_2, ...$
Hence we obtain that

$$\limsup_{r\to\infty} \rho_g^{(p,q)}(f)(r) = \liminf_{r\to\infty} \rho_g^{(p,q)}(f)(r) = \lim_{r\to\infty} \rho_g^{(p,q)}(f)(r) = \rho_g^{(p,q)}(f)$$

since

$$\log^{[p-1]} M_g^{-1} M_f(r) = \left[\log^{[q-1]} r\right]^{\sigma(r)} = \left[\log^{[q-1]} r\right]^{\rho_g^{(p,q)}(f)(r)}$$

for a sequence of values of r tending to infinity and

$$\log^{[p-1]} M_g^{-1} M_f(r) < \left[\log^{[q-1]} r\right]^{\rho_g^{(p,q)}(f)(r)}$$

for remaning r's. Therefore it follows that

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(r)}{\left[\log^{[q-1]} r\right]^{\rho_g^{(p,q)}(f)(r)}} = 1.$$

Furthermore, $\rho_g^{(p,q)}(f)(r)$ is differentiable in adjacent intervals and

$$\rho_g^{(p,q)\prime}(r) = 0 \text{ or } \left(\prod_{i=0}^{p+1} \log^{[i]} r\right)^{-1}.$$

Consequently,

$$\lim_{r\to\infty} \rho_g^{(p,q)\prime}(r) \prod_{i=0}^{p} \log^{[i]} r = 0.$$

Once again, continuity of $\rho_g^{(p,q)}(f)(r)$ follows by construction.
Sub Case $\mathbf{C_I}$. Let $\sigma(r) = \rho_g^{(p,q)}(f)$ for at least a sequence of values of r tending to infinity. Now considering $\rho_g^{(p,q)}(f)(r) = \rho_g^{(p,q)}(f)$ for all sufficiently large values of r one can easily verify the existance of the relative (p,q)th proximate order for the case under consideration.

Case II. Assume $q \geq p$. Now let us consider the following function

$$\sigma(r) = \left[\log^{[q-1]} r\right]^{-\rho_g^{(p,q)}(f)} \cdot \log^{[p-1]} M_g^{-1} M_f(r).$$

Therefore it can easily be shown that

$$\limsup_{r\to\infty} \frac{\log\sigma(r)}{\log^{[q]} r} = 0.$$

Now putting $x = \log^{[q]} r$ and $y = \log\sigma(r)$, we obtain that

$$y = \log\sigma\left(\exp^{[q]} x\right).$$

So

$$\limsup_{r\to\infty} \frac{\log\sigma\left(\exp^{[q]} x\right)}{x} = \limsup_{r\to\infty} \frac{\log\sigma(r)}{\log^{[q]} r} = 0$$

which shows that for any abritrary $\varepsilon > 0$ and for large values of x, $x \geq x_0(\varepsilon)$, the entire curve $y = \log\sigma\left(\exp^{[q]} x\right)$ lies below the line $y = \varepsilon x$ and, on the other hand, there are points on the curve with arbitrarily large abscissae lying above the line $y = -\varepsilon x$.

Now we consider the following two sub cases:

Sub Case $\mathbf{A_{II}}$. Let us consider that $\limsup_{r\to\infty} \log\sigma\left(\exp^{[q]} x\right) = +\infty$. Now we construct the smallest convex domain so that it contains the positive ray of the x axis and all the points of the curve $y = \log\sigma\left(\exp^{[q]} x\right)$. Thus the boundary of newly formed domain lying above the x-axis is a continuous curve and we denote it as $y = \delta(x)$. This curve must satisfy the following properties:

(I) The curve is convex from the above,

(II) $\lim_{x\to\infty} \frac{\delta(x)}{x} = 0$,

(III) $\log\sigma\left(\exp^{[q]} x\right) \leq \delta(x)$,

(IV) $\log\sigma\left(\exp^{[q]} x\right) = \delta(x)$ at the extreme points of the curve $y = \delta(x)$ and

(V) The curve $y = \delta(x)$ contains a sequence of extreme points tending to infinity.

Also the curve $y = \delta(x)$ is made differentiable in the neighbourhood of each angular point (if necessary) by making some unessential changes. Thus it is assumed that the curve $y = \delta(x)$ is differentiable everywhere. Hence from (I) and (II), above it follows that $\lim_{x\to\infty} \delta'(x) = 0$ and from (III) we have

$$\log^{[p-1]} M_g^{-1} M_f(r) \leq \left[\log^{[q-1]} r\right]^{\rho_g^{(p,q)}(f)(r)}$$

where

$$\rho_g^{(p,q)}(f)(r) = \rho_g^{(p,q)}(f) + \frac{\delta\left(\log^{[q]} r\right)}{\log^{[q]} r}.$$

Now from (II) it follows that

$$\lim_{r\to\infty} \rho_g^{(p,q)}(f)(r) = \lim_{r\to\infty}\left(\rho_g^{(p,q)}(f) + \frac{\delta\left(\log^{[q]} r\right)}{\log^{[q]} r}\right) = \rho_g^{(p,q)}(f) .$$

Also in view of the properties (IV) and (V) one can easily verify that there exists a sequence of values of r tending to infinity for which

$$\begin{aligned} \log^{[p-1]} M_g^{-1} M_f(r) &= \left[\log^{[q-1]} r\right]^{\rho_g^{(p,q)}(f)(r)} \\ i.e., \ \limsup_{r\to\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(r)}{\left[\log^{[q-1]} r\right]^{\rho_g^{(p,q)}(f)(r)}} &= 1 \end{aligned}$$

and $\lim_{r\to\infty} \rho_g^{(p,q)\prime}(r) \prod_{i=0}^{q} \log^{[i]} r = 0$ holds.

Thus we have constructed the function $\rho_g^{(p,q)}(f)(r)$.

Sub Case $\mathbf{B_{II}}$. In order to generalize the case, let us consider a concave function $\beta(x)$ which satisfies the following properties:

(I) $\lim_{x\to\infty} \beta'(x) = 0$,

(II) $\lim_{x\to\infty} \frac{\beta(x)}{x} = 0$ and

(III) $\limsup_{r\to\infty} \left[\log \sigma\left(\exp^{[q]} x\right) + \beta(x)\right] = \infty$.

With the goal of constructing $\beta(x)$ we go through the following steps:

First we consider a segment a_1 of the line $y = -\varepsilon_1 x$ from the origin to a point x_1 where $\log \sigma\left(\exp^{[q]} x_1\right) > -\varepsilon_1 x_1 + 1$. Having chosen a positive number $\varepsilon_2 < \varepsilon_1$ we draw a segment a_2 of the line $y + \varepsilon_1 x_1 = -\varepsilon_2(x - x_1)$ from the point $(x_1, -\varepsilon_1 x_1)$ to a point $x_2 > x_1$ satisfying $\log \sigma\left(\exp^{[q]} x_2\right) > -\varepsilon_1 x_1 - \varepsilon_2(x_2 - x_1) + 2$. Then we choose a segment a_3 with slope $-\varepsilon_3$ $(0 < \varepsilon_3 < \varepsilon_2)$, etc. The selected $\{\varepsilon_n\}$ is strictly decreasing with $\varepsilon_n \to 0$ but the sequence $\{x_n\}$ of points is strictly increasing with $x_n \to \infty$. The polygonal function $y = \beta_1(x)$ constructed in this manner satisfies

$$\lim_{x\to\infty} \frac{\beta_1(x)}{x} = 0.$$

The function $\beta_1(x)$ can be made everywhere differentiable by changing it in an unessential manner in the neighbourhood of each angular point. The function $\beta(x)$ defined as $\beta(x) = -\beta_1(x)$ has the required properties.

A convex majorant $\beta_2(x)$ for the function $\log \sigma\left(\exp^{[q]} x\right) + \beta(x)$ is now considered and writing

$$\delta(x) = \beta_2(x) - \beta(x)$$

yields

$$\log \sigma\left(\exp^{[q]} x\right) \le \delta(x).$$

Moreover,

$$\log \sigma\left(\exp^{[q]} x_n'\right) = \delta(x_n')$$

on some sequence $\{x_n'\}_1^\infty$ of extreme points, $x_n' \to \infty$.

Also if the function $\rho_g^{(p,q)}(f)(r)$ is defined as

$$\rho_g^{(p,q)}(f)(r) = \rho_g^{(p,q)}(f) + \frac{\delta\left(\log^{[q]} r\right)}{\log^{[q]} r},$$

it can easily be seen that

$$\lim_{x\to\infty} \delta'(x) = 0, \quad \lim_{x\to\infty} \frac{\delta(x)}{x} = 0.$$

Hence $\lim_{r\to\infty} \rho_g^{(p,q)}(f)(r) = \rho_g^{(p,q)}(f)$ and

$$\lim_{r\to\infty} \rho_g^{(p,q)\prime}(r) \prod_{i=0}^{q} \log^{[i]} r = 0.$$

Moreover

$$\log^{[p-1]} M_g^{-1} M_f(r) \le \left[\log^{[q-1]} r\right]^{\rho_g^{(p,q)}(f)(r)}$$

and

$$\log^{[p-1]} M_g^{-1} M_f(r_n) = \left[\log^{[q-1]} r_n\right]^{\rho_g^{(p,q)}(f)(r_n)}$$

for some sequence $\{r_n\}$, $r_n \to \infty$. Therefore

$$\limsup_{r\to\infty} \frac{\log^{[p-1]} M_g^{-1} M_f(r)}{\left[\log^{[q-1]} r\right]^{\rho_g^{(p,q)}(f)(r)}} = 1,$$

and the proof is complete. □

The following theorem's proof can be obtained in the line of Theorem 1.

Theorem 2. *Let f, g be any two entire functions with index-pairs (m, q) and (m, p), respectively where p, q, m are positive integers with $m \ge \max(p, q)$. If the relative (p, q)-th lower order $\lambda_g^{(p,q)}(f)$ of f with respect to g is finite and non zero, then the relative (p, q)th lower proximate order $\lambda_g^{(p,q)}(f)(r)$ of f with respect to g exists.*

Now we recall the that a positive function $\eta(r)$ is called slowly increasing (Srivastava & Kumar, 2009), if $\lim_{r\to\infty} \frac{\eta(nr)}{\eta(r)} = 1$. We will say that $\eta(r)$ is uniform slowly increasing if the aforementioned limit happens to exist uniformly in m on each interval $0 < b \le n < m < \infty$.

The proofs of the following corollary can be carried out using the same techniques involved in (Nandan, Doherey, & Srivastava, 1980).

Corollary 1. *Let $\rho_g^{(p,q)}(f)(r)$ and $\lambda_g^{(p,q)}(f)(r)$ be respectively the relative (p, q)th proximate order and the relative (p, q)th lower proximate order of f with respect to g, and let $\rho_g^{(p,q)}(f)$ and $\lambda_g^{(p,q)}(f)$ be the relative (p, q)-th order and relative (p, q)-th lower order of f with respect to g for any positive integers p and q. Then:*

1. The functions $\frac{\left[\log^{[q-1]} r\right]^{\rho_g^{(p,q)}(f)(r)}}{\left[\log^{[q-1]} r\right]^{\rho_g^{(p,q)}(f)}}$ and $\frac{\left[\log^{[q-1]} r\right]^{\lambda_g^{(p,q)}(f)(r)}}{\left[\log^{[q-1]} r\right]^{\lambda_g^{(p,q)}(f)}}$ are uniform slowly increasing.
2. The functions $\left[\log^{[q-1]} r\right]^{\rho_g^{(p,q)}(f)(r)}$ and $\left[\log^{[q-1]} r\right]^{\lambda_g^{(p,q)}(f)(r)}$ are monotone increasing for sufficiently large values of r.
3. For $0 < l \le k \le m < \infty$ and $r \to \infty$, we have that

$$\frac{\left[\log^{[q-1]}(kr)\right]^{\rho_g^{(p,q)}(f)(kr)} \cdot \left[\log^{[q-1]} r\right]^{\rho_g^{(p,q)}(f)}}{\left[\log^{[q-1]}(kr)\right]^{\rho_g^{(p,q)}(f)} \cdot \left[\log^{[q-1]} r\right]^{\rho_g^{(p,q)}(f)(r)}} \propto 1,$$

$$\frac{\left[\log^{[q-1]}(kr)\right]^{\lambda_g^{(p,q)}(f)(kr)} \cdot \left[\log^{[q-1]} r\right]^{\lambda_g^{(p,q)}(f)}}{\left[\log^{[q-1]}(kr)\right]^{\lambda_g^{(p,q)}(f)} \cdot \left[\log^{[q-1]} r\right]^{\lambda_g^{(p,q)}(f)(r)}} \propto 1$$

hold uniformly in k.

4. For $\gamma < \min\left\{\left(1+\rho_g^{(p,q)}(f)\right), \left(1+\lambda_g^{(p,q)}(f)\right)\right\}$, we have

$$\int_{r_0}^{r}\left[\log^{[q-1]}x\right]^{\rho_g^{(p,q)}(f)(x)-\gamma}\frac{dx}{\prod_{i=0}^{q-2}\log^{[i]}x} = \frac{1}{\rho_g^{(p,q)}(f)-\gamma+1}\left[\log^{[q-1]}r\right]^{\rho_g^{(p,q)}(f)(r)-\gamma+1}+o\left[\log^{[q-1]}r\right]^{\rho_g^{(p,q)}(f)(r)-\gamma+1}$$

and

$$\int_{r_0}^{r}\left[\log^{[q-1]}x\right]^{\lambda_g^{(p,q)}(f)(x)-\gamma}\frac{dx}{\prod_{i=0}^{q-2}\log^{[i]}x} = \frac{1}{\lambda_g^{(p,q)}(f)-\gamma+1}\left[\log^{[q-1]}r\right]^{\lambda_g^{(p,q)}(f)(r)-\gamma+1}+o\left[\log^{[q-1]}r\right]^{\lambda_g^{(p,q)}(f)(r)-\gamma+1}.$$

4. Conclusions

The main aim of the paper is to extend and modify the notion of proximate order (lower proximate order) to relative proximate order (relative lower proximate order) of higher dimentions in case of entire functions.

The results of this paper, in connection with Nevanlinna's Value Distributibution theory of entire functions on the basis of relative (p,q)th proximate order and relative (p,q)th proximate lower order, may have a wide range of applications in Complex Dynamics, Factorization Theory of entire functions of single complex variable, the solution of complex differential equations etc. In fact, Complex Dynamics is a thrust area in modern function theory and it is solely based on the study of fixed points of entire functions as well as the normality of them. Factorization theory of entire functions is another branch of applications of Nevanlinna's theory which deals on how a given entire function can be factorized into simpler entire functions as well as in the study of the properties of the solutions of complex differential equations.

Competing Interests Section

The authors declare that they have no competing interests.

Acknowledgements

Supported by Spanish Ministerio de Economía y Competitividad. Secretaría General de Ciencia y Tecnología e Innovación ESP2013-41078-R.

References

Albuquerque, N., Bernal-González, L., Pellegrino, D., & Seoane-Sepúlveda, J. B. (2014). Peano curves on topological vector spaces. *Linear Algebra Appl., 460*, 81-96. http://dx.doi.org/10.1016/j.laa.2014.07.029

Bernal, L. (1988). Orden relativo de crecimiento de funciones enteras. *Collect. Math., 39*, 209-229.

Boas, R. P. (1954). *Entire functions*. Academic Press, New York.

Chakraborty, B. C., & Roy, C. (2006). Relative order of an entire function. *J. Pure Math., 23*, 151-158.

Datta, S. K., & Biswas, T. (2009). Growth of entire functions based on relative order. *Int. J. Pure Appl. Math.,51*(1), 49-58.

Datta, S. K., & Biswas, T. (2010). Relative order of composite entire functions and some related growth properties. *Bull. Cal. Math. Soc., 102*(3), 259-266.

Datta, S. K., Biswas, T., & Biswas, R. (2013). On relative order based growth estimates of entire functions. *International J. of Math. Sci. & Engg. Appls. (IJMSEA), 7*(II), 59-67.

Datta, S. K., Biswas, T., & Pramanick, D. C. (2012). On relative order and maximum term-related comarative growth rates of entire functions. *Journal Tri. Math. Soc., 14*, 60-68.

Datta, S. K., Biswas, T., & Biswas, R. (2013). Comparative growth properties of composite entire functions in the light of their relative order. *The Mathematics Student, 82*(1-4), 1-8.

Juneja, O. P., Kapoor, G. P., & Bajpai, S. K. (1976). On the (p,q)-order and lower (p,q)-order of an entire function. *J. Reine Angew. Math., 282*, 53-67.

Lahiri, I. (1989). Generalised proximate order of meromorphic functions. *Mat. Vesnik, 41*, 9-16.

Lahiri, B. K., & Banerjee, D. (2002). Generalised relative order of entire functions. *Proc. Nat. Acad. Sci., 72*(A)(IV), 351-371.

Lahiri, B. K., & Banerjee, D. (2005). A note on relative order of entire functions. *Bull. Cal. Math. Soc., 97*(3), 201-206.

Lahiri, B. K., & Banerjee, D. (2005). Entire functions of relative order (p, q). *Soochow Journal of Mathematics, 31*(4), 497-513.

Nandan, K., Doherey, R. P., & Srivastava, R. S. L. (1980). Proximate order of an entire function with index pair (p, q). *Indian J. Pure Appl. Math., 11*(1), 33-39.

Roy, C. (2010). On the relative order and lower relative order of an entire function. *Bull. Cal. Math. Soc., 102*(1), 17-26.

Sánchez Ruiz, L. M., Datta, S. K., Biswas, T., & Mondal, G. K. (2014). On the (p, q)-th Relative Order Oriented Growth Properties of Entire Functions. *Abstract and Applied Analysis*, Article ID 826137, 8 pages. http://dx.doi.org/10.1155/2014/826137

Sato, D. (1963). On the rate of growth of entire functions of fast growth. *Bull. Amer. Math. Soc., 69*, 411-414. http://dx.doi.org/10.1090/S0002-9904-1963-10951-9

Shah, S. M. (1946). On proximate orders of integral functions, *Bull. Amer. Math. Soc.*, 52, 326-328. http://dx.doi.org/10.1090/S0002-9904-1946-08572-9

Srivastava, G. S., Kumar, S. (2009). Approximation of entire functions of slow growth on compact sets, *Archivum Mathematicum*, 45(2), 137-146.

Titchmarsh, E. C. (1968). *The theory of functions* (2nd ed.). Oxford University Press, Oxford.

Valiron, G. (1949). *Lectures on the General Theory of Integral Functions*. Chelsea Publishing Company, N.Y.

Copyrights