Applied General Topology
© Universidad Politécnica de Valencia
Volume 9, No. 1, 2008
pp. 21-32

Exponentiality for the construct of affine sets

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Abstract. The topological construct **SSET** of affine sets over the two-point set S contains many interesting topological subconstructs such as **TOP**, the construct of topological spaces, and **CL**, the construct of closure spaces. For this category and its subconstructs cartesian closedness is studied. We first give a classification of the subconstructs of **SSET** according to their behaviour with respect to exponentiality. We formulate sufficient conditions implying that a subconstruct behaves similar to **CL**. On the other hand, we characterize a conglomerate of subconstructs with behaviour similar to **TOP**. Finally, we construct the cartesian closed topological hull of **SSET**.

2000 AMS Classification: 54A05, 54C35, 18D15

Keywords: topological construct, affine space, cartesian closed category, cartesian closed topological hull, exponential object

1. Introduction

The lack of natural function spaces in a topological construct that is not cartesian closed, has long been recognized as an akward situation for various applications in homotopy theory and topological algebra and for use in infinite dimensional differential calculus. For references to the original sources one might consult [22, 11, 12, 18, 21]. Topological constructs like **TOP** or the larger construct **CL** and several others that are commonly used by topologists, however are not cartesian closed. For the construct **TOP** of topological spaces this problem is extensively studied in the literature, see for example [17, 8, 4, 5, 19]. For the subconstruct **CL** of closure spaces, the author studied cartesian closedness in [6]. To remedy these facts, topologists have applied various methods. Either they dealt with the local problem, the description of exponential objects and with the construction of cartesian closed subconstructs, or they looked for larger cartesian closed constructs. A topological construct is cartesian closed if and only if every object X is exponential in the sense that the functor $X \times -$ preserves coproducts and quotients. The reason for **TOP** not being cartesian closed is that $X\times -$ does not always preserve quotients, except for corecompact X. For **CL** it is just the other way around, $X \times -$ generally does not preserve coproducts, except for X being indiscrete.

TOP as well as **CL** are fully embedded in the construct **SSET** of affine spaces and affine maps which is a host for many other subconstructs that are important to topologists [13, 14] (see next section for the exact definitions). In this paper we investigate the problem of cartesian closedness for **SSET** and we describe the exponential objects and deduce results for its subconstructs. We prove that for a non-indiscrete affine space X, the functor $X \times -$ does not preserve coproducts in **SSET** and we describe a conglomerate of subconstructs of **SSET** (to which **CL** belongs), in which this negative result goes through. On the other hand, we determine a large subconstruct of **SSET** in all topological subconstructs of which (like for instance **TOP**) the functor $X\times -$ does preserve coproducts. In the final section of the paper we describe the cartesian closed topological hull of **SSET**. Remark that, as was observed by E. Giuli [13], our definition of affine spaces and maps, as we recall it in the next section, only differs slightly from the normal Boolean Chu spaces and continuous maps, as introduced by V. Pratt to model concurrent computation. Objects in **SSET** have a structure containing constants. This assumption makes **SSET** into a well-fibred topological construct in the sense of [1], which has the property that cartesian closedness is equivalent to the existence of "nice" function spaces.

2. Exponential objects in SSET and in its subconstructs.

An affine space X over the two point set $S = \{0, 1\}$ is a structured set, where the structure on the underlying set X is a collection of subsets of X. The sets belonging to the structure are called the "open" sets of X. An affine map from $\mathsf{X} \longrightarrow \mathsf{Y}$ is a function f such that inverse images of open sets are open. An affine space can be isomorphically described in a functional way: An affine space (over S) $(X, \mathcal{A})$ consists of a set X and a subset $\mathcal{A}$ of the powerset S^X. An affine map $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ is a function f such that $\beta \circ f \in \mathcal{A}$ for all $\beta \in \mathcal{B}$. In this paper we will use the functional description. We will restrict ourselves to the affine spaces whose affine structure contains the constant functions **0** and **1**. As in [13], the corresponding construct of affine spaces and affine maps will be denoted by **SSET**. In that paper it was proved that **SSET** is a well-fibred topological construct.

An object X in a category with finite products is *exponential* if the functor $X\times -$ has a right adjoint. In a well-fibred topological construct $\mathbf{X}$, this notion can be characterized as follows: X is exponential in $\mathbf{X}$ iff for each $\mathbf{X}$-object Y the set $\mathrm{Hom}_{\mathbf{X}}(X, Y)$ can be supplied with the structure of a $\mathbf{X}$-object - a function space or a power object Y^X- such that

(1) the evaluation map ev: $X \times Y^X \to Y$ is a $\mathbf{X}$-morphism
(2) for each $\mathbf{X}$-object Z and each $\mathbf{X}$-morphism $f : X \times Z \to Y$, the map $f^* : Z \to Y^X$ defined by $f^*(z)(x) = f(x, z)$ is a $\mathbf{X}$-morphism.

It is well known that in the setting of a topological construct **X**, an object X is exponential in **X** iff $X \times -$ preserves final episinks [15], [16]. Moreover, small fibredness of **X** ensures that this is equivalent to the condition that $X \times -$ preserves quotients and coproducts. A well-fibred topological construct **X** is said to be *cartesian closed* (or to have function spaces) if every object is exponential.

Before characterizing the exponential objects in the subcategories of **SSET**, we first prove some useful results for subconstructs of **SSET**.

We first recall the following result from [20].

Proposition 2.1. [20] *Every topological subcategory* **X** *of* **SSET** *is a bicoreflective subcategory of some full bireflective subcategory* **Y** *of* **SSET**.

The following propositions can be proved using techniques similar to those developed for **TOP** in [17].

Proposition 2.2. *Every topological subcategory of* **SSET** *is closed under the formation of retracts in* **SSET**.

Proposition 2.3. *Every non-trivial topological subcategory* **X** *of* **SSET** *contains all complemented topological spaces.*

In order to investigate the interaction of products and coproducts in **SSET** and in its subconstructs, we first look at the construction of coproducts in **SSET** and in its subconstructs.

Let $(X_i, \mathcal{A}_i)_{i\in I}$ be a family of affine sets, then one can easily verify that the coproduct in **SSET** of these objects is given by $(\amalg X_i, \mathcal{A} = \{\underset{i\in I}{\sqcup}\alpha_i : \alpha_i \in \mathcal{A}_i\})$ with

$$\underset{i\in I}{\sqcup}\alpha_i : \underset{i\in I}{\amalg} X_i \to S : (x_i, i) \to \alpha_i(x_i)$$

Proposition 2.4. *Let* **X** *be a non-trivial topological subconstruct of* **SSET** *and* $(X_i, \mathcal{A}_i)_{i\in I}$ *a family of* **X***-objects. For every* $i \in I$ *and every* $\alpha_i \in \mathcal{A}_i$, *the function* $\underset{j\in I}{\sqcup}\beta_j$ *belongs to the affine structure* $\mathcal{A}_{\amalg X_i}$ *on the coproduct* $\underset{i\in I}{\amalg} X_i$ *whenever* $\beta_i = \alpha_i$ *and* $\beta_j = \mathbf{0}$ *for all* $j \neq i$ *or* $\beta_j = \mathbf{1}$ *for all* $j \neq i$.

Proof. Let **Y** be the bireflective subcategory of **SSET** such as in proposition 2.1 and let $(X_i, \mathcal{A}_i)_{i\in I}$ be a family of **X**-objects. For $r = (r_i)_{i\in I} \in \underset{i\in I}{\Pi} X_i$, we define the function

$$f_r : \underset{i\in I}{\amalg} X_i \to \underset{i\in I}{\Pi} X_i \times I : (x_i, i) \to ((y_j)_{j\in I}, i)$$

with $y_i = x_i$ and $y_j = r_j$ for $j \neq i$. Let $\mathcal{A}$ be the initial affine structure for the source $(f_r : \underset{i\in I}{\amalg} X_i \to \underset{i\in I}{\Pi}(X_i, \mathcal{A}_i) \times (I, S^I))_{r\in \underset{i\in I}{\Pi} X_i}$. Then, $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ with

$$\begin{aligned}\mathcal{A}_1 &= \{1_J \circ pr_I \circ f_r \,|\, J \subset I, r \in \coprod_{i\in I} X_i\} \\ &= \bigcup_{J\subset I}\{\bigsqcup_{i\in I}\beta_i \,|\, \beta_i = \mathbf{1} \text{ if } i\in J \text{ and } \beta_i = \mathbf{0} \text{ if } i\notin J\} \\ \mathcal{A}_2 &= \{\alpha_i \circ pr_{X_i} \circ f_r \,|\, i\in I, \alpha_i \in \mathcal{A}_i, r\in \coprod_{i\in I} X_i\} \\ &= \bigcup_{i\in I}\{\bigsqcup_{j\in I}\beta_j \,|\, \beta_i = \alpha_i \in \mathcal{A}_i \text{ and } \beta_j = \mathbf{1} \text{ if } j\neq i\} \cup \\ &\quad \bigcup_{i\in I}\{\bigsqcup_{j\in I}\beta_j \,|\, \beta_i = \alpha_i \in \mathcal{A}_i \text{ and } \beta_j = \mathbf{0} \text{ if } j\neq i\}\end{aligned}$$

From the previous proposition follows that the categories **X** and **Y** contain the discrete affine sets, and in particular (I, S^I). Since **Y** is closed under the formation of initial structures in **SSET**, it follows that $(\coprod_{i\in I} X_i, \mathcal{A})$ belongs to **Y**. For all $i \in I$, the map $j_i : (X_i, \mathcal{A}_i) \to (\coprod_{i\in I} X_i, \mathcal{A})$ is affine. Hence, $(\coprod_{i\in I} X_i, \mathcal{A})$ is coarser than the coproduct $(\coprod_{i\in I} X_i, \mathcal{A}_{\coprod_{\mathbf{Y}} X_i})$ in the category **Y**. Since **X** is a bicoreflective subcategory of **Y**, this implies that $\mathcal{A} \subset \mathcal{A}_{\coprod_{\mathbf{X}} X_i} = \mathcal{A}_{\coprod_{\mathbf{Y}} X_i}$. □

Hence, every non-trivial subcategory of **SSET** for which the affine structures are closed under arbitrary suprema is closed under the formation of coproducts in **SSET**.

We will now recall a general method to construct hereditary bicoreflective subcategories of **SSET** [9], [10] ,[13].
In order to define a subconstruct of **SSET** we put an algebra structure on $S = \{0, 1\}$. Recall that an *algebra structure* on the set S is a class of operations

$$\Omega = \{\omega_i : S^{n_i} \to S \,|\, i \in I\}$$

of arbitrary arities. Hence the n_i are arbitrary cardinal numbers, and there is no condition on the size of the indexing system I. We assume that Ω contains the constant operations. For every set X, by point-wise extension, the powerset S^X carries an algebra structure. We denote by **SSET**(Ω) the subconstruct of **SSET** consisting of those affine sets $(X, \mathcal{A})$ for which $\mathcal{A}$ is a Ω-subalgebra of the function algebra S^X. The objects in **SSET**(Ω) are called *affine sets over the algebra* (S, Ω). For example the category **CL** of closure spaces and the category **TOP** of topological spaces can be obtained this way. The construct obtained this way, by considering for Ω the class containing the constant operations and the complementation $^- : S \to S$ defined by $^-(0) = 1, ^-(1) = 0$ will be denoted by **SSET**$(\mathcal{C})$.

Lemma 2.5. **SSET**(Ω) *is a subcategory of* **SSET**$(\mathcal{C})$ *whenever* Ω *contains an operation* $\omega_T : S^T \to S$ *that satisfies the following condition:*
$\exists\, (x_t)_{t\in T}, (y_t)_{t\in T} \in S^T$ *such that* $\omega_T((x_t)_{t\in T}) = 0$, $\omega_T((y_t)_{t\in T}) = 1$ *and* $x_t = 0$ *implies* $y_t = 0$ *for all* $t \in T$.

Proof. Let $(X, \mathcal{A})$ be an $\mathbf{SSET}(\Omega)$-object. For $\alpha \in \mathcal{A}$, define the serie functions $(f_t : X \to S)_{t\in T}$ as follows.

$$f_t = \begin{cases} \mathbf{0} & \text{if } x_t = 0 \\ \mathbf{1} & \text{if } y_t = 1 \\ \alpha & \text{if } x_t = 1, y_t = 0 \end{cases}$$

Since $\mathcal{A}$ contains all constant functions, we have that $f_t \in \mathcal{A}$ for all $t \in T$. Then, $\mathcal{A}$ contains the function $\omega_T \circ \prod_{t\in T} f_t : X \to S : x \to \omega_T(f_t(x))_{t\in T}$.
If $\alpha(x) = 1$, then $\omega_T \circ \prod_{t\in T} f_t(x) = \omega_T((x_t)_{t\in T}) = 0 = {}^{-}\circ\, \alpha(x)$.
If $\alpha(x) = 0$, then $\omega_T \circ \prod_{t\in T} f_t(x) = \omega_T((y_t)_{t\in T}) = 1 = {}^{-}\circ\, \alpha(x)$. We can conclude that ${}^{-}\circ\, \alpha = \omega_T \circ \prod_{t\in T} f_t \in \mathcal{A}$ and thus $(X, \mathcal{A})$ is an $\mathbf{SSET}(\mathcal{C})$-object. □

Lemma 2.6. *If* $\mathbf{X}$ *is a non-trivial topological subcategory of* $\mathbf{SSET}$ *and* D_2 *is the two point discrete space, then for every non-indiscrete object* $(X, \mathcal{A})$ *the following holds:*

$$(X, \mathcal{A}) \times \mathsf{D}_2 \in \mathbf{X} \;\Rightarrow\; (X, \mathcal{A}) \text{ is not an exponential object in } \mathbf{X}$$

Proof. For a non-constant function $\alpha \in \mathcal{A}$, $\mathbf{0} \sqcup \alpha$ is an element of $\mathcal{A}_{X\sqcup X}$, while it is not contained in $\mathcal{A}_{X\times D_2}$. □

The previous negative result has important consequences with respect to exponential objects in **SSET** and to cartesian closedness of topological subconstructs.

Corollary 2.7. *If* $\mathbf{X}$ *is a topological subconstruct of* $\mathbf{SSET}$ *which is finitely productive in* $\mathbf{SSET}$*, then the class of exponential objects in* $\mathbf{X}$ *coincides with the class of indiscrete spaces.*

We now characterize a conglomerate of subconstructs of **SSET**, which are not finitely productive in **SSET**, in which the class of exponential objects also coincides with the class of indiscrete objects.

Proposition 2.8. *For every category* $\mathbf{SSET}(\Omega)$ *that is not a subcategory of* $\mathbf{SSET}(\min : S^2 \to S, \max : S^2 \to S)$ *the exponential objects are exactly the indiscrete affine sets.*

Proof. Suppose that $\mathbf{SSET}(\Omega)$ has a non-indiscrete exponential object $(X, \mathcal{A})$. Then we have that $(X, \mathcal{A}) \sqcup (X, \mathcal{A})$ is isomorphic to $(X, \mathcal{A}) \times \mathsf{D}_2$. By proposition 2.4, it follows that $\mathbf{0} \sqcup \alpha$ belongs to $\mathcal{A}_{X\times \mathsf{D}_2}$ for every $\alpha \in \mathcal{A}$. The product $\mathcal{A}_{X\times \mathsf{D}_2}$ is the smallest Ω-subalgebra of $S^{X\times \mathsf{D}_2}$ containing $\mathcal{B} = \{\, \alpha \circ pr_X \mid \alpha \in \mathcal{A}\} \cup \{\, pr_{\mathsf{D}_2}, \overline{pr_{\mathsf{D}_2}} : (x, a) \to \overline{a}\}$.
Hence, $\mathbf{0} \sqcup \alpha = \omega_\alpha((\gamma \circ pr_X)_{\gamma\in\mathcal{A}}, pr_{\mathsf{D}_2}, \overline{pr_{\mathsf{D}_2}})$ with $\omega_\alpha : S^{\mathcal{A}\cup S} \to S$ a composition of algebraic operations of Ω. For a non-constant function $\alpha \in \mathcal{A}$, there exists $x, y \in X$ such that $\alpha(x) = 1$ and $\alpha(y) = 0$. Define $(f_\gamma : S \times S \to S)_{\gamma\in\mathcal{A}}$ as follows:

- If $\gamma(x) = \gamma(y)$, put f_γ the constant function with value $\gamma(x)$.
- If $\gamma(x) = 1$ and $\gamma(y) = 0$, put $f_\gamma = pr_1 : S \times S \to S : (a, b) \to a$

- If $\gamma(x) = 0$ and $\gamma(y) = 1$, put $f_\gamma = ^- \circ pr_1 : S \times S \to S : (a, b) \to \overline{a}$

Then, for $b \in S$ we have:

- $\omega_\alpha \circ (\prod_{\gamma \in \mathcal{A}} f_\gamma \times pr_2 \times \overline{pr_2})(0, b) = \omega_\alpha((\gamma(y))_{\gamma\in\mathcal{A}}, b, \bar{b}) =$
  $\omega_\alpha((\gamma \circ pr_X)_{\gamma\in\mathcal{A}}, pr_{\mathsf{D}_2}, \overline{pr_{\mathsf{D}_2}})(y, b) = \mathbf{0} \sqcup \alpha\,(y, b) = 0 = \min(0, b)$.
- $\omega_\alpha \circ (\prod_{\gamma \in \mathcal{A}} f_\gamma \times pr_2 \times \overline{pr_2})(1, b) = \omega_\alpha((\gamma(x))_{\gamma\in\mathcal{A}}, b, \bar{b}) =$
  $\omega_\alpha((\gamma \circ pr_X)_{\gamma\in\mathcal{A}}, pr_{\mathsf{D}_2}, \overline{pr_{\mathsf{D}_2}})(x, b) = \mathbf{0} \sqcup \alpha\,(x, b) = b = \min(1, b)$.

This implies that $\min(a, b) = \omega_\alpha \circ (\prod_{\gamma \in \mathcal{A}} f_\gamma \times pr_2 \times \overline{pr_2})(a, b)$, which means that the operation $\min : S \times S \to S$ can be written in terms of the operation ω_α, the complementation $^-$ and the constant functions. If $\mathbf{SSET}(\Omega)$ is a subcategory of $\mathbf{SSET}(\mathcal{C})$, it now follows that $\mathbf{SSET}(\Omega)$ is a subcategory of $\mathbf{SSET}(\min : S^2 \to S)$.
For the categories $\mathbf{SSET}(\Omega)$ that are not embedded in $\mathbf{SSET}(\mathcal{C})$, it follows from lemma 2.5 that:

(1) $\omega_\alpha((\gamma(x))_{\gamma\in\mathcal{A}}, 0, 0) = 0$,
because $\omega_\alpha((\gamma(x))_{\gamma\in\mathcal{A}}, 0, 1) = \mathbf{0} \sqcup \alpha(x, 0) = 0$
(2) $\omega_\alpha((0)_{\gamma\in\mathcal{A}}, 1, 0) = 0$,
because $\omega_\alpha((\gamma(y))_{\gamma\in\mathcal{A}}, 1, 0) = \mathbf{0} \sqcup \alpha(y, 1) = 0$
(3) $\omega_\alpha((0)_{\gamma\in\mathcal{A}}, 0, 0) = 0$

By defining $f_\gamma = pr_1 : S \times S \to S : (a, b) \to a$ if $\gamma(x) = 1$ and otherwise $f_\gamma = \mathbf{0}$, we have that $\omega_\alpha \circ (\prod_{\gamma \in \mathcal{A}} f_\gamma \times pr_2 \times \mathbf{0})(a, b) = \min(a, b)$. Indeed, for $b \in S$ we have

- $\omega_\alpha \circ (\prod_{\gamma \in \mathcal{A}} f_\gamma \times pr_2 \times \mathbf{0})(0, b) = \omega_\alpha((0)_{\gamma\in\mathcal{A}}, b, 0) = 0$
- $\omega_\alpha \circ (\prod_{\gamma \in \mathcal{A}} f_\gamma \times pr_2 \times \mathbf{0})(1, b) = \omega_\alpha((\gamma(x))_{\gamma\in\mathcal{A}}, b, 0) = b$

Where the last equation follows from previous observation (1) and the fact that $\omega_\alpha((\gamma(x))_{\gamma\in\mathcal{A}}, 1, 0) = 0 \sqcup \alpha(x, 1) = 1$

So in each category $\mathbf{SSET}(\Omega)$ which has an non-indiscrete exponential object, $\mathcal{A}$ is closed under finite minima for every object $(X, \mathcal{A})$. It can be proved in a similar way that $\mathcal{A}$ is closed under finite maxima. One can easily prove that the indiscrete affine sets are exponential. So we can conclude that the exponential objects of $\mathbf{SSET}(\Omega)$ are exactly the indiscrete affine sets. □

Thus for the categories $\mathbf{SSET}$, $\mathbf{CL}$ and $\mathbf{SSET}(\mathcal{C})$ the exponential objects are exactly the indiscrete objects. From the proof of previous proposition follows that in all the categories $\mathbf{SSET}(\Omega)$ which are not embedded in $\mathbf{SSET}(\min : S^2 \to S, \max : S^2 \to S)$, the functor $X \times -$ does not preserve coproducts for non-indiscrete objects X. In $\mathbf{SSET}(\min : S^2 \to S, \max : S^2 \to S)$ itself, the functor $X \times -$ preserves coproducts for some non-indiscrete objects X. In the following proposition these objects are characterized.

Proposition 2.9. *In the construct* $\mathbf{SSET}(\min : S^2 \to S, \max : S^2 \to S)$*, we have that the functor* $(X, \mathcal{A}) \times -$ *preserves coproducts if and only if* $\mathcal{A}$ *is a finite set.*

Proof. It can be easily verified that for $(X, \mathcal{A})$, with $\mathcal{A}$ a finite set, $(X, \mathcal{A}) \times \coprod_{i \in I} (Y_i, \mathcal{B}_i)$ is isomorphic to $\coprod_{i \in I} (X, \mathcal{A}) \times (Y_i, \mathcal{B}_i)$ for every collection $\mathbf{SSET}(\min : S^2 \to S, \max : S^2 \to S)$-objects $(Y_i, \mathcal{B}_i)_{i \in I}$.
Suppose now that the functor $(X, \mathcal{A}) \times -$ preserves coproducts, then we have that $(X, \mathcal{A}) \times (\mathcal{A}, S^{\mathcal{A}})$ is isomorphic to $\coprod_{\alpha \in \mathcal{A}} (X, \mathcal{A})$. We consider the function

$$\bigsqcup_{\alpha \in \mathcal{A}} \alpha : \coprod_{\alpha \in \mathcal{A}} (X, \mathcal{A}) \to S : (x, \alpha) \to \alpha(x)$$

Since $\mathbf{SSET}(\min : S^2 \to S, \max : S^2 \to S)$ is a bicoreflective subcategory of $\mathbf{SSET}$, this function $\bigsqcup_{\alpha \in \mathcal{A}} \alpha$ belongs to $\mathcal{A}_{\coprod_{\alpha \in \mathcal{A}} (X, \mathcal{A})}$ and thus $\bigsqcup_{\alpha \in \mathcal{A}} \alpha$ belongs to $\mathcal{A}_{X \times (\mathcal{A}, S^{\mathcal{A}})}$. The product $\mathcal{A}_{X \times (\mathcal{A}, S^{\mathcal{A}})}$ is the smallest subalgebra containing $\mathcal{B} = \{ \alpha \circ pr_X \mid \alpha \in \mathcal{A}\} \cup \{ f \circ pr_{\mathcal{A}} \mid f \in S^{\mathcal{A}}\}$. Hence, there exists a finite set I and for every $i \in I$ there exist $\alpha_i \in \mathcal{A}, f_i \in S^{\mathcal{A}}$ such that $\bigsqcup_{\alpha \in \mathcal{A}} \alpha = \max_{i \in I} \min(\alpha_i \circ pr_X, f_i \circ pr_{\mathcal{A}})$. For $\beta \in \mathcal{A}$, set $I_\beta = \{i \in I \mid f_i(\beta) = 1\}$. For every $x \in X$, we have: $\beta(x) = \bigsqcup_{\alpha \in \mathcal{A}} \alpha \ (x, \beta) = \max_{i \in I} \min(\alpha_i(x), f_i(\beta)) = \max_{i \in I_\beta} \alpha_i(x)$. Since the set I is finite, this implies that $\mathcal{A}$ also shall be finite. □

3. Subconstructs in which the functor $X \times -$ preserves coproducts

The categories considered in the previous section fail to be cartesian closed because the functor $X \times -$ does not preserve coproducts. From the last proposition, it follows that the condtion, $\mathbf{SSET}(\Omega)$ is a subcategory of $\mathbf{SSET}(\min : S^2 \to S, \max : S^2 \to S)$, is not a sufficient condition such that the functor $X \times -$ preserves coproducts for all objects X. In this section we formulate a sufficient condition. It is known [17] that in **TOP** and in all its topological subconstructs the functor $X \times -$ preserves coproducts for all objects X. We generalize these results to a larger subconstruct of **SSET**.

Definition 3.1. *Let* **D** *be the full subcategory of* **SSET** *with objects all affine sets* $(X, \mathcal{A})$ *that satisfy the following two conditions.*

(D1) $\alpha \in \mathcal{A}, \beta \in \mathcal{A} \Rightarrow \min(\alpha, \beta) \in \mathcal{A}$
(D2) $\{\alpha_i \mid i \in I\} \subset \mathcal{A}$ *and* $\min(\alpha_i, \alpha_j) = \mathbf{0}$ *for each* $i \neq j \Rightarrow \max_{i \in I} \alpha_i \in \mathcal{A}$

It is clear that **TOP** is a subcategory of this category **D**. For a collection $\mathcal{B} \subset S^X$ we can define a **D**-structure $\mathcal{A}$ on X as follows. Let $\mathcal{C}$ consist of all finite minima of elements of $\mathcal{B} \cup \{\mathbf{0}, \mathbf{1}\}$. By adding to $\mathcal{C}$ the maxima of collections functions $(\alpha_i)_{i \in I}$ of $\mathcal{C}$ that satisfy the condition $\min(\alpha_i, \alpha_j) = \mathbf{0}$ for each $i \neq j$, we get a **D**-structure $\mathcal{A}$. Moreover, $\mathcal{A}$ is the smallest **D**-structure on X containing $\mathcal{B}$. $\mathcal{B}$ is called the subbase of $\mathcal{A}$ and $\mathcal{C}$ the base of $\mathcal{A}$.

Proposition 3.2. **D** *is a topological category.*

Proof. For a family of functions $(f_i : X \to (X_i, \mathcal{A}_i))_{i \in I}$ with $(X_i, \mathcal{A}_i) \in \mathbf{D}$ the **D**-structure $\mathcal{A}$ generated by the subbase $\mathcal{B} = \{\alpha_i \circ f_i \,|\, \alpha_i \in \mathcal{A}_i, i \in I\}$ is the unique initial structure on X for the given source. □

Proposition 3.3. **D** *is a bicoreflective subcategory of* **SSET**

Proof. For an affine set $(X, \mathcal{A})$ the bicoreflection is given by

$$1_X : (X, \mathcal{A}') \to (X, \mathcal{A})$$

with $\mathcal{A}'$ the **D**-structure generated by the subbase $\mathcal{A}$. □

Remark that **D** is not a hereditary subcategory of **SSET** as follows from the next example.

Example 3.4. Let $X = \{0, 1, 2, 3\}$ and $\mathcal{A} = \{\mathbf{0}, \mathbf{1}, 1_{\{1,3\}}, 1_{\{2,3\}}, 1_{\{3\}}\}$, then $(X, \mathcal{A})$ is a **D**-object. Then $(Y, \mathcal{A}|_Y) = (\{0, 1, 2\}, \{\mathbf{0}, \mathbf{1}, 1_{\{1\}}, 1_{\{2\}}\})$ is the **SSET**-subspace of $(X, \mathcal{A})$ with underlying set $\{0, 1, 2\}$. $\min(1_{\{1\}}, 1_{\{2\}}) = \mathbf{0}$ and $\max(1_{\{1\}}, 1_{\{2\}}) = 1_{\{1,2\}} \notin \mathcal{A}|_Y$, so $(Y, \mathcal{A}|_Y)$ does not belong to the category **D**.

Hence, there is no algebraic structure Ω on S such that **D** has the form $\mathbf{SSET}(\Omega)$.

Proposition 3.5. *If* $\mathbf{SSET}(\Omega)$ *is a subcategory of* **D***, then it is a subcategory of* **TOP** *or a subcategory of* $\mathbf{SSET}(\mathcal{C})$.

Proof. For an arbitrary set I, take $\infty \notin I$ and define for every $i \in I$ the function $\alpha_i : I \cup \{\infty\} \to S$ with $\alpha_i(i) = 1$ and $\alpha_i(x) = 0$ for $x \neq i$.
Let $\mathcal{A}$ be the smallest Ω-subalgebra of $S^{I \cup \{\infty\}}$ containing $\{\alpha_i \,|\, i \in I\}$. Then, $(I \cup \{\infty\}, \mathcal{A})$ is an $\mathbf{SSET}(\Omega)$-object.
Since $\mathbf{SSET}(\Omega)$ is a subcategory of **D**, we have that $\max\limits_{i \in I} \alpha_i$ belongs to $\mathcal{A}$. Hence, $\max\limits_{i \in I} \alpha_i = \omega_I(\alpha_i)_{i \in I}$, with $\omega_I : S^I \to S$ a composition of algebraic operations of Ω. This gives the following information about ω_I:

- $\forall j \in I : \omega_I(\alpha_i(j))_{i \in I} = \max(\alpha_i(j))_{i \in I} = 1$
- $\omega_I(0)_{i \in I} = \omega_I(\alpha_i(\infty)_{i \in I}) = \max(\alpha_i(\infty)_{i \in I}) = \max(0)_{i \in I} = 0$

Now two cases can arise:

(1) $\forall x \neq (0)_{i \in I} \in S^I : \omega_I(x) = 1$.
In this case $\omega_I = \max\limits_{i \in I}$

(2) There exists a $x \neq (0)_{i \in I} \in S^I$ such that $\omega_I(x) = 0$. Choose $j \in I$ such that $pr_j(x) \neq 0$ and define $(y_i)_{i \in I} \in S^I$ with $y_j = 1$ and $y_i = 0$ for $i \neq j$. Then we have $\omega_I(x) = 0, \omega_I(y_i)_{i \in I} = \omega_I(\alpha_i(j))_{i \in I} = 1$ and $pr_i(x) = 0$ implies $y_i = 0$. It then follows from lemma 2.5 that $\mathbf{SSET}(\Omega)$ is a subcategory of $\mathbf{SSET}(\mathcal{C})$.

We can conclude that either for every set I, $\omega_I = \max\limits_{i \in I}$ and thus $\mathbf{SSET}(\Omega)$ is a subcategory of **TOP** or $\mathbf{SSET}(\Omega)$ is a subcategory of $\mathbf{SSET}(\mathcal{C})$. □

It can be verified that coproducts are universal in **D**, i.e. coproducts are preserved under pullbacks along arbitrary morphisms. From proposition 2.4 and the condition (D2) follows that **D** and its subconstructs are closed under the formation of coproducts in **SSET**. Combining this with 2.2, the following theorem can be proved.

Theorem 3.6. *In every topological subcategory of* **D** *coproducts are preserved by the functor* $X \times -$*.*

Corollary 3.7. *The exponential objects of a subcategory of* **D** *are the objects* X *for which the functor* $X \times -$ *preserves quotients.*

4. Cartesian closed topological hull of SSET

In section 2, we proved that for finitely productive subcategories of **SSET**, products do not distribute over coproducts. If we want to work in a cartesian closed construct in which products are formed similarly to the ones in **SSET**, we have to consider a larger scope. We will look for cartesian closed topological constructs that are larger than **SSET** and in which **SSET** is finally densely embedded. We know from [7] that quotients in **SSET** are productive. In fact we have the same situation as for the category **CL** [6]. For **CL** a cartesian closed extension was constructed in [6] using the method presented by J. Adámek and J. Reiterman in [2]. We first look if this method is also applicable to **SSET**.

Definition 4.1. *For affine sets* $(X, \mathcal{A})$ *and* $(Y, \mathcal{B})$ *we consider the collection of functions* $\mathcal{N} = \{\Gamma_\beta | \beta \in \mathcal{B}\}$ *on* $\mathrm{Hom}(X, Y)$ *with* $\Gamma_\beta : \mathrm{Hom}(X, Y) \to S$ *defined by* $\Gamma_\beta(f) = 1$ *iff* $\beta \circ f = \mathbf{1}$*.*

Analogous to **CL**, we can prove the following result for this structure on the Hom-sets of **SSET**.

Proposition 4.2. *If* $M \subseteq Hom(X, Y)$ *is a subset endowed with an affine structure* $\mathcal{M}$ *such that the evaluation map* $\mathrm{ev}: (X, \mathcal{A}) \times (M, \mathcal{M}) \to (Y, \mathcal{B})$ *is an affine map, then the following conditions hold:*

(1) $\mathcal{N}|_M \subseteq \mathcal{M}$

(2) $\mathrm{ev}: X \times (M, \mathcal{N}|_M) \to Y$ *is affine.*

This shows that **SSET** is a type of category as considered in 4.3 of [2]. Consider the following superconstruct **K** of **SSET**. Objects of **K** are triples $(X, \mathfrak{A}, \mathcal{A})$ where X is a set, $\mathfrak{A}$ is a cover of X such that

$$U' \subseteq U, U \in \mathfrak{A} \Rightarrow U' \in \mathfrak{A}$$

and $\mathcal{A}$ is an affine structure on X which is $\mathfrak{A}$-final in the sense that $(i : (U, \mathcal{A}|_U) \to (X, \mathcal{A}))_{U \in \mathfrak{A}}$ is final in **SSET**.
The members of $\mathfrak{A}$ are called *generating sets*. A morphism in **K**,

$$f : (X, \mathfrak{A}, \mathcal{A}) \to (Y, \mathfrak{B}, \mathcal{B})$$

is a function that is affine (with respect to $(X, \mathcal{A})$ and $(Y, \mathcal{B})$) and preserves the generating sets: $U \in \mathfrak{A} \Rightarrow f(U) \in \mathfrak{B}$.

SSET is fully embedded in **K** by identifying $(X, \mathcal{A})$ with $(X, \mathcal{P}(X), \mathcal{A})$.
By the general theorem in [2] it follows that **K** is a cartesian closed topological category in which **SSET** is finally densely embedded.
A cartesian closed well-fibred topological construct **Y** is called a *cartesian closed topological hull* (CCT hull) of a construct **X** if **Y** is a finally dense extension of **X** with the property that any finally dense embedding of **X** into a cartesian closed topological construct can be uniquely extended to **Y**. Starting from the cartesian closed extension **K** of **SSET**, we will now construct the cartesian closed topological hull of **SSET**. Here again we will work as we did for **CL**. In particular, we apply the construction of the cartesian closed hull, using power-closed collections, described by J. Adámek, J. Reiterman and G.E. Strecker in II.2 and II.3 in [3]. We first recall from [3] some definitions and the construction of the CCT-hull applied to categories with productive quotients.

Definition 4.3. *Let* **X** *be a construct and let* H, K *be* **X***-objects and* X *a set. A function* $h : X \times H \to K$ *is called a multimorphism if for each* $x \in X$, $h(x, -) : H \to K$ *defined by* $h(x,-)(y) = h(x, y)$ *is a morphism.*

Definition 4.4. *Let* **X** *be a well-fibred topological construct. A collection* $\mathcal{C}$ *of objects* $(A, \mathcal{U})$ *with* $A \subseteq X$ *is said to be power-closed in a set* X *provided that* $\mathcal{C}$ *contains each object* $(A_0, \mathcal{U}_0)$ *with* $A_0 \subseteq X$ *with the following property:*
Given a multimorphism $h : X \times H \to K$ *such that for each* $(A, \mathcal{U}) \in \mathcal{C}$ *the restriction* $h|_A : (A, \mathcal{U}) \times H \to K$ *is a morphism, then the restriction* $h|_{A_0} : (A_0, \mathcal{U}_0) \times H \to K$ *is also a morphism.*
We denote by $\mathsf{PC}(\mathbf{K})$ *the category of power-closed collections. Objects are pairs* $(X, \mathcal{C})$*, where* X *is a set and* $\mathcal{C}$ *is a power-closed collection in* X*. Morphisms* $f : (X, \mathcal{C}) \to (Y, \mathcal{D})$ *are functions from* X *to* Y *such that for each* $(A, \mathcal{U}) \in \mathcal{C}$ *the final object of the restriction* $f_A : (A, \mathcal{U}) \to f(A)$ *is in* $\mathcal{D}$.

Theorem 4.5. [3] *Any well-fibred topological construct with productive quotients has a CCT hull. Moreover, this hull is precisely the category of power-closed collections.*

Next we define a suitable subconstruct of the cartesian closed extension **K** of **SSET**.

Definition 4.6. *Let* $\mathbf{K}^*$ *be the full subconstruct of* **K** *whose objects are the* **K***-objects* $(X, \mathfrak{A}, \mathcal{A})$ *that satisfy the following condition:*
If $V \subset X \notin \mathfrak{A}$*, then there exists a set* $Z \subseteq X$ *with* $V \cap Z \neq \varnothing$, $V \not\subset Z$ *and such that:* $\forall U \in \mathfrak{A} : U \cap Z = \varnothing$ *or* $U \subseteq Z$.

In order to prove that $\mathbf{K}^*$ is the CCT hull of **SSET** we establish an isomorphism between $\mathbf{K}^*$ and the category of power-closed collections of **SSET**.

Proposition 4.7. *For each object* $(X, \mathfrak{A}, \mathcal{A})$ *of* $\mathbf{K}^*$ *the collection of affine spaces* $\mathcal{C}_X = \{(U, \mathcal{B}) \,|\, U \in \mathfrak{A}, \mathcal{A}|_U \subseteq \mathcal{B}\}$ *is power-closed.*

Proof. If $(X_0, \mathcal{A}_0)$ is an affine set with $X_0 \subseteq X$ and $(X_0, \mathcal{A}_0) \notin \mathcal{C}_X$, then either $X_0 \notin \mathfrak{A}$ or $\mathcal{A}|_{X_0} \not\subseteq \mathcal{A}_0$. If $\mathcal{A}_0$ is not finer than $\mathcal{A}|_{X_0}$, there exists an $\alpha \in \mathcal{A}$ such that $\alpha|_{X_0} \notin \mathcal{A}_0$. For an arbitrary affine space H, the function $\alpha \circ pr_X : X \times H \to \mathsf{S}$ with S the Sierpinski space is a multimorphism. For $(U, \mathcal{B}) \in \mathcal{C}_X$ the restriction $h|_U = \alpha|_U \circ pr_U : (U, \mathcal{B}) \times H \to \mathsf{S}$ is affine and the restriction $h|_{X_0} = \alpha|_{X_0} \circ pr_{X_0} : (X_0, \mathcal{A}_0) \times H \to \mathsf{S}$ is not affine.
If $X_0 \notin \mathfrak{A}$, then there exists a subset Z of X, not containing X_0 and intersecting X_0 such that for all $U \in \mathfrak{A}$ we have that $U \cap Z = \varnothing$ or $U \subseteq Z$.
Take an affine set H that has a non-constant function $\gamma \in \mathcal{A}_H$. The function $h = \min(1_Z \circ pr_X, \gamma \circ pr_H) : X \times H \to \mathsf{S}$ is a multimorphism. For $(U, \mathcal{B}) \in \mathcal{C}_X$ the restriction $h|_U$ is either the constant function $\mathbf{0}$ or $\gamma \circ pr_H$. Hence, the restrictions $h|_U : (U, \mathcal{B}) \times H \to \mathsf{S}$ are affine for all $(U, \mathcal{B}) \in \mathcal{C}_X$. Since $X_0 \cap Z \neq \varnothing$ and $X_0 \not\subseteq Z$, we have that $h|_{X_0} \neq \alpha \circ pr_H$ and $h|_{X_0} \neq \beta \circ pr_{X_0}$ for all $\alpha \in \mathcal{A}_H$ and $\beta \in \mathcal{A}_{X_0}$. Therefore the restriction $h|_{X_0} : (X_0, \mathcal{A}_0) \times H \to \mathsf{S}$ is not affine. □

Proposition 4.8. *If $\mathcal{C}$ is a power-closed collection of* **SSET**-*objects in a set X then there exists a unique* $\mathbf{K}^*$-*object $(X, \mathfrak{A}, \mathcal{A})$ such that $\mathcal{C} = \mathcal{C}_X$.*

Proof. For a power-closed collection $\mathcal{C}$ of **SSET**-objects in X, we can prove analogously to **CL** [6] that $(X, \mathfrak{A}, \mathcal{A})$ where $\mathfrak{A} = \{U \subseteq X \mid (U, \mathcal{B}) \in \mathcal{C} \text{ for some} \mathcal{B}\}$ and $\mathcal{A}$ the final structure determined by the sink of inclusion maps $(i : (U, \mathcal{B}) \to X)_{(U,\mathcal{B}) \in \mathcal{C}}$ is a $\mathbf{K}^*$-object such that $\mathcal{C} = \mathcal{C}_X$. □

Theorem 4.9. $\mathbf{K}^*$ *is the CCT hull of* **SSET**.

Proof. It follows from the two previous propositions that the functor $F : \mathbf{K}^* \to \mathsf{PC(SSET)}$ defined by $F(X \xrightarrow{f} X') = (X, \mathcal{C}_X) \xrightarrow{f} (X', \mathcal{C}_{X'})$ is bijective on objects. In a similar way as for **CL** in [6], one can prove that F is an isomorphism. From 4.5 it then follows that $\mathbf{K}^*$ is the CCT-hull of **SSET**. □

References

[1] J. Adámek, H. Herrlich and G.E. Strecker, *Abstract and concrete categories* (Wiley, New York, 1990).

[2] J. Adámek. and J. Reiterman, *Cartesian closed hull of the category of uniform spaces*, Topology Appl. **19** (1985), 261–276.

[3] J. Adámek, J. Reitermann and G.E. Strecker, *Realization of cartesian closed topological hulls*, Manuscripta Math. **53** (1985), 1–33 .

[4] P. Antoine, *Extension minimale de la catégorie des espaces topologiques*, C. R. Acad. Sci. Paris Sér. A-B **262** (1966), A1389–A1392.

[5] G. Bourdaud, *Espaces d'Antoine et semi-espaces d'Antoine*, Cahiers Topologie Géom. Différentielle **16**, no. 2 (1975), 107–133.

[6] V. Claes, E. Lowen-Colebunders and G. Sonck, *Cartesian closed topological hull of the construct of closure spaces*, Theory Appl. Categories **8**, no. 18 (2001), 481–489.

[7] V. Claes and E. Lowen-Colebunders, *Productivity of Zariski-compactness for constructs of affine spaces*, Topology Appl. **153**, no. 5-6 (2005), 747–755.

[8] B. J. Day and G.M. Kelly, *On topological quotients preserved by pullback or products*, Proc. Camb. Phil. Soc. **67** (1970), 553–558.

[9] Y. Diers, *Categories of algebraic sets*, Appl. Categ. Structures **4** (1996), 329–341.

[10] Y. Diers, *Affine algebraic sets relative to an algebraic theory*, J. Geom. **65** (1999), 54–76.
[11] E. Dubuc and H. Porta, *Convenient categories of topological algebras, and their duality theory*, J. Pure Appl. Algebra **1**, no. 3 (1971), 281–316.
[12] A. Frölicher, *Cartesian closed categories and analysis of smooth maps*, In Categories in continuum physics (Buffalo, N.Y., 1982), volume 1174 of Lecture Notes in Math., pages 43–51. (Springer, Berlin, 1986).
[13] E. Giuli, *On classes of T_0 spaces admitting completions*, Appl. Gen. Topol. **4**, no. 1 (2003), 143–155.
[14] E. Giuli, *The structure of affine algebraic sets*, In Categorical structures and their applications, pages 113–120 (World Sci. Publishing, River Edge, NJ, 2004).
[15] H. Herrlich, *Cartesian closed topological categories*, Math. Colloq. Univ. Cape Town **9** (1974), 1–16.
[16] H. Herrlich, *Categorical topology 1971-1981*, Sigma Ser. Pure Math. **3**, 279–383 (Heldermann Verlag, Berlin, 1983).
[17] H. Herrlich, *Are there convenient subcategories of* **Top***?*, Topology Appl. **15**, no. 3 (1983), 263–271.
[18] A. Kriegl, *A Cartesian closed extension of the category of smooth Banach manifolds*, In Categorical topology (Toledo, Ohio, 1983), volume 5 of Sigma Ser. Pure Math., pages 323–336 (Heldermann, Berlin, 1984).
[19] A. Machado, *Espaces d'Antoine et pseudo-topologies*, Cahiers Topologie Géom. Différentielle **14** (1973), 309–327.
[20] H. Müller, *Úber die vertauschbarkeit von reflexionen und corefectionen* (Bielefeld, 1974).
[21] L. D. Nel, *Infinite-dimensional calculus allowing nonconvex domains with empty interior*, Monatsh. Math. **110**, no. 2 (1990), 145–166.
[22] N. E. Steenrod, *A convenient category of topological spaces*, Michigan Math. J. **14** (1967), 133–152.

Received July 2006

Accepted December 2006

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