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Function Spaces and Strong Variants of Continuity

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Abstract. It is shown that if domain is a sum connected space and range is a T_0-space, then the notions of strong continuity, perfect continuity and cl-supercontinuity coincide. Further, it is proved that if X is a sum connected space and Y is Hausdorff, then the set of all strongly continuous (perfectly continuous, cl-supercontinuous) functions is closed in Y^X in the topology of pointwise convergence. The results obtained in the process strengthen and extend certain results of Levine and Naimpally.

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1. Introduction

Strong variants of continuity occur frequently in many areas of mathematics, more so in many subdisciplines of topology and analysis. In this paper we shall be concerned with three such variants of continuity: strongly continuous functions introduced by Levine [5], perfectly continuous functions due to Noiri [8], and cl-supercontinuous functions defined by Reilly and Vamanamurthy [9] under the nomenclature of 'clopen maps' and studied by Singh [10]. Naimpally [7] showed that in contrast to continuous functions, the set of strongly continuous functions is closed in the topology of pointwise convergence if the domain is locally connected and range is Hausdorff. In this paper we extend Naimpally's result to a larger framework and further prove that in the proposed framework the set of perfectly continuous functions as well as the set of cl-supercontinuous functions is closed in the topology of pointwise convergence. For our purpose the class of sum connected spaces [4] turns out to be handy

and provides an appropriate framework. A space X is sum connected if each $x \in X$ has a connected neighbourhood or equivalently each component of X is open in X. The category of sum connected spaces is the coreflective hull of the category of connected spaces and includes all locally connected spaces as well. Sum connected spaces are also closely related to natural cover of Brown [1] and Franklin [2] and Michael's Theorem [6] on existence of selections is also closely related to them.
For terms used in the paper but not defined, we refer the reader to [3].

2. Definitions, Preliminary, Observations and Examples

Definitions 2.1. *A function $f : X \to Y$ from a topological space X into a topological space Y is said to be*

(1) ***strongly continuous*** [5] *if $f(\bar{A}) \subset f(A)$ for all $A \subset X$.*
(2) ***perfectly continuous*** [8] *if for every open set $V \subset Y$, $f^{-1}(V)$ is clopen in X.*
(3) ***cl-supercontinuous***[1] [10] *if for each open set V containing $f(x)$ there exists a clopen set U containing x such that $f(U) \subset V$.*

Definitions 2.2. *A space X is said to be*

(1) *an **ultra-Hausdorff space** [11] if for each pair of distinct points in X, there is a clopen set containing one but missing the other;*
(2) *a **k-space** [3] if $A \subset X$ is closed if and only if $A \cap K$ is closed in K for every compact set K in X.*

Definition 2.3. *A family ϑ of functions from a topological space X into a uniform space (Y, v) is said to be **equicontinuous** at a point $x \in X$ if for each member V of v there exists a neighbourhood U of x such that $f(U) \subset V[f(x)]$ for each $f \in \vartheta$. The family ϑ is said to be equicontinuous if it is equicontinuous at every point.*

Definition 2.4. *A family ϑ of functions from a topological space X into a topological space Y is said to be **evenly continuous** if for each $x \in X$, each $y \in Y$ and each neighbourhood V of y there is a neighbourhood U of x and a neighbourhood W of y such that such that $f(U) \subset V$ whenever $f(x) \in W$.*

Results on strongly continuous functions 2.5.
Let X and Y be topological spaces.

(a) *If X is discrete, then every function $f : X \to Y$ is strongly continuous.*
(b) *If X is connected, then a function $f : X \to Y$ is strongly continuous if and only if f is constant.*
(c) *If $f : X \to Y$ is strongly continuous, then the collection $P = \{f^{-1}(y) : y \in Y\}$ is a partition of X into pairwise disjoint clopen subsets of X. So the topology on X is finer than a partition topology (every open set is closed). Moreover, the quotient space X/P is discrete.*

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[1] 'cl-supercontinuous functions' are called 'clopen maps' in [9].

(d) *If $f : X \to Y$ is a strongly continuous injection, then X is discrete. Further, the hypothesis of injectivity cannot be relaxed. For let X be the set of positive integers equipped with odd-even topology* ([12, p. 43]) *and Y be the set of positive integers with the discrete topology. Let $f : X \to Y$ be defined by*

$$f(x) = \begin{cases} k & \textit{if } \; x = 2k \\ k & \textit{if } \; x = 2k - 1 \end{cases}$$

Then clearly f is a strongly continuous surjection but X is not discrete.

(e) *If $f : X \to Y$ is a strongly continuous open surjection, then Y is discrete.*

(f) *If $f : X \to Y$ is a strongly continuous open bijection, then X and Y are discrete spaces of same cardinality.*

Results on perfectly continuous functions 2.6.

(a) *If X is endowed with a partition topology, then every continuous function defined on X is perfectly continuous.*

(b) *A space X is endowed with a partition topology if and only if the identity mapping defined on X is perfectly continuous. If partition topology on X is not discrete, then the identity mapping on X is not strongly continuous.*

Results on cl-supercontinuous functions 2.7.

(a) *If either of the spaces X and Y is a zero dimensional space, then every continuous function $f : X \to Y$ is cl-supercontinuous.*

3. Results

Theorem 3.1 ([5]). *For a function $f : X \to Y$, the following statements are equivalent.*

(a) *f is strongly continuous.*

(b) *$f^{-1}(B)$ is a clopen subset of X for every subset B of Y.*

(c) *$f(A') \subset f(A)$ for every subset A of X, where A' denotes the set of all limit points of A.*

Theorem 3.2. *If $f : X \to Y$ is cl-supercontinuous function into a T_0-space Y, then $f(C)$ is a singleton for every nonempty connected subset C of X.*

Proof. Assume contrary and let C be a connected subset of X such that $f(C)$ is not a singleton. Let $f(x)$, $f(y)$ be any two distinct points of $f(C)$. Since Y is a T_0-space, there is an open set V containing one of the points $f(x)$ and $f(y)$ and missing the other. To be precise, let $f(x) \in V$. Then, by $f(x) \in V$, $f^{-1}(V) \cap C$ is nonempty proper cl-open subset of C, contradicting the fact that C is connected. □

Corollary 3.3. *Let $f : X \to Y$ be a cl-supercontinuous function into a T_0-space Y. If X is sum connected, then f is constant on each component of X. In particular, if X is connected, then f is constant.*

Theorem 3.4. *Let $f : X \to Y$ be a function from a sum connected space X into a T_0-space Y. Then the following statements are equivalent.*

(a) *f is strongly continuous.*
(b) *f is perfectly continuous.*
(c) *f is cl-supercontinuous.*

Proof. The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are trivial.
To prove that $(c) \Rightarrow (a)$ we show that $f(A') \subset f(A)$ for every subset A of X. To this end, let $x \in A'$. Let C be the component of X containing x. Since X is sum connected, C is open in X and so $C \cap A \neq \phi$. By Theorem 3.2 $f(C)$ is a singleton and so $f(x) = f(C) = f(C \cap A) \subset f(A)$. Thus $f(A') \subset f(A)$ and so in view of Theorem 3.1 f is strongly continuous. □

Corollary 3.5. *If X is a connected space or a locally connected space and Y is a T_0-space, then the notions of strong continuity, perfect continuity and cl-supercontinuity coincide.*

Let $L = L(X, Y)$, $P = (X, Y)$ and $S = S(X, Y)$ denote the function space of all cl-supercontinuous, respectively perfectly continuous and strongly continuous functions from X into Y with the topology of pointwise convergence.

Theorem 3.6. *Let Y be a Hausdorff space and let $g \in \bar{L}$ (where closure is taken in the topology of pointwise convergence). Then for each nonempty connected subset C of X, $g(C)$ is a single point.*

Proof. Suppose that there exists a nonempty connected subset C of X such that $g(C)$ contains at least two points $g(x)$ and $g(y)$. Since X is a Hausdorff space, there exist disjoint open sets U and V containing $g(x)$ and $g(y)$, respectively. Since $g \in \bar{L}$, there is a net $\{f_\alpha \mid \alpha \in \Lambda\} \subset L$ such that $f_\alpha(x) \to g(x)$ for each $x \in X$. So there exists a $\alpha_0 \in \Lambda$ such that for all $\alpha \geq \alpha_0$, $f_\alpha(x) \in U$ and $f_\alpha(y) \in V$. But $f_\alpha(x) = f_\alpha(y)$ on C. This contradicts that Y is a Hausdorff space and proves the result. □

Theorem 3.7. *Let X be a sum connected space and Y be Hausdorff. Then $L = P = S$ is closed in the topology of pointwise convergence.*

Proof. This is immediate in view of Theorems 3.1,3.4 and 3.6. □

Proposition 3.8. *If X is a sum connected space and Y is a compact Hausdorff space, then $L = P = S$ is compact in the topology of pointwise convergence.*

Proof. This is immediate in view of Theorem 3.7, the fact that $\overline{L[x]}$ is compact for each $x \in X$ and ([3, Theorem1, p.218]). □

Proposition 3.9. *If X is a sum connected and (Y, v) is a Hausdorff uniform space, then $L = P = S$ is equicontinuous and consequently evenly continuous.*

Proof. For each $x \in X$, let C_x denote the component of X containing x. Since X is sum connected, C_x is open. In view of Theorem 3.6 for each $f \in L$, $f(C_x) = f(x)$. Clearly $f(C_x) \subset V[f(x)]$ for all $V \in v$ and all $f \in L$. Thus L is equicontinuous. By [3, p. 237] L is evenly continuous. □

Proposition 3.10. *Let X be a sum connected space and (Y, v) be a Hausdorff uniform space. Then the topology of pointwise convergence for $L = P = S$ coincides with the topology of uniform convergence on compacta. Further, if X is a k-space and $\overline{L[x]}$ is compact for each $x \in X$, then $L = P = S$ is compact in the topology of pointwise convergence.*

Proof. By Proposition 3.9, L is equicontinuous. For an equicontinuous family the topology of pointwise convergence is jointly continuous and so coincides with the topology of uniform convergence on compacta ([3, Theorem 15, p. 232]). So by Ascoli theorem ([3, Theorem 18, p. 234]) it follows that $L = P = S$ is compact in the topology of pointwise convergence. $\square$

Theorem 3.11. *Let X be a sum connected space and let Y be a Hausdorff space. Then $L = P = S$ is evenly continuous in the topology of pointwise convergence. Further, if in addition X is a regular locally compact space, Y a regular space, and $\overline{L[x]}$ is compact for each $x \in X$. Then $L = P = S$ is compact in the topology of pointwise convergence.*

Proof. For each $x \in X$, each $y \in Y$ and each neighbourhood V of y, let $U = C_x$ be the component of X containing x. Since X is sum connected, $C_x = U$ is open in X. Let $W = V$. Now by Theorem 3.6 $f(U) = f(C_x) = f(x) \in V$ whenever $f(x) \in W$. So the family $L = P = S$ is evenly continuous. Again, by Theorem 3.7, $L = P = S$ is closed in the topology of pointwise convergence. Hence by ([3, Theorem 19, p. 235]) the topology of pointwise convergence on $L = P = S$ is jointly continuous and therefore coincides with the compact open topology. The compactness of $L = P = S$ is immediate in view of Ascoli theorem pertaining to evenly continuous families ([3, Theorem 21, p. 236]). $\square$

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