The Čech number of $C_p(X)$ when $X$ is an ordinal space

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Abstract. The Čech number of a space $Z$, $\check{C}(Z)$, is the pseudocharacter of $Z$ in $\beta Z$. In this article we obtain, in ZFC and assuming SCH, some upper and lower bounds of the Čech number of spaces $C_p(X)$ of realvalued continuous functions defined on an ordinal space $X$ with the pointwise convergence topology.

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1. Notations and Basic results

In this article, every space $X$ is a Tychonoff space. The symbols $\omega$ (or $\mathbb{N}$), $\mathbb{R}$, $I$, $\mathbb{Q}$ and $\mathbb{P}$ stand for the set of natural numbers, the real numbers, the closed interval $[0,1]$, the rational numbers and the irrational numbers, respectively. Given two spaces $X$ and $Y$, we denote by $C(X,Y)$ the set of all continuous functions from $X$ to $Y$, and $C_p(X,Y)$ stands for $C(X,Y)$ equipped with the topology of pointwise convergence, that is, the topology in $C(X,Y)$ of subspace of the Tychonoff product $Y^X$. The space $C_p(X,\mathbb{R})$ is denoted by $C_p(X)$. The restriction of a function $f$ with domain $X$ to $A \subset X$ is denoted by $f|A$. For a space $X$, $\beta X$ is its Stone-Čech compactification.

Recall that for $X \subset Y$, the pseudocharacter of $X$ in $Y$ is defined as

$$\Psi(X,Y) = \min\{|U| : U \text{ is a family of open sets in } Y \text{ and } X = \bigcap U\}.$$ 

Definition 1.1.

1. The Čech number of a space $Z$ is $\check{C}(Z) = \Psi(Z,\beta Z)$.
2. The $k$-covering number of a space $Z$ is $k cov(Z) = \min\{|K| : K \text{ is a compact cover of } Z\}$.

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We have that (see Section 1 in [8]): \( \hat{C}(Z) = 1 \) if and only if \( Z \) is locally compact; \( \hat{C}(Z) \leq \omega \) if and only if \( Z \) is Čech-complete; \( \hat{C}(Z) = \text{kcov}(\beta Z \setminus Z) \); if \( Y \) is a closed subset of \( Z \), then \( \text{kcov}(Y) \leq \text{kcov}(Z) \) and \( \hat{C}(Y) \leq \hat{C}(Z) \); if \( f : Z \to Y \) is an onto continuous function, then \( \text{kcov}(Y) \leq \text{kcov}(Z) \); if \( f : Z \to Y \) is perfect and onto, then \( \text{kcov}(Y) = \text{kcov}(Z) \) and \( \hat{C}(Y) = \hat{C}(Z) \); if \( bZ \) is a compactification of \( Z \), then \( \hat{C}(Z) = \Psi(Z, bZ) \).

We know that \( \hat{C}(C_p(X)) \leq \aleph_0 \) if and only if \( X \) is countable and discrete ([7]), and \( \hat{C}(C_p(X, I)) \leq \aleph_0 \) if and only if \( X \) is discrete ([9]).

For a space \( X \), \( ec(X) \) (the essential cardinality of \( X \)) is the smallest cardinality of a closed and open subspace \( Y \) of \( X \) such that \( X \setminus Y \) is discrete. Observe that, for such a subspace \( Y \) of \( X \), \( \hat{C}(C_p(X, I)) = \hat{C}(C_p(Y, I)) \). In [8] it was pointed out that \( ec(X) \leq \hat{C}(C_p(X, I)) \) and \( \hat{C}(C_p(X)) = |X| \cdot \hat{C}(C_p(X, I)) \) always hold. So, if \( X \) is discrete, \( \hat{C}(C_p(X)) = |X| \), and if \( |X| = ec(X) \), \( \hat{C}(C_p(X)) = \hat{C}(C_p(X, I)) \).

Consider in the set of functions from \( \omega \) to \( \omega \), \( ^*\omega \), the partial order \( \leq ^* \) defined by \( f \leq ^* g \) if \( f(n) \leq g(n) \) for all but finitely many \( n \in \omega \). A collection \( D \) of \( ^*\omega \) is dominating if for every \( h \in ^*\omega \) there is \( f \in D \) such that \( h \leq ^* f \). As usual, we denote by \( \mathfrak{d} \) the cardinal number \( \text{min}(|D| : D \text{ is a dominating subset of } ^*\omega) \). It is known that \( \mathfrak{d} = \text{cov}(\mathcal{P}) \) (see [3]); so \( \mathfrak{d} = \hat{C}(\mathbb{Q}) \). Moreover, \( \omega_1 \leq \mathfrak{d} \leq \mathfrak{c} \), where \( \mathfrak{c} \) denotes the cardinality of \( \mathbb{R} \).

We will denote a cardinal number \( \tau \) with the discrete topology simply as \( \tau \); so, the space \( \tau^\omega \) is the Tychonoff product of \( \kappa \) copies of the discrete space \( \tau \). The cardinal number \( \tau \) with the order topology will be symbolized by \( [0, \tau] \).

In this article we will obtain some upper and lower bounds of \( \hat{C}(C_p(X, I)) \) when \( X \) is an ordinal space; so this article continues the efforts made in [1] and [8] in order to clarify the behavior of the number \( \hat{C}(C_p(X, I)) \) for several classes of spaces \( X \).

For notions and concepts not defined here the reader can consult [2] and [4].

2. The Čech number of \( C_p(X) \) when \( X \) is an ordinal space

For an ordinal number \( \alpha \), let us denote by \([0, \alpha]\) and \([0, \alpha)\) the set of ordinals \( < \alpha \) and the set of ordinals \( \leq \alpha \), respectively, with its order topology. Observe that for every ordinal number \( \alpha \leq \omega \), \([0, \alpha)\) is a discrete space, so, in this case, \( \hat{C}(C_p([0, \alpha), I)) = 1 \). If \( \omega < \alpha < \omega_1 \), then \([0, \alpha)\) is a countable metrizable space, hence, by Theorem 7.4 in [1], \( \hat{C}(C_p([0, \alpha), I)) = \mathfrak{d} \). We will analyze the number \( \hat{C}(C_p([0, \alpha), I)) \) for an arbitrary ordinal number \( \alpha \).

We are going to use the following symbols:

**Notations 2.1.** For each \( n < \omega \), we will denote as \( E_n \) the collection of intervals

\[
[0, 1/2^{n+1}), (1/2^{n+2}, 3/2^{n+2}), (1/2^{n+1}, 2/2^{n+1}), (3/2^{n+2}, 5/2^{n+2}), ...
\]

\[
..., ((2^n+2 - 2)/2^{n+2}, (2^n+2 - 1)/2^{n+2}), ((2^n+1 - 1)/2^{n+1}, 1].
\]

Observe that \( E_n \) is an irreducible open cover of \([0, 1]\) and each element in \( E_n \) has diameter \( = 1/2^{n+1} \). For a set \( S \) and a point \( y \in S \), we will use the symbol \([yS]^{<\omega} \) in order to denote the collection of finite subsets of \( S \) containing \( y \).
Moreover, if $\gamma$ and $\alpha$ are ordinal numbers with $\gamma \leq \alpha$, $[\gamma, \alpha]$ is the set of ordinal numbers $\lambda$ which satisfy $\gamma \leq \lambda \leq \alpha$. The expression $\alpha_0 < \alpha_1 < \ldots < \alpha_n < \ldots / \gamma$ will mean that the sequence $(\alpha_n)_{n<\omega}$ of ordinal numbers is strictly increasing and converges to $\gamma$.

**Lemma 2.2.** Let $\gamma$ be an ordinal number such that there is $\omega < \alpha_0 < \alpha_1 < \ldots < \alpha_n < \ldots / \gamma$. Then $\check{C}(C_p([0, \gamma]), I) \leq \check{C}(C_p([0, \gamma]), I) \cdot \text{kcov}([\gamma]^\omega)$.

**Proof.** For $m < \omega$, $F \in [\gamma[\alpha_m, \gamma]]^{<\omega} = \{M \subset [\alpha_m, \gamma] : |M| < \aleph_0$ and $\gamma \in M\}$ and $n < \omega$, define

$$B(m, F, E) = \bigcup_{E \in E_n} B(m, F, E)$$

where $B(m, F, E) = \prod_{x \in [0, \gamma]} J_x$ with $J_x = E$ if $x \in F$, and $J_x = I$ otherwise. (So, $B(m, F, n)$ is open in $I^{[0, \gamma]}$.) Define

$$B(m, n) = \bigcap\{B(m, F, n) : F \in [\gamma[\alpha_m, \gamma]]^{<\omega}\}.$$

Observe that $B(m, n)$ is the intersection of at most $|\gamma|$ open sets $B(m, F, n)$.

**Claim:** $G$ is the set of all functions $g : [0, \gamma] \rightarrow [0, 1]$ which are continuous at $\gamma$.

Proof of the claim: Let $g : [0, \gamma] \rightarrow [0, 1]$ be continuous at $\gamma$. Given $n < \omega$ there is $E \in E_n$ such that $g(\gamma) \in E$. Since $g$ is continuous at $\gamma$, there is $\beta < \gamma$ so that $g(t) \in E$ if $t \in [\beta, \gamma]$. Fix $m < \omega$ so that $\beta < \alpha_m$. For every $F \in [\gamma[\alpha_m, \gamma]]^{<\omega}$ we have that $g \in B(m, F, E) \subset B(m, F, n)$; hence, $g \in B(m, n)$ and $G = \bigcap_{n<\omega} G(n)$.

Now, let $h \in G$. We are going to prove that $h$ is continuous at $\gamma$. Assume the contrary, that is, there exist $\epsilon > 0$ and a sequence $t_0 < t_1 < \ldots < t_n < \ldots / \gamma$ such that

$$|f(t_j) - f(\gamma)| \geq \epsilon,$$

for every $j < \omega$. Fix $n < \omega$ such that $1/2^{n+1} < \epsilon$.

Since $h \in G$, then $h \in G(n)$ and there is $m \geq 0$ such that $h \in B(m, n)$. Choose $t_{n_p} > \alpha_m$ and take $F = \{t_{n_p}, \gamma\}$. Thus $h \in B(m, F, n)$, but if $E \in E_n$ and $h(\gamma) \in E$, then $h(t_{n_p}) \in E$, which is a contradiction. So, the claim has been proved.

Now, we have $I^{[0, \gamma]} \setminus G = \bigcup_{n<\omega} (I^{[0, \gamma]} \setminus G(n))$, and

$$I^{[0, \gamma]} \setminus G(n) = \bigcap_{m<\omega} \bigcup_{E \in \Gamma_n} ((I^{[0, \gamma]} \setminus B(m, F, n))).$$

So $I^{[0, \gamma]} \setminus G(n)$ is an $F[\beta, \gamma]$-set. By Corollary 3.4 in [8], $\text{kcov}(I^{[0, \gamma]} \setminus G(n)) \leq \text{kcov}([\gamma]^\omega)$. Hence, $\check{C}(G) = \text{kcov}(I^{[0, \gamma]} \setminus G(n)) \leq \aleph_0 \cdot \text{kcov}([\gamma]^\omega)$. Thus, it follows that

$$\check{C}(C_p([0, \gamma]), I) \leq \check{C}(C_p([0, \gamma]), I) \cdot \text{kcov}([\gamma]^\omega).$$

\qed
Lemma 2.3. If $\gamma < \alpha$, then $\mathcal{C}(C_p([0, \gamma], I)) \leq \mathcal{C}(C_p([0, \alpha], I))$.

Proof. First case: $\gamma = \beta + 1$.

In this case, $[0, \gamma) = [0, \beta]$ and the function $\phi : [0, \alpha) \to [0, \beta]$ defined by $\phi(x) = x$ if $x \leq \beta$ and $\phi(x) = \beta$ if $x > \beta$ is a quotient. So, $\phi^\# : C_p([0, \beta], I) \to C_p([0, \alpha], I)$ defined by $\phi^\#(f) = f \circ \phi$, is a homeomorphism between $C_p([0, \beta], I)$ and a closed subset of $C_p([0, \alpha], I)$ (see [2], pages 13, 14). Then, in this case, $\mathcal{C}(C_p([0, \gamma], I)) \leq \mathcal{C}(C_p([0, \alpha], I))$.

Now, in order to finish the proof of this Lemma, it is enough to show that for every limit ordinal number $\alpha$, $\mathcal{C}(C_p([0, \alpha], I)) \leq \mathcal{C}(C_p([0, \alpha], I))$.

Let $\kappa = \text{cof}(\alpha)$, and $\alpha_0 < \alpha_1 < \ldots < \alpha_\lambda < \ldots / \alpha$ with $\lambda < \kappa$. For each of these $\lambda$, we know, because of the proof of the first case, that $\kappa_\lambda = \mathcal{C}(C_p([0, \alpha_\lambda], I)) \leq \mathcal{C}(C_p([0, \alpha], I))$. Let, for each $\lambda < \kappa$, $\{V_\xi^\lambda : \xi < \kappa_\lambda\}$ be a collection of open subsets of $I^{[0, \alpha_\lambda]}$ such that $C_p([0, \alpha_\lambda], I) = \bigcap_{\xi \in \kappa_\lambda} V_\xi^\lambda$.

For each $\lambda < \kappa$ and each $\xi < \kappa_\lambda$, we take $W_\lambda^\xi = V_\xi^\lambda \times I^{(\alpha_\lambda, \alpha)}$. Each $W_\lambda^\xi$ is open in $I^{(\alpha_\lambda, \alpha)}$ and $\bigcap_{\lambda < \kappa} \bigcap_{\xi < \kappa_\lambda} W_\lambda^\xi = C_p([0, \alpha], I)$. Therefore, $\mathcal{C}(C_p([0, \alpha], I)) \leq \kappa \cdot \sup_{\lambda < \kappa} \kappa_\lambda \leq \kappa \cdot \mathcal{C}(C_p([0, \alpha], I))$. But $\kappa \leq |\alpha| = \text{ec}([0, \alpha]) \leq \mathcal{C}(C_p([0, \alpha], I))$.

Then, $\mathcal{C}(C_p([0, \alpha], I)) \leq \mathcal{C}(C_p([0, \alpha], I))$. \hfill $\Box$

Lemma 2.4. Let $\alpha$ be a limit ordinal number $> \omega$. Then

$$\mathcal{C}(C_p([0, \alpha], I)) = |\alpha| \cdot \sup_{\gamma < \alpha} \mathcal{C}(C_p([0, \gamma], I)).$$

In particular, $\mathcal{C}(C_p([0, \alpha], I)) = \sup_{\gamma < \alpha} \mathcal{C}(C_p([0, \gamma], I))$ if $\text{cof}(\alpha) < \alpha$.

Proof. By Lemma 2.3, $\sup_{\gamma < \alpha} \mathcal{C}(C_p([0, \gamma], I)) \leq \mathcal{C}(C_p([0, \alpha], I))$, and, by Corollary 4.8 in [8], $|\alpha| \leq \mathcal{C}(C_p([0, \alpha], I))$.

For each $\gamma < \alpha$, we write $\kappa_\gamma$ instead of $\mathcal{C}(C_p([0, \gamma], I))$. Let $\{V_\xi^\gamma : \lambda < \kappa_\gamma\}$ be a collection of open sets in $I^\gamma$ such that $C_p([0, \gamma], I) = \bigcap_{\lambda < \kappa_\gamma} V_\lambda^\gamma$. Now we put $W_\lambda^\gamma = V_\lambda^\gamma \times I^{(\gamma, \alpha)}$. We have that $W_\lambda^\gamma$ is open for every $\gamma < \alpha$ and every $\lambda < \gamma$, and $C_p([0, \alpha], I) = \bigcap_{\gamma < \alpha} \bigcap_{\lambda < \gamma} W_\lambda^\gamma$. So, $\mathcal{C}(C_p([0, \alpha], I)) = |\alpha| \cdot \sup_{\gamma < \alpha} \mathcal{C}(C_p([0, \gamma], I))$. \hfill $\Box$

In order to prove the following result it is enough to mimic the prove of 5.12.(c) in [5].

Lemma 2.5. If $\alpha$ is an ordinal number with $\text{cof}(\alpha) > \omega$ and $f \in C_p([0, \alpha], I))$, then there is $\gamma_0 < \alpha$ for which $f \upharpoonright [\gamma_0, \alpha]$ is a constant function.

Lemma 2.6. If $\alpha$ is an ordinal number with cofinality $> \omega$, then $\mathcal{C}(C_p([0, \alpha], I)) = \mathcal{C}(C_p([0, \alpha], I))$.

Proof. Let $\kappa = \mathcal{C}(C_p([0, \alpha], I))$. There are open sets $V_\lambda (\lambda < \kappa)$ in $I^{(0, \alpha)}$ such that $C_p([0, \alpha], I) = \bigcap_{\lambda < \kappa} V_\lambda$. For each $\lambda < \kappa$, we take $W_\lambda = V_\lambda \times I^{(\alpha)}$. Each $W_\lambda$ is open in $I^{(0, \alpha)}$ and $\bigcap_{\lambda < \kappa} W_\lambda = \{f : [0, \alpha] \to I \mid f \upharpoonright [0, \alpha] \in C_p([0, \alpha], I))\}$.

For each $(\gamma, \xi, E) \in \alpha \times \alpha \times E_n$, we take $B(\gamma, \xi, E) = \prod_{\lambda < \alpha} J_\lambda$ where $J_\lambda = E$ if $\lambda \in \{\xi + \gamma, \alpha\}$, and $J_\lambda = I$ otherwise. Let $B(\gamma, \xi, n) = \bigcup_{E \in E_n} B(\gamma, \xi, E)$. This implies that

$$\mathcal{C}(C_p([0, \alpha], I)) = \mathcal{C}(C_p([0, \alpha], I)).$$
Finally, we define $B(\gamma) = \bigcup_{\xi < \alpha} B(\gamma, \xi, n)$, which is an open subset of $I^{[0, \alpha]}$. We denote by $M$ the set $\bigcap_{\lambda < \kappa} W_\lambda \cap \bigcap_{\gamma < \alpha} B(\gamma)$. We are going to prove that $C_p([0, \alpha], I) \subseteq M$.

Let $f \in C_p([0, \alpha], I)$. We know that $f \in \bigcap_{\lambda < \kappa} W_\lambda$, so we only have to prove that $f \in \bigcap_{\gamma < \alpha} B(\gamma)$. For $n < \omega$, there is $E \in \mathcal{E}_n$ such that $f(\alpha) \in E$. Since $f \in C([0, \alpha], I)$, there are $\gamma_0 < \alpha$ and $r_0 \in I$ such that $f(\lambda) = r_0$ if $\gamma_0 \leq \lambda < \alpha$. Let $\chi < \alpha$ such that $\chi + \gamma \geq \gamma_0$. Thus, $f \in B(\gamma, \chi, n) \subseteq B(\gamma)$. Therefore, $C_p([0, \alpha], I) \subseteq M$.

Take an element $f$ of $M$. Since $f \in \bigcap_{\lambda < \kappa} W_\lambda$, $f$ is continuous at every $\gamma < \alpha$, thus $\|f\|_{[0, \alpha]} = r_0$ for a $\gamma_0 < \alpha$ and an $r_0 \in I$.

For each $n < \omega$, and each $\gamma \geq \gamma_0$, $f \in B(\gamma, \xi, n)$ for some $\xi < \alpha$. Then, $|r_0 - f(\alpha)| = |f(\gamma + \xi) - f(\alpha)| < 1/2^n$. But, these relations hold for every $n$. So, $f(\alpha)$ must be equal to $r_0$, and this means that $f$ is continuous at every point.

Therefore, $\hat{C}(C_p([0, \alpha], I)) \leq |\alpha| \cdot \hat{C}(C_p([0, \alpha], I))$. Since $\hat{C}(C_p([0, \alpha], I)) \geq \text{cc}(\alpha) = |\alpha|$, $\hat{C}(C_p([0, \alpha], I)) \leq \hat{C}(C_p([0, \alpha], I))$. Finally, Lemma 2.3 gives us the inequality $\hat{C}(C_p([0, \alpha], I)) \leq \hat{C}(C_p([0, \alpha], I))$. □

**Theorem 2.7.** For every ordinal number $\alpha > \omega$,

$$|\alpha| \cdot \mathfrak{d} \leq \hat{C}(C_p([0, \alpha], I)) \leq \text{cov}(|\alpha|^\omega).$$

**Proof.** Because of Theorem 7.4 in [1], Corollary 4.8 in [8] and Lemma 2.3 above, $|\alpha| \cdot \mathfrak{d} \leq \hat{C}(C_p([0, \alpha], I))$.

Now, if $\omega < \alpha < \omega_1$, we have that $\hat{C}(C_p([0, \alpha], I)) \leq \text{cov}(|\alpha|^\omega)$ because of Corollary 4.2 in [1].

We are going to finish the proof by induction. Assume that the inequality $\hat{C}(C_p([0, \gamma], I)) \leq \text{cov}(|\gamma|^\omega)$ holds for every $\omega < \gamma < \alpha$. By Lemma 2.4 and inductive hypothesis, if $\alpha$ is a limit ordinal, then

$$\hat{C}(C_p([0, \alpha], I)) \leq |\alpha| \cdot \sup_{\gamma < \alpha} \text{cov}(|\gamma|^\omega) \leq \text{cov}(|\alpha|^\omega).$$

If $\alpha = \gamma_0 + 2$, then $\hat{C}(C_p([0, \alpha], I)) = \hat{C}(C_p([0, \gamma_0 + 1], I)) \leq \text{cov}(|\gamma_0 + 1|^\omega) = \text{cov}(|\alpha|^\omega)$.

Now assume that $\alpha = \gamma_0 + 1$, $\gamma_0$ is a limit and $\text{cof}(\gamma_0) = \omega$. We know by Lemma 2.2 that $\hat{C}(C_p([0, \gamma_0 + 1], I)) \leq \hat{C}(C_p([0, \gamma_0], I) \cdot \text{cof}(\gamma_0) = |\alpha|)$. So, by inductive hypothesis we obtain what is required.

The last possible case: $\alpha = \gamma_0 + 1$, $\gamma_0$ is limit and $\text{cof}(\gamma_0) > \omega$.

By Lemma 2.6, we have $\hat{C}(C_p([0, \gamma_0 + 1], I)) = |\alpha| \cdot \hat{C}(C_p([0, \gamma_0], I))$. By inductive hypothesis, $\hat{C}(C_p([0, \gamma_0], I) \leq \text{cov}(|\alpha|^\omega)$. Since $|\alpha| \leq \text{cov}(|\alpha|^\omega)$, we conclude that $\hat{C}(C_p([0, \alpha], I)) \leq \text{cov}(|\alpha|^\omega)$. □

As a consequence of Proposition 3.6 in [8] (see Proposition 2.11, below) and the previous Theorem, we obtain:

**Corollary 2.8.** For an ordinal number $\omega < \alpha < \omega_\omega$, $\hat{C}(C_p([0, \alpha], I)) = |\alpha| \cdot \mathfrak{d}$. 
In particular, we have:

**Corollary 2.9.** \( \hat{C}(C_p([0, \omega_1], I)) = \hat{C}(C_p([0, \omega_1], I)) = \emptyset \).

By using similar techniques to those used throughout this section we can also prove the following result.

**Corollary 2.10.** For every ordinal number \( \alpha > \omega \) and every \( 1 \leq n < \omega \),
\[
|\alpha| \cdot \mathfrak{d} \leq \hat{C}(C_p([0, \alpha]^n, I)) \leq \text{cov}(|\alpha|^n).
\]

For a generalized linearly ordered topological space \( X \), \( \chi(X) \leq \text{ec}(X) \), so \( \chi(X) \leq \hat{C}(C_p(X, I)) \), where \( \chi(X) \) is the character of \( X \). This is not the case for every topological space, even if \( X \) is a countable \( \text{EG} \)-space, as was pointed out by O. Okunev to the authors. Indeed, let \( X \) be a countable dense subset of \( C_p(I) \). We have that \( \chi(X) = \chi(C_p(I)) = \epsilon \) and \( \hat{C}(C_p(X, I)) = \emptyset \).

So, it is consistent with \( \text{ZFC} \) that there is a countable \( \text{EG} \)-space \( X \) with \( \chi(X) > \hat{C}(C_p(X, I)) \).

One is tempted to think that for every linearly ordered space \( X \), the relation \( \hat{C}(C_p(X, I)) \leq \text{cov}(\chi(X)^\omega) \) is plausible. But this illusion vanishes quickly: in fact, when \( \mathfrak{d} < 2^\omega \) and \( X \) is the double arrow, then \( X \) has countable character and \( \text{ec}(X) = |X| = 2^\omega \). Hence, \( \hat{C}(C_p(X, I)) \geq 2^\omega > \mathfrak{d} = \text{cov}(\chi(X)^\omega) \) (compare with Theorem 2.7, above, and Corollary 7.7 in [1]).

In [8] the following was remarked:

**Proposition 2.11.**

1. For every cardinal number \( \omega \leq \tau < \omega_\omega \), \( \text{cov}(\tau^\omega) = \tau \cdot \mathfrak{d} \).
2. For every cardinal \( \tau \geq \lambda \), \( \text{cov}((\tau^+)\lambda) = \tau^+ \cdot \text{cov}(\tau^\lambda) \).
3. If \( \text{cf}(\tau) > \lambda \), then \( \text{cov}(\tau^\lambda) = \tau \cdot \sup\{\text{cov}(\mu^\lambda) : \mu < \tau\} \).

**Lemma 2.12.** For every cardinal number \( \kappa \) with \( \text{cof}(\kappa) = \omega \), we have that \( \text{cov}(\kappa^\omega) > \kappa \).

**Proof.** Let \( \{K_\lambda : \lambda < \kappa\} \) be a collection of compact subsets of \( \kappa^\omega \). Let \( \alpha_0 < \alpha_1 < \ldots < \alpha_n < \ldots \) be an strictly increasing sequence of cardinal numbers converging to \( \kappa \). We are going to prove that \( \bigcup_{\lambda < \kappa} K_\lambda \) is a proper subset of \( \kappa^\omega \). Denote by \( \pi_n : \kappa^\omega \rightarrow \kappa \) the \( n \)-projection. Since \( \pi_n \) is continuous and \( K_\lambda \) is compact, \( \pi_n(K_\lambda) \) is a compact subset of the discrete space \( \kappa \), so, it is finite. Thus, we have that \( |\bigcup_{\lambda < \kappa_0} \pi_n(K_\lambda)| \leq \alpha_n < \kappa \) for each \( n < \omega \). Hence, for every \( n < \omega \), we can take \( \xi_n \in \kappa \setminus \bigcup_{\lambda < \alpha_n} \pi_n(K_\lambda) \). Consider the point \( \xi = (\xi_n)_{n < \omega} \) of \( \kappa^\omega \). We claim that \( \xi \notin \bigcup_{\lambda < \kappa} K_\lambda \). Indeed, assume that \( \xi \in K_{\lambda_0} \). There is \( n < \omega \) such that \( \lambda_0 < \alpha_n \). So, \( \xi_n \in \bigcup_{\lambda < \alpha_n} \pi_n(K_\lambda) \) which is not possible. \( \square \)

Recall that the Singular Cardinals Hypothesis (SCH) is the assertion:

For every singular cardinal number \( \kappa \), if \( 2^{\text{cof}(\kappa)} < \kappa \), then \( \kappa^{\text{cof}(\kappa)} = \kappa^+ \).

A proposition, apparently weaker than SCH, is: “for every cardinal number \( \kappa \) with \( \text{cof}(\kappa) = \omega \), if \( 2^\omega < \kappa \), then \( \kappa^\omega = \kappa^+ \)”. But this last assertion is equivalent to SCH as was settled by Silver (see [6], Theorem 23).
Proposition 2.13. If we assume SCH and \( c \leq (\omega_\omega)^+ \), and if \( \tau \) is an infinite cardinal number, then

\[
(*) \quad k\text{cov}(\tau^\omega) = \begin{cases} 
\tau \cdot \omega & \text{if } \omega \leq \tau < \omega_\omega \\
\tau & \text{if } \tau > \omega_\omega \text{ and } \text{cof}(\tau) > \omega \\
\tau^+ & \text{if } \tau > \omega \text{ and } \text{cof}(\tau) = \omega 
\end{cases}
\]

Proof. Our proposition is true for every \( \omega \leq \tau < \omega_\omega \) because of (1) in Proposition 2.11.

Assume now that \( \kappa \geq \omega_\omega \) and that (\*) holds for every \( \tau < \kappa \). We are going to prove the assertion for \( \kappa \).

Case 1: \( \text{cof}(\kappa) = \omega \). By Lemma 2.12, \( k\text{cov}(\kappa^\omega) > \kappa \). On the other hand, \( k\text{cov}(\kappa^\omega) < \kappa^\omega \).

First two subcases: Either \( c < \omega_\omega \) or \( \kappa > \omega_\omega \). In both subcases, we can apply SCH and conclude that \( k\text{cov}(\kappa^\omega) = \kappa^+ \).

Third subcase: \( c = (\omega_\omega)^+ \) and \( \kappa = \omega_\omega \). In this case we have \( k\text{cov}((\omega_\omega)^\omega) \leq (\omega_\omega)^\omega \leq c^\omega = c = (\omega_\omega)^+ \). Moreover, by Lemma 2.12, \( (\omega_\omega)^+ \leq k\text{cov}((\omega_\omega)^\omega) \). Therefore, \( k\text{cov}((\omega_\omega)^\omega) = (\omega_\omega)^+ \).

Case 2: \( \text{cof}(\kappa) > \omega \). By Proposition 2.11 (3), \( k\text{cov}(\kappa^\omega) = \kappa \cdot \sup\{k\text{cov}(\mu^\omega) : \omega \leq \mu < \kappa \} \). By inductive hypothesis we have that for each \( \mu < \kappa \)

\[
(***) \quad k\text{cov}(\mu^\omega) = \begin{cases} 
\mu \cdot \omega & \text{if } \omega \leq \mu < \omega_\omega \\
\mu & \text{if } \mu > \omega_\omega \text{ and } \text{cof}(\mu) > \omega \\
\mu^+ & \text{if } \mu > \omega \text{ and } \text{cof}(\mu) = \omega 
\end{cases}
\]

First subcase: \( \kappa \) is a limit cardinal. For every \( \mu < \kappa \), \( k\text{cov}(\mu^\omega) < \kappa \) (because of (***) and because we assumed that \( \kappa > (\omega_\omega)^+ \geq c \geq \omega \)); and so \( \sup\{k\text{cov}(\mu^\omega) : \mu < \kappa \} = \kappa \). Thus, \( k\text{cov}(\kappa^\omega) = \kappa \).

Second subcase: Assume now that \( \kappa = \mu_0^+ \). In this case, by Proposition 2.11, \( k\text{cov}(\kappa^\omega) = \kappa \cdot k\text{cov}(\mu_0^\omega) \). Because of (***) and because \( \mu_0 \geq \omega_\omega \), \( k\text{cov}(\mu_0^\omega) \leq \kappa \). We conclude that \( k\text{cov}(\kappa^\omega) = \kappa \). \( \Box \)

Proposition 2.14. Let \( \kappa \) be a cardinal number with \( \text{cof}(\kappa) = \omega \). Then

\[
\check{C}(C_p([0, \kappa], I)) > \kappa.
\]

Proof. Let \( 0 = \alpha_0 < \alpha_1 < \cdots < \alpha_n < \cdots \) be a strictly increasing sequence of cardinal numbers converging to \( \kappa \). Assume that \( \{V_\lambda : \lambda < \kappa \} \) is a collection of open sets in \( I^{[0, \kappa]} \) which satisfies \( C_p([0, \kappa], I) \subset \bigcap_{\lambda < \kappa} V_\lambda \). We are going to prove that \( \bigcap_{\lambda < \kappa} V_\lambda \) contains a function \( h : [0, \kappa] \to I \) which is not continuous. In order to construct \( h \), we are going to define, by induction, the following sequences:
(i) elements $t_0, \ldots, t_n, \ldots$ which belong to $[0, \kappa]$ such that
   1. $0 = t_0 < t_1 < \cdots < t_n < \ldots$,
   2. $t_i \geq \alpha_i$ for each $0 \leq i < \omega$,
   3. each $t_i$ is an isolated ordinal, and
   4. $\kappa = \lim(t_n)$;

(ii) subsets $G_0, \ldots, G_n, \ldots \subseteq [0, \kappa]$ with $|G_i| \leq \alpha_i$ for every $i < \omega$, and such that each function which equals $0$ in $G_i$ and $1$ in $\{t_0, \ldots, t_i\}$ belongs to $\bigcap_{\lambda < \alpha} V_\lambda$ for every $0 \leq i < \omega$ and $(\bigcup_n G_n) \cap \{t_0, \ldots, t_n, \ldots\} = \emptyset$;

(iii) functions $f_0, f_1, \ldots, f_n, \ldots$ such that $f_0 \equiv 0$, and $f_i$ is the characteristic function defined by $\{t_0, \ldots, t_i-1\}$ for each $0 < i < \omega$.

Let $f_0$ be the constant function equal to $0$. Assume that we have already defined $t_0, \ldots, t_{s-1}$, $G_0, \ldots, G_{s-1}$ and $f_0, \ldots, f_{s-1}$. We now choose an isolated point $t_s \in [\alpha_s, \kappa) \setminus G_0 \cup \ldots \cup G_{s-1}$ (this is possible because $|G_0 \cup \ldots \cup G_{s-1}| < \kappa$). Consider the characteristic function defined by $\{t_0, \ldots, t_{s-1}, t_s\}$, $f_s$. This function is continuous, so it belongs to $\bigcap_{\lambda < \alpha_s} V_\lambda$. For each $\lambda < \alpha_s$, there is a canonical open set $A_\lambda^s$ of the form $[f_s(x_1^s, \ldots, x_n^s(\lambda)); 1/m^s(\lambda)] = \{f \in I^{[0,\kappa]} : |f_s(x^s_1) - f(x)| < 1/m^s(\lambda) \forall 1 \leq i \leq n^s(\lambda)\}$ satisfying $f_s \in A_\lambda^s \subset V_\lambda$. For each $\lambda < \alpha_s$, we take $F_\lambda = [x_1^s, \ldots, x_n^s(\lambda)]$. Put $G_s = \bigcup_{\lambda < \alpha_s} F_\lambda \setminus \{t_0, \ldots, t_s\}$. It happens that $\{f \in I^{[0,\kappa]} : f(x) = 0 \forall x \in G_s$ and $f(t_i) = 1 \forall 0 < i < s\}$ is a subset of $\bigcap_{\lambda < \alpha_s} V_\lambda$. This finishes the inductive construction of the required sequences.

Now, consider the function $h : [0, \kappa] \rightarrow [0, 1]$ defined by $h(x) = 0$ if $x \not\in \{t_0, \ldots, t_n, \ldots\}$, and $h(t_n) = 1$ for every $n < \omega$. This function $h$ is not continuous at $\kappa$ because $h(\kappa) = 0$, $\kappa = \lim(t_n)$, and $h(t_n) = 1$ for all $n < \omega$.

Now, take $\alpha_0 \in \kappa$. There exists $l < \omega$ such that $\alpha_0 < \alpha_l$. Since $h$ is equal to $0$ in $G_l$ and $1$ in $\{t_0, \ldots, t_l\}$, then $h \in \bigcap_{\lambda < \alpha_l} V_\lambda$. Therefore, $h \in V_{\omega_\kappa}$. So, $C_p([0, \kappa], I)$ is not equal to $\bigcap_{\lambda < \kappa} V_\lambda$. This means that $\dot{C}(C_p([0, \kappa], I)) > \kappa$. □

**Theorem 2.15.** $SCH + \varepsilon \leq (\omega_\kappa)^+$ implies:

\[
\dot{C}(C_p([0, \alpha], I)) = \begin{cases} \\
1 & \text{if } \alpha \leq \omega \\
|\alpha| \cdot \varepsilon & \text{if } \alpha > \omega \text{ and } \omega \leq |\alpha| < \omega_\omega \\
|\alpha| & \text{if } |\alpha| > \omega_\omega \text{ and } \text{cof}(|\alpha|) > \omega \\
|\alpha|^+ & \text{if } \text{cof}(|\alpha|) = \omega \text{ and } \alpha \text{ is a cardinal number} > \omega_\omega \\
|\alpha| & \text{if } |\alpha| = \omega_\omega \text{ and } \varepsilon < (\omega_\kappa)^+ \\
|\alpha|^+ & \text{if } |\alpha| = \omega_\omega \text{ and } \varepsilon = (\omega_\kappa)^+ \\
\end{cases}
\]

**Proof.** If $\alpha \leq \omega$, $C_p([0, \alpha], I) = I^{[0,\alpha]}$, so $\dot{C}(C_p([0, \alpha], I)) = 1$.

If $\alpha > \omega$ and $\omega \leq |\alpha| < \omega_\omega$, we obtain our result because of Theorem 2.7 and Proposition 2.13.

If $|\alpha| > \omega_\omega$ and $\text{cof}(|\alpha|) > \omega$, by Theorem 2.7 and Proposition 2.13,

\[
|\alpha| \cdot \varepsilon = |\alpha| \leq \dot{C}(C_p([0, \alpha], I)) \leq \text{kcov}(|\alpha|^\omega) = |\alpha|.
\]
Thus, if $\text{cof}(|\alpha|) = \omega$ and $\alpha$ is a cardinal number $> \omega_\omega$, by Lemma 2.4,
\[
\check{C}(C_p([0, \alpha), I)) = |\alpha| \cdot \sup_{\gamma < \alpha} \check{C}(C_p([0, \gamma), I))
\]
The number $\alpha$ is a limit ordinal and for every $\gamma < \alpha$,
\[
\check{C}(C_p([0, \gamma), I)) \leq |\gamma|^+ \cdot \mathfrak{d}.
\]
Since $\mathfrak{d} \leq (\omega_\omega)^+ < |\alpha|$, then $\check{C}(C_p([0, \alpha), I)) = |\alpha|$.

By Lemma 2.4 and Theorem 2.7, if $|\alpha| = \omega_\omega$, then
\[
\omega_\omega \cdot \mathfrak{d} \leq \check{C}(C_p([0, \alpha), I)) = |\alpha| \cdot \sup_{\gamma < \alpha} \check{C}(C_p([0, \gamma), I)) \leq |\alpha| \cdot \sup_{\gamma < \alpha} (|\gamma|^+ \cdot \mathfrak{d}).
\]
Thus, if $|\alpha| = \omega_\omega$ and $\mathfrak{d} < (\omega_\omega)^+$, $\check{C}(C_p([0, \alpha), I)) = |\alpha|$. Assume now that $\text{cof}(|\alpha|) = \omega$, $|\alpha| > \omega_\omega$ and $\alpha$ is not a cardinal number. There exists a cardinal number $\kappa$ such that $\kappa = |\alpha|$ and $[0, \alpha) = [0, \kappa) \oplus [\kappa + 1, \alpha)$. So, $\check{C}(C_p([0, \alpha), I)) = \check{C}(C_p([0, \kappa), I)) \cdot \check{C}(C_p(\kappa, 1, \alpha), I)) = \check{C}(C_p([0, \kappa), I))$ (see Proposition 1.10 in [8] and Lemma 2.3). By Theorem 2.7 and Proposition 2.14, $\kappa \cdot \mathfrak{d} \leq \check{C}(C_p([0, \kappa), I)) \leq \kappa^+$. Being $\kappa$ a cardinal number $> \omega_\omega$ with cofinality $\omega$, it must be $> (\omega_\omega)^+$; so $\kappa > \mathfrak{d}$ and, then, $\kappa \leq \check{C}(C_p([0, \kappa), I)) \leq \kappa^+$. Now we use Proposition 2.14, and conclude that $\check{C}(C_p([0, \alpha), I)) = \kappa^+ = |\alpha|^+$. Finally, assume that $|\alpha| = \omega_\omega$ and $\mathfrak{d} = (\omega_\omega)^+$. By Theorems 2.7 and Proposition 2.13 we have
\[
|\alpha| \cdot \mathfrak{d} \leq \check{C}(C_p([0, \alpha), I)) \leq k_{cov}(|\alpha|^+) = (\omega_\omega)^+.
\]
And we conclude: $\check{C}(C_p([0, \alpha), I)) = |\alpha|^+$. \hfill \Box

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