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Product metrics and boundedness

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Abstract. This paper looks at some possible ways of equipping a countable product of unbounded metric spaces with a metric that acknowledges the boundedness characteristics of the factors.

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1. Introduction

Let $\langle X, d\rangle$ be an unbounded metric space. A net $\langle x_\lambda\rangle_{\lambda\in\Lambda}$ in X based on a directed set Λ is called *convergent to infinity in distance* if eventually $\langle x_\lambda\rangle$ stays outside of each d-bounded set: whenever B is contained in some d-ball, there exists $\lambda_0 \in \Lambda$ such that $\lambda \geqslant \lambda_0 \Rightarrow x_\lambda \notin B$. With $S_\alpha(x)$ representing the open ball of radius α and center x, this condition can be reformulated in any of these equivalent ways:

(1) $\forall x \in X$ and $\alpha > 0$, $\langle x_\lambda\rangle$ is eventually outside of $S_\alpha(x)$;
(2) $\forall x \in X$ we have $\lim_\lambda d(x_\lambda, x) = \infty$;
(3) $\exists x_0 \in X$ with $\lim_\lambda d(x_\lambda, x_0) = \infty$.

Now if $\{\langle X_n, d_n\rangle : n \leqslant n_0\}$ is a finite family of metric spaces, there are a number of standard ways to give the product $\prod_{n=1}^{n_0} X_n$ a metric compatible with the product topology, the most familiar of which are these [11, pg. 111]:

(1) $\rho_1(x, w) = \max\{d_n(\pi_n(x), \pi_n(w)) : n \leqslant n_0\}$;
(2) $\rho_2(x, w) = \sum_{n=1}^{n_0} d_n(\pi_n(x), \pi_n(w))$;
(3) $\rho_3(x, w) = \sqrt{\sum_{n=1}^{n_0} d_n(\pi_n(x), \pi_n(w))^2}$.

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All three of the metrics determine the same class of unbounded sets, and a net $\langle x_\lambda \rangle$ in the product is convergent to infinity in ρ_i-distance for each i if and only if

$$(i)\ \forall x \in \prod_{n=1}^{n_0} X_n \text{ we have } \lim_\lambda \max_{n \leqslant n_0} d_n(\pi_n(x_\lambda), \pi_n(x)) = \infty,$$

or equivalently,

$$(ii)\ \exists x_0 \in \prod_{n=1}^{n_0} X_n \text{ with } \lim_\lambda \max_{n \leqslant n_0} d_n(\pi_n(x_\lambda), \pi_n(x_0)) = \infty.$$

For example, in $\mathbb{R}^2$ equipped with any of the standard metrics, the sequence $\langle (j, 0) \rangle$ is deemed convergent to infinity even though the second coordinate sequence is constant. Thus, while convergence in the product with respect to each of the standard product metrics to a finite point amounts to convergence in each coordinate, this is not the case with respect to convergence to infinity in distance. This lack of symmetry is a little odd.

Something entirely different occurs when considering a countably infinite family of unbounded metric spaces $\{\langle X_n, d_n \rangle : n \in \mathbb{N}\}$. The standard way to define a metric on $\prod_{n=1}^\infty X_n$ equipped with the product topology is this [11, 8]:

$$\rho_\infty(x, w) = \sum_{n=1}^\infty 2^{-n} \min\{1, d_n(\pi_n(x), \pi_n(w))\}.$$

The standard product metric is of course a bounded metric and all the boundedness features of the coordinate spaces are obliterated. In particular, no sequence in the countable product can converge to infinity in ρ_∞-distance. While one can dispense with the weights in finitely many factors and permit convergence to infinity in a restricted setting, this construction, while having the desirable local comportment, is myopic, speaking both figuratively and literally. For a product metric expressed as a supremum but with the same limitations, see [7, pg. 190].

It is natural to consider, in the case of countably infinitely many coordinates, the natural analogs of conditions (i) and (ii) above, namely,

$$(i')\ \forall x \in \prod_{n=1}^{\infty} X_n \text{ we have } \lim_\lambda \sup_{n \in \mathbb{N}} d_n(\pi_n(x_\lambda), \pi_n(x)) = \infty,$$

$$(ii')\ \exists x_0 \in \prod_{n=1}^{\infty} X_n \text{ with } \lim_\lambda \sup_{n \in \mathbb{N}} d_n(\pi_n(x_\lambda), \pi_n(x_0)) = \infty.$$

As in the case of finitely many factors, the existence of some coordinate for which $\langle \pi_n(x_\lambda) \rangle$ converges to infinity in d_n-distance is sufficient but not necessary for convergence in the of sense (i'). Actually, it is easier to understand what it means for (i') to fail than for it to hold.

Proposition 1.1. *Let $\langle x_\lambda \rangle_{\lambda\in\Lambda}$ be a net in a product of unbounded metric spaces $\prod_{n=1}^{\infty} X_n$. The following conditions are equivalent:*

(1) *Condition (i') does not hold for $\langle x_\lambda \rangle_{\lambda\in\Lambda}$;*
(2) *there exists a cofinal subset Λ_0 of Λ and $\alpha > 0$ such that $\forall n \in \mathbb{N}$ we have*
diam $(\{\pi_n(x_\lambda) : \lambda \in \Lambda_0\}) < \alpha$.

Proof. Suppose (i') does not hold; pick x in the product and $\alpha > 0$ such that

$$\liminf_\lambda \sup_{n\in\mathbb{N}} d_n(\pi_n(x_\lambda), \pi_n(x)) < \frac{\alpha}{2}.$$

We can then find Λ_0 cofinal in Λ such that $\forall n \in \mathbb{N}\ \forall \lambda \in \Lambda_0$ we have

$$d_n(\pi_n(x_\lambda), \pi_n(x)) < \frac{\alpha}{2},$$

and so

$$\forall n \in \mathbb{N}\ \forall \lambda \in \Lambda_0\ \pi_n(x_\lambda) \in S_{\frac{\alpha}{2}}(\pi_n(x)),$$

from which (2) follows. Conversely, if (2) holds, fix $\lambda \in \Lambda_0$ and set $x_1 = x_{\lambda_0}$. Then $\forall \lambda \in \Lambda_0$ we have

$$\sup_{n\in\mathbb{N}} d_n(\pi_n(x_\lambda), \pi_n(x_1)) \leqslant \alpha,$$

and as a result,

$$\liminf_\lambda \sup_{n\in\mathbb{N}} d_n(\pi_n(x_\lambda), \pi_n(x_1)) \leqslant \alpha,$$

so that condition (i') fails. □

On the other hand, condition (ii') is much too weak to be useful, for if x_0 is a given point of the product and x_1 is a second point satisfying $\sup_{n\in\mathbb{N}} d_n(\pi_n(x_1), \pi_n(x_0)) = \infty$, then the constant sequence each of whose terms is x_1 obviously satisfies (ii') but not (i').

There are two main objectives of this note. First, while (i') may be worthy of study as a generalization of convergence to infinity with respect to the ℓ_∞-metric, we intend to show that no metric exists on $\prod_{n=1}^{\infty} X_n$ - compatible with the product topology or otherwise - with respect to which convergence in the sense of (i') corresponds to convergence to infinity in distance. In other words, it is impossible to find a metric compatible with *any* metrizable topology on the product of a countably infinite collection of unbounded metric spaces such that convergence of nets to infinity in distance generalizes what occurs with respect our standard metrics when there are only finitely many factors. Second, we display an unbounded metric compatible with the product topology with respect to which convergence to infinity in distance means convergence to infinity in distance in *all* coordinates.

2. An Alternate Product Metric

We address our objectives in reverse order. To construct our metric, we use a standard device [8, pg. 347] : if $\langle X, d\rangle$ is a metric space and f is a continuous real-valued function on X then $d_f : X \to [0, \infty)$ defined by

$$d_f(x, w) = d(x, w) + |f(x) - f(w)|$$

is a metric on X equivalent to d.

Theorem 2.1. *Let $\{\langle X_n, d_n\rangle : n \in \mathbb{N}\}$ be a family of unbounded metric spaces. Then there exists an unbounded metric ρ on $\prod_{n=1}^{\infty} X_n$ compatible with the product topology such that a net in the product is convergent to infinity in ρ-distance if and only if it is convergent to infinity coordinatewise with respect to each of the coordinate metrics d_n.*

Proof. We start with the standard bounded metric ρ_∞ on the product and modify it by a continuous real-valued function f as indicated above. Formally, we define f to be an infinite sum of nonnegative continuous functions $\{f_k : k \in \mathbb{N}\}$ each defined on $\prod_{n=1}^{\infty} X_n$. Fix $x_0 \in \prod_{n=1}^{\infty} X_n$, and for each k, we define f_k by the formula

$$f_k(x) = \min_{n \leqslant k} \min\{1,\ d_n(\pi_n(x), S_k(\pi_n(x_0))\}.$$

The following three properties of f_k are evident from the definition:

(1) f_k is a continuous function with respect to the product topology;
(2) $\forall x \in \prod_{n=1}^{\infty} X_n\ \forall k \in \mathbb{N},\ 0 \leqslant f_{k+1}(x) \leqslant f_k(x) \leqslant 1$;
(3) if $\exists n \leqslant k$ such that $\pi_n(x) \in S_k(\pi_n(x_0))$, then $f_k(x) = 0$.

In addition we have the following key property:

(4) $\forall x \in \prod_{n=1}^{\infty} X_n\ \exists k_0 \in \mathbb{N}$ such that f_{k_0} vanishes on some neighborhood of x.

To establish property (4), choose k_0 such that $\pi_1(x) \in S_{k_0}(\pi_1(x_0))$. Then for each $w \in \pi_1^{-1}[S_{k_0}(\pi_1(x_0)]$, a product neighborhood of x, we have $f_{k_0}(w) = 0$. From properties (2) and (4), setting

$$E_k = \{x : f_k(x) > 0\} \quad (k \in \mathbb{N}),$$

we see that the family $\{E_k : k \in \mathbb{N}\}$ is locally finite. It now follows that $f : \prod_{n=1}^{\infty} X_n \to [0, \infty)$ defined by $f = f_1 + f_2 + f_3 + \cdots$ is real-valued and continuous.

We are now ready to define the desired metric ρ on the product:

$$\rho(x, w) := \rho_\infty(x, w) + |f(x) - f(w)|.$$

As we indicated earlier, ρ is compatible with the product topology. Now a net $\langle x_\lambda\rangle_{\lambda\in\Lambda}$ in the product converges to infinity in ρ-distance if and only if $\lim_\lambda \rho(x_\lambda, x_0) = \infty$, and since for all λ $\rho_\infty(x_\lambda, x_0) \leqslant 1$, this occurs if and only $\lim_\lambda f(x_\lambda) = \infty$. We first show, assuming $\lim_\lambda f(x_\lambda) = \infty$, that for each $n \in \mathbb{N}$, $\langle \pi_n(x_\lambda)\rangle$ converges to infinity in d_n-distance. To this end, fix $n \in \mathbb{N}$, say $n = n_0$. We will show that if k_0 is an arbitrary positive integer, then for all λ sufficiently large,

$$\pi_{n_0}(x_\lambda) \notin S_{k_0}(\pi_{n_0}(x_0))$$

There is no loss in generality in assuming $k_0 > n_0$. Pick $\lambda_0 \in \Lambda$ such that $\lambda \geqslant \lambda_0 \Rightarrow f(x_\lambda) > k_0 - 1$. Fix $\lambda \geqslant \lambda_0$; if $\pi_{n_0}(x_\lambda) \in S_{k_0}(\pi_{n_0}(x_0))$ were true, then by properties (2) and (3) $\forall k \geqslant k_0$ we have $f_k(x_\lambda) = 0$. As a result, we have

$$f(x_\lambda) = \sum_{k=1}^{\infty} f_k(x_\lambda) = \sum_{k=1}^{k_0-1} f_k(x_\lambda) \leqslant k_0 - 1$$

This contradiction shows that for all $\lambda \geqslant \lambda_0$ we have $\pi_{n_0}(x_\lambda) \notin S_{k_0}(\pi_{n_0}(x_0))$ as required.

Conversely, suppose $\forall n$ that $\langle \pi_n(x_\lambda) \rangle$ converges to infinity in d_n-distance. Again fix $k_0 \in \mathbb{N}$; we intend to show that eventually $f(x_\lambda) \geqslant k_0$. Pick $\lambda_0 \in \Lambda$ such that condition $(*)$ below holds:

$$(*) \qquad \forall \lambda \geqslant \lambda_0 \; \forall n \leqslant k_0 + 1, \; \pi_n(x_\lambda) \notin S_{k_0+1}(\pi_n(x_0)).$$

Now fix $\lambda \geqslant \lambda_0$. By $(*), \forall k \leqslant k_0 \; \forall n \leqslant k$ we have

$$d_n(\pi_n(x_\lambda), S_k(\pi_n(x_0)) \geqslant 1,$$

and as a result $\forall k \leqslant k_0$ we have $f_k(x_\lambda) = 1$. We conclude that

$$f(x_\lambda) \geqslant \sum_{k=1}^{k_0} f_k(x_\lambda) = k_0$$

as required. □

The proof presented above goes through in the case that the product is finite, say, $\prod_{n=1}^{n_0} X_n$, by slightly altering the definition of each f_k as follows:

$$f_k(x) = \min_{n \leqslant n_0} \big\{ \min\{1, d_n(\pi_n(x), S_k(\pi_n(x_0))\} \big\}.$$

When the finite product is $\mathbb{R}^{n_0}$, the author's metric of choice is the following one:

$$\rho(x, w) = \min\{1, \max_{n \leqslant n_0} |\pi_n(x) - \pi_n(w)|\} + |\min_{n \leqslant n_0} |\pi_n(x)| - \min_{n \leqslant n_0} |\pi_n(w)||.$$

3. Convergence to Infinity in Distance and Bornologies

There is another way to approach the question of the existence of the metric that Theorem 2.1 provides, following an axiomatic approach to boundedness developed by S.-T. Hu [10, 11] over 50 years ago. Hu discovered that the family $\mathscr{B}_d$ of bounded sets determined by an unbounded metric d on a metrizable space X had certain characteristic properties. First, the bounded sets form a *bornology* [9, 1, 3, 4, 12]; that is, they form of a cover of X that is closed under taking finite unions and subsets. Second, X is not itself in the bornology. Third, $\mathscr{B}_d$ has a countable base $\{B_n : n \in \mathbb{N}\}$, i.e., each bounded set is contained in some B_n. Finally, for each element $B \in \mathscr{B}_d$, there exists B' in the bornology with $\mathrm{cl}(B) \subseteq \mathrm{int}(B')$. Conversely, if $\mathscr{A}$ is a bornology with a countable base on a noncompact metrizable space X, $X \notin \mathscr{A}$, and $\forall A \in \mathscr{A}\ \exists A' \in \mathscr{A}$ with $\mathrm{cl}(A) \subseteq \mathrm{int}(A')$, then there exists a compatible unbounded metric d such that $\mathscr{A} = \mathscr{B}_d$. A bornology that satisfies Hu's axioms or coincides with the power set $\mathscr{P}(X)$ of X (the bornology of a bounded metric) is called a *metric bornology*[3]. For example, the bornology consisting of the subsets of X with compact closure is a metric bornology if and only if X is locally compact and separable [14]. As is well-known, X is compact if and only if there is exactly one metrizable bornology, namely $\mathscr{P}(X)$. It can be shown [1] that if X is noncompact and metrizable, there is actually an uncountable family of compatible metrics $\{d : d \in D\}$ whose associated metric bornologies $\{\mathscr{B}_d : d \in D\}$ are distinct. In particular, the usual metric on the real line $\mathbb{R}$ is just one of many (in terms of boundedness) compatible with the usual topology.

With Hu's result in mind, let's return to the context of a product of a family $\{\langle X_n, d_n\rangle : n \in \mathbb{N}\}$ of unbounded metric spaces. Again fixing x_0 in the product, consider the bornology on $\prod_{n=1}^{\infty} X_n$ having as a countable base all sets of the form

$$\begin{aligned}\triangle(k,F) &:= \{x : \exists n \in F\ \pi_n(x) \in S_k(\pi_n(x_0))\}\\ &= \bigcup_{n\in F} \pi_n^{-1}(S_k(\pi_n(x_0))) \qquad (k \in \mathbb{N},\ F \text{ a finite subset of } \mathbb{N}).\end{aligned}$$

It is easy to verify that Hu's axioms all are verified, and in particular that relative to the product topology, one has

$$\mathrm{cl}(\triangle(k,F)) \subseteq \mathrm{int}(\triangle(k+1,F)) = \triangle(k+1,F).$$

Now if ρ is an unbounded metric whose bounded sets coincide with this bornology, and a net $\langle x_\lambda\rangle_{\lambda\in\Lambda}$ converges to infinity in ρ-distance, then for each n and k, the net is outside $\triangle(k,\{n\})$ eventually which means that $\langle \pi_n(x_\lambda)\rangle$ converges to infinity in d_n-distance. On the other hand, if for each fixed k and n, $\langle \pi_n(x_\lambda)\rangle$ is outside $S_k(\pi_n(x_0))$ eventually, then for any finite set of integers F, eventually $\langle x_\lambda\rangle$ is outside of $\triangle(k,F)$, and so $\langle x_\lambda\rangle$ converges to infinity in ρ-distance.

By definition a net in $\langle X, d\rangle$ is convergent to infinity in d-distance if it is eventually outside of each element of $\mathscr{B}_d$. Abstracting from this, given a bornology $\mathscr{B}$ on X, we say $\langle x_\lambda\rangle_{\lambda\in\Lambda}$ is *convergent to infinity with respect to* $\mathscr{B}$ if for each $B \in \mathscr{B}$ there exists $\lambda_0 \in \Lambda$ such that $\lambda \geqslant \lambda_0 \Rightarrow x_\lambda \notin B$. Observe, that there is no loss of generality in defining this notion for bornologies rather than for covers, and that nets cannot simultaneously converge to infinity with respect to a bornology and to a finite point if and only if X is *locally bounded* [10]: each $x \in X$ has a neighborhood in the bornology (see also [6, Proposition 2.7]). Local boundedness of course implies that each compact set is in the bornology [10]. This all leads naturally to an investigation of extensions of the space and their relation to bornologies that is outside the scope of this paper (see [2, 5, 6]).

To show that convergence to infinity with respect to a bornology is more generally a useful notion, we offer three simple propositions.

Proposition 3.1. *Let X be a metrizable space and let $\mathscr{F}$ be the bornology of finite subsets of X. Then a sequence $\langle x_j \rangle$ is convergent to infinity with respect to $\mathscr{F}$ if and only if $\langle x_j \rangle$ has no constant subsequence.*

Proposition 3.2. *Let X be a metrizable space and let $\mathscr{B}$ be the bornology of subsets of X with compact closure. Then a sequence $\langle x_j \rangle$ is convergent to infinity with respect to $\mathscr{B}$ if and only if $\langle x_j \rangle$ has no convergent subsequence.*

Proof. If $\langle x_j \rangle$ has a convergent subsequence $\langle x_{j_n} \rangle$ to a point p, then the original sequence is not eventually outside the compact set $\{p, x_{j_1}, x_{j_2}, x_{j_3}, \ldots\}$. Sufficiency is obvious. □

Proposition 3.3. *Let X be a normed linear space and let $\mathscr{B}$ be the bornology of weakly bounded subsets of X. Then a net $\langle x_\lambda \rangle_{\lambda \in \Lambda}$ is convergent to infinity with respect to $\mathscr{B}$ if and only if $\forall \alpha > 0\ \exists \lambda_0 \in \Lambda$ such that $\lambda > \lambda_0 \Rightarrow ||x_\lambda|| > \alpha$.*

Proof. Recall that $A \subseteq X$ is weakly bounded if $\forall f \in X^*, f(A)$ is a bounded set of scalars. Evidently, the weakly bounded sets so defined also form a bornology. Now the Uniform Boundedness Principle of functional analysis [13], when applied to the Banach space X^* equipped with the usual operator norm, says that each weakly bounded subset of X is norm bounded. As the converse is obviously true, the bornology of weakly bounded sets coincides with the metric bornology determined by the norm. □

The next proposition and its corollary show that the relative size of two bornologies is determined by the set of nets that converge to infinity with respect to them.

Proposition 3.4. *Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be bornologies on a metrizable space X. The following conditions are equivalent:*

(1) $\mathscr{B}_2 \subseteq \mathscr{B}_1$;
(2) *whenever a net $\langle x_\lambda \rangle_{\lambda \in \Lambda}$ converges to infinity with respect to $\mathscr{B}_1$, then $\langle x_\lambda \rangle_{\lambda \in \Lambda}$ converges to infinity with respect to $\mathscr{B}_2$.*

Proof. Only the implication (2) $\Rightarrow$ (1) requires proof. Suppose (1) fails; then there exists $B_2 \in \mathscr{B}_2$ that is not a subset of any element of $\mathscr{B}_1$. Now since $\mathscr{B}_1$ is closed under finite unions, it is directed by inclusion. For each $B \in \mathscr{B}_1$, pick $x_B \in B_2 \cap B^c$. Then the net $\langle x_B \rangle_{B \in \mathscr{B}_1}$ converges to infinity with respect to $\mathscr{B}_1$ but not with respect to $\mathscr{B}_2$. □

Corollary 3.5. *Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be bornologies on a metrizable space X. The following conditions are equivalent:*

(1) $\mathscr{B}_1 = \mathscr{B}_2$;
(2) *$\mathscr{B}_1$ and $\mathscr{B}_2$ determine the same nets convergent to infinity.*

The next example shows that the same set of *sequences* can converge to infinity for distinct bornologies that do not have countable bases.

Example 3.6. In the real line $\mathbb{R}$, consider these two bornologies:

$$\mathscr{B}_1 = \{A \cup F : A \text{ is a countable subset of } \mathbb{N}^c \text{ and } F \text{ is finite}\},$$

$$\mathscr{B}_2 = \{A \cup E \cup F : A \text{ is a countable subset of } \mathbb{N}^c,\ E \subseteq (0,1),\ \text{and } F \text{ is finite}\}.$$

Observe that neither has a countable base. While $\mathscr{B}_2$ properly contains $\mathscr{B}_1$, the bornologies determine the same sequences convergent to infinity. Specifically, $\langle x_j \rangle$ converges to infinity with respect to either if and only if $\langle x_j \rangle$ has no constant subsequence and eventually is in $\mathbb{N}$.

Using Hu's axioms and Corollary 3.5, we can directly verify that convergence to infinity as described by condition (i') in the introduction is not convergence to infinity with respect to any metric on the product. Now convergence of nets in this sense is obviously convergence to infinity for a bornology $\mathscr{B}$ on $\prod_{n=1}^{\infty} X_n$ having as a base all finite unions of sets of the form

$$B(w,k) := \{x : \forall n \in \mathbb{N}\ \pi_n(x) \in S_k(\pi_n(w))\} = \prod_{n=1}^{\infty} S_k(\pi_n(w))$$

where k runs over $\mathbb{N}$ and w runs over $\prod_{n=1}^{\infty} X_n$ (note the family of all sets of the form $\prod_{n=1}^{\infty} S_k(\pi_n(w))$ is not directed by inclusion). We claim that this $\mathscr{B}$ does not have a countable base. If it did we could find a sequence of the form $\langle (w_j, k_j) \rangle$ such that

$$\{\cup_{j=1}^{n} B(w_j, k_j) : n \in \mathbb{N}\}$$

forms a countable base for $\mathscr{B}$. In fact, no such countable family even forms a cover of the product. To see this, take $x \in \prod_{n=1}^{\infty} X_n$ where $\pi_n(x) \notin \cup_{j=1}^{n} S_{k_j}(\pi_n(w_j))$.

A stronger notion than convergence to infinity in distance coordinatewise is that the convergence be uniform coordinatewise, according to the following definition.

Definition 3.7. *A net $\langle x_\lambda \rangle_{\lambda \in \Lambda}$ in a product $\prod_{n=1}^{\infty} X_n$ of unbounded metric spaces is said to converge coordinatewise to infinity in distance uniformly with respect to $x_0 \in \prod_{n=1}^{\infty} X_n$ if $\forall \alpha > 0\ \exists \lambda_0 \in \Lambda$ such that whenever $\lambda \geqslant \lambda_0$, we have*

$$inf_{n \in \mathbb{N}}\ d_n(\pi_n(x_\lambda), \pi_n(x_0)) > \alpha.$$

Definition 3.7 was formulated for a countably infinite product only because for a finite product, the concept is no stronger than convergence to infinity in distance coordinatewise which we have already discussed.

Example 3.8. A sequence $\langle x_j \rangle$ in a countably infinite product can converge to infinity in distance coordinatewise but not uniformly with respect to any x_0 in the space. For our product take $\mathbb{N}^\infty$, that is, the product of countably many copies of the positive integers each equipped with the usual metric of the line. For each j define $x_j \in \mathbb{N}^\infty$ by

$$\pi_n(x_j) = \begin{cases} j - n + 1 & \text{if } n \leqslant j \\ 1 & \text{if } n > j. \end{cases}$$

As a particular case, $\langle x_4 \rangle$ is the sequence 4, 3, 2, 1, 1, 1, … . For each coordinate index n we have $\lim_{j\to\infty}\pi_n(x_j) = \lim_{j\to\infty} j-n+1 = \infty$, establishing coordinatewise convergence to infinity with respect to the usual metric. We claim that assuming that the convergence is uniform with respect to some x_0 leads to a contradiction.

If this occurs, then in particular $\exists j_0 \in \mathbb{N}$ such that whenever $j \geqslant j_0$, we have

$$(*) \qquad \inf_{n\in\mathbb{N}}|\pi_n(x_j) - \pi_n(x_0)| > 1,$$

and in particular,

$$|\pi_{j_0}(x_{j_0}) - \pi_{j_0}(x_0)| > 1.$$

Now $\pi_{j_0}(x_{j_0}) = 1$, and so $\pi_{j_0}(x_0) \in \{3, 4, 5, \ldots\}$. Set $k = \pi_{j_0}(x_0) - 1$; then $j_0 + k > j_0$ and we compute

$$|\pi_{j_0}(x_{j_0+k}) - \pi_{j_0}(x_0)| = |(j_0 + k) - j_0 + 1 - (k + 1)| = 0$$

and a contradiction to $(*)$ is obtained as claimed.

If a net $\langle x_\lambda \rangle_{\lambda\in\Lambda}$ in a product $\prod_{n=1}^{\infty} X_n$ of unbounded metric spaces converges coordinatewise to infinity in distance uniformly with respect to x_0, then this is also true if we replace x_0 by any x with

$$\sup_{n\in\mathbb{N}} d_n(\pi_n(x), \pi_n(x_0)) < \infty.$$

On the other hand, given any sequence $\langle x_j \rangle$ in the product, coordinatewise convergence to infinity uniformly with respect to *all* points w in the product is impossible: for example, a "bad point" w with respect to $\langle x_j \rangle$ is defined by $\pi_j(w) = \pi_j(x_j)$ for $j = 1, 2, 3, \ldots$. Hopefully, this discussion will give the reader a feeling for the nature of the dependence of this mode of convergence on x_0.

Now convergence of $\langle x_\lambda \rangle_{\lambda\in\Lambda}$ to infinity as described by Definition 3.7 is clearly convergence to infinity with respect to a bornology $\mathscr{B}$ on $\prod_{n=1}^{\infty} X_n$ having a countable base consisting of those product open sets of the form

$$B_k := \{x : \exists n \in \mathbb{N}\ \pi_n(x) \in S_k(\pi_n(x_0))\} = \cup_{n\in\mathbb{N}}\ \pi_n^{-1}(S_k(\pi_n(x_0))) \quad (k \in \mathbb{N}).$$

Since each B_k is in fact dense with respect to the product topology, Hu's axioms are not satisfied, so that by Corollary 3.5 there is no metric ρ compatible with the product topology such that convergence to infinity in ρ-distance equates with convergence as described by Definition 3.7. But the situation is salvageable, provided we are willing to relinquish the product topology in favor of a stronger metrizable one, namely, the topology determined by the bounded metric

$$\rho_{uc}(x, w) = \min\big\{1,\ \sup\{d_n(\pi_n(x), \pi_n(w)) : n \in \mathbb{N}\}\big\}.$$

When all $\langle X_n, d_n\rangle$ are the same unbounded metric space $\langle X, d\rangle$, so that our product is of the form X^∞, this is the metric of uniform convergence for sequences in $\langle X, d\rangle$. Using the formula

$$\mathrm{cl}_{uc}(B_k) = \{x : \inf_{n\in\mathbb{N}} d_n(\pi_n(x), S_k(\pi_n(x_0))) = 0\}$$

and keeping in mind that the product topology is coarser than the ρ_{uc}-topology, is easy to check that the bornology on $\prod_{n=1}^{\infty} X_n$ with base $\{B_k : k \in \mathbb{N}\}$ satisfies Hu's axioms with respect to the ρ_{uc}-topology. Thus, we can remetrize the product equipped with this stronger topology in a way that convergence to infinity in distance for the metric equates with Definition 3.7. We leave it to the imagination of the reader to come up with possible formulas for such a metric.

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