Generalizations of $Z$-supercontinuous functions and $D_\delta$-supercontinuous functions

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**Abstract.** Two new classes of functions, called ‘almost $z$-supercontinuous functions’ and ‘almost $D_\delta$-supercontinuous functions’ are introduced. The class of almost $z$-supercontinuous functions properly includes the class of $z$-supercontinuous functions (Indian J. Pure Appl. Math. 33(7), (2002), 1097-1108) as well as the class of almost clopen maps due to Ekici (Acta. Math. Hungar. 107(3), (2005), 193-206) and is properly contained in the class of almost $D_\delta$-supercontinuous functions which in turn constitutes a proper subclass of the class of almost strongly $\theta$-continuous functions due to Noiri and Kang (Indian J. Pure Appl. Math. 15(1), (1984), 1-8) and which in its turn include all $\delta$-continuous functions of Noiri (J. Korean Math. Soc. 16 (1980), 161-166). Characterizations and basic properties of almost $z$-supercontinuous functions and almost $D_\delta$-supercontinuous functions are discussed and their place in the hierarchy of variants of continuity is elaborated. Moreover, properties of almost strongly $\theta$-continuous functions are investigated and sufficient conditions for almost strongly $\theta$-continuous functions to have $u_\theta$-closed ($\theta$-closed) graph are formulated.

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1. INTRODUCTION

Among several of the variants of continuity in the literature, some are stronger than continuity and some are weaker than continuity and yet others are independent of continuity. In this paper we introduce two new variants of continuity which represent generalizations of the notions of \( z \)-supercontinuity and \( D_\delta \)-supercontinuity and are independent of continuity and coincide with \( z \)-supercontinuity and \( D_\delta \)-supercontinuity, respectively if the range is a semiregular space. The class of almost \( z \)-supercontinuous functions besides containing the class of \( z \)-supercontinuous functions contains the class of almost clopen (≡ almost cl-supercontinuous [34]) functions defined by Ekici [4].

Characterizations and basic properties of almost \( z \)-supercontinuous (almost \( D_\delta \)-supercontinuous) functions are elaborated in Section 3 and their place in the hierarchy of variants of continuity is discussed. Section 4 is devoted to the study of the behaviour of separation axioms under almost \( z \)-supercontinuous (almost \( D_\delta \)-supercontinuous) functions. In Section 5, characterizations and properties of almost strongly \( \theta \)-continuous functions are elaborated. Section 6 is devoted to separation axioms and sufficient conditions for almost strongly \( \theta \)-continuous functions to have \( u_\theta \)-closed (\( \theta \)-closed) graphs are obtained.

2. PRELIMINARIES AND BASIC DEFINITIONS

A subset \( S \) of a space \( X \) is said to be an \( H \)-set [36] or quasi \( H \)-closed relative to \( X \) [28] (respectively \( N \)-closed relative to \( X \) [1]) if for every cover \( \{ U_\alpha | \alpha \in \Lambda \} \) of \( S \) by open sets of \( X \), there exists a finite subset \( \Lambda_0 \) of \( \Lambda \) such that \( S \subset \bigcup \{ U_\alpha | \alpha \in \Lambda_0 \} \) (respectively \( S \subset \bigcup \{ (U_\alpha)^o | \alpha \in \Lambda_0 \} \)). A space \( X \) is said to be quasi \( H \)-closed [28] (respectively nearly compact [32]) if the set \( S \) is quasi \( H \)-closed relative to \( X \) (respectively \( N \)-closed relative to \( X \)). A space \( X \) is said to be quasicompact [5] if every cover of \( X \) by cozero sets admits a finite subcover.

A space \( X \) is said to be \( \delta \)-completely regular [13] (almost completely regular [31]) if for each regular \( G_\delta \)-set (regularly closed set) \( F \) and a point \( x \) not in \( F \) there exists a continuous function \( f : X \to [0, 1] \) such that \( f(x) = 0 \) and \( f(F) = 1 \).

A subset \( A \) of a space \( X \) is called a regular \( G_\delta \)-set [21] if \( A \) is an intersection of a sequence of closed sets whose interiors contain \( A \), i.e., of \( A = \bigcap_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} F_n^o \), where each \( F_n \) is a closed subset of \( X \). The complement of a regular \( G_\delta \)-set is called a regular \( F_\sigma \)-set.

A space \( X \) is called a \( D_\delta \)-completely regular ([15], [16]) if it has a base of regular \( F_\sigma \)-sets.

**Definition 2.1.** A function \( f : X \to Y \) from a topological space \( X \) into a topological space \( Y \) is said to be almost \( z \)-supercontinuous (almost \( D_\delta \)-supercontinuous) if for each \( x \in X \) and each open set \( V \) containing \( f(x) \), there exists a cozero set (regular \( F_\sigma \)-set) \( U \) containing \( x \) such that \( f(U) \subset (V)^o \).
Definition 2.2. A set $G$ is said to be $\delta$-open [36] ($d_\delta$-open [13], $z$-open [12]) if for each $x \in G$, there exists a regular open set (regular $F_\sigma$-set, cozero set) $H$ such that $x \in H \subseteq G$, or equivalently, $G$ can be obtained as an arbitrary union of regular open sets (regular $F_\sigma$-sets, cozero sets). The complement of a $\delta$-open ($d_\delta$-open, $z$-open) set will be referred to as a $\delta$-closed ($d_\delta$-closed, $z$-closed) set.

Definition 2.3. Let $X$ be a topological space and let $A \subseteq X$. A point $x \in X$ is called a $\delta$-adherent [36] ($\theta$-adherent [36], $u_\theta$-adherent ([9], [10]), $d_\delta$-adherent [13], $z$-adherent [12]) point of $A$ if every regular open set (closed neighborhood, $\theta$-open set, regular $F_\sigma$-set, cozero set) containing $x$ has non-empty intersection with $A$. Let $A_\delta$ denote the set of all $\delta$-adherent points ($\text{cl}_{d\delta} A$ the set of all $\delta$-adherent points, $A_{u\theta}$ the set of all $u_\theta$-adherent points, $[A]_{d\delta}$ the set of all $d_\delta$-adherent points, $A_z$ the set of all $z$-adherent points) of a set $A$. The set $A$ is $\delta$-closed ($\theta$-closed, $d_\delta$-closed, $z$-closed) if $A = A_\delta$ ($A = \text{cl}_{d\delta} A$ or $A = A_{u\theta}$, $A = [A]_{d\delta}$, $A = A_z$).

Lemma 2.4 ([8], [11]). A subset $A$ of a topological space $X$ is $\theta$-open if and only if for each $x \in A$, there is an open set $U$ such that $x \in U \subseteq \overline{U} \subseteq A$.

Definition 2.5. A space $X$ is called $\theta$-compact [10] ($D_\delta$-compact [14]) if every $\theta$-open cover (cover by regular $F_\sigma$-sets) of $X$ has a finite subcover.

Definitions 2.6. A function $f : X \to Y$ from a topological space $X$ into a topological space $Y$ is said to be

(a) strongly continuous [18] if $f(\overline{A}) \subseteq f(A)$ for each subset $A$ of $X$.

(b) perfectly continuous ([25], [26]) if $f^{-1}(V)$ is clopen in $X$ for every open set $V \subseteq Y$.

(c) almost perfectly continuous (equiv regular set connected [3]) if $f^{-1}(V)$ is clopen for every regular open set $V$ in $Y$.

(d) $\text{cl}$-supercontinuous [34] (equiv clopen continuous [29]) if for each open set $V$ containing $f(x)$ there is a clopen set $U$ containing $x$ such that $f(U) \subseteq V$.

(e) almost $\text{cl}$-supercontinuous[17] (equiv almost clopen continuous[4]) if for each $x \in X$ and each regular open set $V$ containing $f(x)$ there is a clopen set $U$ containing $x$ such that $f(U) \subseteq V$.

(f) $z$-supercontinuous [12] if for each $x \in X$ and for each open set $V$ containing $f(x)$, there exists a cozero set $U$ containing $x$ such that $f(U) \subseteq V$.

(g) strongly $\theta$-continuous [24] if for each $x \in X$ and for each open set $V$ containing $f(x)$, there exists an open set $U$ containing $x$ such that $f(U) \subseteq V$.

(h) supercontinuous [22] if for each $x \in X$ and for each open set $V$ containing $f(x)$, there exists an open set $U$ containing $x$ such that $f(U) \subseteq V$.

(i) almost strongly $\theta$-continuous [27] if for each $x \in X$ and for each open set $V$ containing $f(x)$, there exists an open set $U$ containing $x$ such that $f(U) \subseteq V$.

(j) $\delta$-continuous [24] if for each $x \in X$ and for each open set $V$ containing $f(x)$, there exists an open set $U$ containing $x$ such that $f(U) \subseteq V$.

(k) almost continuous [33] if for each $x \in X$ and for each open set $V$ containing $f(x)$, there exists an open set $U$ containing $x$ such that $f(U) \subseteq V$.
(l) faintly continuous [20] if for each \( x \in X \) and for each \( \theta \)-open set \( V \) containing \( f(x) \), there exists an open set \( U \) containing \( x \) such that \( f(U) \subset V \).

(m) \( D_\delta \)-supercontinuous [13] if for each \( x \in X \) and for each open set \( V \) containing \( f(x) \), there exists a regular \( F_\sigma \) set \( U \) containing \( x \) such that \( f(U) \subset V \).

The following diagram well illustrates the relationships that exist among almost \( z \)-supercontinuous functions, almost \( D_\delta \)-supercontinuous functions and various variants of continuity defined above.

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strongly continuous
  ↓
perfectly continuous ← almost perfectly continuous (= regular set connected)
  ↓
clopen map (=\( e \)-supercontinuous) ← almost clopen map (= almost \( e \)-supercontinuous)
  ↓
z-supercontinuous ← almost \( z \)-supercontinuous
  ↓
\( D_\delta \)-supercontinuous ← almost \( D_\delta \)-supercontinuous
  ↓
strongly \( \theta \)-continuous ← almost strongly \( \theta \)-continuous
  ↓
supercontinuous
  ↓
continuous ← \( \delta \)-continuous
  ↓
almost continuous
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However, none of the above implications in general is reversible. Kohli and Kumar [12] showed that a strongly \( \theta \)-continuous function need not be \( z \)-supercontinuous function. Noiri and Kang [27] gave examples to show that a \( \delta \)-continuous function need not be almost strongly \( \theta \)-continuous and that almost strongly \( \theta \)-continuous function need not be strongly \( \theta \)-continuous. Moreover, Noiri [24] showed that an almost continuous function need not be \( \delta \)-continuous.

**Example 2.7.** Let \( X = N = Y \) be the set of positive integers equipped with cofinite topology. The identity function on \( X \) is almost \( z \)-supercontinuous but not \( D_\delta \)-supercontinuous.

**Example 2.8.** Let \( X = Y \) be the mountain chain space due to Heldermann [6] which is a regular space. The identity map from \( X \) onto \( Y \) is strongly \( \theta \)-continuous but not almost \( D_\delta \)-supercontinuous.

**Example 2.9.** Let \( X = \{x_1, x_2, x_3, x_4\} \) and \( \Gamma = \{X, \phi, \{x_3\}, \{x_1, x_2\}, \{x_1, x_2, x_3\}\} \)
Let \( Y = \{y_1, y_2, y_3, y_4\} \) and \( \sigma = \{Y, \phi, \{y_1\}, \{y_3\}, \{y_1, y_2\}, \{y_1, y_3\}, \{y_1, y_2, y_3\}, \{y_1, y_3, y_4\}\} \)
Define a function \( f : (X, \Gamma) \rightarrow (Y, \sigma) \) as follows: \( f(x_1) = f(x_2) = y_2 \) and \( f(x_3) = f(x_4) = y_1 \) Then \( f \) is an almost \( z \)-supercontinuous functions but not continuous.
Example 2.10. Let \( A = K \cup \{a_+, a_-\} \) be the space due to Hewitt [7] which is \( D_3 \)-completely regular. The identity function defined on \( A \) is \( D_3 \)-supercontinuous but not almost \( z \)-supercontinuous.

Example 2.11. Let \( X \) denote the real line endowed with usual topology. The identity function defined on \( X \) is almost \( z \)-supercontinuous but not almost \( c \)-supercontinuous (=almost clopen).

Examples 2.8 and 2.9 show that the notions of almost \( z \)-supercontinuous function (almost \( D_3 \)-supercontinuous function) and continuous function are independent of each other.

3. Characterizations and Basic Properties of almost \( z \)-Supercontinuous and \( D_3 \)-Supercontinuous Functions

Proposition 3.1. For a function \( f : X \to Y \) from a topological space \( X \) into a topological space \( Y \), the following statements are equivalent:

(a) \( f \) is almost \( z \)-supercontinuous (almost \( D_3 \)-supercontinuous).
(b) The inverse image of every regular open subset of \( Y \) is \( z \)-open (\( d_3 \)-open) in \( X \).
(c) The inverse image of every regular closed subset of \( Y \) is \( z \)-closed (\( d_3 \)-closed) in \( X \).
(d) The inverse image of every \( \delta \)-open subset of \( Y \) is \( z \)-open (\( d_3 \)-open) in \( X \).
(e) The inverse image of every \( \delta \)-closed subset of \( Y \) is \( z \)-closed (\( d_3 \)-closed) in \( X \).

Proof. It is easy using definitions. \( \square \)

Theorem 3.2. For a function \( f : X \to Y \) the following statement are equivalent.

(a) \( f \) is almost \( z \)-supercontinuous.
(b) \( f(A) \subset (f(A))_\delta \) for every \( A \subset X \).
(c) \( (f^{-1}(B))_\delta \subset f^{-1}(B) \) for every \( B \subset Y \).

Proof. (a) \( \Rightarrow \) (b). Let \( y = f(x) \) for some \( x \in A \). To show that \( f(x) \in (f(A))_\delta \), let \( V \) be any regular open set containing \( f(x) \). Then there exists a cozero set \( U \) containing \( x \) such that \( f(U) \subset V \). Since \( x \in A \), \( U \cap A \neq \emptyset \) and so \( f(U \cap A) \neq \emptyset \) which in turn implies that \( f(U) \cap f(A) \neq \emptyset \) and hence \( V \cap f(A) \neq \emptyset \). Thus \( f(x) \in (f(A))_\delta \). Hence \( f(A) \subset (f(A))_\delta \) for every \( A \subset X \).

(b) \( \Rightarrow \) (c). Let \( B \subset Y \). Then \( f((f^{-1}(B))_\delta) \subset f(f^{-1}(B))_\delta \subset B \) and so it follows that \( (f^{-1}(B))_\delta \subset f^{-1}(B) \).

(c) \( \Rightarrow \) (a). Let \( F \) be any \( \delta \)-closed set in \( Y \). Then \( (f^{-1}(F))_\delta \subset f^{-1}(F) \). Since \( f^{-1}(F) \subset (f^{-1}(F))_\delta \subset (f^{-1}(F))_\delta \), so \( f^{-1}(F) = (f^{-1}(F))_\delta \) which in turn implies that \( f \) is almost \( z \)-supercontinuous. \( \square \)

Theorem 3.3. For a function \( f : X \to Y \) the following statement are equivalent.

(a) \( f \) is almost \( D_3 \)-supercontinuous.
(b) \( f([A]_{d_3}) \subset (f(A))_\delta \) for every \( A \subset X \).
(c) \( [f^{-1}(B)]_{d_3} \subset f^{-1}(B) \) for every \( B \subset Y \).
Proof. (a)$\Rightarrow$(b). Let $y = f(x)$ for some $x \in [A]_{d_A}$. To show that $y \in (f(A))_{d_A}$, let $V$ be a regular open set containing $f(x)$. Since $f$ is almost $D_\delta$-supercontinuous, there is a regular $F_\sigma$-set $U$ containing $x$ such that $f(U) \subset V$. Since $x \in [A]_{d_A}$, $U \cap A \neq \emptyset$ and hence $f(U \cap A) \neq \emptyset$ which in turn implies that $f(U) \cap f(A) \neq \emptyset$. Thus $V \cap f(A) \neq \emptyset$ and so $y \in (f(A))_{d_A}$ for every $y \in X$.

(b)$\Rightarrow$(c). Let $B \subset Y$. Then $f([f^{-1}(B)]_{d_B}) \subset (f(f^{-1}(B)))_\delta \subset B_\delta$ and so it follows that $[f^{-1}(B)]_{d_B} \subset f^{-1}(B_\delta)$.

(c)$\Rightarrow$(a). Let $F$ be any $\delta$-closed set in $Y$. Then $[f^{-1}(F)]_{d_B} \subset f^{-1}(F_\delta) = f^{-1}(F)$. Since $f^{-1}(F) \subset [f^{-1}(F)]_{d_B}$, $f^{-1}(F) = [f^{-1}(F)]_{d_B}$ and so $f^{-1}(F)$ is $d_\delta$-closed. It follows that $f$ is almost $D_\delta$-supercontinuous.

Definition 3.4. A filterbase $F$ is said to $z$-converge[12] ($d_\delta$-converge[13], $\delta$-converge[36]) to a point $x$, written as $F \overset{z}{\to} x$, if every cozero set (regular $F_\sigma$-set, regular open set) containing $x$ contains a member of $F$.

Theorem 3.5. A function $f : X \to Y$ is almost $z$-supercontinuous (almost $D_\delta$-supercontinuous) if and only if $f(F) \overset{\delta}{\to} f(x)$ for each $x \in X$ and each filter $F$ in $X$ that $z$-converges ($d_\delta$-converges) to $x$.

Proof. We shall prove the result in the case of almost $z$-supercontinuous functions only. Suppose that $f$ is almost $z$-supercontinuous and let $F$ be a filter in $X$ that $z$-converges to $x$. Let $W$ be a regular open set containing $f(x)$. Then $x \in f^{-1}(W)$ and $f^{-1}(W)$ is $z$-open. Let $H$ be a cozero set such that $x \in H \subset f^{-1}(W)$ and so $f(H) \subset W$. Since $F$ $z$-converges to $x$, there exists $U \in F$ such that $U \subset H$ and hence $f(U) \subset f(H) \subset W$. Thus, $f(F) \overset{\delta}{\to} f(x)$.

Conversely, let $W$ be a regular open set containing $f(x)$. Now, the filter $F$ generated by the filterbase $B_\delta$ consisting of cozero sets containing $x$, $z$-converges to $x$. Since by hypothesis $f(F) \overset{\delta}{\to} f(x)$, there exists a member $f(F)$ of $f(F)$ such that $f(F) \subset W$. Choose $B \in B_\delta$ such that $B \subset W$. Since $B$ is a cozero set containing $x$ and since $f(B) \subset f(F) \subset W$, $f$ is almost $z$-supercontinuous.

Remark 3.6. It is routine to verify that almost $z$-supercontinuity (almost $D_\delta$-supercontinuity) is invariant under restrictions and composition of functions and enlargement of range. Moreover, the composition $gof$ is almost $z$-supercontinuous whenever $f : X \to Y$ is almost $z$-supercontinuous and $g : Y \to Z$ is $\delta$-continuous. Furthermore, if $gof$ is almost $z$-supercontinuous and $f$ is a surjection which maps $z$-open sets to $z$-open sets, then $g$ is almost $z$-supercontinuous.

The following lemma is due to Singal and Singal [33] and will be used in the sequel.

Lemma 3.7 (Singal and Singal [33]). Let $\{X_\alpha : \alpha \in I\}$ be a family of spaces and let $X = \prod X_\alpha$ be the product space. If $x = (x_\alpha) \in X$ and $V$ is a regular open subset of $X$ containing $x$, then there exists a basic regular open set $IV_\alpha$ such that $x \in IV_\alpha \subset V$, where $V_\alpha$ is regular open in $X_\alpha$ for each $\alpha \in I$ and $V_\alpha = X_\alpha$ for all $\alpha \in I$ except for a finite number of indices $\alpha_i, i = 1, 2, \ldots, n$. 

Theorem 3.8. Let \( \{f_\alpha : X_\alpha \to Y_\alpha\} \) be a family of almost \( z \)-supercontinuous (almost \( D_3 \)-supercontinuous) functions. Let \( X = \Pi X_\alpha \) and \( Y = \Pi Y_\alpha \). Then \( f : X \to Y \) defined by \( f((x_\alpha)) = (f_\alpha(x_\alpha)) \) for each \( (x_\alpha) \in X \) is almost \( z \)-supercontinuous (almost \( D_3 \)-supercontinuous).

Proof. Let \( (x_\alpha) \in X \) and \( W \) be a regular open set in \( Y \) containing \( f((x_\alpha)) \). By Lemma 3.7 there exists a basic regular open set \( V = \Pi V_\alpha \) such that \( f(x) \in V \subset W \), where each \( V_\alpha \) is a regular open set in \( Y_\alpha \) and \( V_\alpha = Y_\alpha \) for \( \alpha \in \Delta \) except for \( \alpha = \alpha_1, \alpha_2, \ldots, \alpha_n \). For each \( i = 1, 2, \ldots, n \), in view of almost \( z \)-supercontinuity (almost \( D_3 \)-supercontinuity) of \( f_\alpha \), there exists a cozero set (regular \( F_\gamma \)-set) \( U_\alpha \), containing \( x_\alpha \), such that \( f_\alpha(U_\alpha) \subset V_\alpha \). Let \( U = \bigcup U_\alpha \), where \( U_\alpha = X_\alpha \) for \( \alpha \neq \alpha_i, (i = 1, 2, \ldots, n) \). Then \( U \) is a cozero set (regular \( F_\gamma \)-set) in \( X \) such that \( (x_\alpha) \in U \) and \( f(U) \subset V \). Thus \( f \) is almost \( z \)-supercontinuous (almost \( D_3 \)-supercontinuous). □

Theorem 3.9. Let \( f : X \to Y \) be any function. If \( \{U_\alpha : \alpha \in \Delta\} \) is a cover of \( X \) by cozero sets (regular \( F_\delta \)-sets) and for each \( \alpha, f_\alpha = f|U_\alpha : U_\alpha \to Y \) is almost \( z \)-supercontinuous (almost \( D_3 \)-supercontinuous), then \( f \) is almost \( z \)-supercontinuous (almost \( D_3 \)-supercontinuous).

Proof. Let \( V \) be a regular open set in \( Y \). Then \( f^{-1}(V) = \bigcup f_\alpha^{-1}(V) : \alpha \in \Delta \) and since each \( f_\alpha \) is almost \( z \)-supercontinuous (almost \( D_3 \)-supercontinuous), each \( f_\alpha^{-1}(V) \) is \( z \)-open (\( d_3 \)-open) in \( U_\alpha \) and hence in \( X \). Thus \( f^{-1}(V) \) being the union of \( z \)-open (\( d_3 \)-open) sets is \( z \)-open (\( d_3 \)-open). Thus \( f \) is almost \( z \)-supercontinuous (almost \( D_3 \)-supercontinuous). □

Theorem 3.10. Let \( f : X \to Y \) be a function and \( g : X \to X \times Y \), defined by \( g(x) = (x, f(x)) \) for each \( x \in X \), be the graph function. Then \( g \) is almost \( z \)-supercontinuous if and only if \( f \) is almost \( z \)-supercontinuous and \( X \) is an almost completely regular space.

Proof. Suppose that \( g \) is almost \( z \)-supercontinuous. Let \( V \) be a regular open set in \( Y \). Then \( p_Y^{-1}(V) = X \times V \) is a regular open set in \( X \times Y \), where \( p_Y \) is the projection from \( X \times Y \) onto \( X \). Therefore \( f^{-1}(V) = (p_{Y \circ g})^{-1}(V) = g^{-1}(p_Y^{-1}(V)) = g^{-1}(X \times V) \) is \( z \)-open and so \( f \) is almost \( z \)-supercontinuous. To prove that \( X \) is an almost completely regular space, let \( F \) be a regular closed set and suppose that \( x \notin F \). Then \( x \in X \setminus F \) and \( g(x) \in (X \setminus F) \times Y \) which is a regularly open set in \( X \times Y \). So there exists a cozero set \( W \) in \( X \) such that \( g(W) \subset (X \setminus F) \times Y \). Hence \( x \in W \subset X \setminus F \). Thus \( X \) is an almost completely regular space.

To prove sufficiency, let \( x \in X \) and \( W \) be a regular open set containing \( g(x) \). By Lemma 3.7 there exist regular open sets \( U \subset X \) and \( V \subset Y \) such that \( (x, f(x)) \in U \times V \subset W \). Since \( X \) is almost completely regular, there exists a cozero set \( G_1 \) in \( X \) containing \( x \) such that \( x \in G_1 \subset U \). Since \( f \) is almost \( z \)-supercontinuous, there exists a cozero set \( G_2 \) in \( X \) containing \( x \) such that \( f(G_2) \subset V \). Let \( G = G_1 \cap G_2 \). Then \( G \) is a cozero set containing \( x \) and \( g(G) \subset U \times V \subset W \). This proves that \( g \) is almost \( z \)-supercontinuous. □
Definition 4.3 ([30]). A space \( X \) is said to be almost regular if for each regular closed set \( A \) and each point \( x \notin A \), there exist disjoint open sets \( U \) and \( V \) such that \( x \in U, A \subseteq V \).

**Theorem 4.4.** Let \( f : X \rightarrow Y \) be an almost \( D_\delta \)-supercontinuous open bijection onto a space \( Y \). Then \( Y \) is an almost regular space. Further, if \( Y \) is a semiregular space, then \( Y \) is a regular space.

**Proof.** Let \( B \) be any regularly closed set in \( Y \) and let \( y \notin B \). Then \( f^{-1}(B) \cap f^{-1}(y) = \phi \). Since \( f \) is almost \( D_\delta \)-supercontinuous, by Proposition 3.1 \( f^{-1}(B) \) is \( d_\delta \)-closed and so \( f^{-1}(B) = \bigcap_{\alpha \in \Lambda} Z_\alpha \), where each \( Z_\alpha \) is a regular \( G_\delta \)-set. Since \( f \) is one-one, \( f^{-1}(y) \) is a singleton and so there exists \( \alpha_0 \in \Lambda \), such that \( f^{-1}(y) \notin Z_{\alpha_0} \). Since \( Z_{\alpha_0} \) is a regular \( G_\delta \)-set, \( Z_{\alpha_0} = \bigcap_{i=1}^{\infty} H_i = \bigcap_{i=1}^{\infty} H_i^0 \), where each \( H_i \) is a closed set.

So there exists an integer \( j \) such that \( f^{-1}(y) \notin H_j \). Then \( X \setminus H_j \) and \( H_j^0 \) are disjoint open sets containing \( f^{-1}(y) \) and \( f^{-1}(B) \), respectively. Since \( f \) is an open bijection, \( f(X \setminus H_j) \) and \( f(H_j^0) \) are disjoint open sets containing \( y \) and \( B \), respectively. Thus \( Y \) is an almost regular space. Since a semiregular almost regular space is regular, the last assertion is immediate. □
5. Characterizations and some basic properties of almost strongly \( \theta \)-continuous functions

**Proposition 5.1.** A function \( f : X \rightarrow Y \) is almost strongly \( \theta \)-continuous if and only if for each \( x \in X \) and each regular open set \( V \) containing \( f(x) \), there exists a \( \theta \)-open set \( U \) containing \( x \) such that \( f(U) \subseteq V \).

Proof. It is easy using definitions. \( \square \)

**Theorem 5.2.** For a function \( f : X \rightarrow Y \) the following statement are equivalent.

1. \( f \) is almost strongly \( \theta \)-continuous.
2. \( f(A_{\delta}) \subseteq (f(A))_{\delta} \) for each \( A \subseteq X \).
3. \( (f^{-1}(B))_{\delta} \subseteq f^{-1}(B_{\delta}) \) for every \( B \subseteq Y \).

Proof. (a) \( \Rightarrow \) (b). Since \( (f(A))_{\delta} \) is \( \delta \)-closed in \( Y \), by [27, Theorem 3.1, \( (f) \)], \( f^{-1}((f(A))_{\delta}) \) is \( \delta \)-closed in \( X \). Again, since \( A \subseteq f^{-1}((f(A))_{\delta}) \), \( A_{\delta} \subseteq (f^{-1}(f(A)))_{\delta} = f^{-1}((f(A))_{\delta}) \) and so \( f(A_{\delta}) \subseteq (f(A))_{\delta} \).

(b) \( \Rightarrow \) (c). Let \( B \subseteq Y \). Then, by hypothesis \( f((f^{-1}(B))_{\delta}) \subseteq f(f^{-1}(B))_{\delta} \subseteq B_{\delta} \) and so it follows that \( (f^{-1}(B))_{\delta} \subseteq f^{-1}(B_{\delta}) \).

(c) \( \Rightarrow \) (a). Let \( F \) be any \( \delta \)-closed set in \( Y \). Then \( (f^{-1}(F))_{\delta} \subseteq f^{-1}(F_{\delta}) = f^{-1}(F) \) which implies that \( f^{-1}(F) = (f^{-1}(F))_{\delta} \) and so \( f^{-1}(F) \) is \( \theta \)-closed. This proves that \( f \) is almost strongly \( \theta \)-continuous. \( \square \)

**Definition 5.3.** ([9], [10]): A filter \( F \) is said to \( u_\theta \)-converge to a point \( x \), written as \( F \underset{u_\theta}{\rightarrow} x \), if every \( \theta \)-open set containing \( x \) contains a member of \( F \).

**Theorem 5.4.** A function \( f : X \rightarrow Y \) is almost strongly \( \theta \)-continuous if and only if \( f(F) \underset{\delta}{\rightarrow} f(x) \) for each \( x \in X \) and each filter in \( X \) which \( u_\theta \)-converges to a point \( x \).

Proof. Suppose that \( f \) is almost strongly \( \theta \)-continuous and let \( F \underset{u_\theta}{\rightarrow} x \). Let \( W \) be a regular open set in \( Y \) containing \( f(x) \). Then by Proposition 5.1, \( f^{-1}(W) \) is a \( \theta \)-open set in \( X \). Since \( F \underset{u_\theta}{\rightarrow} x \), there exists \( F \in F \) such that \( F \subseteq f^{-1}(W) \) and so \( f(F) \subseteq W \). This shows that \( f(F) \underset{\delta}{\rightarrow} f(x) \).

Conversely, let \( V \) be a regular open subset of \( Y \) containing \( f(x) \). Now let \( F \) be the filter generated by the filterbase \( \mathcal{V}_x \) consisting of all \( \theta \)-open sets containing \( x \). By hypothesis \( f(F) \underset{\delta}{\rightarrow} f(x) \) and so there exists a member \( f(F) \) of \( f(\mathcal{V}_x) \) such that \( f(F) \subseteq V \). Choose \( U \in \mathcal{V}_x \) such that \( U \subseteq \mathcal{F} \) which implies that \( f(U) \subseteq f(F) \) and \( f(F) \subseteq V \). Hence \( f \) is almost strongly \( \theta \)-continuous. \( \square \)

**Theorem 5.5.** If \( f : X \rightarrow Y \) is faintly continuous and \( g : Y \rightarrow Z \) is almost strongly \( \theta \)-continuous. Then \( g \circ f \) is almost strongly \( \theta \)-continuous.

Proof. Let \( V \) be a regular open set in \( Z \). By almost strongly \( \theta \)-continuity of \( g, g^{-1}(V) \) is \( \theta \)-open in \( Y \). So \( (g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)) \) is open in \( X \), since \( f \) is faintly continuous. Hence \( g \circ f \) is almost continuous. \( \square \)

**Theorem 5.6.** Let \( f : X \rightarrow Y \) be an almost continuous function defined on a completely regular space \( X \). Then \( f \) is almost \( z \)-supercontinuous.
Proof. Let $V$ be a regular open set containing $f(x)$. Since $f$ is almost continuous, $f^{-1}(V)$ is open. Again since $X$ is completely regular space, $f^{-1}(V)$ is $z$-open. Hence $f$ is almost $z$-supercontinuous. \hfill $\square$

**Corollary 5.7.** If $f : X \rightarrow Y$ is a $\delta$-continuous function defined on a completely regular space $X$, then $f$ is almost $z$-supercontinuous.

*Proof.* A $\delta$-continuous function is almost continuous. \hfill $\square$

**Corollary 5.8.** Let $f : X \rightarrow Y$ be an almost strongly $\theta$-continuous function defined on a completely regular space $X$, then $f$ is almost $z$-supercontinuous.

*Proof.* An almost strongly $\theta$-continuous function is a $\delta$-continuous function and hence almost continuous. \hfill $\square$


6. SEPARATION AXIOMS AND ALMOST STRONGLY $\theta$-CONTINUOUS FUNCTIONS

**Definition 6.1** ([2], [10]). A subset $S$ of a space $X$ is said to be $\theta$-set if for every cover $\{U_\alpha : \alpha \in \Lambda\}$ of $S$ by $\theta$-open subsets of $X$, there exists a finite subset $\Lambda_0$ of $\Lambda$ such that $S \subseteq \bigcup \{U_\alpha : \alpha \in \Lambda_0\}$.

**Theorem 6.2.** If $f : X \rightarrow Y$ is almost strongly $\theta$-continuous and $A$ is a $\theta$-set in $X$, then $f(A)$ is $N$-closed relative to $Y$.

*Proof.* Let $\{U_\alpha : \alpha \in \Lambda\}$ be a cover of $f(A)$ by regular open sets in $Y$. Since $f$ is almost strongly $\theta$-continuous, $\{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a cover of $A$, by $\theta$-open sets in $X$. Since $A$ is $\theta$-set in $X$, so $A \subseteq \bigcup \{f^{-1}(U_\alpha) : \alpha \in \Lambda_0\}$ for some finite subset $\Lambda_0$ of $\Lambda$. Thus $f(A) \subseteq \bigcup \{U_\alpha : \alpha \in \Lambda_0\}$. Hence $f(A)$ is $N$-closed relative to $Y$. \hfill $\square$

**Corollary 6.3.** An almost strongly $\theta$-continuous image of a $\theta$-compact space is nearly compact.

**Corollary 6.4.** An almost strongly $\theta$-continuous image of an $\alpha$-compact space is nearly compact.

**Definition 6.5** ([2], [35]). A topological space $X$ is said to be $\theta$-Hausdorff if each pair of distinct points are contained in disjoint $\theta$-open sets.

**Theorem 6.6.** Let $f : X \rightarrow Y$ be an almost strongly $\theta$-continuous injection into a Hausdorff space $Y$. Then $X$ is $\theta$-Hausdorff.

*Proof.* Let $x \neq y$ be two points in $X$. Since $f$ is one-one, $f(x) \neq f(y)$. Since $Y$ is Hausdorff, there exist disjoint open sets $U$ and $V$ containing $f(x)$ and $f(y)$, respectively. Now, $U \cap V = \emptyset$ which implies that $\overline{U \cap V} = \emptyset$ and so $(\overline{U})^\theta \cap V = \emptyset$ which in turn implies that $(\overline{U})^\theta \cap \overline{V} = \emptyset$ and thus, $(\overline{U})^\theta \cap (\overline{V})^\theta = \emptyset$. Let $V_1 = (\overline{U})^\theta$ and $V_2 = (\overline{V})^\theta$, which are regular open sets such that $V_1 \cap V_2 = \emptyset$. By almost strongly $\theta$-continuity of $f$, $f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint $\theta$-open sets containing $x$ and $y$, respectively. Hence $X$ is $\theta$-Hausdorff. \hfill $\square$

**Definition 6.7.** A space $X$ is said to be a $\delta T_0$-space [17] if for each pair of distinct points $x$ and $y$ in $X$ there exists a regular open set containing one of the points $x$ and $y$ but not the other.
Theorem 6.8. Let \( f : X \to Y \) be an almost strongly \( \theta \)-continuous injection into a \( \delta T_0 \)-space. Then \( X \) is a Hausdorff space.

Proof. Let \( x_1 \) and \( x_2 \) be two distinct points in \( X \). Then \( f(x_1) \neq f(x_2) \). Since \( Y \) is a \( \delta T_0 \)-space, there exists a regular open set \( V \) containing one of the points \( f(x_1) \) or \( f(x_2) \) but not the other. To be precise, assume that \( f(x_1) \in V \). Since any union of \( \theta \)-open sets is \( \theta \)-open, in view of Proposition 5.1 it follows that \( f^{-1}(V) \) is a \( \theta \)-open set containing \( x_1 \). By Lemma 2.4 there exists an open set \( U \) such that \( x_1 \in U \subset \overline{U} \subset f^{-1}(V) \). Then \( U \) and \( X \setminus \overline{U} \) are disjoint open sets containing \( x_1 \) and \( x_2 \), respectively and so \( X \) is Hausdorff. □

Functions with closed graphs are important in functional analysis and several other areas of mathematics. Several variants of closed graphs occur in literature (see for example [19], [23]).

Definition 6.9 ([19]). The graph \( G(f) \) of \( f : X \to Y \) is called \( \theta \)-closed with respect to \( X \) if for each \( (x, y) \notin G(f) \) there exist open sets \( U \) and \( V \) containing \( x \) and \( y \), respectively such that \( \overline{(U \times V)} \cap G(f) = \emptyset \).

Definition 6.10 ([19]). The graph \( G(f) \) of \( f : X \to Y \) is called \( \theta \)-closed with respect to \( X \times Y \) if for each \( (x, y) \notin G(f) \), there exist open sets \( U \) and \( V \) containing \( x \) and \( y \), respectively such that \( (U \times V) \cap G(f) = \emptyset \).

Definition 6.11. The graph \( G(f) \) of \( f : X \to Y \) is called \( u_{\theta} \)-closed with respect to \( X \times Y \) if for each \( (x, y) \notin G(f) \), there exist \( \theta \)-open sets \( U \) and \( V \) containing \( x \) and \( y \), respectively such that \( (U \times V) \cap G(f) = \emptyset \).

Theorem 6.12. Let \( f : X \to Y \) be a function whose graph is \( u_{\theta} \)-closed with respect to \( X \times Y \). If \( K \) is a \( \theta \)-set in \( Y \), then \( f^{-1}(K) \) is \( \theta \)-closed in \( X \).

Proof. Let \( f : X \to Y \) be a function whose graph \( G(f) \) is \( u_{\theta} \)-closed with respect to \( X \times Y \). Let \( x \in X \setminus f^{-1}(K) \). For each \( y \in K \setminus f^{-1}(K) \), there exist \( \theta \)-open sets \( U_y \) and \( V_y \) containing \( x \) and \( y \), respectively such that \( f(U_y) \cap V_y = \emptyset \). The family \( \{V_y : y \in K \} \) is a cover of \( K \) by \( \theta \)-open sets of \( Y \). Since \( K \) is a \( \theta \)-set, so \( K \cup \{V_y : y \in K \} \) for some finite subset \( K_0 \) of \( K \). Let \( U = \cap \{U_y : y \in K_0 \} \). Then \( U \) is \( \theta \)-open set containing \( x \) and \( f(U) \cap K = \emptyset \) which implies that \( U \cap f^{-1}(K) = \emptyset \) and hence \( x \notin f^{-1}(K) \). This shows that \( f^{-1}(K) \) is \( \theta \)-closed in \( X \). □

Corollary 6.13 ([27]). Let \( f : X \to Y \) be a function whose graph is \( \theta \)-closed with respect to \( X \times Y \). If \( K \) is quasi \( H \)-closed relative to \( Y \), then \( f^{-1}(K) \) is \( \theta \)-closed in \( X \).

Proof. Since \( K \) is quasi \( H \)-closed relative to \( Y \), it is a \( \theta \)-set in \( Y \) (see [10]). □

Theorem 6.14. If \( f : X \to Y \) is an almost strongly \( \theta \)-continuous function and \( Y \) is a Hausdorff space, then \( G(f) \), the graph of \( f \) is \( \theta \)-closed with respect to \( X \times Y \).

Proof. Let \( x \in X \) and let \( y \neq f(x) \). Since \( Y \) is Hausdorff, there exist disjoint open sets \( V \) and \( W \) containing \( y \) and \( f(x) \), respectively. So \( V \) and \( W \) are disjoint sets containing \( y \) and \( f(x) \), respectively. Since \( f \) is almost strongly
respectively. Since $f$ is $\theta$-continuous, there is an open set $U$ containing $x$ such that $f(U) \subseteq (W)^\theta$.

Then $f(U) \subseteq (W)^\theta \subseteq Y \setminus V$. Consequently, $U \times V$ contains no point of $G(f)$. Hence $G(f)$ is $\theta$-closed with respect to $X \times Y$. □

**Corollary 6.15.** If $f: X \to Y$ is an almost strongly $\theta$-continuous function and $Y$ is Hausdorff, then $G(f)$, the graph of $f$, is $\theta$-closed with respect to $X$. 

**Theorem 6.16.** If $f: X \to Y$ is an almost strongly $\theta$-continuous function and $Y$ is an almost regular Hausdorff space, then $G(f)$, the graph of $f$, is weakly $\theta$-closed with respect to $X \times Y$.

**Proof.** Let $x \in X$ and let $y \neq f(x)$. Since $Y$ is Hausdorff, there exist disjoint open sets $V_1$ and $W_1$ containing $y$ and $f(x)$, respectively. Thus, there exist disjoint regular open sets $V = (V_1)^\theta$ and $W = (W_1)^\theta$ containing $y$ and $f(x)$, respectively. Since $f$ is almost strongly $\theta$-continuous, by Proposition 5.1, there exists a $\theta$-open set $U$ containing $x$ such that $f(U) \subseteq W$ and so $f(U) \subseteq W \subseteq Y \setminus V$. Thus $U \times V$ contains no point of $G(f)$. Since $Y$ is almost regular, $V$ is a $\theta$-open set. Thus $U \times V$ is a $\theta$-open set and $(U \times V) \cap G(f) = \emptyset$. Hence $G(f)$ is weakly $\theta$-closed with respect to $X \times Y$. □

**References**


Generalizations of Z-supercontinuous functions and $D_4$-supercontinuous functions


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