Applied General Topology

Volume 9, No. 2, 2008

pp. 239-251

Generalizations of Z-supercontinuous functions and D_δ-supercontinuous functions

J. K. Kohli, D. Singh and Rajesh Kumar

Abstract. Two new classes of functions, called 'almost z-supercontinuous functions' and 'almost D_δ-supercontinuous functions' are introduced. The class of almost z-supercontinuous functions properly includes the class of z-supercontinuous functions (Indian J. Pure Appl. Math. 33(7), (2002), 1097-1108) as well as the class of almost clopen maps due to Ekici (Acta. Math. Hungar. 107(3), (2005), 193-206) and is properly contained in the class of almost D_δ-supercontinuous functions which in turn constitutes a proper subclass of the class of almost strongly θ-continuous functions due to Noiri and Kang (Indian J. Pure Appl. Math. 15(1), (1984), 1-8) and which in its turn include all δ-continuous functions of Noiri (J. Korean Math. Soc. 16 (1980), 161-166). Characterizations and basic properties of almost z-supercontinuous functions and almost D_δ-supercontinuous functions are discussed and their place in the hierarchy of variants of continuity is elaborated. Moreover, properties of almost strongly θ-continuous functions are investigated and sufficient conditions for almost strongly θ-continuous functions to have u_θ-closed (θ-closed) graph are formulated.

2000 AMS Classification: Primary: 54C05, 54C08, 54C10; Secondary: 54D10, 54D15, 54D20.

Keywords: (almost) z-supercontinuous function, (almost) D_δ-supercontinuous function, (almost) strongly θ-continuous function, almost continuous function, δ-continuous function, faintly continuous function, , u_θ-closed graph , θ-closed graph, u_θ-limit point, θ-limit point, z-convergence.

1. Introduction

Among several of the variants of continuity in the literature, some are stronger than continuity and some are weaker than continuity and yet others are independent of continuity. In this paper we introduce two new variants of continuity which represent generalizations of the notions of z-supercontinuity and D_δ -supercontinuity and are independent of continuity and coincide with z-supercontinuity and D_δ-supercontinuity, respectively if the range is a semiregular space. The class of almost z-supercontinuous functions besides containing the class of z-supercontinuos functions contains the class of almost clopen ($\equiv$ almost cl-supercontinuous [34]) functions defined by Ekici [4].
Characterizations and basic properties of almost z-supercontinuous (almost D_δ-supercontinuous) functions are elaborated in Section 3 and their place in the hierarchy of variants of continuity is discussed. Section 4 is devoted to the study of the behaviour of separation axioms under almost z-supercontinuous (almost D_δ-supercontinuous) functions. In Section 5, characterizations and properties of almost strongly θ-continuous functions are elaborated. Section 6 is devoted to separation axioms and sufficient conditions for almost strongly θ-continuous functions to have u_θ-closed (θ-closed) graphs are obtained.

2. Preliminaries and Basic Definitions

A subset S of a space X is said to be an *H-set* [36] or *quasi H-closed relative to* X [28] (respectively N-closed relative to X [1]) if for every cover $\{ U_\alpha | \alpha \in \Lambda \}$ of S by open sets of X, there exists a finite subset Λ_o of Λ such that $S \subset \cup \{ \overline{U}_\alpha | \alpha \in \Lambda_o \}$ (respectively $S \subset \cup \{ (\overline{U}_\alpha)^o | \alpha \in \Lambda_o \}$). A space X is said to be *quasi H-closed* [28] (respectively *nearly compact* [32]) if the set X is quasi H-closed relative to X (respectively N-closed relative to X). A space X is said to be *quasicompact* [5] if every cover of X by cozero sets admits a finite subcover.
A space X is said to be *δ-completely regular* [13] (*almost completely regular* [31]) if for each regular G_δ -set (regularly closed set) F and a point x not in F there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(F) = 1$.
A subset A of a space X is called a *regular G_δ-set* [21] if A is an intersection of a sequence of closed sets whose interiors contain A, i.e., if $A = \bigcap_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} F_n^o$, where each F_n is a closed subset of X. The complement of a regular G_δ-set is called a *regular F_σ-set.*
A space X is called a *D_δ-completely regular* ([15], [16]) if it has a base of regular F_σ-sets.

Definition 2.1. *A function $f : X \to Y$ from a topological space X into a topological space Y is said to be* almost z-supercontinuous (almost D_δ-supercontinuous) *if for each $x \in X$ and each open set V containing $f(x)$, there exists a cozero set (regular F_σ-set) U containing x such that $f(U) \subset (\overline{V})^o$.*

Definition 2.2. *A set G is said to be* δ-open *[36]* (d_δ-open [13], z-open [12]) *if for each $x\in G$, there exists a regular open set (regular F_σ-set, cozero set) H such that $x\in H\subset G$, or equivalently, G can be obtained as an arbitrary union of regular open sets (regular F_σ-sets, cozero sets). The complement of a δ -open (d_δ-open, z-open) set will be referred to as a* δ-closed (d_δ-closed, z-closed*) set.*

Definition 2.3. *Let X be a topological space and let $A\subset X$. A point $x\in X$ is called a* δ-adherent *[36] (*θ-adherent [36], u_θ-adherent ([9], [10]), d_δ-adherent [13], z-adherent [12]) *point of A if every regular open set (closed neighborhood, θ-open set, regular F_σ-set, cozero set) containing x has non-empty intersection with A. Let A_δ denote the set of all δ-adherent points ($cl_\theta A$ the set of all θ-adherent points, A_{u_θ} the set of all u_θ-adherent points, $[A]_{d_\delta}$ the set of all d_δ-adherent points, A_z the set of all z-adherent points) of a set A. The set A is δ-closed (θ-closed, d_δ-closed, z-closed) if $A = A_\delta$ ($A = cl_\theta A$ or $A = A_{u_\theta}, A = [A]_{d_\delta}, A = A_z$).*

Lemma 2.4 ([8], [11])**.** *A subset A of a topological space X is θ-open if and only if for each $x\in A$, there is an open set U such that $x\in U\subset\overline{U}\subset A$.*

Definition 2.5. *A space X is called* θ-compact *[10] (*D_δ-compact *[14]) if every θ-open cover (cover by regular F_σ -sets) of X has a finite subcover.*

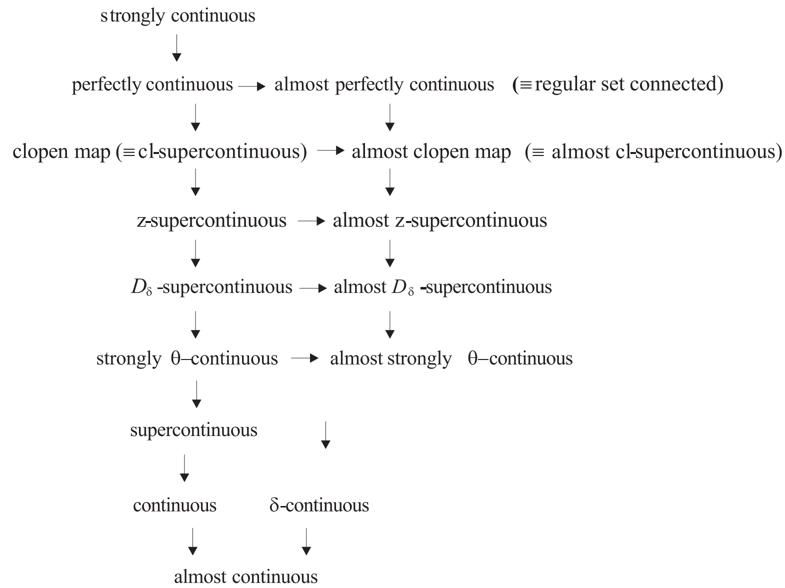
Definitions 2.6. *A function $f : X\to Y$ from a topological space X into a topological space Y is said to be*

(a) strongly continuous *[18] if $f(\overline{A}) \subset f(A)$ for each subset A of X.*
(b) perfectly continuous*([25], [26]) if $f^{-1}(V)$ is clopen in X for every open set $V\subset Y$.*
(c) almost perfectly continuous ($\equiv$ regular set connected [3]) *if $f^{-1}(V)$ is clopen for every regular open set V in Y.*
(d) cl-supercontinuous [34] ($\equiv$ clopen continuous [29]) *if for each open set V containing $f(x)$ there is a clopen set U containing x such that $f(U)\subset V$.*
(e) almost cl-supercontinuous[17] ($\equiv$ almost clopen continuous[4]) *if for each $x\in X$ and each regular open set V containing $f(x)$ there is a clopen set U containing x such that $f(U)\subset V$.*
(f) z-supercontinuous *[12] if for each $x\in X$ and for each open set V containing $f(x)$, there exists a cozero set U containing x such that $f(U)\subset V$.*
(g) strongly θ-continuous *[24] if for each $x\in X$ and for each open set V containing $f(x)$, there exists an open set U containing x such that $f(\overline{U})\subset V$.*
(h) supercontinuous *[22] if for each $x\in X$ and for each open set V containing $f(x)$, there exists an open set U containing x such that $f(\overline{U})^o\subset V$.*
(i) almost strongly θ-continuous *[27] if for each $x\in X$ and for each open set V containing $f(x)$, there exists an open set U containing x such that $f(\overline{U})\subset(\overline{V})^o$.*
(j) δ-continuous *[24] if for each $x\in X$ and for each open set V containing $f(x)$, there exists an open set U containing x such that $f(\overline{U})^o\subset(\overline{V})^o$.*
(k) almost continuous *[33] if for each $x\in X$ and for each open set V containing $f(x)$, there exists an open set U containing x such that $f(U)\subset(\overline{V})^o$.*

(l) faintly continuous *[20] if for each $x \in X$ and for each θ-open set V containing $f(x)$, there exists an open set U containing x such that $f(U) \subset V$.*

(m) D_δ-supercontinuous *[13] if for each $x \in X$ and for each open set V containing $f(x)$, there exists a regular F_σ set U containing x such that $f(U) \subset V$.*

The following diagram well illustrates the relationships that exist among almost z-supercontinuous functions, almost D_δ -supercontinuous functions and various variants of continuity defined above.



However, none of the above implications in general is reversible. Kohli and Kumar [12] showed that a strongly θ-continuous function need not be z-supercontinuous function. Noiri and Kang [27] gave examples to show that a δ-continuous function need not be almost strongly θ-continuous and that almost strongly θ-continuous function need not be strongly θ-continuous. Moreover, Noiri [24] showed that an almost continuous function need not be δ-continuous.

Example 2.7. *Let $X = N = Y$ be the set of positive integers equipped with cofinite topology. The identity function on X is almost z-supercontinuous but not D_δ-supercontinuous.*

Example 2.8. *Let $X = Y$ be the mountain chain space due to Heldermann [6] which is a regular space. The identity map from X onto Y is strongly θ-continuous but not almost D_δ-supercontinuous.*

Example 2.9. *Let $X = \{x_1, x_2, x_3, x_4\}$ and $\Gamma = \{X, \phi, \{x_3\}, \{x_1, x_2\}, \{x_1, x_2, x_3\}\}$ Let $Y = \{y_1, y_2, y_3, y_4\}$ and $\sigma = \{Y, \phi, \{y_1\}, \{y_3\}, \{y_1, y_2\}, \{y_1, y_3\}, \{y_1, y_2, y_3\}, \{y_1, y_3, y_4\}\}$*
Define a function $f : (X, \Gamma) \to (Y, \sigma)$ as follows: $f(x_1) = f(x_2) = y_2$ and $f(x_3) = f(x_4) = y_1$ Then f is an almost z-supercontinuous functions but not continuous.

Example 2.10. *Let $A = K\cup\{a_{+,}a_{-}\}$ be the space due to Hewitt [7] which is D_δ-completely regular. The identity function defined on A is D_δ-supercontinuous but not almost z-supercontinuous.*

Example 2.11. *Let X denote the real line endowed with usual topology. The identity function defined on X is almost z-supercontinuous but not almost cl-supercontinuous (=almost clopen).*

Examples 2.8 and 2.9 show that the notions of almost z-supercontinuous function (almost D_δ-supercontinuous function) and continuous function are independent of each other.

3. Characterizations and Basic Properties of almost z-Supercontinuous and D_δ-Supercontinuous Functions

Proposition 3.1. *For a function $f : X\rightarrow Y$ from a topological space X into a topological space Y, the following statements are equivalent:*

(a) *f is almost z-supercontinuous (almost D_δ-supercontinuous).*
(b) *The inverse image of every regular open subset of Y is z-open (d_δ-open) in X.*
(c) *The inverse image of every regular closed subset of Y is z-closed (d_δ-closed) in X.*
(d) *The inverse image of every δ-open subset of Y is z-open (d_δ-open) in X.*
(e) *The inverse image of every δ-closed subset of Y is z-closed (d_δ-closed) in X.*

Proof. It is easy using definitions. □

Theorem 3.2. *For a function $f : X\rightarrow Y$ the following statement are equivalent.*

(a) *f is almost z-supercontinuous.*
(b) *$f(A_z)\subset(f(A))_\delta$ for every $A\subset X$.*
(c) *$(f^{-1}(B))_z\subset f^{-1}(B_\delta)$ for every $B\subset Y$.*

Proof. *(a)⇒(b)*. Let $y = f(x)$ for some $x\in A_z$. To show that $f(x)\in(f(A))_\delta$, let V be any regular open set containing $f(x)$. Then there exists a cozero set U containing x such that $f(U)\subset V$. Since $x\in A_z, U\cap A\neq\phi$ and so $f(U\cap A)\neq\phi$ which in turn implies that $f(U)\cap f(A)\neq\phi$ and hence $V\cap f(A)\neq\phi$. Thus $f(x)\in(f(A))_\delta$. Hence $f(A_z)\subset(f(A))_\delta$ for every $A\subset X$.
(b)⇒(c). Let $B\subset Y$. Then $f((f^{-1}(B))_z)\subset(f(f^{-1}(B)))_\delta\subset B_\delta$ and so it follows that $(f^{-1}(B))_z\subset f^{-1}(B_\delta)$.
(c)⇒(a). let F be any δ-closed set in Y. Then $(f^{-1}(F))_z\subset f^{-1}(F_\delta) = f^{-1}(F)$. Since $f^{-1}(F)\subset(f^{-1}(F))_\delta\subset(f^{-1}(F))_z$, so $f^{-1}(F) = (f^{-1}(F))_z$ which in turn implies that f is almost z-supercontinuous. □

Theorem 3.3. *For a function $f : X\rightarrow Y$ the following statement are equivalent.*

(a) *f is almost D_δ-supercontinuous.*
(b) *$f([A]_{d_\delta})\subset(f(A))_\delta$ for every $A\subset X$.*
(c) *$[f^{-1}(B)]_{d_\delta}\subset f^{-1}(B_\delta)$ for every $B\subset Y$.*

Proof. $(a)\Rightarrow(b)$. Let $y = f(x)$ for some $x\in[A]_{d_\delta}$. To show that $y\in(f(A))_\delta$, let V be a regular open set containing $f(x)$. Since f is almost D_δ-supercontinuous, there is a regular F_σ-set U containing x such that $f(U)\subset V$. Since $x\in[A]_{d_\delta}, U\cap A\neq\phi$ and hence $f(U\cap A)\neq\phi$ which in turn implies that $f(U)\cap f(A)\neq\phi$. Thus $V\cap f(A)\neq\phi$ and so $y\in(f(A))_\delta$ for every $A\subset X$.

$(b)\Rightarrow(c)$. Let $B\subset Y$. Then $f([f^{-1}(B)]_{d_\delta})\subset(f(f^{-1}(B)))_\delta\subset B_\delta$ and so it follows that $[f^{-1}(B)]_{d_\delta}\subset f^{-1}(B_\delta)$.

$(c)\Rightarrow(a)$. let F be any δ-closed set in Y. Then $[f^{-1}(F)]_{d_\delta}\subset f^{-1}(F_\delta) = f^{-1}(F)$. Since $f^{-1}(F)\subset[f^{-1}(F)]_{d_\delta}, f^{-1}(F) = [f^{-1}(F)]_{d_\delta}$ and so $f^{-1}(F)$ is d_δ-closed. It follows that f is almost D_δ-supercontinuous. □

Definition 3.4. *A filterbase $\mathcal{F}$ is said to* z-converge*[12] (*d_δ-converge[13], δ-converge[36]*) to a point x, written as $\mathcal{F} \xrightarrow{z} x(\mathcal{F} \xrightarrow{d_\delta} x, \mathcal{F} \xrightarrow{\delta} x)$, if every cozero set (regular F_σ-set, regular open set) containing x contains a member of $\mathcal{F}$.*

Theorem 3.5. *A function $f : X\rightarrow Y$ is almost z-supercontinuous (almost D_δ-supercontinuous) if and only if $f(\mathcal{F}) \xrightarrow{\delta} f(x)$ for each $x\in X$ and each filter $\mathcal{F}$ in X that z-converges (d_δ-converges) to x.*

Proof. We shall prove the result in the case of almost z-supercontinuous functions only. Suppose that f is almost z-supercontinuous and let $\mathcal{F}$ be a filter in X that z-converges to x. Let W be a regular open set containing $f(x)$. Then $x\in f^{-1}(W)$ and $f^{-1}(W)$ is z-open. Let H be a cozero set such that $x\in H\subset f^{-1}(W)$ and so $f(H)\subset W$. Since $\mathcal{F}$ z-converges to x, there exists $U\in\mathcal{F}$ such that $U\subset H$ and hence $f(U)\subset f(H)\subset W$. Thus, $f(\mathcal{F}) \xrightarrow{\delta} f(x)$.
Conversely, let W be a regular open set containing $f(x)$. Now, the filter $\mathcal{F}$ generated by the filterbase $\mathcal{B}_x$ consisting of cozero sets containing x, z-converges to x. Since by hypothesis $f(\mathcal{F}) \xrightarrow{\delta} f(x)$, there exists a member $f(F)$ of $f(\mathcal{F})$ such that $f(F)\subset W$. Choose $B\in\mathcal{B}_x$ such that $B\subset F$. Since B is a cozero set containing x and since $f(B)\subset f(F)\subset W, f$ is almost z-supercontinuous. □

Remark 3.6. *It is routine to verify that almost z-supercontinuity (almost D_δ-supercontinuity) is invariant under restrictions and composition of functions and enlargement of range. Moreover, the composition gof is almost z-supercontinuous whenever $f : X\rightarrow Y$ is almost z-supercontinuous and $g : Y\rightarrow Z$ is δ-continuous. Furthermore, if gof is almost z-supercontinuous and f is a surjection which maps z-open sets to z-open sets, then g is almost z-supercontinuous.*

The following lemma is due to Singal and Singal [33] and will be used in the sequel.

Lemma 3.7 (Singal and Singal [33])**.** *Let $\{X_\alpha : \alpha\in I\}$ be a family of spaces and let $X = \prod X_\alpha$ be the product space. If $x = (x_\alpha)\in X$ and V is a regular open subset of X containing x, then there exists a basic regular open set ΠV_α such that $x\in\Pi V_\alpha\subset V$, where V_α is regular open in X_α for each $\alpha\in I$ and $V_\alpha = X_\alpha$ for all $\alpha\in I$ except for a finite number of indices $\alpha_i, i = 1, 2, \ldots, n$.*

Theorem 3.8. *Let $\{f_\alpha : X_\alpha \rightarrow Y_\alpha\}$ be a family of almost z-supercontinuous (almost D_δ -supercontinuous) functions. Let $X = \Pi X_\alpha$ and $Y = \Pi Y_\alpha$. Then $f : X \rightarrow Y$ defined by $f((x_\alpha)) = (f_\alpha(x_\alpha))$ for each $(x_\alpha) \in X$ is almost z-supercontinuous (almost D_δ-supercontinuous).*

Proof. Let $(x_\alpha) \in X$ and W be a regular open set in Y containing $f((x_\alpha))$. By Lemma 3.7 there exists a basic regular open set $V = \Pi V_\alpha$ such that $f(x) \in V \subset W$, where each V_α is a regular open set in Y_α and $V_\alpha = Y_\alpha$ for $\alpha \in \Delta$ except for $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_n$. For each $i = 1, 2, \ldots, n$, in view of almost z-supercontinuity (almost D_δ-supercontinuity) of f_{α_i} there exists a cozero set (regular F_δ-set) U_{α_i} containing x_{α_i} such that $f_{\alpha_i}(U_{\alpha_i}) \subset V_{\alpha_i}$. Let $U = \prod U_\alpha$, where $U_\alpha = X_\alpha$ for $\alpha \neq \alpha_i, (i = 1, 2, \ldots, n)$. Then U is a cozero set (regular F_δ-set) in X such that $(x_\alpha) \in U$ and $f(U) \subset V \subset W$. Thus f is almost z-supercontinuous (almost D_δ -supercontinuous). □

Theorem 3.9. *Let $f : X \rightarrow Y$ be any function. If $\{U_\alpha : \alpha \in \Delta\}$ is a cover of X by cozero sets (regular F_δ-sets) and for each $\alpha, f_\alpha = f|U_\alpha : U_\alpha \rightarrow Y$ is almost z-supercontinuous (almost D_δ-supercontinuous), then f is almost z-supercontinuous (almost D_δ-supercontinuous).*

Proof. Let V be a regular open set in Y. Then $f^{-1}(V) = \cup\{f_\alpha^{-1}(V) : \alpha \in \Delta\}$ and since each f_α is almost z-supercontinuous (almost D_δ-supercontinuous), each $f_\alpha^{-1}(V)$ is z-open (d_δ-open) in U_α and hence in X. Thus $f^{-1}(V)$ being the union of z-open (d_δ -open) sets is z-open (d_δ-open). Thus f is almost z-supercontinuous (almost D_δ-supercontinuous). □

Theorem 3.10. *Let $f : X \rightarrow Y$ be a function and $g : X \rightarrow X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$, be the graph function. Then g is almost z-supercontinuous if and only if f is almost z-supercontinuous and X is an almost completely regular space.*

Proof. Suppose that g is almost z-supercontinuous. Let V be a regular open set in Y. Then $p_y^{-1}(V) = X \times V$ is a regular open set in $X \times Y$, where p_y is the projection from $X \times Y$ onto Y. Therefore $f^{-1}(V) = (p_y o g)^{-1}(V) = g^{-1}(p_y^{-1}(V)) = g^{-1}(X \times V)$ is z-open and so f is almost z-supercontinuous. To prove that X is an almost completely regular space, let F be a regular closed set and suppose that $x \notin F$. Then $x \in X \setminus F$ and $g(x) \in (X \setminus F) \times Y$ which is a regularly open set in $X \times Y$. So there exists a cozero set W in X such that $g(W) \subset (X \setminus F) \times Y$. Hence $x \in W \subset X \setminus F$. Thus X is an almost completely regular space.

To prove sufficiency, let $x \in X$ and let W be a regular open set containing $g(x)$. By Lemma 3.7 there exist regular open sets $U \subset X$ and $V \subset Y$ such that $(x, f(x)) \in U \times V \subset W$. Since X is almost completely regular, there exists a cozero set G_1 in X containing x such that $x \in G_1 \subset U$. Since f is almost z-supercontinuous, there exists a cozero set G_2 in X containing x such that $f(G_2) \subset V$. Let $G = G_1 \cap G_2$. Then G is a cozero set containing x and $g(G) \subset U \times V \subset W$. This proves that g is almost z-supercontinuous. □

Proposition 3.11. *Let $f : X \to Y$ be a function defined on a δ-completely regular space X. Then the graph function $G(f)$ is almost D_δ-supercontinuous if and only if f is almost D_δ-upercontinuous.*

Proof. It is easy using definitions. □

4. Separation axioms related to almost z-supercontinuous functions and D_δ-supercontinuous functions

Theorem 4.1. *An almost z-supercontinuous (almost D_δ -supercontinuous) image of a quasicompact (D_δ-compact) space is nearly compact.*

Proof. Let $f : X \to Y$ be an almost z-supercontinuous (almost D_δ -supercontinuous) surjection from a quasicompact (D_δ -compact) space X onto a space Y. Let $\mathcal{V} = \{V_\alpha : \alpha \in \Lambda\}$ be a cover of Y by regularly open sets (regular F_δ-sets) in Y. Since f is almost z-supercontinuous (almost D_δ -supercontinuous), each $f^{-1}(V_\alpha)$ is z-open (d_δ-open) in X and so is a union of cozero sets (regular F_δ-sets). This in turn yields a cover $\mathcal{G}$ of X consisting of cozero sets (regular F_δ-sets). Since X is quasicompact (D_δ-compact), there is a finite subcollection $\{C_1, \ldots, C_n\}$ of $\mathcal{G}$ which covers X. Suppose $C_i \subset f^{-1}(V_{\alpha_i})$ for some $\alpha_i \in \Lambda (i = 1, \ldots, n)$. Then $\{V_{\alpha_1}, \ldots, V_{\alpha_n}\}$ is a finite subcollection of $\mathcal{V}$ which covers Y. Thus Y is nearly compact. □

Corollary 4.2. *Let $f : X \to Y$ be an almost z-supercontinuous (almost D_δ -supercontinuous) surjection from a quasicompact (D_δ -compact) space onto a semiregular space Y. Then Y is compact.*

Proof. A semiregular nearly compact space is compact. □

Definition 4.3 ([30])**.** *A space X is said to be* almost regular *if for each regular closed set A and each point $x \notin A$, there exist disjoint open sets U and V such that $x \in U, A \subset V$.*

Theorem 4.4. *Let $f : X \to Y$ be an almost D_δ-supercontinuous open bijection onto a space Y. Then Y is an almost regular space. Further, if Y is a semiregular space, then Y is a regular space.*

Proof. Let B be any regularly closed set in Y and let $y \notin B$. Then $f^{-1}(B) \cap f^{-1}(y) = \phi$. Since f is almost D_δ-supercontinuous, by Proposition 3.1 $f^{-1}(B)$ is d_δ-closed and so $f^{-1}(B) = \bigcap_{\alpha \in \Lambda} Z_\alpha$, where each Z_α is a regular G_δ-set. Since f is one-one, $f^{-1}(y)$ is a singleton and so there exists $\alpha_o \in \Lambda$, such that $f^{-1}(y) \notin Z_{\alpha_o}$. Since Z_{α_o} is a regular G_δ-set, $Z_{\alpha_o} = \bigcap_{i=1}^{\infty} H_i = \bigcap_{i=1}^{\infty} H_i^o$, where each H_i is a closed set. So there exists an integer j such that $f^{-1}(y) \notin H_j$. Then $X \setminus H_j$ and H_j^o are disjoint open sets containing $f^{-1}(y)$ and $f^{-1}(B)$, respectively. Since f is an open bijection, $f(X \setminus H_j)$ and $f(H_j^o)$ are disjoint open sets containing y and B, respectively. Thus Y is an almost regular space. Since a semiregular almost regular space is regular, the last assertion is immediate. □

5. Characterizations and some basic properties of almost strongly θ-continuous functions

Proposition 5.1. *A function $f : X{\to}Y$ is almost strongly θ-continuous if and only if for each $x{\in}X$ and each regular open set V containing $f(x)$, there exists a θ-open set U containing x such that $f(U){\subset}V$.*

Proof. It is easy using definitions. □

Theorem 5.2. *For a function $f : X{\to}Y$ the following statement are equivalent.*
(1) *f is almost strongly θ-continuous.*
(2) *$f(A_{u_\theta}){\subset}(f(A))_\delta$ for each $A{\subset}X$.*
(3) *$(f^{-1}(B))_{u_\theta}{\subset}f^{-1}(B_\delta)$ for every $B{\subset}Y$.*

Proof. $(a){\Rightarrow}(b)$. Since $(f(A))_\delta$ is δ-closed in Y, by [27, Theorem 3.1, (f)], $f^{-1}((f(A))_\delta)$ is θ-closed in X. Again, since $A{\subset}f^{-1}((f(A))_\delta)$, $A_{u_\theta}{\subset}(f^{-1}(f(A))_\delta)_{u_\theta} = f^{-1}((f(A))_\delta)$ and so $f(A_{u_\theta}){\subset}(f(A))_\delta$.
$(b){\Rightarrow}(c)$. Let $B{\subset}Y$. Then, by hypothesis $f((f^{-1}(B))_{u_\theta}){\subset}(f(f^{-1}(B))_\delta{\subset}B_\delta$ and so it follows that $(f^{-1}(B))_{u_\theta}{\subset}f^{-1}(B_\delta)$.
$(c){\Rightarrow}(a)$. Let F be any δ-closed set in Y. Then $(f^{-1}(F))_{u_\theta}{\subset}f^{-1}(F_\delta) = f^{-1}(F)$ which implies that $f^{-1}(F) = (f^{-1}(F))_{u_\theta}$ and so $f^{-1}(F)$ is θ-closed. This proves that f is almost strongly θ-continuous. □

Definition 5.3. *([9], [10]): A filter $\mathcal{F}$ is said to u_θ-converge to a point x, written as $\mathcal{F} \overset{u_\theta}{\to} x$, if every θ-open set containing x contains a member of $\mathcal{F}$.*

Theorem 5.4. *A function $f : X{\to}Y$ is almost strongly θ-continuous if and only if $f(\mathcal{F}) \overset{\delta}{\to} f(x)$ for each $x{\in}X$ and each filter in X which u_θ-converges to a point x.*

Proof. Suppose that f is almost strongly θ-continuous and let $\mathcal{F} \overset{u_\theta}{\to} x$. Let W be a regular open set in Y containing $f(x)$. Then by Proposition 5.1, $f^{-1}(W)$ is a θ-open set in X. Since $\mathcal{F} \overset{u_\theta}{\to} x$, there exists $F{\in}\mathcal{F}$ such that $F{\subset}f^{-1}(W)$ and so $f(F){\subset}W$. This shows that $f(\mathcal{F}) \overset{\delta}{\to} f(x)$.
Conversely, let V be a regular open subset of Y containing $f(x)$. Now let $\mathcal{F}$ be the filter generated by the filterbase $\mathcal{V}_x$ consisting of all θ-open sets containing x. By hypothesis $f(\mathcal{F}) \overset{\delta}{\to} f(x)$ and so there exists a member $f(F)$ of $f(\mathcal{V}_x)$ such that $f(F){\subset}V$. Choose $U{\in}\mathcal{V}_x$ such that $U{\subset}F$ which implies that $f(U){\subset}f(F)$ and $f(F){\subset}V$ and so $f(U){\subset}V$. Hence f is almost strongly θ-continuous. □

Theorem 5.5. *If $f : X{\to}Y$ is faintly continuous and $g : Y{\to}Z$ be almost strongly θ-continuous. Then gof is almost continuous.*

Proof. Let V be a regular open set in Z. By almost strongly θ-continuity of $g, g^{-1}(V)$ is θ-open in Y. So $(gof)^{-1}(V) = f^{-1}(g^{-1}(V))$ is open in X, since f is faintly continuous. Hence gof is almost continuous. □

Theorem 5.6. *Let $f : X{\to}Y$ be an almost continuous function defined on a completely regular space X. Then f is almost z-supercontinuous.*

Proof. Let V be a regular open set containing $f(x)$. Since f is almost continuous, $f^{-1}(V)$ is open. Again since X is completely regular space, $f^{-1}(V)$ is z-open. Hence f is almost z-supercontinuous. □

Corollary 5.7. *If $f : X \to Y$ is a δ-continuous function defined on a completely regular space X, then f is almost z-supercontinuous.*

Proof. A δ-continuous function is almost continuous. □

Corollary 5.8. *Let $f : X \to Y$ be an almost strongly θ-continuous function defined on a completely regular space X, then f is almost z-supercontinuous.*

Proof. An almost strongly θ-continuous function is a δ -continuous function and hence almost continuous. □

6. SEPARATION AXIOMS AND ALMOST STRONGLY θ-CONTINUOUS FUNCTIONS

Definition 6.1 ([2], [10])**.** *A subset S of a space X is said to be* θ-set *if for every cover $\{U_\alpha | \alpha \in \Lambda\}$ of S by θ-open subsets of X, there exists a finite subset Λ_o of Λ such that $S \subset \cup \{U_\alpha | \alpha \in \Lambda_o\}$.*

Theorem 6.2. *If $f : X \to Y$ is almost strongly θ-continuous and A is a θ-set in X, then $f(A)$ is N-closed relative to Y.*

Proof. Let $\{U_\alpha : \alpha \in \Lambda\}$ be a cover of $f(A)$ by regular open sets in Y. Since f is almost strongly θ-continuous, $\{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a cover of A, by θ-open sets in X. Since A is θ-set in X, so $A \subset \cup \{f^{-1}(U_\alpha) : \alpha \in \Lambda_o\}$ for some finite subset Λ_o of Λ. Thus $f(A) \subset \cup \{U_\alpha : \alpha \in \Lambda_o\}$. Hence $f(A)$ is N-closed relative to Y. □

Corollary 6.3. *An almost strongly θ-continuous image of a θ-compact space is nearly compact.*

Corollary 6.4. *An almost strongly θ-continuous image of an almost compact space is nearly compact.*

Definition 6.5 ([2], [35])**.** *A topological space X is said to be* θ-Hausdorff *if each pair of distinct points are contained in disjoint θ-open sets.*

Theorem 6.6. *Let $f : X \to Y$ be an almost strongly θ-continuous injection into a Hausdorff space Y. Then X is θ-Hausdorff.*

Proof. Let $x \neq y$ be two points in X. Since f is one-one, $f(x) \neq f(y)$. Since Y is Hausdorff, there exist disjoint open sets U and V containing $f(x)$ and $f(y)$, respectively. Now, $U \cap V = \phi$ which implies that $\overline{U} \cap V = \phi$ and so $(\overline{U})^o \cap V = \phi$ which in turn implies that $(\overline{U})^o \cap \overline{V} = \phi$ and thus, $(\overline{U})^o \cap (\overline{V})^o = \phi$. Let $V_1 = (\overline{U})^o$ and $V_2 = (\overline{V})^o$, which are regular open sets such that $V_1 \cap V_2 = \phi$. By almost strongly θ-continuity of $f, f^{-1}(V_1)$ and $f^{-1}(V_2)$ are disjoint θ-open sets containing x and y, respectively. Hence X is θ-Hausdorff. □

Definition 6.7. *A space X is said to be a δT_0-space [17] if for each pair of distinct points x and y in X there exists a regular open set containing one of the points x and y but not the other.*

Theorem 6.8. *Let $f : X \to Y$ be an almost strongly θ-continuous injection into a δT_0-space. Then X is a Hausdorff space.*

Proof. Let x_1 and x_2 be two distinct points in X. Then $f(x_1) \neq f(x_2)$. Since Y is a δT_0-space, there exists a regular open set V containing one of the points $f(x_1)$ or $f(x_2)$ but not the other. To be precise, assume that $f(x_1) \in V$. Since any union of θ-open sets is θ-open, in view of Proposition 5.1 it follows that $f^{-1}(V)$ is a θ-open set containing x_1. By Lemma 2.4 there exists an open set U such that $x_1 \in U \subset \overline{U} \subset f^{-1}(V)$. Then U and $X \backslash \overline{U}$ are disjoint open sets containing x_1 and x_2, respectively and so X is Hausdorff. □

Functions with closed graphs are important in functional analysis and several other areas of mathematics. Several variants of closed graphs occur in literature (see for example [19], [23]).

Definition 6.9 ([19])**.** *The graph $G(f)$ of $f : X \to Y$ is called* θ-closed with respect to X *if for each $(x, y) \notin G(f)$ there exist open sets U and V containing x and y, respectively such that $(\overline{U} \times V) \cap G(f) = \phi$.*

Definition 6.10 ([19])**.** *The graph $G(f)$ of $f : X \to Y$ is called* θ-closed with respect to $X \times Y$ *if for each $(x, y) \notin G(f)$, there exist open sets U and V containing x and y, respectively such that $(U \times \overline{V}) \cap G(f) = \phi$*

Definition 6.11. *The graph $G(f)$ of $f : X \to Y$ is called* u_θ-closed with respect to $X \times Y$ *if for each $(x, y) \notin G(f)$, there exist θ-open sets U and V containing x and y, respectively such that $(U \times V) \cap G(f) = \phi$.*

Theorem 6.12. *Let $f : X \to Y$ be a function whose graph is u_θ-closed with respect to $X \times Y$. If K is a θ-set in Y, then $f^{-1}(K)$ is θ-closed in X.*

Proof. Let $f : X \to Y$ be a function whose graph $G(f)$ is u_θ-closed with respect to $X \times Y$. Let $x \in X \backslash f^{-1}(K)$. For each $y \in K, (x, y) \notin G(f)$, there exist θ-open sets U_y and V_y containing x and y, respectively such that $f(U_y) \cap V_y = \phi$. The family $\{V_y | y \in K\}$ is a cover of K by θ-open sets of Y. Since K is a θ-set, so $K \subset \cup \{V_y | y \in K_o\}$ for some finite subset K_o of K. Let $U = \cap \{U_y | y \in K_o\}$. Then U is θ-open set containing x and $f(U) \cap K = \phi$ which implies that $U \cap f^{-1}(K) = \phi$ and hence $x \notin (f^{-1}(K))_{u_\theta}$. This shows that $f^{-1}(K)$ is θ-closed in X. □

Corollary 6.13 ([27])**.** *Let $f : X \to Y$ be a function whose graph is θ-closed with respect to $X \times Y$. If K is quasi H-closed relative to Y, then $f^{-1}(K)$ is θ-closed in X.*

Proof. Since K is quasi H-closed relative to Y, it is a θ-set in Y (see [10]). □

Theorem 6.14. *If $f : X \to Y$ is an almost strongly θ-continuous function and Y is a Hausdorff space, then $G(f)$, the graph of f is θ-closed with respect to $X \times Y$.*

Proof. Let $x \in X$ and let $y \neq f(x)$. Since Y is Hausdorff, there exist disjoint open sets V and W containing y and $f(x)$, respectively. So $\overline{V}$ and $(\overline{W})^o$ are disjoint sets containing y and $f(x)$, respectively. Since f is almost strongly

θ-continuous, so there is an open set U containing x such that $f(\overline{U})\subset(\overline{W})^o$. Then $f(\overline{U})\subset(\overline{W})^o\subset Y\backslash\overline{V}$. Consequently, $\overline{U}\times\overline{V}$ contains no point of $G(f)$. Hence $G(f)$ is θ-closed with respect to $X\times Y$. □

Corollary 6.15. *If $f: X\rightarrow Y$ is an almost strongly θ-continuous function and Y is Hausdorff, then $G(f)$, the graph of f is θ-closed with respect to X.*

Theorem 6.16. *If $f: X\rightarrow Y$ is an almost strongly θ-continuous function and Y is an almost regular Hausdorff space, then $G(f)$, the graph of f is u_θ-closed with respect to $X\times Y$.*

Proof. Let $x\in X$ and let $y\neq f(x)$. Since Y is Hausdorff, there exist disjoint open sets V_1 and W_1 containing y and $f(x)$, respectively. Thus, there exist disjoint regular open sets $V=(\overline{V}_1)^o$ and $W=(\overline{W}_1)^o$ containing y and $f(x)$, respectively. Since f is almost strongly θ-continuous, by Proposition 5.1, there exists a θ-open set U containing x such that $f(U)\subset W$ and so $f(U)\subset W\subset Y\backslash V$. Thus $U\times V$ contains no point of $G(f)$. Since Y is almost regular, V is a θ-open set. Thus $U\times V$ is a θ-open set and $(U\times V)\cap G(f)=\phi$. Hence $G(f)$ is u_θ -closed with respect to $X\times Y$. □

References

[1] D. Carnahan, *Locally nearly compact spaces*, Boll. Mat. Un. Ital. **6**, no. 4 (1972), 146–153.

[2] A. K. Das, *A note on θ-Hausdorff spaces*, Bull. Cal. Math. Soc. **97**, no. 1(2005), 15–20.

[3] J. Dontchev, M. Ganster and I. Reilly, *More on almost s-continuity*, Indian J. Math. **41** (1999), 139–146.

[4] E. Ekici, *Generalization of perfectly continuous, regular set-connected and clopen functions*, Acta. Math. Hungar. **107**, no. 3 (2005), 193–206.

[5] Z-Frolik, *Generalizations of compact and Lindelöf spaces*, Czechoslovak Math J. **13**, no. 84 (1959), 172–217 (Russian) MR 21 # 3821.

[6] N. C. Heldermann, *Developability and some new regularity axioms*, Can. J. Math. **33**, no. 3(1981), 641–668.

[7] E. Hewitt, *On two problems of Urysohn*, Ann. of Math. **47**, no.3 (1946), 503–509.

[8] J. K. Kohli and A. K. Das, *New normality axioms and decompositions of normality*, Glasnik Mat. **37**, no. 57 (2002), 105–114.

[9] J. K. Kohli and A. K. Das, *On functionally θ-normal spaces*, Applied General Topology **6**, no. 1 (2005), 1–14.

[10] J. K. Kohli and A. K. Das, *A class of spaces containing all generalized absolutely closed* (*almost compact*) *spaces*, Applied General Topology **7**, no. 2 (2006), 233–244.

[11] J. K. Kohli, A. K. Das and R. Kumar, *Weakly functionally θ-normal spaces, θ-shrinking of covers and partition of unity*, Note di Matematica **19** (1999), 293–297.

[12] J. K. Kohli and R. Kumar, *z-supercontinuous functions*, Indian J. Pure Appl. Math. **33**, no. 7 (2002), 1097–1108.

[13] J. K. Kohli and D. Singh, *D_δ-supercontinuous functions*, Indian J.Pure Appl. Math. **34**, no. 7 (2003), 1089–1100.

[14] J. K. Kohli and D. Singh, *Between compactness and quasicompactness*, Acta Math. Hungar. **106**, no. 4 (2005), 317–329.

[15] J. K. Kohli and D. Singh, *Between weak continuity and set connectedness*, Studii Si Cercetari Stintifice Seria Mathematica **15** (2005), 55–65.

[16] J. K. Kohli and D. Singh, *Between regularity and complete regularity and a factorization of complete regularity*, Studii Si Cercetari Stintifice Seria Mathematica, **17** (2007), 125–134.

[17] J. K. Kohli and D. Singh, *Almost cl-supercontinuous functions*, Applied General Topology, to appear.

[18] N. Levine, *Strong continuity in topological spaces*, Amer. Math. Monthly **67** (1960), 269.

[19] P. E. Long and L. Herrington, *Strongly θ-continuous functions*, J. Korean Math. Soc. **8** (1981), 21–28.

[20] P. E. Long and L. Herrington, *The T_θ-topology and faintly continuous functions*, Kyungpook Math. J. **22** (1982), 7–14.

[21] J. Mack, *Countable paracompactness and weak normality properties*, Trans. Amer. Math. Soc. **148** (1970), 265–272.

[22] B. M. Munshi and D. S. Bassan, *Supercontinuous mappings*, Indian J. Pure Appl. Math. **13** (1982), 229–236.

[23] T. Noiri, *On functions with strongly closed graph*, Acta Math. Hungar. **32** (1978), 1–4.

[24] T. Noiri, *On δ-continuous functions*, J. Korean Math. Soc. **18** (1980), 161–166.

[25] T. Noiri, *Supercontinuity and some strong forms of continuity*, Indian J. Pure. Appl. Math. **15**, no. 3 (1984), 241–250.

[26] T. Noiri, *Strong forms of continuity in topological spaces*, Suppl. Rendiconti Circ. Mat. Palermo, II **12** (1986), 107–113.

[27] T. Noiri and S. M. Kang, *On almost strongly θ-continuous functions*, Indian J. Pure Appl. Math. **15**, no. 1 (1984), 1–8.

[28] J. R. Porter and J. Thomas, *On H-closed spaces and minimal Hausdorff spaces*, Trans. Amer. Math. Soc. **138** (1969), 159–170.

[29] I. L. Reilly and M. K. Vamanamurthy, *On super-continuous mappings*, Indian J. Pure. Appl. Math. **14**, no. 6 (1983), 767–772.

[30] M. K. Singal and S. P. Arya, *On almost regular spaces*, Glasnik Mat. **4**, no. 24 (1969), 89–99.

[31] M. K. Singal and S. P. Arya, *On almost normal and almost completely regular spaces*, Glasnik Mat. **5**, no. 25 (1970), 141–152.

[32] M. K. Singal and A. Mathur, *On nearly compact spaces*, Boll. Un. Mat. Ital. **2**, no. 4 (1969), 702–710.

[33] M. K. Singal and A. R. Singal, *Almost continuous mappings*, Yokohama Math. J. **16** (1968), 63–73.

[34] D. Singh, *cl-supercontinuous functions*, Applied General Topology **8**, no. 2 (2007), 293–300.

[35] S. Sinharoy and S. Bandyopadhyay, *On θ-completely regular spaces and locally $\theta - H$-closed spaces*, Bull. Cal. Math. Soc. **87** (1995), 19–28.

[36] N. K. Veličko, *H-closed topological spaces*, Amer. Math. Soc. Transl. **78**, no. 2 (1968), 103–118.

Received May 2007
Accepted January 2008

J. K. Kohli (jk_kohli@yahoo.com)

Dep. of Mathematics, Hindu College, University of Delhi, Delhi 110 007, India

D. Singh (dstopology@rediffmail.com)

Dep. of Mathematics, Sri Aurobindo College, University of Delhi-South Campus, Delhi 110 017, India

Rajesh Kumar (rkumar2704@yahoo.co.in)

Dep. of Mathematics, Rajdhani College, University of Delhi, Delhi 110 015, India