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*-half completeness in quasi-uniform spaces

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Abstract. Romaguera and Sánchez-Granero (2003) have introduced the notion of T_1 *-half completion and used it to see when a quasi-uniform space has a *-compactification. In this paper, for any quasi-uniform space, we construct a *-half completion, called standard *-half completion. The constructed *-half completion coincides with the usual uniform completion in the uniform spaces and is the unique (up to quasi-isomorphism) T_1 *-half completion of a symmetrizable quasi-uniform space. Moreover, it constitutes a *-compactification for *-Cauchy bounded quasi-uniform spaces. Finally, we give an example which shows that the standard *-half completion differs from the bicompletion construction.

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1. Introduction and preliminaries

The problem of constructing compactifications of quasi-uniform spaces has been investigated by several authors ([4, 3.47], [5], [7]). This notion of quasi-uniform compactification is by definition Hausdorff. Moreover, a point symmetric totally bounded T_1 quasi-uniform space may have many totally bounded compactifications (see [5, page 34]) . Contrary to this notion, Romaguera and Sánchez-Granero have introduced the notion of *-compactification of a T_1 quasi-uniform space (see [8], [10] and [11]) and prove that: (a) Each T_1 quasi-uniform space having a T_1 *-compactification has an (up to quasi-isomorphism) unique T_1 *-compactification ([11, Corollary of Theorem 1]); and (b) All the Wallman-type compactifications of a T_1 topological space can be characterized in terms of the *-compactification of its point symmetric totally transitive compatible quasi-uniformities ([9, Theorem 1]). The proof of (a) is achieved with the help of the notion of T_1 *-half completion of a quasi-uniform space, which is introduced in [11]. Following ([11, Theorem 1]), if a quasi-uniform space $(X,\mathcal{U})$ is T_1

*-half completable (it has a T_1 *-half completion), then any T_1 *-half completion of $(X,\mathcal{U})$ is unique up to a quasi-isomorphism. In this paper, we prove that every quasi-uniform space has a *-half completion, called standard *-half completion, which in the case of a uniform space coincides with the usual one. We also give an example which shows that the standard *-half completion and the bicompletion are in general different. While a quasi-uniform space may have many *-half completions, here we prove that a symmetrizable quasi-uniform space has an (up to a quasi-isomorphism) unique *-half completion. We also prove that the standard *-half completion constitutes a *-compactification for *-Cauchy bounded quasi-uniform spaces.

Let us recall that a quasi-uniformity on a (nonempty) set X is a filter $\mathcal{U}$ on $X \times X$ such that for each $U \in \mathcal{U}$, (i) $\Delta(X) = \{(x,x) | x \in X\} \subseteq U$, and (ii) $V \circ V \subseteq U$ for some $V \in \mathcal{U}$, where $V \circ V = \{(x,y) \in X \times X|$ there is $z \in X$ such that $(x,z) \in V$ and $(z,y) \in V\}$. The pair $(X,\mathcal{U})$ is called a *quasi-uniform space*. If $\mathcal{U}$ is a quasi-uniformity on a set X, then $\mathcal{U}^{-1} = \{U^{-1} | U \in \mathcal{U}\}$ is also a quasi-uniformity on X called the *conjugate* of $\mathcal{U}$. Given a quasi-uniformity $\mathcal{U}$ on X, $\mathcal{U}^\star = \mathcal{U} \bigvee \mathcal{U}^{-1}$ will denote the coarsest uniformity on X which is finer than $\mathcal{U}$. If $U \in \mathcal{U}$, the entourage $U \cap U^{-1}$ of $\mathcal{U}^\star$ will be denoted by $U^\star$. The topology $\tau(\mathcal{U})$ induced by the quasi-uniformity $\mathcal{U}$ on X is $\{G \subseteq X|$ for each $x \in G$ there is $U \in \mathcal{U}$ such that $U(x) \subseteq G$ $\}$ where $U(x) = \{y \in X | (x,y) \in U\}$. If (X,τ) is a topological space and $\mathcal{U}$ is a quasi-uniformity on X such that $\tau = \tau(\mathcal{U})$ we say that $\mathcal{U}$ is compatible with τ. Let $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ be two quasi-uniform spaces. A mapping $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is said to be *quasi-uniformly continuous* if for each $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $(f(x), f(y)) \in V$ whenever $(x,y) \in U$. A bijection $f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ is called a *quasi-isomorphism* if f and f^{-1} are quasi-uniformly continuous. In this case we say that $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are *quasi-isomorphic*. A filter $\mathcal{B}$ is called *$\mathcal{U}^\star$-Cauchy* if and only if for each $U \in \mathcal{U}$ there exists $B \in \mathcal{B}$ such that $B \times B \subseteq U$ (see [4, page 48]). A net $(x_a)_{a \in A}$ is called *$\mathcal{U}^\star$-Cauchy net* if for each $U \in \mathcal{U}$ there exists an $a_U \in A$ such that $(x_a, x_{a'}) \in U$ whenever $a \geq a_U, a' \geq a_U$. We call a_U *extreme index* of $(x_a)_{a \in A}$ for U and x_{a_U} *extreme point* of $(x_a)_{a \in A}$ for U. A quasi-uniform space $(X,\mathcal{U})$ is *half complete* if every $\mathcal{U}^\star$-Cauchy filter is $\tau(\mathcal{U})$-convergent [2]. Following to [11, Theorem 1], a **-half completion* of a T_1 quasi-uniform space $(X,\mathcal{U})$ is a half complete T_1 quasi-uniform space $(Y,\mathcal{V})$ that has a $\tau(\mathcal{V}^\star)$-dense subspace quasi-isomorphic to $(X,\mathcal{U})$. In [11, Definition 3] also the authors introduce and study the notion of a *-compactification a T_1 quasi-uniform space. A **-compactification* of a T_1 quasi-uniform space $(X,\mathcal{U})$ is a compact T_1 quasi-uniform space $(Y,\mathcal{V})$ that has a $\tau(\mathcal{V}^\star)$-dense subspace quasi-isomorphic to $(X,\mathcal{U})$.

2. The *-half-completion

According to Doitchinov [3, Definition 1], a net $(y_\beta)_{\beta \in B}$ is called *a conet* of the net $(x_a)_{a \in A}$, if for any $U \in \mathcal{U}$ there are $a_U \in A$ and $\beta_U \in B$ such that

$(y_\beta, x_a) \in U$ whenever $a \geq a_U$ and $\beta \geq \beta_U$. In this case, we write $(y_\beta, x_a) \longrightarrow 0$. We denote (x) the constant net $(x_a)_{a\in A}$, for which $x_a = x$ for each $a \in A$.

Definition 2.1 (see [1, Definitions 1.1(3)]). *Let $(X, \mathcal{U})$ be a quasi-uniform space.*

(1) *For every $\mathcal{U}^\star$-Cauchy net $(x_a)_{a\in A}$ we consider a $\mathcal{U}^\star$-Cauchy net $(y_\beta)_{\beta\in B}$ which is a conet of $(x_a)_{a\in A}$, different than $(x_a)_{a\in A}$. In the following, we consider all the nets $\mathcal{A} = \{(x^i_a)_{a\in A_i} | i \in I\}$ that have $(y_\beta)_{\beta\in B}$ as their conet including $(y_\beta)_{\beta\in B}$ itself. In the next, we pick up all the nets $\mathcal{B} = \{(y^j_\beta)_{\beta\in B_j} | j \in J\}$ which are conets of all the elements of $\mathcal{A}$. The ordered couple $(\mathcal{A}, \mathcal{B})$ have the following properties:*

   (a) *for every $U \in \mathcal{U}$ and every $(x^i_a)_{a\in A_i} \in \mathcal{A}$, $(y^j_\beta)_{\beta\in B_j} \in \mathcal{B}$ there are indices a^i_U, β^j_U such that $(y^j_\beta, x^i_a) \in U$ whenever $a \geq a^i_U$ and $\beta \geq \beta^j_U$.*
   We call a^i_U (resp. β^j_U) extreme index of $(x^i_a)_{a\in A_i}$ (resp. $(y^j_\beta)_{\beta\in B_j}$) for U and $x^i_{a^i_U}$ (resp. $y^j_{\beta^j_U}$) extreme point of $(x^i_a)_{a\in A_i}$ (resp. $(y^j_\beta)_{\beta\in B_j}$) for U.

   (b) *$\mathcal{B}$ contains all the conets of all the elements of $\mathcal{A}$ and conversely $\mathcal{A}$ contains all the nets whose conets are all the elements of $\mathcal{B}$.*
   We call the ordered pair $(\mathcal{A}, \mathcal{B})$ h^*-cut, the nets $(x_a)_{a\in A}$ and $(y_\beta)_{\beta\in B}$ generator and co-generator of $(\mathcal{A}, \mathcal{B})$ respectively. We also say that the pair $((y_\beta)_{\beta\in B}, (x_a)_{a\in A})$ generates the h^*-cut $(\mathcal{A}, \mathcal{B})$. It is clear that different pairs of $\mathcal{U}^\star$-Cauchy nets can generate the same h^*-cut.
   The families $\mathcal{A}$ and $\mathcal{B}$ are called classes (first and second respectively) of the h^*-cut $(\mathcal{A}, \mathcal{B})$. In the following, $\widetilde{X}$ denotes the set of all h^*-cuts in X.

   If the above $\mathcal{U}^\star$-Cauchy net $(x_a)_{a\in A}$ has not as conet a $\mathcal{U}^\star$-Cauchy net different from itself, then we relate to it the h^*-cut which generated by the pair $((x_a)_{a\in A}, (x_a)_{a\in A})$.

(2) *To every $x \in X$ we assign an h^*-cut, denoted $\phi(x) = (\mathcal{A}_{\phi(x)}, \mathcal{B}_{\phi(x)})$, which is generated by the pair $((x), (x))$. Clearly, x belongs to both of $\mathcal{A}_{\phi(x)}$ and $\mathcal{B}_{\phi(x)}$. Thus the class $\mathcal{A}_{\phi(x)}$ contains all the nets which converge to x in $\tau_{\mathcal{U}}$ and $\mathcal{B}_{\phi(x)}$ contains nets which converge to x in $\tau_{\mathcal{U}^{-1}}$.*

(3) *Suppose that $\mathcal{K} = \{(x_a)_{a\in A} | (x_a)_{a\in A}$ is a non $\tau(\mathcal{U})$-convergent $\mathcal{U}^\star$-Cauchy net$\}$. Let $X^{\mathcal{K}} = \{\xi \in \widetilde{X} |$ the generator of ξ belongs to $\mathcal{K}\}$. Then we put $\overline{X} = \phi(X) \cup X^{\mathcal{K}}$.*

(4) *We often say for a $\mathcal{U}^\star$-Cauchy net $(x_a)_{a\in A}$ with a conet $(y_\beta)_{\beta\in B}$ and $U \in \mathcal{U}$ that:*

   "finally $((y_\beta)_\beta, (x_a)_a) \in U$"or in symbols "$\tau.((y_\beta)_\beta, (x_a)_a) \in U$ ",

   if there are $a_U \in A$ and $\beta_U \in B$ such that $(y_\beta, x_a) \in U$ whenever $a \geq a_U$, $\beta \geq \beta_U$.

Definition 2.2. *Let $(X,\mathcal{U})$ be a quasi-uniform space, $\xi \in \overline{X}$ and $W \in \mathcal{U}$.*

(1) *We say that a net $(t_\gamma)_{\gamma\in\Gamma}$ is W-close to ξ, if for each net $(x^i_a)_{a\in A_i} \in \mathcal{A}_\xi$ there holds $\tau.((t_\gamma)_\gamma, (x^i_a)_a) \in W$.*

(2) *For each $U \in \mathcal{U}$ denote by $\overline{U}$ the collection of all pairs (ξ', ξ'') for which a co-generator of ξ' is U-close to ξ''.*

The proof of the following result is straightforward, so it is omitted.

Proposition 2.3. *Let $(X,\mathcal{U})$ be a quasi-uniform space and let $(y_\beta)_{\beta\in B}$ be a co-generator of an h^*-cut ξ in $\overline{X}$. Then $(y_\beta)_{\beta\in B}$ belongs to both of the classes $\mathcal{A}_\xi$ and $\mathcal{B}_\xi$.*

As an immediate consequence of Definition 2.2 and Proposition 2.3 we obtain the following proposition.

Proposition 2.4. *Let $(X,\mathcal{U})$ be a quasi-uniform space, $\xi', \xi'' \in \overline{X}$ and $U \in \mathcal{U}$. If $(y_\beta)_{\beta\in B}$, $(y_\gamma)_{\gamma\in\Gamma}$ are co-generators of ξ' and ξ'' respectively, then*

$$(\xi', \xi'') \in \overline{U} \text{ if and only if } \tau.((y_\beta)_\beta, (y_\gamma)_\gamma) \in U.$$

Corollary 2.5. *Let $(X,\mathcal{U})$ be a quasi-uniform space and let $\xi', \xi'' \in \overline{X}$. If $(y_\beta)_{\beta\in B}$, $(y_\gamma)_{\gamma\in\Gamma}$ are co-generators of ξ' and ξ'' respectively, then*

$$\xi' = \xi'' \text{ if and only if } (y_\beta, y_\gamma) \longrightarrow 0 \text{ in } \tau(\mathcal{U}^\star).$$

The following lemma is obvious.

Lemma 2.6. *Let $U, V \in \mathcal{U}$. Then $U \subseteq V$ if and only if $\overline{V} \subseteq \overline{U}$.*

Theorem 2.7. *The family $\overline{\mathcal{U}} = \{\overline{U} | U \in \mathcal{U}\}$ is a base for a quasi-uniformity $\overline{\mathcal{U}}$ on $\overline{X}$.*

Proof. By definitions 2.2 and Proposition 2.3, it follows that the pair (ξ, ξ) belongs to every element of $\overline{\mathcal{U}}$ and by the previous Lemma $\overline{\mathcal{U}}$ is a filter.

Let now $U, W \in \mathcal{U}$ be such that $W \circ W \circ W \subseteq U$ and $\overline{x}, \overline{y} \in \overline{X}$ with $(\overline{x}, \overline{y}) \in \overline{W} \circ \overline{W}$. Then there exists a $\overline{z}$ in $\overline{X}$ such that $(\overline{x}, \overline{z}) \in \overline{W}$ and $(\overline{z}, \overline{y}) \in \overline{W}$. If $(x^{\overline{x}}_a)_{a\in A}$, $(z^{\overline{z}}_\gamma)_{\gamma\in\Gamma}$ and $(y^{\overline{y}}_\beta)_{\beta\in B}$ are co-generators of $\overline{x}$, $\overline{z}$ and $\overline{y}$ respectively, then Definition 2.2 and Proposition 2.3 imply that $\tau.((x^{\overline{x}}_a)_a, (z^{\overline{z}}_\gamma)_\gamma) \in W$ and $\tau.((z^{\overline{z}}_\gamma)_\gamma, (y^{\overline{y}}_\beta)_\beta) \in W$. We note that, for each $(t_\delta)_{\delta\in\Delta} \in \mathcal{A}_{\overline{y}}$, it holds that $\tau.(y^{\overline{y}}_\beta, t_\delta) \longrightarrow 0$. Hence, $\tau.((x^{\overline{x}}_a)_a, (t_\delta)_\delta) \in W \circ W \circ W \subseteq U$ which implies that $(\overline{x}, \overline{y}) \in \overline{U}$. □

Proposition 2.8. *If $\xi \in \overline{X}$ and $(x_a)_{a\in A}$ is a $\mathcal{U}^\star$-Cauchy net which belong to $\mathcal{A}_\xi$, then $\phi(x_a) \longrightarrow \xi$. Dually, if $(y_\beta)_{\beta\in B}$ is a $\mathcal{U}^\star$-Cauchy net which belong to $\mathcal{B}_\xi$, then $\lim\limits_\beta(\phi(y_\beta), \xi) = 0$.*

Proof. Let V, $U \in \mathcal{U}$ such that $V \circ V \subseteq U$. If $(z_\gamma)_{\gamma\in\Gamma}$ is a co-generator of ξ, then $(z_\gamma, x_a) \longrightarrow 0$. Thus there are a_V and γ_V such that $(z_\gamma, x_a) \in V$ for $\gamma \geq \gamma_V$ and $a \geq a_V$. Fix an $a \geq a_V$ and pick a net $(x_\delta)_{\delta\in\Delta}$ of $\mathcal{A}_{\phi(x_a)}$. Then, $x_\delta \longrightarrow x_a$ and so $(x_a, x_\delta) \in V$, whenever $\delta \geq \delta_V$ for some $\delta_V \in \Delta$. Hence, $(z_\gamma, x_\delta) \in U$ for $\gamma \geq \gamma_V$ and $\delta \geq \delta_V$. Hence $(\xi, \phi(x_a)) \in \overline{U}$, whenever $a \geq a_V$.

The proof of the dual is similar. □

Theorem 2.9. *The quasi-uniform space* $(\overline{X},\overline{\mathcal{U}})$ *is a* **-half completion of* $(X,\mathcal{U})$.

Proof. We firstly prove that $(\overline{X},\overline{\mathcal{U}})$ is half-complete, and secondly that the space $(\overline{X},\overline{\mathcal{U}})$ has a $\tau(\overline{\mathcal{U}}^\star)$-dense subspace quasi-isomorphic to $(X,\mathcal{U})$. Indeed, let $(\xi_a)_{a\in A}$ be a $\overline{\mathcal{U}}^\star$-Cauchy net of $\overline{X}$. In the following, for each $a \in A$, $(y^a_\beta)_{\beta\in B_a}$ denotes a co-generator of ξ_a. Suppose that $W \in \mathcal{U}$. Then, there exists $a_{\overline{W}} \in A$ such that $(\xi_\gamma, \xi_a) \in \overline{W}$ whenever $\gamma, a \geq a_{\overline{W}}$. Fix an $a \geq a_{\overline{W}}$ and suppose that $\beta(a, W)$ is the extreme index of $(y^a_\beta)_{\beta\in B_a}$ for W.

We consider the set

$$A^\star = \{(a, W)|a \in A, W \in \mathcal{U}\}$$

ordered by $(a', W') \leq (a'', W'')$ if $a' \leq a''$ and $W'' \subseteq W'$.

We put $y(a, W) = y^a_{\beta(a,W)}$ and we prove that the net

$$\{y(a, W)|(a, W) \in A^\star\}$$

is a $\mathcal{U}^\star$-Cauchy net.

Indeed, let $U \in \mathcal{U}$. Pick $V \in \mathcal{U}$ such that $V \circ V \circ V \subseteq U$. Suppose that $(a', W'), (a'', W'') \geq (a_{\overline{V}}, V)$. Then, $(y(a', W'), y^{a'}_{\beta'}) \in (W')^\star \subseteq V^\star$ and $(y(a'', W''), y^{a''}_{\beta''}) \in (W'')^\star \subseteq V^\star$ whenever $\beta' \geq \beta'(a', W')$ and $\beta'' \geq \beta''(a'', W'')$. Since $(\xi_a)_{a\in A}$ is a $\overline{\mathcal{U}}^\star$-Cauchy net of $\overline{X}$, Proposition 2.4 implies that $\tau.((y^{a'}_{\beta'})_{\beta'}, (y^{a''}_{\beta''})_{\beta''}) \in V^\star$ whenever $a', a'' \geq a_{\overline{V}}$. Hence, $(y(a', W'), y(a'', W'')) \in V^\star \circ V^\star \circ V^\star \subseteq U^\star$.

We now prove that $(\xi_a)_{a\in A}$ is $\tau(\overline{\mathcal{U}})$-convergent. We have two cases.

Case 1. $(y(a, W))_{(a,W)\in A^\star}$ $\tau(\mathcal{U})$*-converges to a point* $x \in X$.

In this case, we have that $(\phi(y(a, W)))_{(a,W)\in A^\star}$ $\tau(\overline{\mathcal{U}})$-converges to $\phi(x)$. Since $(y^a_\beta)_{\beta\in B_a}$ belongs to $\mathcal{B}_{\xi_a}$, Proposition 2.8 implies that $(\phi(y(a, W)), \xi_a) \longrightarrow 0$. Hence, from $(\phi(x), \phi(y(a, W))) \longrightarrow 0$ we conclude that $(\xi_a)_{a\in A}$ $\tau(\overline{\mathcal{U}})$-converges to $\phi(x)$.

Case 2. $(y(a, W))_{(a,W)\in A^\star}$ *is a non* $\tau(\mathcal{U})$*-convergent net.*

Let ξ be the h^*-cut in $\overline{X}$ which is generated by $(y(a, W))_{(a,W)\in A^\star}$. It follows, by Proposition 2.8, that $(\xi, \phi(y(a, W))) \to 0$. Since $(y^a_\beta)_{\beta\in B_a}$ belongs to $\mathcal{B}_{\xi_a}$, Proposition 2.8 implies that $(\phi(y(a, W)), \xi_a) \longrightarrow 0$. The rest is obvious.

It remains to prove that $(\phi(X), \overline{\mathcal{U}}/\phi(X) \times \phi(X))$ is a $\tau(\overline{\mathcal{U}}^\star)$-dense subspace of $(\overline{X},\overline{\mathcal{U}})$. Indeed, let $\xi \in \overline{X}$ and let $(y_\beta)_{\beta\in B}$ be a co-generator of it. Then, since the co-generator belongs to both of classes of ξ, Proposition 2.8 implies that $\phi(y_\beta)$ $\tau(\overline{\mathcal{U}}^\star)$-converges to ξ. □

In the sequel the *-half completion $(\overline{X},\overline{\mathcal{U}})$ constructed above will be called *standard* **-half completion* of the space $(X,\mathcal{U})$.

The following example shows that the standard *-half completion and the bicompletion of a quasi-uniform space are in general different.

Example 2.10. Let X be the set consisting of all nonzero real numbers and let d be the quasi-metric on X given by:

$$d(x,y) = \begin{cases} y - x & \text{if } x < y \\ 0 & \text{otherwise} \end{cases}$$

Suppose that $\mathcal{U}$ is the quasi-uniformity generated by d. Let $\mathcal{F}$ be the $\mathcal{U}^{\star}$-Cauchy filter generated by $\{(0,a)|a > 0\}$ and $\mathcal{G}$ be the $\mathcal{U}^{\star}$-Cauchy filter generated by $\{(b,0)|b < 0\}$. Then a new point is defined by the h^{*}-cut $\xi = (\mathcal{A}_{\xi}, \mathcal{B}_{\xi})$, where $\mathcal{A}_{\xi} = \{\mathcal{G}, \mathcal{F}\}$ and $\mathcal{B}_{\xi} = \{\mathcal{F}\}$. Hence, $\overline{X} = \phi(X) \cup \{\xi\}$. Clearly, ξ defines the point 0 in $(\overline{X},\overline{\mathcal{U}})$. On the other hand, there is exactly one minimal $\mathcal{U}^{\star}$-Cauchy filter coarser than $\mathcal{F}$ and $\mathcal{G}$ respectively. More precisely, if $\mathcal{F}_0$ and $\mathcal{G}_0$ are any bases for $\mathcal{F}$ and $\mathcal{G}$ respectively, then $\{U(F_0)\ |F_0 \in \mathcal{F}_0$ and U is a symmetric member of $\mathcal{U}^{\star}\}$ and $\{U(G_0)\ |G_0 \in \mathcal{G}_0$ and U is a symmetric member of $\mathcal{U}^{\star}\}$ are equivalent bases for the minimal $\mathcal{U}^{\star}$-Cauchy filter $\widetilde{\mathcal{H}}$ coarser than $\mathcal{F}$ and $\mathcal{G}$ respectively. Hence, we have $\widetilde{X} = i(X) \cup \{\mathcal{H}\}$. The filter $\mathcal{H}$ defines the point 0 in $(\widetilde{X},\widetilde{\mathcal{U}})$ as well. We conclude the following:

(i) *The bicompletion of* $(X,\mathcal{U})$ *differs from the standard* **-half completion.* Indeed, by the definition of ξ and from the Propositions 2.3 and 2.8, we conclude that $\phi(\mathcal{G})$ and $\phi(\mathcal{F})$ converge to 0 with respect to $\tau(\overline{\mathcal{U}})$ and $\tau(\overline{\mathcal{U}}^{\star})$ respectively. On the other hand, $i(\mathcal{G})$ and $i(\mathcal{F})$ converge to 0 with respect to $\tau(\widetilde{\mathcal{U}}^{\star})$.

(ii) *The standard* **-half completion is not quasi-uniformly isomorphic to its bicompletion.* This is true by (i) and the fact that the bicompletion of $(\overline{X},\overline{\mathcal{U}})$ coincides up to a quasi-isomorphism with the bicompletion of $(X,\mathcal{U})$.

Theorem 2.11. *Let* $(X,\mathcal{U})$ *be a uniform space. Then, the standard* **-half completion* $(\overline{X},\overline{\mathcal{U}})$ *coincides with the usual uniform completion.*

Proof. Let $(X,\mathcal{U})$ be a uniform space and let ξ be an h^{*}-cut in X. Suppose that $(x_a)_{a\in A} \in \mathcal{A}_{\xi}$ and $(y_{\beta})_{\beta\in B} \in \mathcal{B}_{\xi}$. Then $(y_{\beta}, x_a) \longrightarrow 0$ and $(x_a, y_{\beta}) \longrightarrow 0$. Hence the nets and the conets of ξ coincide. Thus, the class of equivalent Cauchy nets, of the uniform case, is identified with an h^{*}-cut and vice versa. Hence the "ground set" of the two completions is the $\overline{X}$. The rest is obvious. □

Next, we give an equivalent definition for nets for the Definition 5 in [11].

Definition 2.12. *Let* $(X,\mathcal{U})$ *be a quasi-uniform space. A* $\mathcal{U}^{\star}$*-Cauchy net* $(x_a)_{a\in A}$ *on* X *is said to be symmetrizable if whenever* $(y_{\beta})_{\beta\in B}$ *is a* $\mathcal{U}^{\star}$*-Cauchy net on* X *such that* $(y_{\beta}, x_a) \longrightarrow 0$*, then* $(x_a, y_{\beta}) \longrightarrow 0$.

Definition 2.13. *A quasi-uniform space* $(X,\mathcal{U})$ *is called symmetrizable if each* $\mathcal{U}^{\star}$*-Cauchy net on* X*, including for each* $x \in X$ *the constant net* (x)*, is symmetrizable.*

It easy to check that a quasi-uniform space is symmetrizable if and only if the bicompletion is T_1. In this case, the space has only one T_0 *-half completion, the bicompletion. From Theorem 2.9 and [11, Theorem 1] we immediate deduce the following result.

Corollary 2.14. *If a T_1 quasi-uniform space is symmetrizable, then it has a T_1 *-half completion which is unique up to a quasi-isomorphism.*

3. Standard *-half completion and *-Compactification

We recall some well known notions from [6].

A net $(x_a)_{a\in A}$ is said to be *frequently in S*, for some subset S of X, if and only if for all $a \in A$ there is some $a' \geq a$ such that $x_{a'} \in S$. A net is said to be *eventually in S* if and only if there is an a_0 in A such that for all $a \geq a_0$, x_a is in S. A point x in X is a cluster point of the net $(x_a)_{a\in A}$ if and only if the net is frequently in all neighborhoods of x. The net $(x_a)_{a\in A}$ converges to x if and only if $(x_a)_{a\in A}$ is eventually in all neighborhoods of x. The tail sets of $(x_a)_{a\in A}$ are the sets T_a (a in A) where $T_a = \{x_{a'} | a' \geq a\}$. Note that the T_a have the finite intersection property, by the directedness of the index set A, so they generate a filter, the *filter of tails* of $(x_a)_{a\in A}$ or *the filter associated with the net* $(x_a)_{a\in A}$. Then a point x is a cluster point of $(x_a)_{a\in A}$ if and only if x is in $cl(T_a)$ for all a (if and only if it is a cluster point of the filter of tails). And $x_a \longrightarrow x$ if and only if the filter of tails converges to x. This already shows that there is a close relationship between convergence of filters and convergence of nets.

Definition 3.1 (see [6, page 81])**.** *A universal net in X is one such that for each $S \subset X$, either the net is eventually in S, or it is eventually in $X \setminus S$.*

From the classical theory we have the following statements.

(a) A net is a universal net if and only if its associated filter is an ultrafilter.
(b) Let $\mathcal{F}$ be the filter associated with the net $(x_a)_{a\in A}$ and $\mathcal{G}$ be a filter with $\mathcal{F} \subset \mathcal{G}$. Then $(x_a)_{a\in A}$ has a subnet whose associated filter is $\mathcal{G}$.

(a) and (b) implies that:

(c) Every net has a universal subnet.
(d) A universal net converges to each of its cluster points.
(e) A space is compact if and only if every universal net is convergent.

Definition 3.2 (see [11, Definition 6])**.** *A quasi-uniform space $(X,\mathcal{U})$ is called *-Cauchy bounded if for each ultrafilter $\mathcal{F}$ on X there is a $\mathcal{U}^\star$-Cauchy filter $\mathcal{G}$ on X such that $(\mathcal{G},\mathcal{F}) \longrightarrow 0$.*

Definition 3.2 admits an equivalent definition for nets.

Definition 3.3. *A quasi-uniform space $(X,\mathcal{U})$ is called *-Cauchy bounded if for each universal net $(x_a)_{a\in A}$ on X there is a $\mathcal{U}^\star$-Cauchy net $(y_\beta)_{\beta\in B}$ on X such that $(y_\beta, x_a) \longrightarrow 0$.*

Theorem 3.4. *Let $(X,\mathcal{U})$ be a *-Cauchy bounded quasi-uniform space. Then the standard *-half completion $(\overline{X},\overline{\mathcal{U}})$ is a *-compactification of the space $(X,\mathcal{U})$.*

Proof. Let $(\xi_a)_{a\in A}$ be a universal net in $(\overline{X},\overline{\mathcal{U}})$. Suppose that for any $a \in A$, $\xi_a = (\mathcal{A}_{\xi_a}, \mathcal{B}_{\xi_a})$. Let $(y^a_\beta)_{\beta\in B_a}$ and $\{y(a,W)|(a,W) \in A^\star\}$ be as in the proof of Theorem 2.9. Then, $\{y(a,W)|(a,W) \in A^\star\}$ is a net in X. By the above statement (c), we have that $(y(a,W))_{(a,W)\in A^\star}$ has a universal subnet, let $\{y(a_k,W_k)|(a_k,W_k) \in A^\star,\ k \in K\}$. Since $(X,\mathcal{U})$ is *-Cauchy bounded, there is a $\mathcal{U}^\star$-Cauchy net $(x_\gamma)_{\gamma\in\Gamma}$ of X such that $(x_\gamma, y(a_k,W_k)) \longrightarrow 0$. Hence $(\phi(x_\gamma), \phi(y(a_k,W_k))) \longrightarrow 0$ in $(\overline{X},\overline{\mathcal{U}})$ (1). On the other hand, since the space $(\overline{X},\overline{\mathcal{U}})$ is half-complete, there exists $\xi \in \overline{X}$ such that $(\phi(x_\gamma))_{\gamma\in\Gamma}$ $\tau(\overline{\mathcal{U}})$-converges to ξ (2). Hence by (1) and (2) we conclude that $\{\phi(y(a_k,W_k))|(a_k,W_k) \in A^\star,\ k \in K\}$ $\tau(\overline{\mathcal{U}})$-converges to ξ. Since $\{\phi(y(a_k,W_k))|(a_k,W_k) \in A^\star,\ k \in K\}$ is a subnet of $\phi(y(a,W))_{(a,W)\in A^\star}$ we conclude that ξ is a cluster point of the latter. Since $(y^a_\beta)_{\beta\in B_a}$ belongs to $\mathcal{B}_{\xi_a}$, Proposition 8 implies that $(\phi(y(a,W)), \xi_a) \longrightarrow 0$. Hence, ξ is a cluster point of $(\xi_a)_{a\in A}$. There also holds that $(\xi_a)_{a\in A}$ is a universal net, thus the above statement (d) implies that it $\tau(\overline{\mathcal{U}})$-converges to ξ. Finally, by the above statement (e) we conclude that the space $(\overline{X},\overline{\mathcal{U}})$ is compact. By Theorem 9, the space $(\overline{X},\overline{\mathcal{U}})$ has a $\tau(\overline{\mathcal{U}}^\star)$-dense subspace quasi-isomorphic to $(X,\mathcal{U})$. Hence $(\overline{X},\overline{\mathcal{U}})$ is a *-compactification of $(X,\mathcal{U})$. □

References

[1] A. Andrikopoulos, *Completeness in quasi-uniform spaces*, Acta Math. Hungar. **105** (2004), 549-565, MR 2005f:54050.

[2] J. Deak, *On the coincidence of some notions of quasi-uniform completeness defined by filter pairs*, Stud. Sci. Math. Hungar. **26** (1991), 411-413, MR 94e:94077.

[3] D. Doitchinov, *A concept of completeness of quasi-uniform spaces*, Topology Appl. **38** (1991), 205-217, MR 92b:54061.

[4] P. Fletcher and W. F. Lindgren, *Quasi-uniform spaces*, Lectures Notes in Pure and Appl. Math. **77** (1978), Marc. Dekker, New York, MR 84h:54026.

[5] P. Fletcher, and W. F. Lindgren, *Compactifications of totally bounded quasi-uniform spaces*, Glasgow Math. J. **28** (1986), 31-36, MR 87f:54037.

[6] J. Kelley, *General Topology*, D.Van Nostrand Company, Inc., Toronto-New York-London, (1955), MR 16, 1136c.

[7] H. Render, *Nonstandard methods of completing quasi-uniform spaces*, Topology Appl. **62** (1995), 101-125, MR 96a:54041.

[8] S. Romaguera and M. A. Sánchez-Granero, **-Compactifications of quasi-uniform paces*, Stud. Sci. Math. Hung. **44** (2007), 307-316.

[9] S. Romaguera and M. A. Sánchez-Granero, *A quasi-uniform characterization of Wallman type compactifications*, Stud. Sci. Math. Hung. **40** (2003), 257-267, MR 2004h:54021.

[10] S. Romaguera and M. A. Sánchez-Granero, *Compactifications of quasi-uniform hyperspaces*, Topology Appl. **127** (2003), 409-423, MR 2003j:54011.

[11] S. Romaguera and M. A. Sánchez-Granero, *Completions and compactifications of quasi-uniform spaces*, Topology Appl. **123** (2002), 363-382, MR 2003c:54051.

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