

F-supercontinuous functions

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ABSTRACT. A strong variant of continuity called ' F -supercontinuity' is introduced. The class of F -supercontinuous functions strictly contains the class of z -supercontinuous functions (Indian J. Pure Appl. Math. 33 (7) (2002), 1097–1108) which in turn properly contains the class of cl -supercontinuous functions (\equiv clopen maps) (Appl. Gen. Topology 8 (2) (2007), 293–300; Indian J. Pure Appl. Math. 14 (6) (1983), 762–772). Further, the class of F -supercontinuous functions is properly contained in the class of R -supercontinuous functions which in turn is strictly contained in the class of continuous functions. Basic properties of F -supercontinuous functions are studied and their place in the hierarchy of strong variants of continuity, which already exist in the mathematical literature, is elaborated. If either domain or range is a functionally regular space (Indagationes Math. 15 (1951), 359–368; 38 (1976), 281–288), then the notions of continuity, F -supercontinuity and R -supercontinuity coincide.

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1. INTRODUCTION

Several strong variants of continuity occur in the lore of mathematical literature which arise in many branches of mathematics and applications of mathematics.

In many situations in topology, analysis and other disciplines continuity is not sufficient and a strong form of continuity is required to meet the demand of a particular situation. The strong variants of continuity with which we shall be dealing in this paper include, among others, are strongly continuous functions [16], perfectly continuous functions [20], clopen maps [21] (\equiv cl -supercontinuous

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functions [23]), z -supercontinuous functions [8], D -supercontinuous functions [10], D^* -supercontinuous functions [22], D_δ -supercontinuous functions [11], strongly θ -continuous functions [19] and supercontinuous functions [18]. The main purpose of this paper is to introduce a new class of functions called ' F -supercontinuous functions', study their basic properties and discuss their place in the hierarchy of strong variants of continuity that already exist in the mathematical literature. The notion of F -supercontinuous functions arise naturally in case either domain or range is a functionally regular space ([1], [2]). It turns out that the class of F -supercontinuous functions properly includes the class of z -supercontinuous functions [8] and is strictly contained in the class of R -supercontinuous functions [14] which in turn is properly contained in the class of continuous functions. Further if either domain or range is a functionally regular space ([1], [2]) then all the three classes of (i) F -supercontinuous functions (ii) R -supercontinuous functions, and (iii) continuous functions coincide. Moreover, if either X or Y is a completely regular space, then all these three classes of functions are identical with the class of z -supercontinuous functions [8]. Furthermore, if either domain or range is zero dimensional space, then all the four above classes of functions coincide with the class of cl-supercontinuous functions ([21], [23]).

Section 2 is devoted to the preliminaries and basic definitions. In Section 3, we introduce the notion of ' F -supercontinuous function' and elaborate on its place in the hierarchy of strong variants of continuity which already exist in the literature. Basic properties of F -supercontinuous functions are studied in Section 4, while properties of graph of an F -supercontinuous function are discussed in Section 5. Interplay of topological properties and F -supercontinuous functions is investigated in Section 6 and the notion of F -quotient topology is formulated in Section 7. Change of topology of a topological space (X, τ) into a functionally regular topology τ_F and a completely regular topology τ_z are considered in Section 8 wherein interrelations between τ , τ_F and τ_z are elaborated and alternative proofs of certain results in the preceding sections are suggested.

2. BASIC DEFINITIONS AND PRELIMINARIES

A collection β of subsets of a space X is called as *open complementary system* [6] if β consists of open sets such that for every $B \in \beta$, there exist $B_1, B_2, \dots \in \beta$ with $B = \cup\{X \setminus B_i : i \in N\}$. A subset A of a space X is called a *strongly open F_σ -set* [6] if there exists a countable open complementary system $\beta(A)$ with $A \in \beta(A)$. The complement of a strongly open F_σ -set is called *strongly closed G_δ -set*. A subset A of a space X is called a *regular G_δ -set* [17] if A is an intersection of a sequence of closed sets whose interiors contain A , i.e., if $A = \bigcap_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} F_n^\circ$, where each F_n is a closed subset of X . The complement of a regular G_δ -set is called a *regular F_σ -set*. An open subset A of a space X is said to be *r -open* [14] if it is expressible as a union of closed sets.

Definition 2.1. A function $f : X \rightarrow Y$ from a topological space X into a topological space Y is said to be

- (a) **strongly continuous** [16] if $f(\bar{A}) \subset f(A)$ for each subset A of X .
- (b) **perfectly continuous** [20] if $f^{-1}(V)$ is clopen in X for every open set $V \subset Y$.
- (c) **cl-supercontinuous** [23] (\equiv **clopen map** [21]) if for each $x \in X$ and each open set V containing $f(x)$ there is a clopen set U containing x such that $f(U) \subset V$.
- (d) **z-supercontinuous** [8] if for each $x \in X$ and for each open set V containing $f(x)$, there exists a cozero set U containing x such that $f(U) \subset V$.
- (e) **D_δ -supercontinuous** [11] if for each $x \in X$ and for each open set V containing $f(x)$, there exists a regular F_σ set U containing x such that $f(U) \subset V$.
- (f) **D -supercontinuous** [10] if for each $x \in X$ and each open set U containing $f(x)$ there exists an open F_σ -set V containing x such that $f(V) \subset U$.
- (g) **D^* -supercontinuous** [22] if for each $x \in X$ and each open set U containing $f(x)$ there exists a strongly open F_σ -set V containing x such that $f(V) \subset U$.
- (h) **strongly θ -continuous** [19] if for each $x \in X$ and for each open set V containing $f(x)$, there exists an open set U containing x such that $f(\bar{U}) \subset V$.
- (i) **R -supercontinuous** [14] if for each $x \in X$ and each open set U containing $f(x)$ there exists an r -open set V containing x such that $f(V) \subset U$.
- (j) **supercontinuous** [18] if for each $x \in X$ and for each open set V containing $f(x)$, there exists a regular open set U containing x such that $f(U) \subset V$.

Definition 2.2. A topological space X is said to be

- (i) **functionally regular** ([1], [2]) if for each closed set A and a point $x \notin A$ there exists a continuous real-valued function f defined on X such that $f(x) \notin f(A)$; or equivalently for each $x \in X$ and each open set U containing x there exists a zero set Z such that $x \in Z \subset U$.
- (ii) **D -regular space** [6] if it has a base of open F_σ -sets.
- (iii) **functionally Hausdorff** [25] If for $x, y \in X$, $x \neq y$ there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f(x) \neq f(y)$.
- (iv) **countably H -closed** if it is Hausdorff and every countable open cover of X has a finite subcollection whose union is dense in X .
- (v) **semiregular** if it has a base of regular open sets.

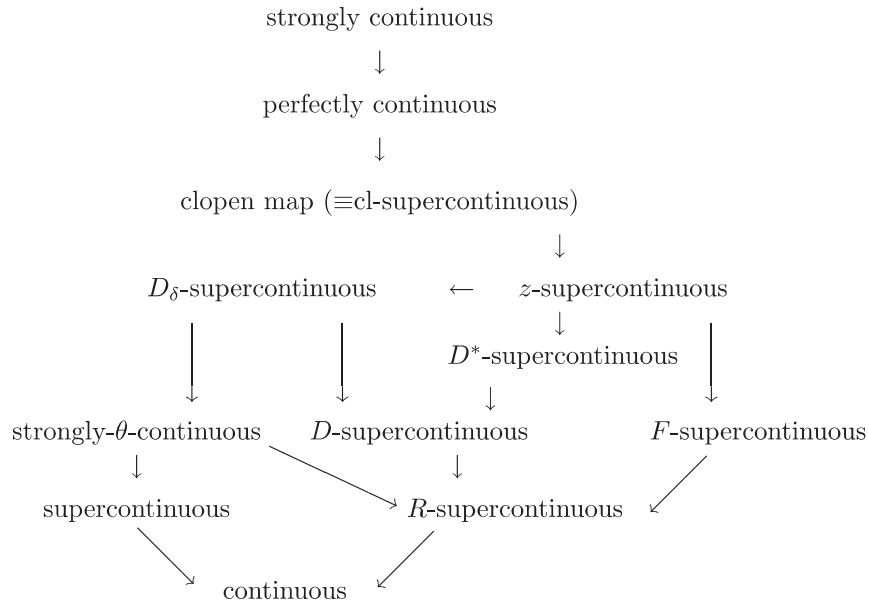
In order to systematize the study of separation by continuous real-valued functions, Van Est and Freudenthal [25] introduced the notion of a functionally regular space and showed its distinctiveness from the standard separation axioms and other separation axioms defined by them. Further properties of functionally regular spaces have been studied by Aull ([1], [2]).

3. *F*-SUPERCONTINUOUS FUNCTIONS

An open set U in a space X is said to be ***F*-open** if for each $x \in U$, there exists a zero set Z in X such that $x \in Z \subset U$, or equivalently, U is expressible as a union of zero sets. The complement of an *F*-open set will be referred to as an ***F*-closed set**.

Definition 3.1. A function $f : X \rightarrow Y$ from a topological space X into a topological space Y is said to be ***F*-supercontinuous** if for each $x \in X$ and each open set U containing $f(x)$ there exists an *F*-open set V containing x such that $f(V) \subset U$.

The following diagram reflects upon the place of *F*-supercontinuous functions in the hierarchy of strong variants of continuity that already exist in the literature. The implications are either well known or immediately follow from definitions.



However, none of the above implications is reversible which is either well known or follows from the following observations and examples.

Observations and Examples

3.2 If either X or Y is a functionally regular space, then every continuous function $f : X \rightarrow Y$ is *F*-supercontinuous and hence *R*-supercontinuous.

3.3 If either X or Y is a completely regular space, then every continuous function $f : X \rightarrow Y$ is *z*-supercontinuous.

3.4 Let $X = Y$ be the regular space due to Hewitt [7] on which every continuous real valued function is constant and let f denote the identity map defined on X . Then f is strongly θ -continuous and so R -supercontinuous but it is not F -supercontinuous.

3.5 Let X be a functionally regular space which is not completely regular and let $Y = X$. Then the identity mapping defined on X is F -supercontinuous but not z -supercontinuous.

3.6 Let X be the space of [3, Exercise 24, p. 139]. Then X is a Hausdorff semiregular space which is not regular. Further, Aull pointed out that the space X is a functionally regular space (see [1, Example 3]). Let f denote the identity mapping defined on X . Then the function f is supercontinuous as well as F -supercontinuous but not strongly θ -continuous.

3.7 Let us denote by X the space of Arens square [24, Example 80, p. 98]. Then X is a Hausdorff space which is not functionally Hausdorff and hence not a functionally regular space. Also, X is semiregular but not regular. So the identity mapping defined on X is supercontinuous but not F -supercontinuous.

3.8 Let us denote by X the space of irregular lattice topology [24, Example 79, p. 97]. Then X is a functionally Hausdorff Lindelöf space which is not a semiregular space. In view of [1, Theorem 3], the space X is a functionally regular space. Let f denote the identity mapping defined on X . Then f is an F -supercontinuous function but not supercontinuous.

3.9 Let us denote by X the real line with the smallest topology generated by the Euclidean topology and the cocountable topology on X . The space X is a functionally regular space, since it is a functionally Hausdorff Lindelöf space (see [1, Theorem 3]). The space X is not D -regular, since X is not a subparacompact space and every Lindelöf, D -regular space is subparacompact (see [4, Theorem 2]). Then the identity mapping defined on X is an F -supercontinuous function but not a D -supercontinuous function.

Proposition 3.10. *Let $f : X \rightarrow Y$ be a continuous function, defined on a functionally Hausdorff Lindelöf space X . Then f is F -supercontinuous.*

Proof. Since a functionally Hausdorff Lindelöf space is functionally regular (see [1]) and since every continuous function defined on a functionally regular space is F -supercontinuous, f is F -supercontinuous. \square

Proposition 3.11. *Let $f : X \rightarrow Y$ be a continuous function. If X is a countably paracompact functionally regular space, then f is F -supercontinuous as well as strongly θ -continuous.*

Proof. Since every continuous function defined on a functionally regular space is F -supercontinuous, so is f . Again, since every countably paracompact functionally regular space is regular (see [1]), and since every continuous function defined on a regular space is strongly θ -continuous, f is strongly θ -continuous. \square

Proposition 3.12. *Let $f : X \rightarrow Y$ be a continuous function defined on a countably compact functionally regular space X . Then f is z -supercontinuous.*

Proof. This is immediate in view of the fact that every countably compact functionally regular space is completely regular (see [1]), and every continuous function defined on a completely regular space is z -supercontinuous. \square

Proposition 3.13. *Let $f : X \rightarrow Y$ be a continuous function defined on a countably H -closed, semiregular, functionally regular space X . Then f is z -supercontinuous.*

Proof. This is immediate from the fact that a countably H -closed, semiregular, functionally regular space X is completely regular (see [1]). \square

4. BASIC PROPERTIES OF F -SUPERCONTINUOUS FUNCTIONS

Theorem 4.1. *For a function $f : X \rightarrow Y$ from a topological space X into a topological space Y , the following statements are equivalent:*

- (a) f is F -supercontinuous.
- (b) The inverse image of every open subset of Y is F -open in X .
- (c) The inverse image of every closed subset of Y is F -closed in X .
- (d) The inverse image of every subbasic open subset of Y is F -open in X .

Proof. It is easy using definitions. \square

Definition 4.2. *Let X be a topological space and let $A \subset X$. A point $x \in X$ is said to be an **F-adherent point** of the set A if every F -open set containing x has non-empty intersection with A . Let A_F denote the set of all F -adherent points of the set A . The set A is F -closed if and only if $A = A_F$.*

Theorem 4.3. *For a function $f : X \rightarrow Y$ the following statements are equivalent.*

- (a) f is F -supercontinuous.
- (b) $f(A_F) \subset f(A)$ for every $A \subset X$.
- (c) $(f^{-1}(B))_F \subset f^{-1}(\overline{B})$ for every $B \subset Y$.

Proof. (a) \Rightarrow (b). Since $\overline{f(A)}$ is closed in Y , by Theorem 4.1 $f^{-1}(\overline{f(A)})$ is an F -closed set in X . Again, since $A \subset f^{-1}(\overline{f(A)})$, $A_F \subset [f^{-1}(\overline{f(A)})]_F = f^{-1}(\overline{f(A)})$ and so $f(A_F) \subset f(f^{-1}(\overline{f(A)})) \subset \overline{f(A)}$.

(b) \Rightarrow (c). Let $B \subset Y$. Then $f((f^{-1}(B))_F) \subset \overline{f(f^{-1}(B))} \subset \overline{B}$ and so it follows that $(f^{-1}(B))_F \subset f^{-1}(\overline{B})$.

(c) \Rightarrow (a). Let B be any closed set in Y . Then $(f^{-1}(B))_F \subset f^{-1}(B)$. Since $f^{-1}(B) \subset \overline{f^{-1}(B)} \subset (f^{-1}(B))_F$, $f^{-1}(B) = (f^{-1}(B))_F$ which in turn implies that f is F -supercontinuous. \square

Definition 4.4. *A filterbase \mathcal{F} is said to **F-converge** to a point x , written as $\mathcal{F} \xrightarrow{F} x$ if every F -open set containing x contains a member of \mathcal{F} .*

Theorem 4.5. *A function $f : X \rightarrow Y$ is F -supercontinuous if and only if for each $x \in X$ and each filter base \mathcal{F} in X that F -converges to x , $f(\mathcal{F}) \rightarrow f(x)$.*

Proof. Suppose that f is F -supercontinuous and let \mathcal{F} be a filter base in X that F -converges to x . To show that the filter base $f(\mathcal{F})$ converges to $f(x)$, let W be any open set containing $f(x)$. Then $x \in f^{-1}(W)$ and $f^{-1}(W)$ is F -open. Since the filter base \mathcal{F} converge to x , there exists $F \in \mathcal{F}$ such that $F \subset f^{-1}(W)$. Then $f(F) \subset f(f^{-1}(W)) \subset W$ and so $f(\mathcal{F}) \rightarrow f(x)$.

Conversely, let W be an open set containing $f(x)$. Now, the filter \mathcal{F} generated by the filterbase \mathcal{B}_x consisting of F -open sets containing x , F -converges to x . Since by hypothesis $f(\mathcal{F}) \rightarrow f(x)$, there exists a member $f(F)$ of $f(\mathcal{F})$ such that $f(F) \subset W$. Choose $B \in \mathcal{B}_x$ such that $B \subset F$. Since B is an F -open set containing x and since $f(B) \subset f(F) \subset W$, f is F -supercontinuous. \square

Theorem 4.6. *If $f : X \rightarrow Y$ is F -supercontinuous and $g : Y \rightarrow Z$ is continuous, then the composition $g \circ f$ is F -supercontinuous. In particular, the composition of F -supercontinuous functions is F -supercontinuous.*

In general F -supercontinuity of $g \circ f$ need not imply even continuity of f . For example, let X be the real line with cofinite topology, Y be the real line with cocompact topology and Z be the real line with indiscrete topology. Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be the identity mappings. Then $g \circ f$ and g are F -supercontinuous. However, f is not continuous.

It is routine to verify that F -supercontinuity is invariant under restrictions and enlargement of range.

Definition 4.7. *A function $f : X \rightarrow Y$ is said to be **F-open** (**F-closed**) if $f(A)$ is open (closed) in Y for every F -open (F -closed) set A in X .*

Theorem 4.8. *Let $f : X \rightarrow Y$ be an F -open (F -closed), F -supercontinuous surjection and $g : Y \rightarrow Z$ be any function. Then the composition $g \circ f$ is F -supercontinuous if and only if g is continuous. Further, if in addition f maps F -open (F -closed) set to F -open (F -closed) set, then g is an F -supercontinuous function.*

Proof. Suppose that $g \circ f$ is F -supercontinuous. To show that g is continuous, let W be an open (closed) subset of Z . Then by Theorem 4.1 $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$ is F -open (F -closed) in X . Since f is an F -open (F -closed) surjection $f(f^{-1}(g^{-1}(W))) = g^{-1}(W)$ is open (closed) in Y and so g is continuous.

Conversely, suppose that g is continuous and let W be an open (closed) set in Z . Then $g^{-1}(W)$ is open (closed) in Y . Since f is F -supercontinuous, $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$ is F -open (F -closed) in X and so $g \circ f$ is F -supercontinuous. \square

For the last assertion we need only note that F -openness (F -closedness) of $g^{-1}(W)$ ensures the F -supercontinuity of g .

Theorem 4.9. *Let $f : X \rightarrow Y$ be any function. Then the following statements are true.*

- (a) *If $\{U_\alpha : \alpha \in \Lambda\}$ is an F -open cover of X and for each α , $f_\alpha = f|U_\alpha$ is F -supercontinuous, then f is F -supercontinuous.*

(b) If $\{F_i : i = 1, \dots, n\}$ is an F -closed cover of X and $f_i = f|_{F_i}$ is F -supercontinuous, then f is F -supercontinuous.

Proof. (a) Let V be any F -open set in Y . Then $f^{-1}(V) = \cup\{f_\alpha^{-1}(V) : \alpha \in \Lambda\}$. Since each f_α is F -supercontinuous, In view of Theorem 4.1 each $f_\alpha^{-1}(V)$ is F -open in U_α and hence in X . Since any union of F -open sets is F -open, $f^{-1}(V)$ is F -open.

(b) Let B be any F -closed set in Y . Then $f^{-1}(B) = \bigcup_{i=1}^n f_i^{-1}(B)$. Since each f_i is F -supercontinuous, by Theorem 4.1 each $f_i^{-1}(B)$ is F -closed in F_i and hence in X . Then $f^{-1}(B)$ being a finite union of F -closed sets is F -closed. So f is F -supercontinuous. \square

Theorem 4.10. Let $\{f_\alpha : X \rightarrow X_\alpha : \alpha \in \Lambda\}$ be a family of functions and let $f : X \rightarrow \prod_{\alpha \in \Lambda} X_\alpha$ be defined by $f(x) = (f_\alpha(x))$ for each $x \in X$. Then f is F -supercontinuous if and only if each $f_\alpha : X \rightarrow X_\alpha$ is F -supercontinuous.

Proof. Let $f : X \rightarrow \prod_{\alpha \in \Lambda} X_\alpha$ be F -supercontinuous. Then the composition $p_\alpha \circ f = f_\alpha$, where p_α denotes the projection of $\prod_{\alpha \in \Lambda} X_\alpha$ onto α^{th} -coordinate space X_α . So in view of Theorem 4.6 each f_α is F -supercontinuous.

Conversely, suppose that each $f_\alpha : X \rightarrow X_\alpha$ is F -supercontinuous. To show that the function f is F -supercontinuous, it is sufficient to show that $f^{-1}(V)$ is F -open for each open set V in the product space $\prod_{\alpha \in \Lambda} X_\alpha$. Since arbitrary unions and finite intersections of F -open sets is F -open, it suffices to prove that $f^{-1}(S)$ is F -open for every subbasic open set S in the product space $\prod_{\alpha \in \Lambda} X_\alpha$. Let $V_\beta \times \prod_{\alpha \neq \beta} X_\alpha$ be a subbasic open set in $\prod_{\alpha \in \Lambda} X_\alpha$. Then $f^{-1}(V_\beta \times \prod_{\alpha \neq \beta} X_\alpha) = f^{-1}(p_\beta^{-1}(V_\beta)) = f_\beta^{-1}(V_\beta)$ is F -open in X . Hence f is F -supercontinuous. \square

Theorem 4.11. For each $\alpha \in \Delta$, let $f_\alpha : X_\alpha \rightarrow Y_\alpha$ be a mapping and let $f : \prod X_\alpha \rightarrow \prod Y_\alpha$ be a mapping defined by $f((x_\alpha)) = (f_\alpha(x_\alpha))$ for each (x_α) in $\prod X_\alpha$. Then f is F -supercontinuous if and only if f_α is F -supercontinuous for each $\alpha \in \Delta$.

Proof. Let $f : \prod X_\alpha \rightarrow \prod Y_\alpha$ be F -supercontinuous. Let V_β be an open subset of Y_β . Then $V_\beta \times (\prod_{\alpha \neq \beta} Y_\alpha)$ is a subbasic open subset of the product space $\prod Y_\alpha$. Since f is F -supercontinuous, $f^{-1}(V_\beta \times \prod_{\alpha \neq \beta} Y_\alpha) = f_\beta^{-1}(V_\beta) \times (\prod_{\alpha \neq \beta} X_\alpha)$ is F -open in $\prod X_\alpha$. Consequently, $f_\beta^{-1}(V_\beta)$ is a F -open set in X_β and hence f_β is a F -supercontinuous.

Conversely, suppose that each $f_\alpha : X_\alpha \rightarrow Y_\alpha$ is F -supercontinuous. Let $V = V_\beta \times (\prod_{\alpha \neq \beta} Y_\alpha)$ be a subbasic open set in $\prod Y_\alpha$. Since each f_α is F -supercontinuous, and since $f^{-1}(V) = f^{-1}(V_\beta \times (\prod_{\alpha \neq \beta} Y_\alpha)) = f_\beta^{-1}(V_\beta) \times (\prod_{\alpha \neq \beta} X_\alpha)$, $f^{-1}(V)$ is F -open, and so f is F -supercontinuous. \square

Theorem 4.12. *Let $f : X \rightarrow Y$ be a function and $g : X \rightarrow X \times Y$, defined by $g(x) = (x, f(x))$ for each $x \in X$, be the graph function. Then g is F -supercontinuous if and only if f is F -supercontinuous and X is functionally regular.*

Proof. To prove necessity, suppose that g is F -supercontinuous. Then the composition $f = p_y \circ g$ is F -supercontinuous, where p_y is the projection from $X \times Y$ onto Y . Let U be any open set in X and let $x \in U$. Then $U \times Y$ is an open set containing $g(x)$. Since g is F -supercontinuous, there exists an F -open set W_x containing x such that $g(W_x) \subset U \times Y$. Thus $x \in W_x \subset U$ and since U is a union of F -open sets, then it is F -open and so X is functionally regular.

To prove sufficiency, let $x \in X$ and let W be an open set containing $g(x)$. There exist open sets $U \subset X$ and $V \subset Y$ such that $(x, f(x)) \in U \times V \subset W$. Since X is functionally regular, there exists an F -open set G_1 in X containing x such that $x \in G_1 \subset U$. Since f is F -supercontinuous, there exists an F -open set G_2 in X containing x such that $f(G_2) \subset V$. Let $G = G_1 \cap G_2$. Then G is an F -open set containing x and $g(G) \subset U \times V \subset W$, which implies that g is F -supercontinuous. \square

The following example shows that the hypothesis that ‘ X is functionally regular’ in Theorem 4.12 cannot be omitted.

Example 4.13. *Let $X = Y = \{a, b, c, d\}$. Let the topology on X be given by $\tau = \{\phi, X, \{a, b\}, \{d\}, \{a, b, d\}\}$ and let Y be equipped with indiscrete topology. Let $f : X \rightarrow Y$ be the constant function which takes the value b . Then f is F -supercontinuous but the graph function $g : X \rightarrow X \times Y$ is not F -supercontinuous.*

Theorem 4.14. *Let $f, g : X \rightarrow Y$ be F -supercontinuous functions from X into a Hausdorff space Y . Then the equalizer $E = \{x \in X : f(x) = g(x)\}$ of the functions f and g is an F -closed set in X .*

Proof. To show that E is F -closed, we shall show that its complement $X \setminus E$ is an F -open subset of X . Let $x \in X \setminus E$. Then $f(x) \neq g(x)$. Since Y is Hausdorff, there exist disjoint open sets V and W containing $f(x)$ and $g(x)$, respectively. Since f and g are F -supercontinuous, $f^{-1}(V)$ and $g^{-1}(W)$ are F -open sets containing x . Then $U = f^{-1}(V) \cap g^{-1}(W)$ is an F -open set containing x which is contained in $X \setminus E$ and so $X \setminus E$ is F -open. \square

Corollary 4.15. *Let X be a Hausdorff space. Then the set of fixed points of every F -supercontinuous function $f : X \rightarrow X$ is an F -closed set.*

Definition 4.16. *A space X is said to be **F -completely regular** if for every F -closed set A and a point x outside A there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f(x) = 0$ and $f(A) = 1$.*

Theorem 4.17. *Let $f : X \rightarrow Y$ be an F -supercontinuous function. If X is F -completely regular, then f is z -supercontinuous.*

Proof. Let $x \in X$ and let V be an open set containing $f(x)$. Since f is F -supercontinuous, there exists an F -open set U containing x such that $f(U) \subset V$. Since X is a F -completely regular space, there exists a continuous function $h : X \rightarrow [0, 1]$ such that $h(x) = 0$ and $h(X \setminus U) = 1$. Then $h^{-1}[0, 1)$ is a cozero set containing x and contained in U and so it is mapped into V by f . This shows that f is z -supercontinuous. \square

5. PROPERTIES OF THE GRAPH OF AN F -SUPERCONTINUOUS FUNCTION

Let $f : X \rightarrow Y$ be an F -supercontinuous function. Since every F -supercontinuous function is continuous, the family $\{1_x, f\}$, where 1_x denotes the identity mapping on X , separates points and separates points from closed sets. Therefore, the mapping $g : X \rightarrow X \times Y$ defined by $g(x) = (x, f(x))$ is an embedding of X into $X \times Y$. Thus X is homeomorphic to its graph $G(f) = g(X)$ and so every topological property enjoyed by X is also enjoyed by its graph $G(f)$.

The next two notions reflect upon the fact that how the graph $G(f)$ of an F -supercontinuous function $f : X \rightarrow Y$ is situated in the product space $X \times Y$.

Definition 5.1. Let $f : X \rightarrow Y$ be a function from a topological space X into a topological space Y . The graph $G(f)$ of f is said to be

- (i) **F -closed with respect to X** if for each $(x, y) \notin G(f)$, there exist open sets U and V containing x and y , respectively such that U is F -open and $(U \times V) \cap G(f) = \phi$.
- (ii) **F -closed with respect to $X \times Y$** if for each $(x, y) \notin G(f)$, there exist F -open sets U and V containing x and y , respectively such that $(U \times V) \cap G(f) = \phi$.

Proposition 5.2. For a topological space X the following are equivalent.

- (a) X is functionally Hausdorff.
- (b) Every pair of distinct points in X are contained in disjoint cozero sets.
- (c) Every pair of distinct points in X are contained in disjoint F -open sets.

Proof. The implication (a) \Rightarrow (b) \Rightarrow (c) are trivial. To prove (c) \Rightarrow (a), let $x, y \in X$, $x \neq y$ and let U and V be disjoint F -open sets containing x and y , respectively. Let A and B be the zero sets in X such that $x \in A \subset U$ and $y \in B \subset V$. Let f, g be the real-valued functions defined on X such that $Z(f) = A$ and $Z(g) = B$, where $Z(f)$ and $Z(g)$ denote the zero sets of f and g , respectively. Let $h : X \rightarrow R$ be the function defined by $h(t) = f(t)/[f(t) + g(t)]$, for $t \in X$. Then h is a continuous function defined on X such that $h(x) = 0$ and $h(y) = 1$. \square

Theorem 5.3. If $f : X \rightarrow Y$ is F -supercontinuous and Y is functionally Hausdorff, then $G(f)$, the graph of f is F -closed with respect to $X \times Y$.

Proof. Let $x \in X$ and let $y \neq f(x)$. Since Y is functionally Hausdorff, there exist disjoint F -open sets V and W containing y and $f(x)$, respectively. By F -supercontinuity of f there exists an F -open set U containing x such that

$f(U) \subset W \subset Y \setminus \bar{V}$. Consequently, $U \times V$ contains no point of $G(f)$. Hence $G(f)$ is F -closed with respect to $X \times Y$. \square

The following result is immediate.

Proposition 5.4. *If $f : X \rightarrow Y$ is F -supercontinuous and Y is Hausdorff, then $G(f)$, the graph of f is F -closed with respect to X .*

6. TOPOLOGICAL PROPERTIES AND F -SUPERCONTINUITY

Theorem 6.1. *Let $f : X \rightarrow Y$ be an F -supercontinuous open bijection. Then X and Y are homeomorphic functionally regular spaces.*

Proof. Let U be an open set in X and let $x \in U$. Since f is an open map, $f(U)$ is an open set containing $f(x)$. Since f is F -supercontinuous, there exists an F -open set G containing x such that $f(G) \subset f(U)$. Now, $x \in f^{-1}(f(G)) \subset f^{-1}(f(U))$. Since f is a bijection, $f^{-1}(f(G)) = G$ and $f^{-1}(f(U)) = U$. Thus $x \in G \subset U$. So U being a union of F -open sets is F -open. Thus X is a functionally regular space. Since f is a homeomorphism and functional regularity is a topological property, Y is functionally regular. \square

Theorem 6.2. *Let $f : X \rightarrow Y$ be an F -supercontinuous injection into a T_0 -space, then X is a functionally Hausdorff space.*

Proof. Let x and y be two distinct points in X . Then $f(x) \neq f(y)$. Since Y is a T_0 -space, there exists an open set V containing one of the points $f(x)$ or $f(y)$ but not the other. To be precise, assume that $f(x) \in V$. Since f is F -supercontinuous, $f^{-1}(V)$ is an F -open set containing x but not y . So there exists a zero set Z_x such that $x \in Z_x \subset f^{-1}(V)$. Let $\varphi : X \rightarrow [0,1]$ be the continuous function such that $Z(\varphi) = Z_x$ the zero set of φ . Then $\varphi(x) = 0 \neq \varphi(y)$. Let G and H be disjoint open sets in $[0,1]$ containing $\varphi(x)$ and $\varphi(y)$ respectively. It follows that $\varphi^{-1}(G)$ and $\varphi^{-1}(H)$ are disjoint cozero sets in X containing x and y , respectively. So X is a functionally Hausdorff space. \square

Corollary 6.3. *Let $f : X \rightarrow Y$ be a z -supercontinuous injection into a T_0 -space, then X is a functionally Hausdorff space.*

Definition 6.4 ([13]). *A space X is said to be **F -compact** if every F -open cover of X has a finite subcover.*

Theorem 6.5. *If $f : X \rightarrow Y$ is an F -supercontinuous surjection from an F -compact space X onto Y , then Y is compact.*

Proof. Let $\beta = \{V_\alpha | \alpha \in \Delta\}$ be an open cover of Y . In view of F -supercontinuity of f , $\{f^{-1}(V_\alpha) : \alpha \in \Delta\}$ is an F -open cover of X . Since X is F -compact, there exists a finite subset $\{\alpha_1, \dots, \alpha_n\}$ of Δ such that $\bigcup_{i=1}^n f^{-1}(V_{\alpha_i}) = X$. Since f is surjection, $\{V_{\alpha_1}, \dots, V_{\alpha_n}\}$ is a finite subcover of Y . \square

Definition 6.6 ([13]). A space X is said to be **weakly F -normal** if every pair of disjoint F -closed sets are contained in disjoint open sets.

Theorem 6.7 ([13]). Let $f : X \rightarrow Y$ be an F -supercontinuous closed surjection. If X is a weakly F -normal space, then Y is a normal space.

7. F -QUOTIENT TOPOLOGY AND F -QUOTIENT SPACES

Let $f : X \rightarrow Y$ be a surjection from a topological space X onto a set Y . The quotient topology on Y is the finest topology on Y , which makes f continuous. Several variants of quotient topology have been defined in the literature (see [8, 10, 11, 15, 22, and 23]) which in general are weaker than quotient topology and coincide with the quotient topology if the domain is suitably augmented. For interrelations among these variants of quotient topology we refer the interested reader to [15]. In this section we introduce the notion of F -quotient topology which in general lies strictly between the quotient topology and the z -quotient topology [8].

We may recall that a set U in a space X is said to be z -open if it is expressible as a union of cozero sets in X .

Definition 7.1. Let $p : X \rightarrow Y$ be a surjection from a topological space X onto a set Y .

- (i) The collection τ of all subsets $A \subset Y$ such that $p^{-1}(A)$ is z -open in X is a topology on Y and is called **z -quotient topology** [8] and the map p is called the **z -quotient map**.
- (ii) The collection τ of all subsets $A \subset Y$ such that $p^{-1}(A)$ is F -open in X is a topology on Y and is called **F -quotient topology** and the map p is called the **F -quotient map**.

Clearly, z -quotient topology $\subset F$ -quotient topology \subset quotient topology.

However, none of the above inclusions are reversible as is well exhibited by the following examples.

Example 7.2. Let (X, τ) be the space of Example 3.8. Then X is a functionally regular space which is not a completely regular space. Let $Y = X$ and let p denote the identity map defined on X . Then p is an F -supercontinuous function which is not z -supercontinuous. The F -quotient topology on Y is identical with τ while z -quotient topology is strictly coarser than τ .

Example 7.3. Let X be the space of all positive integers endowed with the prime integer topology σ [24, Example 61, p. 82]. Then X is a Hausdorff space which is not a functionally Hausdorff space and hence not a functionally regular space. Let $Y = X$ and let p denote the identity map defined on X . Then the quotient topology on Y is same as prime integer topology σ but F -quotient topology on Y is strictly coarser than σ .

Theorem 7.4. Let $p : X \rightarrow Y$ be a surjection from a topological space X onto a topological space (Y, τ) , where τ is the F -quotient topology on Y . Then p

is F -supercontinuous. Moreover, τ is the finest topology on Y which makes $p : X \rightarrow Y$, F -supercontinuous.

Proof. F -supercontinuity of p is an immediate consequence of the definition of F -quotient topology. Now let τ_1 be a topology on Y such that $p : X \rightarrow (Y, \tau_1)$ is F -supercontinuous. Let G be a τ_1 open set in Y . By F -supercontinuity of p , $p^{-1}(G)$ is F -open in X . Now by the definition of F -quotient topology, G is τ -open and hence $\tau_1 \subset \tau$. \square

In contrast with the quotient space, the following result shows that a function out of an F -quotient space is continuous if and only if its composition with the F -quotient map is F -supercontinuous.

Theorem 7.5. *Let $p : X \rightarrow Y$ be an F -quotient map. Then a function $g : Y \rightarrow Z$ is continuous if and only if $g \circ p$ is F -supercontinuous.*

Proof. Let U be an open set in Z and $g \circ p$ is F -supercontinuous, then $(g \circ p)^{-1}(U) = p^{-1}(g^{-1}(U))$, is F -open in X . Since p is an F -quotient map, $g^{-1}(U)$ is open in Y . Hence, g is continuous.

The converse is immediate. \square

8. CHANGE OF TOPOLOGY

If the topology of domain of an F -supercontinuous function is changed in an appropriate way, then f is simply a continuous function. For, let (X, τ) be a topological space, and let β denote the collection of all F -open subsets of (X, τ) . Since the intersection of two F -open sets is F -open, the collection β is a base for a topology τ_F on X . Indeed $\beta = \tau_F$ and $\tau_F \subset \tau$. The space (X, τ) is functionally regular if and only if $\tau_F = \tau$.

Further, if B_z denotes the collection of all cozero subsets of (X, τ) , then B_z is a base for a topology τ_z on X . It is immediate that $\tau_z \subset \tau_F \subset \tau$ and the space X is completely regular if and only if $\tau = \tau_z$. Thus for a completely regular space $\tau_z = \tau_F = \tau$.

Throughout the section, the symbol τ_F will have the same meaning as in the above paragraph.

Remark 8.1. *Any topological property which is invariant under continuous bijection will be transferred from (X, τ) to (X, τ_F) . The list of such properties is fairly long. In particular, if (X, τ) is compact Lindelöf or countably compact, pseudocompact or quasicompact [5], D -compact or D^* -compact or D_δ -compact [12], separable, connected or pathwise connected, then so is (X, τ_F) .*

Theorem 8.2. *A function $f : (X, \tau) \rightarrow (Y, \mathfrak{S})$ is F -supercontinuous if and only if $f : (X, \tau_F) \rightarrow (Y, \mathfrak{S})$ is continuous.*

Many of the results studied in preceding sections follow now from above theorem and the corresponding standard properties of continuous functions.

Theorem 8.3. *Let (X, τ) be a topological space. Then the following statements are equivalent.*

- (a) (X, τ) is functionally regular.
 (b) Every continuous function from (X, τ) into a space (Y, \mathfrak{S}) is F -supercontinuous.

Proof. (a) \Rightarrow (b) is obvious.

(b) \Rightarrow (a): Take $(Y, \mathfrak{S}) = (X, \tau)$. Then the identity function 1_x on X is continuous, and hence F -supercontinuous. Hence by Theorem 8.2, $1_x : (X, \tau_F) \rightarrow (X, \tau)$ is continuous. Since $U \in \tau$ implies $1_x^{-1}(U) = U \in \tau_F$, therefore $\tau \subset \tau_F$. Thus it follows that $\tau = \tau_F$, and so (X, τ) is a functionally regular. \square

Definition 8.4 ([9]). A function $f : X \rightarrow Y$ from a topological space X into a topological space Y is said to be **F -continuous** if for each $x \in X$ and each F -open set U containing $f(x)$ there exists an open set V containing x such that $f(V) \subset U$.

Theorem 8.5. Let $f : (X, \tau) \rightarrow (Y, \mathfrak{S})$ be a function. Then

- (a) f is F -continuous if and only if $f : (X, \tau) \rightarrow (Y, \mathfrak{S}_F)$ is continuous.
 (b) f is F -open if and only if $f : (X, \tau_F) \rightarrow (Y, \mathfrak{S})$ is open.

In view of Theorems 8.2 and 8.3, Theorem 4.8 can be restated as follows. If $f : (X, \tau_F) \rightarrow (Y, \mathfrak{S})$ is a continuous open surjection and $g : (Y, \mathfrak{S}) \rightarrow (Z, \nu)$ is a function, then g is continuous if and only if $g \circ f$ is continuous.

Moreover, F -quotient topology on Y determined by the surjection $f : (X, \tau) \rightarrow Y$ in Section 7 coincides with usual quotient topology on Y determined by $f : (X, \tau_F) \rightarrow Y$.

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