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Topologies on function spaces and hyperspaces

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Abstract. Let Y and Z be two fixed topological spaces, $\mathcal{O}(Z)$ the family of all open subsets of Z, $C(Y,Z)$ the set of all continuous maps from Y to Z, and $\mathcal{O}_Z(Y)$ the set $\{f^{-1}(U) : f \in C(Y,Z) \text{ and } U \in \mathcal{O}(Z)\}$. In this paper, we give and study new topologies on the sets $C(Y,Z)$ and $\mathcal{O}_Z(Y)$ calling $(\mathcal{A}, \mathcal{A}_0)$-splitting and $(\mathcal{A}, \mathcal{A}_0)$-admissible, where $\mathcal{A}$ and $\mathcal{A}_0$ families of spaces.

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1. Preliminaries

Let Y and Z be two fixed topological spaces. By $C(Y,Z)$ we denote the set of all continuous maps from Y to Z. If t is a topology on the set $C(Y,Z)$, then the corresponding topological space is denoted by $C_t(Y,Z)$.

Let X be a space. To each map $g : X \times Y \to Z$ which is continuous in $y \in Y$ for each fixed $x \in X$, we associate the map $g^* : X \to C(Y,Z)$ defined as follows: for every $x \in X$, $g^*(x)$ is the map from Y to Z such that $g^*(x)(y) = g(x,y)$, $y \in Y$. Obviously, for a given map $h : X \to C(Y,Z)$, the map $h^\diamond : X \times Y \to Z$ defined by $h^\diamond(x,y) = h(x)(y)$, $(x,y) \in X \times Y$, satisfies $(h^\diamond)^* = h$ and is continuous in y for each fixed $x \in X$. Thus, the above association (defined in [7]) between the mappings from $X \times Y$ to Z that are continuous in y for each fixed $x \in X$, and the mappings from X to $C(Y,Z)$ is one-to-one.

In 1946 R. Arens [1] introduced the notion of an admissible topology: a topology t on $C(Y,Z)$ is called *admissible* if the map $e : C_t(Y,Z) \times Y \to Z$, called *evaluation map*, defined by $e(f,y) = f(y)$, is continuous.

In 1951 R. Arens and J. Dugundji [2] introduced the notion of a splitting topology: a topology t on $C(Y,Z)$ is called *splitting* if for every space X, the continuity of a map $g : X \times Y \to Z$ implies the continuity of the map

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$g^* : X \to C_t(Y,Z)$. On the set $C(Y,Z)$ there exists the greatest splitting topology, denoted here by t_{gs} (see [2]). They also proved that a topology t on $C(Y,Z)$ is admissible if and only if for every space X, the continuity of a map $h : X \to C_t(Y,Z)$ implies that of the map $h^\diamond : X \times Y \to Z$

If in the above definitions it is assumed that the space X belongs to a fixed class $\mathcal{A}$ of topological spaces, then the topology t is called $\mathcal{A}$-*splitting* or $\mathcal{A}$-*admissible*, respectively (see [8]). In the case where $\mathcal{A} = \{X\}$ we write X-splitting (respectively, X-admissible) instead of $\{X\}$-splitting (respectively, $\{X\}$-admissible).

Let X be a space. In what follows by $\mathcal{O}(X)$ we denote the family of all open subsets of X. Also, for two fixed topological spaces Y and Z we denote by $\mathcal{O}_Z(Y)$ the set $\{f^{-1}(U) : f \in C(Y,Z) \text{ and } U \in \mathcal{O}(Z)\}$.

The *Scott topology* $\Omega(Y)$ on $\mathcal{O}(Y)$ (see, for example, [11]) is defined as follows: a subset $I\!H$ of $\mathcal{O}(Y)$ belongs to $\Omega(Y)$ if:

(α) the conditions $U \in I\!H$, $V \in \mathcal{O}(Y)$, and $U \subseteq V$ imply $V \in I\!H$, and
(β) for every collection of open sets of Y, whose union belongs to $I\!H$, there are finitely many elements of this collection whose union also belongs to $I\!H$.

The *strong Scott topology* $\Omega^s(Y)$ on $\mathcal{O}(Y)$ (see [12]) is defined as follows: a subset $I\!H$ of $\mathcal{O}(Y)$ belongs to $\Omega^s(Y)$ if:

(α) the conditions $U \in I\!H$, $V \in \mathcal{O}(Y)$, and $U \subseteq V$ imply $V \in I\!H$, and
(β) for every open cover of Y there are finitely many elements of this cover whose union also belongs to $I\!H$.

The *Isbell topology* t_{Is} (respectively, *strong Isbell topology* t_{sIs}) on $C(Y,Z)$ (see, for example, [13] and [12]) is the topology, which has as a subbasis the family of all sets of the form:

$$(I\!H, U) = \{f \in C(Y,Z) : f^{-1}(U) \in I\!H\},$$

where $I\!H \in \Omega(Y)$ (respectively, $I\!H \in \Omega^s(Y)$) and $U \in \mathcal{O}(Z)$.

The *compact open topology* (see [7]) on $C(Y,Z)$, denoted here by t_{co}, is the topology for which the family of all sets of the form

$$(K, U) = \{f \in C(Y,Z) : f(K) \subseteq U\},$$

where K is a compact subset of Y and U is an open subset of Z, form a subbase. It is known that $t_{co} \subseteq t_{Is}$ (see, for example, [13]).

A subset K of a space X is said to be *bounded* if every open cover of X has a finite subcover for K (see [12]).

A space X is called *corecompact* (see [11]) if for every $x \in X$ and for every open neighborhood U of x, there exists an open neighborhood V of x such that the subset V is bounded in the space U (see [11]).

Below, we give some well known results:

(1) The Isbell topology and, hence, the compact open topology, and the point open topology (denoted here by t_{po}) on $C(Y,Z)$ are always splitting (see, for example, [2], [3], and [13]).
(2) The compact open topology on $C(Y,Z)$ is admissible if Y is a regular locally compact space. In this case the compact open topology is also the greatest splitting topology (see [2]).
(3) The Isbell topology on $C(Y,Z)$ is admissible if Y is a corecompact space. In this case the Isbell topology is also the greatest splitting topology (see, for example, [12] and [14]).
(4) A topology larger than a admissible topology is also admissible (see [2]).
(5) A topology smaller than a splitting topology is also splitting (see [2]).
(6) The strong Isbell topology on $C(Y,Z)$ is admissible if Y is a locally bounded space (see [12]).

For a summary of all the above results and some open problems on function spaces see [10]. Also, [4] and [5] are other papers related to this area.

In what follows if $\varphi : X \to Y$ is a map and $X_0 \subseteq X$, then by $\varphi|_{X_0} : X_0 \to Y$ we denote the restriction of the map φ on the set X_0. Also, if $h : X \times Y \to Z$ is a map and $X_0 \subseteq X$, then by $h|_{X_0 \times Y}$ we denote the restriction of the map h on the set $X_0 \times Y$.

In Sections 2 and 3 we give and study new topologies on the sets $C(Y,Z)$ and $\mathcal{O}_Z(Y)$ calling $(\mathcal{A},\mathcal{A}_0)$-splitting and $(\mathcal{A},\mathcal{A}_0)$-admissible, where $\mathcal{A}$ and $\mathcal{A}_0$ families of spaces.

2. $(\mathcal{A},\mathcal{A}_0)$-splitting and $(\mathcal{A},\mathcal{A}_0)$-admissible topologies on the set $C(Y,Z)$

Note 1. *Let $\mathcal{A}$ be a family of topological spaces. For every $X \in \mathcal{A}$ we denote by X_0 a subspace of X and by $\mathcal{A}_0$ the family of all such subspaces X_0. In all paper by $(\mathcal{A},\mathcal{A}_0)$ we denote the family of all pairs (X,X_0) such that $X \in \mathcal{A}$, $X_0 \in \mathcal{A}_0$, and X_0 is a subspace of X.*

Definition 2.1. *A topology t on $C(Y,Z)$ is called $(\mathcal{A},\mathcal{A}_0)$-splitting if for every pair $(X,X_0) \in (\mathcal{A},\mathcal{A}_0)$, the continuity of a map $g : X \times Y \to Z$ implies the continuity of the map $g^*|_{X_0} : X_0 \to C_t(Y,Z)$, where $g^* : X \to C_t(Y,Z)$ the map which is defined in preliminaries.*

A topology t on $C(Y,Z)$ is called $(\mathcal{A},\mathcal{A}_0)$-admissible if for every pair $(X,X_0) \in (\mathcal{A},\mathcal{A}_0)$, the continuity of a map $h : X \to C_t(Y,Z)$ implies that of the map $h^\diamond|_{X_0 \times Y} : X_0 \times Y \to Z$, where $h^\diamond : X \times Y \to Z$ the map which is defined in preliminaries.

In the case where $\mathcal{A} = \{X\}$ and $\mathcal{A}_0 = \{X_0\}$, where X_0 is a subspace of X, we write (X,X_0)-splitting (respectively, (X,X_0)-admissible) instead of $(\{X\},\{X_0\})$-splitting (respectively, $(\{X\},\{X_0\})$-admissible).

Clearly, the following theorem is true.

Theorem 2.2. *The following statements are true:*

(1) *Every splitting (respectively, admissible) topology on $C(Y,Z)$ is $(\mathcal{A},\mathcal{A}_0)$-splitting (respectively, $(\mathcal{A},\mathcal{A}_0)$-admissible), where $\mathcal{A}$ and $\mathcal{A}_0$ are arbitrary families of spaces such that every element $X_0 \in \mathcal{A}_0$ is a subspace of an element $X \in \mathcal{A}$.*

(2) *Every $\mathcal{A}$-splitting (respectively, $\mathcal{A}$-admissible) topology on $C(Y,Z)$ is $(\mathcal{A},\mathcal{A}_0)$-splitting (respectively, $(\mathcal{A},\mathcal{A}_0)$-admissible), where $\mathcal{A}$ and $\mathcal{A}_0$ are arbitrary families of spaces such that every element $X_0 \in \mathcal{A}_0$ is a subspace of an element $X \in \mathcal{A}$.*

Example 2.3.

(1) The point-open, the compact open, and the Isbell topologies are $(\mathcal{A},\mathcal{A}_0)$-splitting, where $\mathcal{A}$ and $\mathcal{A}_0$ are arbitrary families of spaces such that every element $X_0 \in \mathcal{A}_0$ is a subspace of an element $X \in \mathcal{A}$.

(2) If Y is a regular locally compact space, then the compact-open topology is $(\mathcal{A},\mathcal{A}_0)$-admissible, where $\mathcal{A}$ and $\mathcal{A}_0$ are arbitrary families of spaces such that every element $X_0 \in \mathcal{A}_0$ is a subspace of an element $X \in \mathcal{A}$.

(3) If Y is a corecompact space, then the Isbell topology is $(\mathcal{A},\mathcal{A}_0)$-admissible, where $\mathcal{A}$ and $\mathcal{A}_0$ are arbitrary families of spaces such that every element $X_0 \in \mathcal{A}_0$ is a subspace of an element $X \in \mathcal{A}$.

(4) If Y is a locally bounded space, then the strong Isbell topology is $(\mathcal{A},\mathcal{A}_0)$-admissible, where $\mathcal{A}$ and $\mathcal{A}_0$ are arbitrary families of spaces such that every element $X_0 \in \mathcal{A}_0$ is a subspace of an element $X \in \mathcal{A}$.

(5) Let X be a space, $x_0 \in X$, X_0 the subspace $\{x_0\}$ of X, and t an arbitrary topology on $C(Y,Z)$ which it is not X-splitting. Then, the topology t is (X,X_0)-splitting. It is clear that this topology t is not splitting.

(6) Let X be a space, $x_0 \in X$, X_0 the subspace $\{x_0\}$ of X, and t an arbitrary topology on $C(Y,Z)$ which it is not X-admissible. Then, the topology t is (X,X_0)-admissible. It is clear that this topology t is not admissible.

Theorem 2.4. *The following statements are true:*

(1) *A topology smaller than an $(\mathcal{A},\mathcal{A}_0)$-splitting topology is also $(\mathcal{A},\mathcal{A}_0)$-splitting.*

(2) *A topology larger than an $(\mathcal{A},\mathcal{A}_0)$-admissible topology is also $(\mathcal{A},\mathcal{A}_0)$-admissible.*

Proof. We prove only the statement (1). The proof of (2) is similar. Let t_1 be an $(\mathcal{A},\mathcal{A}_0)$-splitting topology on $C(Y,Z)$ and t_2 a topology on $C(Y,Z)$ such that $t_2 \subseteq t_1$. We prove that the topology t_2 is a $(\mathcal{A},\mathcal{A}_0)$-splitting topology. Indeed, let $(X,X_0) \in (\mathcal{A},\mathcal{A}_0)$ and let $g : X \times Y \to Z$ be a continuous map. Since the topology t_1 is $(\mathcal{A},\mathcal{A}_0)$-splitting, the map $g^*|_{X_0} : X_0 \to C_{t_1}(Y,Z)$ is continuous. Also, since $t_2 \subseteq t_1$, the identical map $id : C_{t_1}(Y,Z) \to C_{t_2}(Y,Z)$ is continuous. So, the map $g^*|_{X_0} : X_0 \to C_{t_2}(Y, Z)$ is continuous as a composition of continuous maps. Thus, the topology t_2 is $(\mathcal{A}, \mathcal{A}_0)$-splitting. □

Definition 2.5. *Let $(\mathcal{A}^1, \mathcal{A}^1_0)$ and $(\mathcal{A}^2, \mathcal{A}^2_0)$ two pairs of spaces, where $\mathcal{A}^1$ (respectively, $\mathcal{A}^2$) and $\mathcal{A}^1_0$ (respectively, $\mathcal{A}^2_0$) are arbitrary families of spaces such that every element $X_0 \in \mathcal{A}^1_0$ (respectively, every element $X_0 \in \mathcal{A}^2_0$) is a subspace of an element $X \in \mathcal{A}^1$ (respectively, of an element $X \in \mathcal{A}^2$). We say that the pairs $(\mathcal{A}^1, \mathcal{A}^1_0)$ and $(\mathcal{A}^2, \mathcal{A}^2_0)$ are equivalent if a topology t on $C(Y, Z)$ is $(\mathcal{A}^1, \mathcal{A}^1_0)$-splitting if and only if t is $(\mathcal{A}^2, \mathcal{A}^2_0)$-splitting, and t is $(\mathcal{A}^1, \mathcal{A}^1_0)$-admissible if and only if t is $(\mathcal{A}^2, \mathcal{A}^2_0)$-admissible. In this case we write*

$$(\mathcal{A}^1, \mathcal{A}^1_0) \sim (\mathcal{A}^2, \mathcal{A}^2_0).$$

Theorem 2.6. *For every pair $(\mathcal{A}, \mathcal{A}_0)$, where $\mathcal{A}$ and $\mathcal{A}_0$ are arbitrary families of spaces such that every element $X_0 \in \mathcal{A}_0$ is a subspace of an element $X \in \mathcal{A}$, there exists a pair $(X(\mathcal{A}), X(\mathcal{A}_0))$, where $X(\mathcal{A})$ is a space and $X(\mathcal{A}_0)$ is a subspace of $X(\mathcal{A})$ such that*

$$(\mathcal{A}, \mathcal{A}_0) \sim (X(\mathcal{A}), X(\mathcal{A}_0)).$$

Proof. Let T^c_{sp} be the set of all topologies on $C(Y, Z)$ which are not $(\mathcal{A}, \mathcal{A}_0)$-splitting and let T^c_{ad} the set of all topologies on $C(Y, Z)$ which are not $(\mathcal{A}, \mathcal{A}_0)$-admissible. For each $t \in T^c_{sp}$ there exists in $(\mathcal{A}, \mathcal{A}_0)$ a pair $(X^{sp}_t, X^{sp}_{t,0})$ such that t is not $(X^{sp}_t, X^{sp}_{t,0})$-splitting. Similarly, for each $t \in T^c_{ad}$ there exists in $(\mathcal{A}, \mathcal{A}_0)$ a pair $(X^{ad}_t, X^{ad}_{t,0})$ such that t is not $(X^{ad}_t, X^{ad}_{t,0})$-admissible. Let

$$\mathcal{A}' = \{X^{sp}_t : t \in T^c_{sp}\} \cup \{X^{ad}_t : t \in T^c_{ad}\}$$

and

$$\mathcal{A}'_0 = \{X^{sp}_{t,0} : t \in T^c_{sp}\} \cup \{X^{ad}_{t,0} : t \in T^c_{ad}\}.$$

Of course, we can suppose that the spaces from $\mathcal{A}'$ and $\mathcal{A}'_0$ are pair-wise disjoint. Let $X(\mathcal{A})$ and $X(\mathcal{A}_0)$ be the free union of all the spaces from $\mathcal{A}'$ and $\mathcal{A}'_0$, respectively. We prove that the pair $(X(\mathcal{A}), X(\mathcal{A}_0))$ is the required pair.

Let t be an $(\mathcal{A}, \mathcal{A}_0)$-splitting topology on $C(Y, Z)$. We prove that this topology is $(X(\mathcal{A}), X(\mathcal{A}_0))$-splitting. Indeed, let $g : X(\mathcal{A}) \times Y \to Z$ be a continuous map. It suffices to prove that the map

$$g^*|_{X(\mathcal{A}_0)} : X(\mathcal{A}_0) \to C_t(Y, Z)$$

is continuous. Let $X \in \mathcal{A}' \subseteq \mathcal{A}$. Then, the restriction $g|_{X\times Y}$ of the map g on $X \times Y \subseteq X(\mathcal{A}) \times Y$ is also a continuous map and, therefore, since the topology t is $(\mathcal{A}, \mathcal{A}_0)$-splitting we have that the map $(g|_{X\times Y})^*|_{X_0} : X_0 \to C_t(Y, Z)$ is continuous. Since $X(\mathcal{A}_0)$ is the free union of all the spaces from $\mathcal{A}'_0$ and $(g|_{X\times Y})^*|_{X_0} = (g^*|_{X(\mathcal{A}_0)})|_{X_0}$, it follows that the map $g^*|_{X(\mathcal{A}_0)} : X(\mathcal{A}_0) \to C_t(Y, Z)$ is continuous. Thus, the topology t on $C(Y, Z)$ is $(X(\mathcal{A}), X(\mathcal{A}_0))$-splitting.

Now, let t be an $(X(\mathcal{A}), X(\mathcal{A}_0))$-splitting topology on $C(Y, Z)$. We prove that t is $(\mathcal{A}, \mathcal{A}_0)$-splitting. We suppose that t is not $(\mathcal{A}, \mathcal{A}_0)$-splitting. Then, $t \in T^c_{sp}$ and, therefore, t is not $(X^{sp}_t, X^{sp}_{t,0})$-splitting for some pair $(X^{sp}_t, X^{sp}_{t,0}) \in (\mathcal{A}, \mathcal{A}_0)$. Thus, there exists a continuous map $g : X^{sp}_t \times Y \to Z$ such that the map $g^*|_{X^{sp}_{t,0}} : X^{sp}_{t,0} \to C_t(Y,Z)$ is not continuous. Since the space $X(\mathcal{A})$ is the free union of all the spaces from the family $\mathcal{A}'$, the map g can be extended to a continuous map $g_1 : X(\mathcal{A}) \times Y \to Z$. Since the map $g^*|_{X^{sp}_{t,0}}$ is not continuous, $X^{sp}_{t,0} \in \mathcal{A}'_0$, and the space $X(\mathcal{A}_0)$ is the free union of all spaces from $\mathcal{A}'_0$ we have that the map

$$g^*|_{X(\mathcal{A}_0)} : X(\mathcal{A}_0) \to C_t(Y,Z)$$

is not continuous, which contradicts our assumption that t is a $(X(\mathcal{A}), X(\mathcal{A}_0))$-splitting topology. Thus, a topology t on $C(Y,Z)$ is $(\mathcal{A},\mathcal{A}_0)$-splitting if and only if it is $(X(\mathcal{A}), X(\mathcal{A}_0))$-splitting.

Similarly, a topology t on $C(Y,Z)$ is $(\mathcal{A},\mathcal{A}_0)$-admissible if and only if is $(X(\mathcal{A}), X(\mathcal{A}_0))$-admissible. Hence,

$$(\mathcal{A},\mathcal{A}_0) \sim (X(\mathcal{A}), X(\mathcal{A}_0)).$$

□

Theorem 2.7. *There exists the greatest $(\mathcal{A},\mathcal{A}_0)$-splitting topology, where $\mathcal{A}$ and $\mathcal{A}_0$ are arbitrary families of spaces such that every element $X_0 \in \mathcal{A}_0$ is a subspace of an element $X \in \mathcal{A}$.*

Proof. Let $\{t_i : i \in I\}$ be the family of all $(\mathcal{A},\mathcal{A}_0)$-splitting topologies on $C(Y,Z)$. We consider the topology $t = \vee\{t_i : i \in I\}$. Clearly, t is $(\mathcal{A},\mathcal{A}_0)$-splitting and $t_i \subseteq t$, for every $i \in I$. Thus, t is the greatest $(\mathcal{A},\mathcal{A}_0)$-splitting topology. □

Note 2. *In what follows we denote by $t(\mathcal{A},\mathcal{A}_0)$ the greatest $(\mathcal{A},\mathcal{A}_0)$-splitting topology on $C(Y,Z)$,*

Theorem 2.8. *The following statements are true:*

(1) *If $(\mathcal{A},\mathcal{A}_0) = \cup\{(\mathcal{A}^i,\mathcal{A}^i_0) : i \in I\}$, then*

$$t(\mathcal{A},\mathcal{A}_0) = \cap\{t(\mathcal{A}^i,\mathcal{A}^i_0) : i \in I\}.$$

(2) $t(\mathcal{A},\mathcal{A}_0) = \cap\{t(X,X_0) : (X,X_0) \in (\mathcal{A},\mathcal{A}_0)\}$.

(3) *If $(\mathcal{A},\mathcal{A}_0) = \cap\{(\mathcal{A}^i,\mathcal{A}^i_0) : i \in I\}$, then*

$$\vee\{t(\mathcal{A}^i,\mathcal{A}^i_0) : i \in I\} \subseteq t(\mathcal{A},\mathcal{A}_0).$$

Proof. (1) Since $(\mathcal{A},\mathcal{A}_0) = \cup\{(\mathcal{A}^i,\mathcal{A}^i_0) : i \in I\}$ we have that every topology which is $(\mathcal{A},\mathcal{A}_0)$-splitting is also $(\mathcal{A}^i,\mathcal{A}^i_0)$-splitting, for every $i \in I$. Thus, the topology $t(\mathcal{A},\mathcal{A}_0)$ is $(\mathcal{A}^i,\mathcal{A}^i_0)$-splitting and, therefore,

$$t(\mathcal{A},\mathcal{A}_0) \subseteq t(\mathcal{A}^i,\mathcal{A}^i_0),$$

for every $i \in I$. So, we have

$$t(\mathcal{A},\mathcal{A}_0) \subseteq \cap\{t(\mathcal{A}^i,\mathcal{A}^i_0) : i \in I\}.$$

Now, we prove the converse relation, that is

$$\cap\{t(\mathcal{A}^i,\mathcal{A}^i_0) : i \in I\} \subseteq t(\mathcal{A},\mathcal{A}_0).$$

For the above relation it suffices to prove that the topology $\cap\{t(\mathcal{A}^i,\mathcal{A}_0^i) : i \in I\}$ is $(\mathcal{A},\mathcal{A}_0)$-splitting. Let $(X,X_0) \in (\mathcal{A},\mathcal{A}_0)$ and let $g : X \times Y \to Z$ be a continuous map. We prove that the map

$$g^*|_{X_0} : X_0 \to C_{\cap\{t(\mathcal{A}^i,\mathcal{A}_0^i):i\in I\}}(Y,Z)$$

is continuous. Since $(X,X_0) \in (\mathcal{A},\mathcal{A}_0)$, there exists $i \in I$ such that $(X,X_0) \in (\mathcal{A}^i,\mathcal{A}_0^i)$. This means that the map

$$g^*|_{X_0} : X_0 \to C_{t(\mathcal{A}^i,\mathcal{A}_0^i)}(Y,Z)$$

is continuous. Also, since $\cap\{t(\mathcal{A}^i,\mathcal{A}_0^i) : i \in I\} \subseteq t(\mathcal{A}^i,\mathcal{A}_0^i)$, the identical map

$$id : C_{t(\mathcal{A}^i,\mathcal{A}_0^i)}(Y,Z) \to C_{\cap\{t(\mathcal{A}^i,\mathcal{A}_0^i):i\in I\}}(Y,Z)$$

is continuous. So, the map

$$g^*|_{X_0} : X_0 \to C_{\cap\{t(\mathcal{A}^i,\mathcal{A}_0^i):i\in I\}}(Y,Z)$$

is continuous as a composition of continuous maps. Thus, the topology

$$\cap\{t(\mathcal{A}^i,\mathcal{A}_0^i) : i \in I\}$$

is $(\mathcal{A},\mathcal{A}_0)$-splitting.

(2) The proof of this is a corollary of the statement (1).

(3) The proof of this follows by the fact that the topology

$$\vee\{t(\mathcal{A}^i,\mathcal{A}_0^i) : i \in I\}$$

is $(\mathcal{A},\mathcal{A}_0)$-splitting. □

Theorem 2.9. *Let t be an $(\mathcal{A},\mathcal{A}_0)$-admissible topology on $C(Y,Z)$. If*

$$(C_t(Y,Z), C_t(Y,Z)) \in (\mathcal{A},\mathcal{A}_0),$$

then t is admissible and $t(\mathcal{A},\mathcal{A}_0) \subseteq t$.

Proof. Let $id \equiv h : C_t(Y,Z) \to C_t(Y,Z)$ be the identical map. Clearly, this map is continuous. Since

$$(C_t(Y,Z), C_t(Y,Z)) \in (\mathcal{A},\mathcal{A}_0)$$

and t is $(\mathcal{A},\mathcal{A}_0)$-admissible, the map $h^\diamond|_{C_t(Y,Z)} \equiv h^\diamond : C_t(Y,Z) \times Y \to Z$ is continuous. Hence, the topology t is admissible.

Now, since the map $h^\diamond \equiv g : C_t(Y,Z) \times Y \to Z$ is continuous,

$$(C_t(Y,Z), C_t(Y,Z)) \in (\mathcal{A},\mathcal{A}_0),$$

and the topology $t(\mathcal{A},\mathcal{A}_0)$ is $(\mathcal{A},\mathcal{A}_0)$-splitting, the map

$$g^*|_{C_t(Y,Z)} = id : C_t(Y,Z) \to C_{t(\mathcal{A},\mathcal{A}_0)}(Y,Z)$$

is also continuous. Thus, $t(\mathcal{A},\mathcal{A}_0) \subseteq t$. □

Corollary 2.10. *Let t be an $(\mathcal{A},\mathcal{A}_0)$-splitting and $(\mathcal{A},\mathcal{A}_0)$-admissible topology on $C(Y,Z)$. If $(C_t(Y,Z), C_t(Y,Z)) \in (\mathcal{A},\mathcal{A}_0)$, then $t(\mathcal{A},\mathcal{A}_0) = t$.*

Proof. By Theorem 2.9, $t(\mathcal{A},\mathcal{A}_0) \subseteq t$. Also, since the topology t is $(\mathcal{A},\mathcal{A}_0)$-splitting, $t \subseteq t(\mathcal{A},\mathcal{A}_0)$. Thus, $t(\mathcal{A},\mathcal{A}_0) = t$. □

Theorem 2.11. *Let Y be a regular locally compact space, $\mathcal{A}$ the family of all T_i-spaces, $i = 0, 1, 2, 3, 3\frac{1}{2}$, $\mathcal{A}_0$ an arbitrary family of spaces containing subspaces of spaces of $\mathcal{A}$, $C_{t_{co}}(Y, Z) \in \mathcal{A}_0$, and $Z \in \mathcal{A}$. Then, we have $t(\mathcal{A}, \mathcal{A}_0) = t_{co} = t_{Is}$.*

Proof. Since Y is a regular locally compact space, the compact open topology coincides with the Isbell topology on $C(Y, Z)$ and it is admissible. Hence, t_{co} is $(\mathcal{A}, \mathcal{A}_0)$-admissible. Also, the topology t_{co} is splitting and, therefore, t_{co} is $(\mathcal{A}, \mathcal{A}_0)$-splitting. Since $Z \in \mathcal{A}$, we have that $C_{t_{co}}(Y, Z) \in \mathcal{A}$ (see preliminaries) and, therefore, $(C_{t_{co}}(Y, Z), C_{t_{co}}(Y, Z)) \in (\mathcal{A}, \mathcal{A}_0)$. Thus, by Corollary 2.10 we have that $t(\mathcal{A}, \mathcal{A}_0) = t_{co}$. □

Theorem 2.12. *Let Y be a regular locally compact space, $\mathcal{A}$ the family of all topological spaces whose weight is not greater than a certain fixed infinite cardinal, $\mathcal{A}_0$ an arbitrary family of spaces containing subspaces of spaces of $\mathcal{A}$, $C_{t_{co}}(Y, Z) \in \mathcal{A}_0$, and $Y, Z \in \mathcal{A}$. Then, we have $t(\mathcal{A}, \mathcal{A}_0) = t_{co} = t_{Is}$.*

Proof. The proof of this theorem is similar to the proof of Theorem 2.11 and follows by Corollary 2.10 and Theorem 3.4.16 of [6]. □

Theorem 2.13. *Let Y be a regular second-countable locally compact space, $\mathcal{A}$ the family of all metrizable spaces, $\mathcal{A}_0$ an arbitrary family of spaces containing subspaces of spaces of $\mathcal{A}$, $C_{t_{co}}(Y, Z) \in \mathcal{A}_0$, and $Z \in \mathcal{A}$. Then, we have $t(\mathcal{A}, \mathcal{A}_0) = t_{co} = t_{Is}$.*

Proof. The proof of this theorem is similar to the proof of Theorem 2.11 and follows by Corollary 2.10 and Exercices 4.2.H and 3.4.E(c) of [6]. □

Theorem 2.14. *Let Y be a regular locally compact Lindelöf space, $\mathcal{A}$ the family of all completely metrizable spaces, $\mathcal{A}_0$ an arbitrary family of spaces containing subspaces of spaces of $\mathcal{A}$, $C_{t_{co}}(Y, Z) \in \mathcal{A}_0$, and $Z \in \mathcal{A}$. Then, we have $t(\mathcal{A}, \mathcal{A}_0) = t_{co} = t_{Is}$.*

Proof. The proof of this theorem is similar to the proof of Theorem 2.11 and follows by Corollary 2.10 and Exercice 4.3.F(a) of [6]. □

Theorem 2.15. *Let Y be a corecompact space, $\mathcal{A}$ the family of all T_i-spaces, where $i = 0, 1, 2$, $\mathcal{A}_0$ an arbitrary family of spaces containing subspaces of spaces of $\mathcal{A}$, $C_{t_{Is}}(Y, Z) \in \mathcal{A}_0$, and $Z \in \mathcal{A}$. Then, we have $t(\mathcal{A}, \mathcal{A}_0) = t_{Is}$.*

Proof. Since Y is corecompact, the Isbell topology t_{Is} on $C(Y, Z)$ is admissible. Hence the topology t_{Is} is $(\mathcal{A}, \mathcal{A}_0)$-admissible. Also, the topology t_{Is} is splitting and, therefore, t_{Is} is $(\mathcal{A}, \mathcal{A}_0)$-splitting. Since $Z \in \mathcal{A}$, we have that $C_{t_{Is}}(Y, Z) \in \mathcal{A}$ (see preliminaries) and, therefore, $(C_{t_{Is}}(Y, Z), C_{t_{Is}}(Y, Z)) \in (\mathcal{A}, \mathcal{A}_0)$. Thus, by Corollary 2.10 we have that $t(\mathcal{A}, \mathcal{A}_0) = t_{Is}$. □

Theorem 2.16. *Let Y be a corecompact space, $\mathcal{A}$ the family of all second-countable spaces, $\mathcal{A}_0$ an arbitrary family of spaces containing subspaces of spaces of $\mathcal{A}$, $C_{t_{Is}}(Y, Z) \in \mathcal{A}_0$, and $Y, Z \in \mathcal{A}$. Then, we have $t(\mathcal{A}, \mathcal{A}_0) = t_{Is}$.*

Proof. The proof of this theorem is similar to the proof of Theorem 2.15 and follows by Corollary 2.10 and the fact that $C_{t_{Is}}(Y,Z) \in \mathcal{A}$ (see [12]). □

3. On dual topologies

Note 3. *Let Y and Z be two fixed topological spaces. By $\mathcal{O}_Z(Y)$ we denote the set*

$$\{f^{-1}(U) : f \in C(Y,Z) \text{ and } U \in \mathcal{O}(Z)\}.$$

Let $\mathbb{H} \subseteq \mathcal{O}_Z(Y)$, $\mathcal{H} \subseteq C(Y,Z)$, and $U \in \mathcal{O}(Z)$. We set

$$(\mathbb{H}, U) = \{f \in C(Y,Z) : f^{-1}(U) \in \mathbb{H}\}$$

and

$$(\mathcal{H}, U) = \{f^{-1}(U) : f \in \mathcal{H}\}.$$

Definition 3.1. *(See [9]) Let τ be a topology on $\mathcal{O}_Z(Y)$. The topology on $C(Y,Z)$, for which the set*

$$\{(\mathbb{H}, U) : \mathbb{H} \in \tau,\ U \in \mathcal{O}(Z)\}$$

is a subbasis, is called dual to τ and is denoted by $t(\tau)$.

Now, let t be a topology on $C(Y,Z)$. The topology on $\mathcal{O}_Z(Y)$, for which the set

$$\{(\mathcal{H}, U) : \mathcal{H} \in t,\ U \in \mathcal{O}(Z)\}$$

is a subbasis, is called dual to t and is denoted by $\tau(t)$.

We observe that if τ is a topology on $\mathcal{O}_Z(Y)$ and σ a subbasis for τ, then the set $\{(\mathbb{H}, U) : \mathbb{H} \in \sigma,\ U \in \mathcal{O}(Z)\}$ is a subbasis for $t(\tau)$ (see Lemma 2.5 in [9]). Also, if t is a topology on $C(Y,Z)$ and s a subbasis for t, then the set $\{(\mathcal{H}, U) : \mathcal{H} \in s,\ U \in \mathcal{O}(Z)\}$ is a subbasis for $\tau(t)$ (see Lemma 2.6 in [9]).

Note 4. *Let X be a space and $g : X \times Y \to Z$ a continuous map. If $g_x : Y \to Z$ is the map for which $g_x(y) = g(x,y)$, for every $y \in Y$, then by $\overline{g}$ we denote the map of $X \times \mathcal{O}(Z)$ into $\mathcal{O}_Z(Y)$, for which $\overline{g}(x,U) = g_x^{-1}(U)$ for every $x \in X$ and $U \in \mathcal{O}(Z)$.*

Now, let $h : X \to C(Y,Z)$ be a map. By $\overline{h}$ we denote the map of $X \times \mathcal{O}(Z)$ into $\mathcal{O}_Z(Y)$, for which $\overline{h}(x,U) = (h(x))^{-1}(U)$ for every $x \in X$ and $U \in \mathcal{O}(Z)$.

Definition 3.2. *Let τ be a topology on $\mathcal{O}_Z(Y)$. We say that a map $M : X \times \mathcal{O}(Z) \to \mathcal{O}_Z(Y)$ is continuous with respect to the first variable if for every fixed element U of $\mathcal{O}(Z)$, the map $M_U : X \to (\mathcal{O}_Z(Y), \tau)$, for which $M_U(x) = M(x,U)$ for every $x \in X$, is continuous.*

Definition 3.3. *A topology τ on $\mathcal{O}_Z(Y)$ is called $(\mathcal{A}, \mathcal{A}_0)$-splitting if for every $(X, X_0) \in (\mathcal{A}, \mathcal{A}_0)$ the continuity of a map $g : X \times Y \to Z$ implies the continuity with respect to the first variable of the map $\overline{g}|_{X_0 \times \mathcal{O}(Z)} : X_0 \times \mathcal{O}(Z) \to (\mathcal{O}_Z(Y), \tau)$.*

A topology τ on $\mathcal{O}_Z(Y)$ is called $(\mathcal{A}, \mathcal{A}_0)$-admissible if for every $(X, X_0) \in (\mathcal{A}, \mathcal{A}_0)$ and for every map $h : X \to C(Y,Z)$ the continuity with respect to the first variable of the map $\overline{h} : X \times \mathcal{O}(Z) \to (\mathcal{O}_Z(Y), \tau)$ implies the continuity of *the map* $h^\diamond|_{X_0\times Y} : X_0 \times Y \to Z$ *defined by* $h^\diamond|_{X_0\times Y}(x,y) = h(x)(y)$, $(x,y) \in X_0 \times Y$.

Theorem 3.4. *A topology* τ *on* $\mathcal{O}_Z(Y)$ *is* $(\mathcal{A},\mathcal{A}_0)$*-splitting if and only if the topology* $t(\tau)$ *on* $C(Y,Z)$ *is* $(\mathcal{A},\mathcal{A}_0)$*-splitting.*

Proof. Suppose that the topology τ on $\mathcal{O}_Z(Y)$ is $(\mathcal{A},\mathcal{A}_0)$-splitting, that is for every pair $(X,X_0) \in (\mathcal{A},\mathcal{A}_0)$ the continuity of a map $g : X \times Y \to Z$ implies the continuity with respect to the first variable of the map

$$\overline{g}|_{X_0\times\mathcal{O}(Z)} : X_0 \times \mathcal{O}(Z) \to (\mathcal{O}_Z(Y),\tau).$$

We prove that the topology $t(\tau)$ on $C(Y,Z)$ is $(\mathcal{A},\mathcal{A}_0)$-splitting. Let $(X,X_0) \in (\mathcal{A},\mathcal{A}_0)$ and $g : X \times Y \to Z$ be a continuous map. We need to prove that $g^*|_{X_0} : X_0 \to C_{t(\tau)}(Y,Z)$ is a continuous map.

Let $x \in X_0$ and $(\mathbb{H},U)$ be an open neighborhood of $(g^*|_{X_0})(x)$ in $C_{t(\tau)}(Y,Z)$. We must find an open neighborhood V of x in X_0 such that $(g^*|_{X_0})(V) \subseteq (\mathbb{H},U)$. We have that $((g^*|_{X_0})(x))^{-1}(U) \in \mathbb{H}$. Since $(g^*|_{X_0})(x) = g_x$, we have $g_x^{-1}(U) \in \mathbb{H}$, that is, $\overline{g}(x,U) \in \mathbb{H}$. Since the map

$$\overline{g}|_{X_0\times\mathcal{O}(Z)} : X_0 \times \mathcal{O}(Z) \to (\mathcal{O}_Z(Y),\tau).$$

is continuous with respect to the first variable, the map $(\overline{g}|_{X_0\times\mathcal{O}(Z)})_U : X_0 \to (\mathcal{O}_Z(Y),\tau)$ is continuous. Also, $(\overline{g}|_{X_0\times\mathcal{O}(Z)})_U(x) \in \mathbb{H}$. Thus, there exists an open neighborhood V of x in X_0 such that $(\overline{g}|_{X_0\times\mathcal{O}(Z)})_U(V) \subseteq \mathbb{H}$.

Let $x' \in V$. Then, $(\overline{g}|_{X_0\times\mathcal{O}(Z)})_U(x') \in \mathbb{H}$, that is, $g_{x'}^{-1}(U) \in \mathbb{H}$ or $(g^*|_{X_0})(x') \in (\mathbb{H},U)$. Thus, $(g^*|_{X_0})(V) \subseteq (\mathbb{H},U)$, which means that the map $g^*|_{X_0}$ is continuous.

Conversely, suppose that $t(\tau)$ is $(\mathcal{A},\mathcal{A}_0)$-splitting. We prove that τ is $(\mathcal{A},\mathcal{A}_0)$-splitting. Let (X,X_0) be an element of $(\mathcal{A},\mathcal{A}_0)$ and $g : X\times Y \to Z$ a continuous map. It is sufficient to prove that $\overline{g}|_{X_0\times\mathcal{O}(Z)} : X_0 \times \mathcal{O}(Z) \to (\mathcal{O}_Z(Y),\tau)$ is continuous with respect to the first variable.

Let U be a fixed element of $\mathcal{O}(Z)$. Consider the map $(\overline{g}|_{X_0\times\mathcal{O}(Z)})_U : X_0 \to (\mathcal{O}_Z(Y),\tau)$. Let $x \in X_0$, $\mathbb{H} \in \tau$, and $(\overline{g}|_{X_0\times\mathcal{O}(Z)})_U(x) = g_x^{-1}(U) \in \mathbb{H}$. We need to find an open neighborhood V of x in X_0 such that $(\overline{g}|_{X_0\times\mathcal{O}(Z)})_U(V) \subseteq \mathbb{H}$.

Consider the open set $(\mathbb{H},U)$ of the space $C_{t(\tau)}(Y,Z)$. Since

$$(\overline{g}|_{X_0\times\mathcal{O}(Z)})_U(x) = g_x^{-1}(U) \in \mathbb{H},$$

we have $g_x \in (\mathbb{H},U)$. Since $t(\tau)$ is $(\mathcal{A},\mathcal{A}_0)$-splitting, the map $g^*|_{X_0} : X_0 \to C_{t(\tau)}(Y,Z)$ is continuous. Hence, there exists an open neighborhood V of x in X_0 such that $(g^*|_{X_0})(V) \subseteq (\mathbb{H},U)$.

Let $x' \in V$. Then, $(g^*|_{X_0})(x') = g_{x'} \in (\mathbb{H},U)$, that is, $g_{x'}^{-1}(U) \in \mathbb{H}$ or $(\overline{g}|_{X_0\times\mathcal{O}(Z)})_U(x') \in \mathbb{H}$. Thus, $(\overline{g}|_{X_0\times\mathcal{O}(Z)})_U(V) \subseteq \mathbb{H}$, which means that the map $(\overline{g}|_{X_0\times\mathcal{O}(Z)})_U$ is continuous. □

Theorem 3.5. *A topology* t *on* $C(Y,Z)$ *is* $(\mathcal{A},\mathcal{A}_0)$*-splitting if and only if the topology* $\tau(t)$ *on* $\mathcal{O}_Z(Y)$ *is* $(\mathcal{A},\mathcal{A}_0)$*-splitting.*

Proof. The proof of this theorem is similar to the proof of Theorem 3.4. □

Example 3.6.

(1) The topologies $\tau(t_{co})$ and $\tau(t_{Is})$ are $(\mathcal{A},\mathcal{A}_0)$-splitting for every pair $(\mathcal{A},\mathcal{A}_0)$. This follows by the fact that the topologies t_{co} and t_{Is} are splitting and, therefore, $(\mathcal{A},\mathcal{A}_0)$-splitting.

(2) Let Z be the Sierpinski space, $\Omega(Y)$ the Scott topology, and $\Omega_Z(Y)$ the relative topology of $\Omega(Y)$ on $\mathcal{O}_Z(Y)$. Then, the topology $t(\Omega_Z(Y))$ coincides with the Isbell topology on $C(Y,Z)$. Hence, the topology $t(\Omega_Z(Y))$ is splitting and, therefore, $(\mathcal{A},\mathcal{A}_0)$-splitting. Thus, the topology $\tau(t(\Omega_Z(Y)))$ on $\mathcal{O}_Z(Y)$ is $(\mathcal{A},\mathcal{A}_0)$-splitting.

Theorem 3.7. *A topology τ on $\mathcal{O}_Z(Y)$ is $(\mathcal{A},\mathcal{A}_0)$-admissible if and only if the topology $t(\tau)$ on $C(Y,Z)$ is $(\mathcal{A},\mathcal{A}_0)$-admissible.*

Proof. Suppose that the topology τ on $\mathcal{O}_Z(Y)$ is $(\mathcal{A},\mathcal{A}_0)$-admissible, that is for every space $(X,X_0)\in(\mathcal{A},\mathcal{A}_0)$ and for every map $h: X\to C(Y,Z)$ the continuity with respect to the first variable of the map $\overline{h}: X\times\mathcal{O}(Z)\to(\mathcal{O}_Z(Y),\tau)$ implies the continuity of the map $h^{\diamond}|_{X_0\times Y}: X_0\times Y\to Z$. We prove that $t(\tau)$ is $(\mathcal{A},\mathcal{A}_0)$-admissible. Let $(X,X_0)\in(\mathcal{A},\mathcal{A}_0)$ and $h: X\to C_{t(\tau)}(Y,Z)$ be a continuous map. It is sufficient to prove that the map $h^{\diamond}|_{X_0\times Y}: X_0\times Y\to Z$ is continuous. Clearly, it suffices to prove that the map $\overline{h}: X\times\mathcal{O}(Z)\to(\mathcal{O}_Z(Y),\tau)$ is continuous with respect to the first variable.

Let $x\in X, U\in\mathcal{O}(Z)$ and $\mathbb{H}\in\tau$ such that $\overline{h}_U(x)=\overline{h}(x,U)=(h(x))^{-1}(U)\in\mathbb{H}$. We prove that there exists an open neighborhood V of x in X such that $\overline{h}_U(V)\subseteq\mathbb{H}$. Consider the open set $(\mathbb{H},U)$ of the space $C_{t(\tau)}(Y,Z)$. Then, $h(x)\in(\mathbb{H},U)$.

Since the map $h: X\to C_{t(\tau)}(Y,Z)$ is continuous, there exists an open neighborhood V of x in X such that $h(V)\subseteq(\mathbb{H},U)$.

Let $x'\in V$. Then $h(x')\in(\mathbb{H},U)$, that is $(h(x'))^{-1}(U)\in\mathbb{H}$ or $\overline{h}_U(x')=\overline{h}(x',U)\in\mathbb{H}$. Thus, $\overline{h}_U(V)\subseteq\mathbb{H}$, which means that $\overline{h}_U$ is continuous.

Conversely, suppose that the topology $t(\tau)$ is $(\mathcal{A},\mathcal{A}_0)$-admissible. We prove that the topology τ is $(\mathcal{A},\mathcal{A}_0)$-admissible. Let (X,X_0) be a pair of $(\mathcal{A},\mathcal{A}_0)$ and $h: X\to C(Y,Z)$ a map such that $\overline{h}: X\times\mathcal{O}(Z)\to(\mathcal{O}_Z(Y),\tau)$ is continuous with respect to the first variable. We need to prove that the map $h^{\diamond}|_{X_0\times Y}: X_0\times Y\to Z$ is continuous.

Since $t(\tau)$ is $(\mathcal{A},\mathcal{A}_0)$-admissible, it is sufficient to prove that the map $h: X\to C_{t(\tau)}(Y,Z)$ is continuous.

Let $x\in X$, $U\in\mathcal{O}(Z)$, and $\mathbb{H}\in\tau$ such that $h(x)\in(\mathbb{H},U)$. Then, $(h(x))^{-1}(U)\in\mathbb{H}$. Since the map $\overline{h}_U: X\to(\mathcal{O}_Z(Y),\tau)$ is continuous, there exists an open neighborhood V of x in X such that $\overline{h}_U(V)\subseteq\mathbb{H}$.

Let $x'\in V$. Then, $\overline{h}_U(x')=(h(x'))^{-1}(U)\in\mathbb{H}$ or $h(x')\in(\mathbb{H},U)$. Thus, $h(V)\subseteq(\mathbb{H},U)$, which means that the map h is continuous. □

Theorem 3.8. *A topology t on $C(Y,Z)$ is $(\mathcal{A},\mathcal{A}_0)$-admissible if and only if the topology $\tau(t)$ on $\mathcal{O}_Z(Y)$ is $(\mathcal{A},\mathcal{A}_0)$-admissible.*

Proof. The proof of this theorem is similar to the proof of Theorem 3.7. □

Example 3.9.

(1) If Y is a regular locally compact space, then the topology $\tau(t_{co})$ is $(\mathcal{A}, \mathcal{A}_0)$-admissible for every pair $(\mathcal{A}, \mathcal{A}_0)$.
(2) If Y is a corecompact space, then the topology $\tau(t_{Is})$ is $(\mathcal{A}, \mathcal{A}_0)$-admissible for every pair $(\mathcal{A}, \mathcal{A}_0)$.
(3) If Y is a locally bounded space, then the topology $\tau(t_{sIs})$ is $(\mathcal{A}, \mathcal{A}_0)$-admissible for every pair $(\mathcal{A}, \mathcal{A}_0)$.
(4) Let $\Omega(Y)$ be the Scott topology on $\mathcal{O}(Y)$. By $\Omega_Z(Y)$ we denote the relative topology of $\Omega(Y)$ on $\Omega_Z(Y)$. If Y is corecompact, then the topology $\Omega_Z(Y)$ is admissible (see Corollary 3.12 of [9]) and, therefore, it is $(\mathcal{A}, \mathcal{A}_0)$-admissible. Thus, the topology $t(\Omega_Z(Y))$ on $C(Y, Z)$ is $(\mathcal{A}, \mathcal{A}_0)$-admissible.

Theorem 3.10. *Let $\mathcal{A}$ and $\mathcal{A}_0$ are arbitrary families of spaces such that every element $X_0 \in \mathcal{A}_0$ is a subspace of an element $X \in \mathcal{A}$. Then in the set $\mathcal{O}_Z(Y)$ there exists the greatest $(\mathcal{A}, \mathcal{A}_0)$-splitting topology.*

Proof. Let $\{\tau_i : i \in I\}$ be the set of all $(\mathcal{A}, \mathcal{A}_0)$-splitting topologies on $\mathcal{O}_Z(Y)$. We consider the topology

$$\tau = \vee\{\tau_i : i \in I\}.$$

It is not difficult to prove that this topology is $(\mathcal{A}, \mathcal{A}_0)$-splitting. By this fact we have that this topology is the required greatest $(\mathcal{A}, \mathcal{A}_0)$-splitting topology. □

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