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Compact self T_1-complementary spaces without isolated points

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Abstract. We present an example of a compact Hausdorff self T_1-complementary space without isolated points. This answers Question 3.11 from [A compact Hausdorff topology that is a T_1-complement of itself, *Fund. Math.* **175** (2002), 163–173] affirmatively.

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1. Introduction

We deal with the concept of complementarity in the lattice of T_1-topologies on a given infinite set. Two elements a, b of an abstract lattice $\{L, \vee, \wedge, \mathbf{0}, \mathbf{1}\}$ with the smallest and greatest elements $\mathbf{0}$ and $\mathbf{1}$, respectively, are called *complementary* if $a \vee b = \mathbf{1}$ and $a \wedge b = \mathbf{0}$. Birkhoff noted in [1] that the family $\mathcal{L}(X)$ of all topologies on a nonempty set X becomes a lattice when the infimum $\tau_1 \wedge \tau_2$ of $\tau_1, \tau_2 \in \mathcal{L}(X)$ is defined to be the intersection $\tau_1 \cap \tau_2$ and the supremum $\tau_1 \vee \tau_2$ is the topology on X with the subbase $\tau_1 \cup \tau_2$. Clearly, the smallest element $\mathbf{0}$ of $\mathcal{L}(X)$ is the coarsest topology $\{\varnothing, X\}$, while the greatest element $\mathbf{1}$ of $\mathcal{L}(X)$ is the discrete topology of X.

In the case of the lattice $\mathcal{L}_1(X)$ of all T_1-topologies on X, the smallest element $\mathbf{0}$ of $\mathcal{L}_1(X)$ is the *cofinite* topology

$$cfin(X) = \{\varnothing\} \cup \{X \setminus F : F \subseteq X,\ F \text{ is finite}\}.$$

Therefore, two topologies $\tau_1, \tau_2 \in \mathcal{L}_1(X)$ are *complementary* in $\mathcal{L}_1(X)$ if $\tau_1 \cap \tau_2 = cfin(X)$ and $\tau_1 \cup \tau_2$ is a subbase for the discrete topology on X. It is said that τ_1 and τ_2 are T_1-*complementary* in this case.

The study of complementarity in $\mathcal{L}_1(X)$ was initiated by A. Steiner and E. Steiner in [6, 8, 7]. Later on, S. Watson used an elaborated combinatorics in

[10] to prove that a set X of cardinality $\mathfrak{c}^+$, where $\mathfrak{c} = 2^\omega$, admits a Tychonoff *self* T_1*-complementary* topology τ. Self T_1-complementarity of τ means that there exists a bijection f of X onto itself such that the topologies τ and $\sigma = \{f^{-1}(U) : U \in \tau\}$ are T_1-complementary.

In [4], D. Shakhmatov and the author applied a recursive construction to show that the Alexandroff duplicate $A(\beta\omega \setminus \omega)$ of $\beta\omega \setminus \omega$ is a T_1-complement of itself. $A(\beta\omega \setminus \omega)$ was the first example of an infinite compact Hausdorff space with this property. It is clear that $|A(\beta\omega \setminus \omega)| = 2^\mathfrak{c} > \mathfrak{c}$, which looks quite similar to the cardinality of Watson's self T_1-complementary space in [10]. The necessity of working with topologies on big sets was explained in [4, Corollary 3.6]—the existence of a compact Hausdorff self T_1-complementary space of cardinality less than or equal to $\mathfrak{c}$ is independent of ZFC.

The concept of T_1-complementarity of topologies is naturally split into *transversality* and T_1*-independence.* Following [5, 9], we say that topologies $\tau_1, \tau_2 \in \mathcal{L}_1(X)$ are *transversal* if $\tau_1 \vee \tau_2$ is the discrete topology, and T_1*-independent* if $\tau_1 \wedge \tau_2$ is the cofinite topology on X. In addition, if the topologies τ_1 and τ_2 are homeomorphic (i.e., τ_2 is obtained from τ_1 by means of a bijection of X), we come to the notions of *self-transversality* and *self* T_1*-independence*, respectively.

A usual way to produce self-transversal topologies is to work with a space that has many isolated points. Indeed, suppose that X is a space with topology τ, $Y \subseteq X$, $|Y| = |X| = |X \setminus Y|$, and each point of Y is isolated in X. Take any bijection $f\colon X \to X$ such that $f(X \setminus Y) = Y$ and put

$$\sigma = \{f^{-1}(U) : U \in \tau\}.$$

It is easy to see that every point of X is isolated either in τ or in σ, so $\tau \vee \sigma$ is the discrete topology on X. In other words, the space (X, τ) is self-transversal. This approach was also adopted in [4, Corollary 3.8] to show that the compact space $A(\beta\omega \setminus \omega)$ is self-transversal (as a part of the proof that the space is self T_1-complementary). This explains Question 3.11 from [4]: *Does there exist a self* T_1*-complementary compact Hausdorff space without isolated points?*

Theorem 2.1 answers this question in the affirmative. Our space (or, better to say, a series of spaces) is $A(\beta\omega \setminus \omega) \times Y$, where Y is any dense-in-itself compact Hausdorff space of cardinality $\mathfrak{c}$. It is worth mentioning that the idea of the proof of Theorem 2.1 is a natural refinement of the arguments in [4] and [2]. Taking Y to be the closed unit interval or the Cantor set, we obtain in ZFC an example of a compact Hausdorff space without isolated points which is a T_1-complement of itself (see Corollary 2.2). Further, assuming that $2^{\aleph_1} = \mathfrak{c}$ and taking $Y = \{0, 1\}^{\omega_1}$, we get an example of a compact Hausdorff space without points of countable character which is again a T_1-complement of itself (see Corollary 2.3). We finish the article with three open problems about possible cardinalities of compact Hausdorff self T_1-complementary spaces.

2. The Alexandroff duplicate of $\beta\omega \setminus \omega$ and products

In what follows K denotes $\beta\omega \setminus \omega$, the remainder of the Čech–Stone compactification of the countable discrete space ω. It is clear that every nonempty open subset of K has cardinality $2^{\mathfrak{c}}$. We will also use the fact that K contains a pairwise disjoint family λ of open sets such that $|\lambda| = \mathfrak{c}$.

The Alexandroff duplicate of K is $A(K)$. It is easy to verify that every infinite closed subset of $A(K)$ has cardinality $2^{\mathfrak{c}}$. The reader can find a detailed discussion of the properties of $A(X)$, for an arbitrary space X, in [3].

Theorem 2.1. *For every compact Hausdorff space Y with $|Y| \le \mathfrak{c}$, the product space $A(K) \times Y$ is self T_1-complementary.*

Proof. Let $Z = A(K) \times Y$. Let also τ be the product topology of Z. By recursion of length $\kappa = 2^{\mathfrak{c}}$ we will construct a bijection $f\colon Z \to Z$ such that

(1) $f \circ f = id_Z$;

(2) the topology $\sigma = \{f(U) : U \in \tau\}$ is T_1-complementary to τ.

Let $K^* = A(K) \setminus K$. One of the main ideas of our construction is to use open fibers $\{x\} \times Y \subseteq Z$, with $x \in K^*$, to guarantee that each point $z \in Z$ will be isolated in $(Z, \tau \vee \sigma)$. More precisely, we will construct the bijection f to satisfy the following additional conditions:

(3) $f(K \times Y) = K^* \times Y$;

(4) for every $x \in K^*$, the image $f(\{x\} \times Y)$ is a discrete subset of $K \times Y$.

Let us show first that every bijection f satisfying conditions (1), (3), and (4) produces the topology $\sigma = f(\tau)$ transversal to τ. Indeed, let $\pi\colon A(K) \times Y \to A(K)$ be the projection. Take a point $z \in Z$ such that $x = \pi(z) \in K^*$. Clearly, $z \in \{x\} \times Y$ and, by (4), $f(\{x\} \times Y)$ is a discrete subset of $K \times Y$. Hence there exists an open set U in Z such that

$$(*) \qquad \{f(z)\} = U \cap f(\{x\} \times Y).$$

Since the point x is isolated in $A(K)$, the set $\{x\} \times Y$ is τ-open in $A(K) \times Y$. Hence $(*)$ implies that $f(z)$ is an isolated point of the space $(Z, \tau \vee \sigma)$. Further, it follows from (1) and (3) that $K \times Y = f(K^* \times Y)$, and we conclude that every point of $K \times Y$ is isolated in $(Z, \tau \vee \sigma)$. Applying f to both parts of $(*)$ and taking into account (1), we obtain the equality $\{z\} = f(U) \cap (\{x\} \times Y)$. This means that every point of $K^* \times Y$ is isolated in $(Z, \tau \vee \sigma)$. We have thus proved that the topology $\tau \vee \sigma$ is discrete, i.e., τ and σ are transversal.

To guarantee the T_1-independence of τ and σ is a more difficult task. We can reformulate the latter relation between τ and σ by saying that $f(F)$ is not τ-closed in Z, for every proper infinite τ-closed set $F \subseteq Z$. Let us describe the recursive construction of the bijection f in detail. In what follows the space Z always carries the topology τ unless the otherwise is specified.

We start with three observations that will be used in our construction of f. The first and the third of them are evident.

Fact 1. *If B is an infinite subset of $A(K)$, then the set $\overline{B} \cap K$ has cardinality $\kappa = 2^{\mathfrak{c}}$, where $\overline{B}$ is the closure of B in $A(K)$.*

Fact 2. *If $C \subseteq Z$ and the set $\pi(C)$ is infinite, then the projection $\pi(\overline{C} \cap (K \times Y))$ has cardinality κ, where $\overline{C}$ is the closure of C in Z.*

Indeed, since the projection π is a closed mapping, we have the equality $\pi(\overline{C}) = \overline{\pi(C)}$. It follows from $|\pi(C)| \geq \omega$ and Fact 1 that the set $\overline{\pi(C)} \cap K$ has cardinality κ. Again, since the mapping π is closed, we see that $\pi^{-1}(x) \cap \overline{C} \neq \varnothing$ for each $x \in \overline{\pi(C)} \cap K$. Hence $|\pi(\overline{C} \cap (K \times Y))| = \kappa$.

Fact 3. *If U is open in Z and $U \cap (K \times Y) \neq \varnothing$, then $|U \setminus (K \times Y)| = \kappa$.*

It is clear that $\chi(K) \leq w(K) = \mathfrak{c}$, $\chi(A(K)) = \chi(K) \leq \mathfrak{c}$, and $w(Y) \leq |Y| \leq \mathfrak{c}$. Therefore, $\chi(z, Z) \leq \mathfrak{c}$ for every $z \in Z$. Since $|K \times Y| = |K| = \kappa$, there exists a base $\mathcal{B}$ for $K \times Y$ in Z with $|\mathcal{B}| \leq \kappa$. In other words, $\mathcal{B}$ is a family of open sets in Z with the property that for every $z \in K \times Y$ and every open neighbourhood O of z in Z, there exists $U \in \mathcal{B}$ such that $z \in U \subseteq O$. Clearly, we can assume that $U \cap (K \times Y) \neq \varnothing$ for each $U \in \mathcal{B}$. Since $\kappa = \kappa^\omega$, we see that $|[Z]^\omega \times \mathcal{B}| = \kappa$, where $[Z]^\omega$ denotes the family of all countably infinite subsets of Z. Let $\{(C_\alpha, U_\alpha) : \alpha < \kappa\}$ be an enumeration of the set $[Z]^\omega \times \mathcal{B}$ such that for every pair $(C, U) \in [Z]^\omega \times \mathcal{B}$, the set $\{\alpha < \kappa : (C, U) = (C_\alpha, U_\alpha)\}$ is cofinal in κ.

Let $\{z_\alpha : \alpha < \kappa\}$ be a faithful enumeration of Z. By recursion on $\alpha < \kappa$ we will construct sets $Z_\alpha \subseteq Z$ and mappings $f_\alpha \colon Z_\alpha \to Z_\alpha$ satisfying the following conditions:

(i_α) $|Z_\alpha| \leq |\alpha| \cdot \mathfrak{c}$;
(ii_α) if $\gamma < \alpha$, then $Z_\gamma \subseteq Z_\alpha$;
(iii_α) $z_\alpha \in Z_{\alpha+1}$;
(iv_α) f_α is a bijection of Z_α onto itself and $f_\alpha \circ f_\alpha = id_{Z_\alpha}$;
(v_α) if $\gamma < \alpha$, then $f_\alpha \restriction_{Z_\gamma} = f_\gamma$;
(vi_α) if $z', z'' \in Z_\alpha$, $\pi(z') = \pi(z'')$, and $z' \neq z''$, then $\pi(f_\alpha(z')) \neq \pi(f_\alpha(z''))$;
(vii_α) $f_{\alpha+1}(U_\alpha \cap Z_{\alpha+1}) \cap \overline{f_{\alpha+1}(C_\alpha \cap Z_{\alpha+1})} \neq \varnothing$ provided that the set $\pi f_\alpha(C_\alpha \cap Z_\alpha)$ is infinite;
(viii_α) $\pi^{-1}(x) \subseteq Z_\alpha$ for each $x \in \pi(Z_\alpha) \cap K^*$;
(ix_α) if $x \in \pi(Z_\alpha) \cap K^*$, then $f_\alpha(\{x\} \times Y)$ is a discrete subset of $K \times Y$;
(x_α) $f_\alpha(Z_\alpha \cap (K \times Y)) \subseteq K^* \times Y$.

Put $Z_0 = \varnothing$ and $f_0 = \varnothing$. Clearly, Z_0 and f_0 satisfy (i_0)–(x_0). Let $\alpha < \kappa$, and suppose that a set $Z_\beta \subseteq Z$ and a mapping f_β of Z_β to itself satisfying conditions (i_β)–(x_β) have already been defined for all $\beta < \alpha$. If $\alpha > 0$ is limit, we put $Z_\alpha = \bigcup\{Z_\beta : \beta < \alpha\}$ and $f_\alpha = \bigcup\{f_\beta : \beta < \alpha\}$. Then the subset Z_α of Z and the mapping $f_\alpha \colon Z_\alpha \to Z_\alpha$ satisfy (i_α)–(x_α), except for (iii_α) and (vii_α) which are valid for all $\beta < \alpha$.

Suppose now that $\alpha = \gamma + 1$. Let $Z'_\gamma = Z_\gamma \cup \{z_\gamma\}$. Since $U_\gamma \cap (K \times Y) \neq \varnothing$, the cardinality of the set $U_\gamma \setminus (K \times Y)$ is κ by Fact 3. It follows from $|Z'_\gamma| \leq |Z_\gamma| + 1 \leq |\gamma+1| \cdot \mathfrak{c} < \kappa$ and $|\pi^{-1}\pi(Z'_\gamma)| \leq |Z'_\gamma| \cdot |Y| < \kappa$ that $|(U_\gamma \setminus (K \times Y)) \setminus \pi^{-1}\pi(Z'_\gamma)| = \kappa$. Therefore, we can pick a point $s_\alpha \in U_\gamma \setminus \pi^{-1}(K \cup \pi(Z'_\gamma))$.

If $\pi f_\gamma(C_\gamma \cap Z_\gamma)$ is infinite, then $\overline{f_\gamma(C_\gamma \cap Z_\gamma)} \cap (K \times Y)$ is a closed subset of Z whose projection to $A(K)$ has cardinality κ by Fact 2. We then use the inequalities $|Z'_\gamma| < \kappa$ and $|Y| \leq \mathfrak{c}$ to pick a point $t_\alpha \in (K \times Y) \cap \overline{f_\gamma(C_\gamma \cap Z_\gamma)} \setminus$

$\pi^{-1}\pi(Z'_\gamma)$. Otherwise pick an arbitrary point $t_\alpha \in \pi^{-1}(K \setminus \pi(Z'_\gamma))$; again, such a point exists because $|\pi(Z'_\gamma)| \leq |Z'_\gamma| < \kappa = |K|$. In either case, $t_\alpha \in K \times Y$.

Suppose that $z_\gamma = (x_\gamma, y_\gamma)$, $s_\alpha = (x'_\alpha, y'_\alpha)$, and $t_\alpha = (x''_\alpha, y''_\alpha)$. Notice that $x'_\alpha \in K^* \setminus \pi(Z'_\gamma)$ and $x''_\alpha \in K \setminus \pi(Z'_\gamma)$. To define Z_α, we consider the following possible cases.

Case 1. $z_\gamma \in Z_\gamma$. Then $Z'_\gamma = Z_\gamma$ and we choose a discrete set $D_\alpha \subseteq K \times \{y''_\alpha\}$ such that $t_\alpha \in D_\alpha$, $\pi(D_\alpha) \cap \pi(Z_\gamma) = \varnothing$, and $|D_\alpha| = |Y|$. This is possible since $x''_\alpha = \pi(t_\alpha) \notin \pi(Z_\gamma)$ and K contains $\mathfrak{c}$ pairwise disjoint nonempty open sets, each of cardinality κ. Put

$$Z_\alpha = Z_\gamma \cup D_\alpha \cup (\{x'_\alpha\} \times Y).$$

It follows from the definition that $\{z_\gamma, s_\alpha, t_\alpha\} \subseteq Z_\alpha$. Since the sets D_α, $\{x'_\alpha\} \times Y$, and Z_γ are pairwise disjoint, there exists an idempotent bijection f_α of Z_α onto itself such that f_α extends f_γ, $f_\alpha(\{x'_\alpha\} \times Y) = D_\alpha$, and $f_\alpha(s_\alpha) = t_\alpha$.

Case 2. $z_\gamma \notin Z_\gamma$. Again, we split this case into two subcases.

Case 2.1. $z_\gamma \in K \times Y$, i.e., $x_\gamma \in K$. Then we choose a discrete subset D_α of $K \times Y$ such that $\{z_\gamma, t_\alpha\} \subseteq D_\alpha$, $D_\alpha \cap Z_\gamma = \varnothing$, the restriction of π to D_α is one-to-one, and $|D_\alpha| = |Y|$. Again, this is possible since neither z_γ nor t_α is in Z_γ and, by the choice of t_α, $x_\gamma = \pi(z_\gamma) \neq \pi(t_\alpha) = x''_\alpha$. As in Case 1, we put

$$Z_\alpha = Z_\gamma \cup D_\alpha \cup (\{x'_\alpha\} \times Y).$$

Then $\{z_\gamma, s_\alpha, t_\alpha\} \subseteq Z_\alpha$. Since the sets D_α, $\{x'_\alpha\} \times Y$, and Z_γ are pairwise disjoint, there exists an idempotent bijection $f_\alpha\colon Z_\alpha \to Z_\alpha$ such that f_α extends f_γ, $f_\alpha(s_\alpha) = t_\alpha$, and $f_\alpha(\{x'_\alpha\} \times Y) = D_\alpha$.

Case 2.2. $x_\gamma \in K^*$. We choose a discrete set $D_\alpha \subseteq K \times \{y''_\alpha\}$ such that $t_\alpha \in D_\alpha$, $\pi(D_\alpha) \cap \pi(Z_\gamma) = \varnothing$, and $|D_\alpha| = |Y|$. Then we put

$$Z_\alpha = Z_\gamma \cup D_\alpha \cup (\{x_\gamma, x'_\alpha\} \times Y).$$

Clearly, $\{z_\gamma, s_\alpha, t_\alpha\} \subseteq Z_\alpha$. Since $\{x_\gamma, x'_\alpha\} \subseteq K^*$ and $\{z_\gamma, s_\alpha\} \cap Z_\gamma = \varnothing$, it follows from ($\mathrm{viii}_\gamma$) that $(\{x_\gamma, x'_\alpha\} \times Y) \cap Z_\gamma = \varnothing$. In addition, the set D_α is disjoint from both Z_γ and $\{x_\gamma, x'_\alpha\} \times Y$, so there exists an idempotent bijection f_α of Z_α onto itself such that f_α extends f_γ, $f_\alpha(\{x_\gamma, x'_\alpha\} \times Y) = D_\alpha$, and $f_\alpha(s_\alpha) = t_\alpha$.

Clearly, conditions (i_α), (ii_α), (iii_γ), (iv_α), (v_α), and (viii_α)–(x_α) hold true. Let us verify conditions (vi_α) and (vii_γ).

We verify (vi_α) only in Case 2.1—the argument in the rest of cases is analogous or even simpler. Suppose that z' and z'' are distinct elements of Z_α such that $\pi(z') = \pi(z'')$. If $\{z', z''\} \subseteq Z_\gamma$, then ($\mathrm{v}_\alpha$) and ($\mathrm{vi}_\gamma$) imply that $\pi(f_\alpha(z')) = \pi(f_\gamma(z')) \neq \pi(f_\gamma(z'')) = \pi(f_\alpha(z''))$. If $\{z', z''\} \subseteq \{x'_\alpha\} \times Y$, then $\pi(f_\alpha(z')) \neq \pi(f_\alpha(z''))$ since $f_\alpha(\{x'_\alpha\} \times Y) = D_\alpha$ and the restriction of π to D_α is one-to-one. The case $\{z', z''\} \subseteq D_\alpha$ is clearly impossible. Finally, suppose that $z' \in Z_\gamma$ and $z'' \in Z_\alpha \setminus Z_\gamma$ (or vice versa). Since $x'_\alpha \notin \pi(Z_\gamma)$, if follows from $\pi(z') = \pi(z'')$ and the definition of Z_α that $z'' \in D_\alpha$. Our choice of f_α implies that $f_\alpha(D_\alpha) = \{x'_\alpha\} \times Y$ because f_α is an idempotent bijection of Z_α onto itself. Hence $\pi(f_\alpha(z'')) = x'_\alpha \notin \pi(Z_\gamma)$ and, therefore, $\pi(f_\alpha(z'')) \neq \pi(f_\alpha(z'))$.

To check (vii$_\gamma$), suppose that $\pi f_\gamma(C_\gamma \cap Z_\gamma)$ is infinite. It follows from our construction that $s_\alpha \in U_\gamma \cap Z_\alpha$ and $f_\alpha(s_\alpha) = t_\alpha \in \overline{f_\gamma(C_\gamma \cap Z_\gamma)}$ which yields $t_\alpha \in f_\alpha(U_\gamma \cap Z_\alpha) \cap \overline{f_\alpha(C_\gamma \cap Z_\alpha)} \neq \varnothing$. The recursive step is completed.

We can now define the bijection $f\colon Z \to Z$. From (iii$_\alpha$) for all $\alpha < \kappa$ it follows that $Z = \bigcup\{Z_\alpha : \alpha < \kappa\}$. Let $f = \bigcup\{f_\alpha : \alpha < \kappa\}$. Since (ii$_\alpha$), (iv$_\alpha$) and (v$_\alpha$) hold for all $\alpha < \kappa$, f is an idempotent bijection of Z onto itself. This means that (1) holds. It also follows from (viii$_\alpha$) and (ix$_\alpha$) for all $\alpha < \kappa$ that $f(K^* \times Y) \subseteq K \times Y$, while (x$_\alpha$) implies that $f(K \times Y) \subseteq K^* \times Y$. Since f is a bijection, we conclude that $f(K^* \times Y) = K \times Y$ and $f(K \times Y) = K^* \times Y$, i.e., (3) holds. Similarly, conditions (viii$_\alpha$) and (ix$_\alpha$) for all $\alpha < \kappa$ together imply the validity of (4).

It was shown before the recursive construction that for any bijection $f\colon Z \to Z$ satisfying (1), (3), and (4), the topologies τ and $\sigma = f(\tau)$ on Z are transversal. It only remains to prove that τ and $\sigma = f(\tau)$ are T_1-independent, for this special bijection f. In other words, we have to verify that for every proper infinite closed subset Φ of Z, the image $f(\Phi)$ is not closed in Z. Let us consider two cases.

Case A. The projection $\pi(\Phi)$ is finite. Since $\Phi \subseteq \pi^{-1}\pi(\Phi)$ and each fiber $\pi^{-1}(x)$ has cardinality $|Y| \leq \mathfrak{c}$, we see that $|\Phi| \leq \mathfrak{c}$. Also, since $\kappa^{\mathfrak{c}} = \kappa$, the cofinality of the cardinal κ is greater than $\mathfrak{c}$. Applying the equality $Z = \bigcup\{Z_\alpha : \alpha < \kappa\}$ and (ii$_\alpha$) for $\alpha < \kappa$, we see that $\Phi \subseteq Z_\beta$ for some $\beta < \kappa$. It is also clear that $\pi^{-1}(x) \cap \Phi$ is infinite for some $x \in A(K)$. Then (vi$_\beta$) yields that the set $\pi(f(\Phi)) = \pi(f_\beta(\Phi))$ is infinite. In its turn, it follows from Fact 2 that the closure of $f(\Phi)$ in Z has cardinality κ and, since $|\Phi| \leq \mathfrak{c}$, the set $f(\Phi)$ cannot be closed in Z.

Case B. The set $\pi(\Phi)$ is infinite. Then $|\Phi| = \kappa$, by Fact 2. Again, we split this case into two subcases.

Case B.1. $(K \times Y) \setminus \Phi \neq \varnothing$. Since $cf(\kappa) > \mathfrak{c} > \omega$, the set $\pi f_\beta(\Phi \cap Z_\beta)$ must be infinite for some $\beta < \kappa$. Indeed, otherwise $\pi f(\Phi)$ is finite and hence $|\Phi| = |f(\Phi)| \leq \mathfrak{c}$, a contradiction. Choose a countable set $C \subseteq \Phi \cap Z_\beta$ such that $\pi f(C)$ is infinite. Take a point $z \in (K \times Y) \setminus \Phi$ and an element $U \in \mathcal{B}$ such that $z \in U \subseteq Z \setminus \Phi$. This is possible because $\mathcal{B}$ is a base for $K \times Y$ in Z. Note that $(C, U) \in [Z]^\omega \times \mathcal{B}$. Since the set $\{\alpha < \kappa : (C, U) = (C_\alpha, U_\alpha)\}$ is cofinal in κ, $(C, U) = (C_\alpha, U_\alpha)$ for some α with $\beta \leq \alpha < \kappa$. From $Z_\alpha \supseteq Z_\beta$ and $C_\alpha = C \subseteq Z_\beta$ we get $C_\alpha \cap Z_\alpha \supseteq C_\alpha \cap Z_\beta = C$ and, since $\pi f(C)$ is infinite, so is $\pi f(C_\alpha \cap Z_\alpha) = \pi f_\alpha(C_\alpha \cap Z_\alpha)$. Then (vii$_\alpha$) shows that $f_{\alpha+1}(U_\alpha \cap Z_{\alpha+1}) \cap \overline{f_{\alpha+1}(C_\alpha \cap Z_{\alpha+1})} \neq \varnothing$. Since f extends f_α and $\Phi \supseteq C = C_\alpha$, it follows that

$$f(U_\alpha) \cap \overline{f(\Phi)} \supseteq f(U_\alpha) \cap \overline{f(C_\alpha)} \supseteq f_{\alpha+1}(U_\alpha \cap Z_{\alpha+1}) \cap \overline{f_{\alpha+1}(C_\alpha \cap Z_{\alpha+1})} \neq \varnothing.$$

Therefore, there exists $z^* \in U_\alpha$ such that $f(z^*) \in \overline{f(\Phi)}$. It follows from $U_\alpha = U \subseteq Z \setminus \Phi$ that $z^* \notin \Phi$. Since f is a bijection of Z, this yields $f(z^*) \notin f(\Phi)$. Thus $f(z^*) \in \overline{f(\Phi)} \setminus f(\Phi)$, that is, the set $f(\Phi)$ is not closed in Z.

Case B.2. $K \times Y \subseteq \Phi$. Suppose to the contrary that $f(\Phi)$ is closed in Z. Since $f(K \times Y) = K^* \times Y$ and the latter set is dense in Z, we see that $K^* \times Y \subseteq f(\Phi) = Z$. This contradicts our choice of Φ as a proper subset of Z.

We have thus proved that $f(\Phi)$ fails to be closed in Z, i.e., the topologies τ and $\sigma = f(\tau)$ are T_1-independent. Since we already know that τ and σ are transversal, this finishes the proof of the theorem. □

Taking Y in Theorem 2.1 to be the Cantor set or the closed unit interval $\mathbb{I} = [0, 1]$, we obtain the following result which answers Question 3.11 from [4] in the affirmative:

Corollary 2.2. *There exists an infinite compact Hausdorff self T_1-complementary space without isolated points.*

Under additional set-theoretic assumptions, one can refine Corollary 2.2 as follows:

Corollary 2.3. *Let κ be a cardinal with $\omega \leq \kappa < \mathfrak{c}$. It is consistent with ZFC that there exists a compact Hausdorff self T_1-complementary space Z such that $\chi(z, Z) \geq \kappa$ for each $z \in Z$.*

Proof. One can assume that $2^\kappa = 2^\omega = \mathfrak{c}$ and take $Y = \mathbb{I}^\kappa$ in Theorem 2.1. □

The following questions remain open.

Problem 2.4. *Let $K = \beta\omega \setminus \omega$. Is the product space $A(K) \times K$ self T_1-complementary?*

Problem 2.5. *Is it true that for every cardinal λ, there exists a compact Hausdorff self T_1-complementary space Z with $|Z| \geq \lambda$?*

Here is a stronger version of the above problem:

Problem 2.6. *Is it true that for every cardinal λ, there exists a compact Hausdorff self T_1-complementary space Z such that $\chi(z, Z) \geq \lambda$ for all $z \in Z$?*

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