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Between continuity and set connectedness

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Abstract. Two new weak variants of continuity called 'R-continuity' and 'F-continuity' are introduced. Their basic properties are studied and their place in the hierarchy of weak variants of continuity, that already exist in the literature, is elaborated. The class of R-continuous functions properly contains the class of continuous functions and is strictly contained in each of the three classes of (1) faintly continuous functions studied by Long and Herrignton (*Kyungpook Math. J.* 22(1982), 7-14); (2) D-continuous functions introduced by Kohli (*Bull. Cal. Math. Soc.* 84 (1992), 39-46), and (3) F-continuous functions which in turn are strictly contained in the class of z-continuous functions studied by Singal and Niemse (*Math. Student* 66 (1997), 193-210). So the class of R-continuous functions is also properly contained in each of the classes of D*-continuous functions, D_δ-continuous function and set connected functions.

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1. Introduction

Functions occur everywhere in mathematics and applications of mathematics. Continuous functions play a prominent role in topology, analysis and many other branches of mathematics. In many situations/applications in geometry, topology, functional analysis and complex analysis, continuity is not sufficient and a condition stronger than continuity is required. On the other hand a condition strictly weaker than continuity is sufficient to meet the demand of a particular situation. Several strong variants of continuity occur in the lore of mathematical literature and applications of mathematics (see for example [6, 12, 13, 14, 19, 20, 24, 27, 30, 32, 34, 38]), while others are weaker than

continuity (see for example [7, 11, 15, 23, 25, 28, 35, 36, 37]), and yet others are independent of continuity (see for example [17, 18, 21, 22, 26, 31]). In this paper we restrict ourselves to the study of weak variants of continuity and introduce two new weak variants of continuity called 'R-continuity' and 'F-continuity' and study their basic properties. We discuss their interrelations and interconnections with other weak variants of continuity that already exist in the literature. Sufficient conditions on range are given for an F-continuous (R-continuous) function to be continuous. The other weak variants of continuity with which we shall be dealing in this paper include among others, almost continuous functions [36], θ-continuous function [7], D-continuous functions [11], faintly continuous functions [28] and z-continuous functions [35]. It turns out that the class of R-continuous functions properly includes the class of continuous functions and is strictly contained in each of the three classes of (i) D-continuous functions [11], (ii) faintly continuous functions [28], and (iii) F-continuous functions; while the class of F-continuous functions properly contains the class of R-continuous functions and is strictly contained in the class of z-continuous functions [35]. This in turn implies that the class of R-continuous functions is also properly contained in each of the classes of D*-continuous functions [37], D_δ-continuous functions [15], z-continuous functions [35] and set connected functions [23].
The organization of the paper is as follows. Section 2 is devoted to preliminaries and basic definitions. In Section 3, we introduce the notions of 'R-continuous function' and 'F-continuous function' and elaborate on their place in the hierarchy of weak variants of continuity that already exist in the literature. Examples are included to reflect upon the distinctiveness of the old and new variants of continuity discussed in the paper. Section 4 is devoted to the study of basic properties of R-continuous functions and F-continuous functions. In Section 5, we discuss the properties of graphs of R-continuous (F-continuous) functions. In Section 6, we consider retopologization of the range of an R-continuous (F-continuous) function and conclude with alternative proofs of certain results of preceding sections. In Section 7, we discuss conditions on the range of a function under which certain weak variants of continuity are identical among themselves and/or coincide with continuity.

2. Preliminaries and Basic Definitions

A collection β of subsets of a space X is called an ***open complementary system*** [8] if β consists of open sets such that for every $B \in \beta$, there exist $B_1, B_2, \ldots, \in \beta$. with $B = \cup\{X \setminus B_i : i \in N\}$. A subset A of a space X is called a ***strongly open*** $\mathbf{F_\sigma}$***-set*** [8] if there exists a countable open complementary system $\beta(A)$ with $A \in \beta(A)$. The complement of a strongly open F_σ-set is called **strongly closed $\mathbf{G_\delta}$-set**. A subset A of a space X is called **regular $\mathbf{G_\delta}$-set** [29] if A is the intersection of a sequence of closed sets whose interiors contain A, i.e., if $A = \bigcap_{n=1}^{\infty} F_n = \bigcap_{n=1}^{\infty} F_n^0$, where each F_n is a closed subset of X (here F_n^0 denotes the interior of F_n). The complement of a regular G_δ-set is called a ***regular*** $\mathbf{F}_{\boldsymbol{\sigma}}$**-set**. A point $x \in X$ is called a ***θ-adherent point*** [40] of $A \subset X$ if every closed neigbourhood of x intersects A. Let $\mathrm{cl}_{\boldsymbol{\theta}}\mathbf{A}$ denote the set of all θ-adherent points of A. The set A is called ***θ*-closed** it $A = \mathrm{cl}_\theta A$. The complement of a θ-closed set is referred to as a ***θ-open set***.

Definition 2.1. *A function $f : X \to Y$ is said to be*

(i) $\mathbf{D}_{\boldsymbol{\delta}}$***-continuous*** [15] *if for each point $x \in X$ and each regular F_σ-set V containing $f(x)$ there is an open set U containing x such that $f(U) \subset V$.*
(ii) $\mathbf{D}^*$***-continuous*** [37] *if for each point $x \in X$ and each strongly open F_σ- set V containing $f(x)$ there is an open set U containing x such that $f(U) \subset V$.*
(iii) ***almost continuous*** [36] *if for each $x \in X$ and each open set V containing $f(x)$ there is an open set U containing x such that $f(U) \subset (\overline{V})^0$.*
(iv) **D*-continuous*** [11] *if for each $x \in X$ and each open F_σ- set V containing $f(x)$ there is an open set U containing x such that $f(U) \subset V$.*
(v) **z*-continuous*** [35] *if for each $x \in X$ and each cozero set V containing $f(x)$ there is an open set U containing x such that $f(U) \subset V$.*
(vi) ***θ-continuous*** [7] *if for each $x \in X$ and each open set V containing $f(x)$ there is an open set U containing x such that $f(\overline{U}) \subset \overline{V}$.*
(vii) ***weakly continuous*** [25] *if for each $x \in X$ and each open set V containing $f(x)$ there exists an open set U containing x such that $f(U) \subset \overline{V}$.*
(viii) ***faintly continuous*** [28] *if for each $x \in X$ and each θ-open set V containing $f(x)$ there exists an open set U containing x such that $f(U) \subset V$.*
(ix) ***quasi θ-continuous function*** [33] *if for each $x \in X$ and each θ-open set V containing $f(x)$ there exists a θ-open set U containing x such that $f(U) \subset V$.*
(x) ***set connected mapping*** [23] *if whenever X is connected between A and B; $f(X)$ is connected between $f(A)$ and $f(B)$.*
(xi) cl-***continuous***** *if $f^{-1}(V)$ is open in X for every clopen set $V \subset Y$.*

Definition 2.2. *A topological space X is said to be*

(i) ***functionally regular*** [2] *if for each closed set A in X and a point $x \notin A$ there exists a continuous real-valued function f defined on X such that $f(x) \notin f(A)$; or equivalently for each $x \in X$ and each open set U containing x there exists a zero set Z in X such that $x \in Z \subset U$.*
(ii) $\mathbf{R_0}$***-space*** [4] *if for each open set U in X and each $x \in U$, $\overline{\{x\}} \subset U$.*
(iii) **D*-regular*** [8] *if it has a base of open F_σ-sets.*
(iv) **D*-completely regular*** [8] *if it has a base of strongly open F_σ-sets*
(v) $\mathbf{D}_{\boldsymbol{\delta}}$***-completely regular*** [16] *if it has a base of regular F_σ-sets.*

Definition 2.3. *A collection $\mathcal{K}$ of subsets of a space X is said to be*

(i) ***closure preserving*** *if* $\mathrm{cl}(\cup\mathcal{L}) = \cup\{\mathrm{cl}(L)|L \in \mathcal{L}\}$ *for any subcollection $\mathcal{L}$ of $\mathcal{K}$; and*

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*cl-continuous functions have been referred to as slightly continuous in ([10], [15]).

(ii) ***hereditarily closure preserving*** *if whenever a subset* $H(K) \subset K$ *is chosen for each* $K \in \mathcal{K}$*, then the resulting collection* $\mathcal{H} = \{H(K)|K \in \mathcal{K}\}$ *is closure preserving.*

3. R-Continuous functions and F-continuous functions

Let X be a topological space. An open subset U of X is said to be **r-open** [20] if for each $x \in U$ there exists a closed set B such that $x \in B \subset U$ or equivalently, if U is expressible as a union of closed sets. An open subset W of X is said to be **F-open** [19] if for each $x \in W$ there exists a zero set Z such that $x \in Z \subset W$ or equivalently, if W is expressible as a union of zero sets.

Definition 3.1. *A function* $f : X \to Y$ *from a topological space* X *into a topological space* Y *is said to be* **F-*continuous*** (**R-*continuous***) *at* $x \in X$*, if for each* F*-open* (r*-open*) *set* V *in* Y *containing* $f(x)$ *there exists an open set* U *containing* x *such that* $f(U) \subset V$*. The function* f *is said to be* **F-*continuous*** (**R-*continuous***) *if it is F-continuous* (*R-continuous*) *at each* $x \in X$*.*

The following diagram (Figure 1) well illustrates the interrelations and interconnections that exist between R-continuity, F-continuity, and certain other weak variants of continuity that already exist in the literature (see Definitions 2.1).

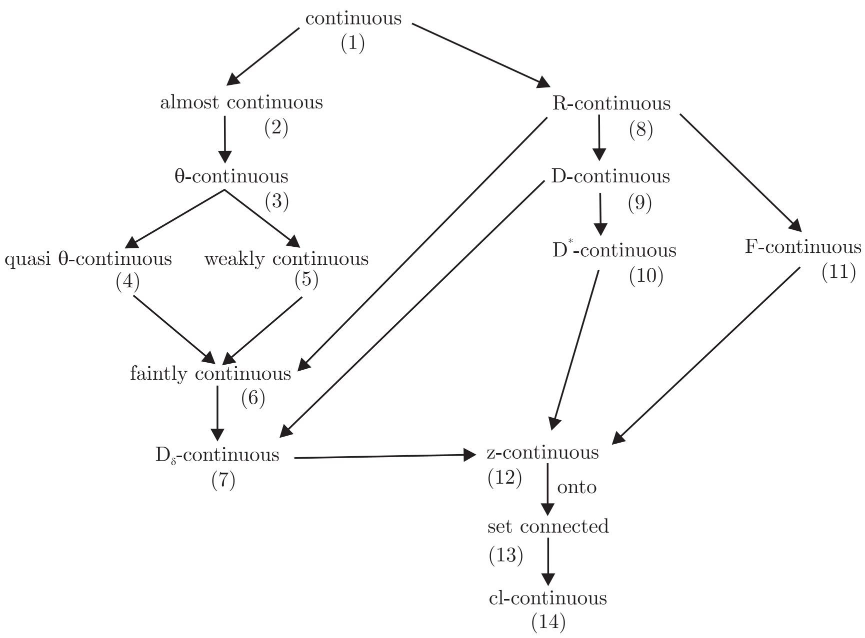

Figure 1

Examples in [1, 11, 15, 35, 36, 37] and the following examples show that none of the above implications is reversible.

Example 3.2. *Let X be the real line endowed with indiscrete topology and Y be the real line equipped with the right order topology* [39, Example 50]. *Let $f : X \to Y$ be the identity function from X onto Y. Then f is an R-continuous function which is not continuous.*

Example 3.3. *Let X be the real line endowed with cofinite topology and let Y be the real line equipped with cocountable topology. Let f be the identity mapping of X onto Y. Then f is D-continuous but not R-continuous.*

Example 3.4. *Let X denote the real line endowed with the usual Euclidean topology $\mathcal{U}$. Let Y denote the real line endowed with the topology generated by $\mathcal{U}$ together with all sets of the form $(a, b) - K$, where $K = \{1/n : n \in \mathbb{Z}^+\}$. Let f denote the identity mapping of X onto Y. Then f is faintly continuous but not R-continuous.*

Example 3.5. *Let $X = Y = \mathbb{Z}^+$, where $\mathbb{Z}^+$ is the set of positive integers. Let X be endowed with indiscrete topology τ and Y be endowed with relative prime integer topology τ^** [39, Example 60]. *Then every real-valued continuous function defined on (Y, τ^*) is constant. Let f denote the identity function from X onto Y. Then f is F-continuous but not R-continuous.*

Example 3.6. *Let X be the real line endowed with usual topology $\mathcal{U}$ and let Y be the real line endowed with the topology $\mathcal{U}^*$ generated by $\mathcal{U}$ and the addition of all sets of the form $\mathbb{Q} \cap U$, where $U \in \mathcal{U}$ and $\mathbb{Q}$ is the set of rationals. The space $(Y, \mathcal{U}^*)$ is a functionally Hausdorff Lindelöf space and so by [2, Theorem 3] it is a functionally regular space. Let $g : (X, \mathcal{U}) \to (Y, \mathcal{U}^*)$ denote the identity mapping. Then g is not an F-continuous function since $\mathbb{Q}$ is F-open in Y but its inverse image is not open in X. However, we claim that g is a z-continuous function. To this end, let $f : (\mathbb{R}, \mathcal{U}^*) \to (\mathbb{R}, \mathcal{U})$ be any continuous function. To show that $f : (\mathbb{R}, \mathcal{U}) \to (\mathbb{R}, \mathcal{U})$ is continuous, let $x \in \mathbb{R}$. The following cases arise:*

Case I: *x is irrational. Clearly f is $\mathcal{U} - \mathcal{U}$ continuous at x.*

Case II: *x is rational. Let $\epsilon > 0$ be given. Then there exist an open interval I containing x such that $f(\mathbb{Q} \cap I) \subset (f(x) - \epsilon/2, f(x) + \epsilon/2)$. Now, since $\mathbb{Q}$ is dense in $(\mathbb{R}, \mathcal{U}^*)$, $\mathrm{cl}_{\mathcal{U}^*}(\mathbb{Q} \cap I) = \mathrm{cl}_{\mathcal{U}^*}(I)$. Therefore, $f(I) \subset f(\mathrm{cl}_{\mathcal{U}^*} I) = f(\mathrm{cl}_{\mathcal{U}^*}(\mathbb{Q} \cap I)) \subset \overline{f(\mathbb{Q} \cap I)} \subset [f(x) - \epsilon/2, f(x) + \epsilon/2] \subset (f(x) - \epsilon, f(x) + \epsilon)$. That is $f(I) \subset (f(x) - \epsilon, f(x) + \epsilon)$. Thus it turns out that the spaces $(\mathbb{R}, \mathcal{U}^*)$ and $(\mathbb{R}, \mathcal{U})$ have same classes of zero sets. Consequently, g is z-continuous.*

Proposition 3.7. *A function $f : X \to Y$ from a topological space into a functionally regular space (R_0-space) Y is continuous if and only if it is F-continuous (R-continuous).*

4. Basic properties of R-continuous functions and F-continuous functions

Theorem 4.1. *Let $f : X \to Y$ be a function from a topological space X into a topological space Y. Then the following statements are equivalent.*

(a) *f is F-continuous (R-continuous)*
(b) *The set $f^{-1}(V)$ is open for every F-open (r-open) subset $V \subset Y$.*
(c) *The set $f^{-1}(B)$ is closed for every F-closed (r-closed) subset $B \subset Y$.*
(d) *For each $x \in X$ and each net $\{x_\alpha\}_{\alpha\in\Lambda}$ which converges to x, the net $\{f(x_\alpha)\}_{\alpha\in\Lambda}$ is eventually in each F-open (r-open) set containing $f(x)$.*

Proof. Easy. □

Theorem 4.2. *Let $f : X \to Y$ be any function. Then the following statements are true.*

(a) *If $f : X \to Y$ is F-continuous (R-continuous) and $A \subset X$, then $f|A : A \to Y$ is F-continuous (R-continuous).*
(b) *If $\{U_\alpha : \alpha \in \Lambda\}$ is an open cover of X and for each $\alpha \in \Lambda, f_\alpha = f|U_\alpha$ is F-continuous (R-continuous), then f is F-continuous (R-continuous).*
(c) *If $\{F_\beta : \beta \in \Lambda\}$ is a hereditarily closure preserving closed cover of X and if for each $\beta \in \Lambda, f_\beta = f|F_\beta$ is F-continuous (R-continuous), then f is F-continuous (R-continuous).*

Proof. (a) Let V be an F-open (r-open) subset of Y. Then $f^{-1}(V)$ is an open set in view of F-continuity (R-continuity) of f. So $(f|A)^{-1}(V) = f^{-1}(V) \cap A$ is an open subset of A.
(b) Let V be an F-open (r-open) subset of Y. Then $f^{-1}(V) = \cup\{f_\alpha^{-1}(V) : \alpha \in \Lambda\}$. Since each f_α is F-continuous (R-continuous), each $f_\alpha^{-1}(V)$ is open in U_α and hence in X. Thus $f^{-1}(V)$ being the union of open sets is open.
(c) Let B be an F-closed (r-closed) subset of Y. Then $f^{-1}(B) = \cup\{f_\beta^{-1}(B) | \beta \in \Lambda\}$. Since each f_β is F-continuous (R-continuous), each $f_\beta^{-1}(B)$ is closed in F_β and hence in X. Again, since $\{F_\beta : \beta \in \Lambda\}$ is a hereditarily closure preserving closed cover of X, the collection $\{f_\beta^{-1}(B) : \beta \in \Lambda\}$ is a closure preserving collection of closed sets. Consequently, their union $f^{-1}(B)$ is closed. □

Theorem 4.3. *If $f : X \to Y$ is continuous and $g : Y \to Z$ is F-continuous (R-continuous), then $g \circ f$ is F-continuous (R-continuous).*

Proof. Let W be an F-open (r-open) subset of Z. Then $g^{-1}(W)$ is open in Y. Again, since f is continuous $f^{-1}(g^{-1}(W)) = (g \circ f)^{-1}(W)$ is open in X. So $g \circ f$ is F-continuous (R-continuous). □

Example 4.4. *In general F-continuity of $g \circ f$ need not imply continuity of f. For example, let X be the real line with Smirnov's deleted sequence topology* [39, Example 64]*, Y be the real line with countable complement extension topology* [39, Example 63] *and Z be the real line equipped with usual topology. Let $f : X \to Y$ and $g : Y \to Z$ be the identity mappings. Then $g \circ f$ and g are F-continuous. However, f is not a continuous function.*

Theorem 4.5. *Let $f:X\to Y$ be a quotient map. Then a function $g:Y\to Z$ is F-continuous (R-continuous) if and only if $g\circ f$ is F-continuous (R-continuous).*

Proof. To prove necessity, let W be an F-closed (r-closed) subset of Z. Then $g^{-1}(W)$ is a closed subset of Y and since f is a quotient map, $(g\circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$ is closed. Hence $g\circ f$ is F-continuous. To prove sufficiency, let V be an F-open (r-open) subset of Z. Then $(g\circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is open in X. Since f is a quotient map, $g^{-1}(V)$ is open in Y and so g is F-continuous (R-continuous). □

Theorem 4.6. *Let $f : X \to Y$ be either an open or closed surjection and let $g : Y \to Z$ be any function such that $g \circ f$ is F-continuous (R-continuous). Then g is F-continuous (R-continuous).*

Proof. Assume that f is open (respectively, closed) and let V be an F-open (r-open) (respectively, F-closed (r-closed)) subset of Z. Since $g\circ f$ is F-continuous (R-continuous), $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ is open (respectively, closed). Since f is an open (respectively, closed) surjection, $f(f^{-1}(g^{-1}(V))) = g^{-1}(V)$ is open (respectively, closed) and consequently, g is F-continuous (R-continuous). □

Theorem 4.7. *Let $f : X \to \prod_{\alpha\in\Lambda} X_\alpha$ be a function into a product space. If f is F-continuous (R-continuous), then each $p_\alpha\circ f$ is F-continuous (R-continuous), where p_α denotes the projection map $p_\alpha : \prod_{\alpha\in\Lambda} X_\alpha \to X_\alpha$.*

Proof. Let F_β be an F-closed (r-closed) subset of X_β. Now $(p_\beta \circ f)^{-1}(F_\beta) = f^{-1}(p_\beta^{-1}(F_\beta)) = f^{-1}\Big(F_\beta\times(\prod_{\alpha\neq\beta} X_\alpha)\Big)$. Since f is F-continuous (R-continuous) and since $F_\beta \times (\prod_{\alpha\neq\beta} X_\alpha)$ is an F-closed (r-closed) set in the product space $\prod_{\alpha\in\Lambda} X_\alpha$, $f^{-1}\Big(F_\beta \times (\prod_{\alpha\neq\beta} X_\alpha)\Big)$ is a closed set in X and consequently $p_\alpha \circ f$ is F-continuous (R-continuous). □

Theorem 4.8. *Let $f_\alpha : X_\alpha \to Y_\alpha$ be a mapping for each $\alpha \in \Lambda$ and let $f : \Pi X_\alpha \to \Pi Y_\alpha$ be the mapping defined by $f((x_\alpha)) = (f_\alpha(x_\alpha))$ for each point (x_α) in ΠX_α. If f is F-continuous (R-continuous), then f_α is F-continuous (R-continuous) for each $\alpha \in \Lambda$.*

Proof. Let F_β be an F-closed (r-closed) subset of Y_β. Then $F_\beta\times(\prod_{\alpha\neq\beta} Y_\alpha)$ is an F-closed (r-closed) subset of the product space $\prod_{\alpha\in\Lambda} Y_\alpha$. Since f is F-continuous (R-continuous), $f^{-1}\Big(F_\beta\times\prod_{\alpha\neq\beta} Y_\alpha\Big) = f_\beta^{-1}(F_\beta)\times(\prod_{\alpha\neq\beta} X_\alpha)$ is closed in $\prod_{\alpha\in\Lambda} X_\alpha$. Consequently, $f_\beta^{-1}(F_\beta)$ is closed in X_β and so f_β is F-continuous (R-continuous). □

Definition 4.9. *Let $f : X \to Y$ be any function. Then the function $g : X \to X \times Y$ defined by $g(x) = (x, f(x))$ is called the graph function with respect to f.*

Theorem 4.10. *Let $f : X \to Y$ be a function such that the graph function $g : X \to X \times Y$ is F-continuous (R-continuous). Then f is F-continuous (R-continuous).*

Proof. Let $x \in X$ and let V be an F-open set containing $f(x)$. Since $Y - V$ is an F-closed (r-closed) set, so is $X \times (Y - V) = X \times Y - p_y^{-1}(V)$, where p_y denotes the projection of $X \times Y$ onto Y. Therefore, $p_y^{-1}(V)$ is an F-open (r-open) subset of $X \times Y$ containing $g(x)$. Since g is F-continuous (R-continuous), there is an open set U containing x, such that $g(U) \subset p_y^{-1}(V)$. It follows that $p_y(g(U)) = f(U) \subset V$ and so f is F-continuous (R-continuous). □

5. Properties of graphs of F-continuous (R-conttnuous) functions

In this section we study how the graph of an F-continuous (R-continuous) function $f : X \to Y$ is situated in the product space $X \times Y$. First we prove the following proposition which will be used in the sequel.

Proposition 5.1. *For a topological space X, the following statements are equivalent.*

(a) *X is functionally Hausdorff.*
(b) *Every pair of distinct points in X are contained in disjoint cozero sets.*
(c) *Every pair of distinct points in X are contained in disjoint F-open sets.*
(d) *For every pair of distinct points in X there exists an F-open set containing one of the points but not the other.*

Proof. The equivalence of (a)-(c) is discussed in [19, Proposition 5.2]. The implication (c) $\Rightarrow$ (d) is trivial. To show that (d) $\Rightarrow$ (a), let $x, y \in X$, $x \neq y$ and let V be an F-open set containing one of the points x and y but not the other. To be precise, suppose that $x \in V$. Since V is an F-open set, there exists a zero set Z such that $x \in Z \subset V$. Let $f : X \to [0, 1]$ be a continuous function with Z as its zero set. Then $f(x) = 0$ and $f(y) \neq 0$ and so X is functionally Hausdorff. □

Definition 5.2. *The graph $G(f)$ of a function $f : X \to Y$ is said to be*

(i) ***cozero closed with respect to $\mathbf{X} \times \mathbf{Y}$*** *if for each $(x, y) \notin G(f)$ there exist cozero sets U and V containing x and y, respectively such that $(U \times V) \cap G(f) = \phi$;*
(ii) ***cozero closed with respect to $\mathbf{X}$*** *if for each $(x, y) \notin G(f)$ there exists a cozero set U containing x and an open set V containing y such that $(U \times V) \cap G(f) = \phi$; and*
(iii) **F-*closed with respect to* $\mathbf{X} \times \mathbf{Y}$** *if for each $(x, y) \notin G(f)$ there exist F-open sets U (in X) and V (in Y) containing x and y respectively such that $(U \times V) \cap G(f) = \phi$.*

Theorem 5.3. *Let $f : X \to Y$ be an injection such that its graph $G(f)$ is cozero closed with respect to X. Then X is functionally Hausdorff.*

Proof. Let $x_1, x_2 \in X, x_1 \neq x_2$. Since f is an injection $(x_1, f(x_2)) \notin G(f)$. In view of cozero closedness of the graph $G(f)$, there exists a cozero set U containing x_1 and an open set V containing $f(x_2)$ such that $(U \times V) \cap G(f) = \phi$ and hence $U \cap f^{-1}(V) = \phi$. Therefore $x_2 \notin U$. Since every cozero set is F-open, in view of Proposition 5.1, X is functionally Hausdorff. □

Theorem 5.4. *Let $f : X \to Y$ be a z-continuous function into a functionally Hausdorff space Y. Then the graph $G(f)$ of f is cozero closed with respect to $X \times Y$.*

Proof. Suppose $(x, y) \notin G(f)$. Then $f(x) \neq y$. Since Y is functionally Hausdorff, in view of Proposition 5.1, there exist disjoint cozero sets V and W containing $f(x)$ and y, respectively. Since f is z-continuous, by [35, Theorem 2.3] $U = f^{-1}(V)$ is a cozero set containing x and $f(U) \subset F \subset Y - W$. Consequently, $U \times V$ contains no point of $G(f)$ and so $G(f)$ is cozero closed with respect to $X \times Y$. □

Corollary 5.5. *Let $f : X \to Y$ be an F-continuous function. If Y is functionally Hausdorff, then f has an F-closed graph with respect to $X \times Y$.*

Definition 5.6 ([26]). *A function $f : X \to Y$ is said to have **strongly closed graph**, if for each $(x, y) \notin G(f)$, there exist open sets U and V containing x and y, respectively such that $U \times \overline{V}$ contains no point of $G(f)$.*

Theorem 5.7. *Let $f : X \to Y$ be an R-continuous function into a Hausdorff space Y. Then f has strongly closed graph.*

Proof. Suppose $(x, y) \notin G(f)$. Then $f(x) \neq y$. In view of Hausdorffness of Y, there are disjoint open sets V and W such that $f(x) \in V$ and $y \in W$ and V is an r-open set. Since f is R-continuous, $f^{-1}(V)$ is open. Then $U = f^{-1}(V)$ is an open set containing x and $f(U) \subset V \subset Y - \overline{W}$ consequently, $U \times \overline{W}$ contains no point of $G(f)$ and so $G(f)$ is strongly closed $X \times Y$. □

Singal and Niemse [35] showed that the equalizer of two z-continuous functions into a functionally Hausdorff space is z-closed ([35, Theorem 4.3]). Consequently, the equalizer of two F-continuous into a functionally Hausdorff space is z-closed and hence F-closed. In contrast the following result shows that the equalizer of two R-continuous functions into a Hausdorff space is closed.

Theorem 5.8. *Let $f : X \to Y$ be F-continuous (R-continuous) injection into a functionally Hausdorff (Hausdorff) space Y. Then X is Hausdorff.*

Proof. Let x and y be any two distinct points in X. Then $f(x) \neq f(y)$. Since Y is functionally Hausdorff (Hausdorff) there exist disjoint F-open (r-open) sets U and V containing $f(x)$ and $f(y)$ respectively. Since f is F-continuous (R-continuous), $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint open sets containing x and y, respectively. Hence X is Hausdorff. □

6. Change of Topology

In this section we show that if the range of an F-continuous (R-continuous) function is retopologized in an appropriate way then f is simply a continuous function. This fact in conjunction with the properties of continuous functions leads to alternative proofs of certain results of Section 4. For let (X, τ) be a topological space. Let τ_F (respectively τ_R) denote the collection of all F-open (respectively r-open) subsets of X. It is easily verified that the collection τ_F (respectively τ_R) is a topology on X. Moreover, since every zero set is a closed set, $\tau_F \subset \tau_R \subset \tau$.

Theorem 6.1. *A space (X, τ) is functionally regular (respectively R_0-space) if and only if $\tau = \tau_F$ (respectively $\tau = \tau_R$).*

Theorem 6.2. *For a topological space (X, τ) the following statements are equivalent.*

(a) *(X, τ) is functionally regular.*
(b) *Every F-continuous function $f : Y \to (X, \tau)$ from a topological space Y into (X, τ) is continuous.*
(c) *The identity mapping $1_X : (X, \tau_F) \to (X, \tau)$ is continuous.*

Proof. (a) $\Rightarrow$ (b). Let $U \in \tau_F$. Since (X, τ) is functionally regular space, U is F-open. In view of F-continuity of $f, f^{-1}(U)$ is open in Y and so f is continuous.
(b) $\Rightarrow$ (c). By the definition of τ_F, the identity mapping $1_X : (X, \tau_F) \to (X, \tau)$ is F-continuous and hence in view of (b), it is continuous.
(c) $\Rightarrow$ (a). By (c) every open set in (X, τ) is F-open and so in view of Theorem 6.1 it is a functionally regular space. □

Theorem 6.3. *For a topological space (X, τ), the following statements are equivalent.*

(a) *(X, τ) is an R_0-space.*
(b) *Every R-continuous function $f : Y \to (X, \tau)$ from a topological space Y into (X, τ) is continuous.*
(c) *The identity mapping $1_X : (X, \tau_R) \to (X, \tau)$ is continuous.*

Proof of Theorem 6.3 is similar to that of Theorem 6.2 and hence omitted.

Many of the results studied in Section 4 now follow from Theorem 6.2 or 6.3 and the corresponding standard properties of continuous functions. Furthermore, as an application of Theorems 6.2 and 6.3 we provide with an alternative proofs of following well known results in the literature.

Theorem 6.4 (Aull [3])**.** *Any product of functionally regular spaces is functionally regular.*

Proof. Let $\{X_\alpha : \alpha \in \Lambda\}$ be any collection of functionally regular spaces. Let $X = \prod X_\alpha$. To show that X is functionally regular, in view of Theorem 6.2 it is sufficient to prove that every F-continuous function $f : Y \to X$ is continuous.

Thus it is sufficient to prove that $\pi_\alpha \circ f$ is continuous for each $\alpha \in \Lambda$, where $\pi_\alpha : X \to X_\alpha$ denote the projection onto the α-th co-ordinate space X_α. To this end, let U_α be an F-open subset of X_α. Let $U_\alpha = \bigcup_{\gamma \in \Lambda_{U_\alpha}} Z_{\alpha_\gamma}$, where each Z_{α_γ} is a zero set in X_α. It is easily verified that $U_\alpha \times \big(\prod_{\beta \neq \alpha} X_\beta\big)$ is an F-open subset of X. Since f is F-continuous, $f^{-1}\Big(U_\alpha \times \big(\prod_{\beta \neq \alpha} X_\beta\big)\Big)$ is open in Y. Again, since $(\pi_\alpha \circ f)^{-1}(U_\alpha) = f^{-1}(\pi_\alpha^{-1}(U_\alpha)) = f^{-1}\Big(U_\alpha \times \big(\prod_{\beta \neq \alpha} X_\beta\big)\Big)$, $\pi_\alpha \circ f$ is F-continuous. Since each X_α is functionally regular, in view of Theorem 6.1 each $\pi_\alpha \circ f$ is continuous. This completes the proof of Theorem 6.4. □

Theorem 6.5. *Any product of R_0-spaces is an R_0-space.*

Proof of Theorem 6.5 ([5]) is similar to that of Theorem 6.4 except for obvious modifications and hence omitted.

7. When do weak forms of continuity imply continuity?

Variants of continuity defined in Sections 2 and 3 (see Definitions 2.1 and 3.1) are distinct from each other and are strictly weaker than continuity in general. However, if the range space Y is suitably augmented, then many of them coincide among themselves and/or identical with continuity. For the convenience of reader, we summarize the following observations which are either easily verified or well elaborated in the literature (see the corresponding papers cited in the references).

7.1. *Y is an R_0-space if and only if continuity and R-continuity are identical notions.*

7.2. *Y is a semiregular space if and only if continuity and almost continuity coincide* [36].

7.3. *Y is a regular space if and only if continuous ≡ almost continuous ≡ $\boldsymbol{\theta}$-continuous ≡ quasi $\boldsymbol{\theta}$-continuous ≡ weakly continuous ≡ faintly continuous ≡ R-continuous* ([7], [25], [28]).

7.4. *Y is a D-regular space if and only if continuous ≡ D-continuous ≡ R-continuous* [11].

7.5. *Y is a D-completely regular space if and only if continuous ≡ D-continuous ≡ D^*-continuous ≡ R-continuous* [37].

7.6. *Y is a functionally regular space if and only if continuous ≡ F-continuous ≡ R-continuous*

7.7. *Y is a D_δ-completely regular space if and only if continuous ≡ almost continuous ≡ $\boldsymbol{\theta}$-continuous ≡ quasi $\boldsymbol{\theta}$-continuous ≡ weakly continuous ≡ faintly continuous ≡ D_δ-continuous ≡ D-continuous ≡ R-continuous* [16].

7.8. *Y is a completely regular space if and only if all the classes of functions from (1) to (12) in the diagram (Figure 1) are identical.*

7.9. *Y is a zero dimensional space if and only if all the classes of functions from (1) to (14) in the diagram (Figure 1) coincide.*

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