Thin subsets of balleans

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ABSTRACT. A ballean is a set endowed with some family of balls in such a way that a ballean can be considered as an asymptotic counterpart of a uniform topological space. We characterize the ideal generated by the family of all thin subsets in an ordinal ballean, and apply this characterization to metric spaces and groups.

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Let \( G \) be a group with the identity \( e \). A subset \( A \subseteq G \) is called thin if \( |gA \cap A| < \aleph_0 \) for every \( g \in G, g \neq e \). For thin subsets, its modifications, applications and references see [4]. We denote by \( T_G \) the family of all thin subsets of \( G \). Then the smallest ideal \( T^*_G \) (in the Boolean algebra of all subsets of \( G \)) containing \( T_G \) is the family of all finite unions of thin subsets. Thus, to characterize \( T^*_G \), we need some test which, for given \( A \subseteq G \) and \( m \in \mathbb{N} \), detect whether \( A \) can be represented as a union of \( \leq m \) thin subsets.

Let \((X, d)\) be a metric space. We say that a subset \( A \subseteq X \) is thin if, for every \( r \in \mathbb{R}^+ \), there exists a bounded subset \( Y \subseteq X \) such that \( A \cap B(x, r) = \{x\} \) for every \( x \in A \setminus Y \), where \( B_d(x, r) = \{y \in X : d(x, y) \leq r\} \). As in the group case, to characterize the ideal \( T^*(X, d) \) generated by the family \( T(X, d) \) of all thin subsets of \((X, d)\), we ask for a test recognizing if a subset \( A \subseteq X \) is a union of \( \leq m \) thin subsets.

It is easy to see that a subset \( A \subseteq G \) is thin if and only if, for every finite subset \( F \) of \( G \) containing \( e \), there exists a finite subset \( Y \) of \( G \) such that \( A \cap Fg = \{g\} \) for every \( x \in A \setminus Y \). Following [11], we say that \( Fg \) is a ball of radius \( F \) around \( g \).

From this point of view, the definitions of the thin subsets in groups and metric spaces are very similar syntactically. To formalize this similarity we use the ballean approach from [5]. A ballean is a set endowed with some family of
its subsets which are called the balls. The property of the family of ball are postulated in such a way that the balleans can be considered as the counterparts of the uniform topological spaces (see Section 1 for precise definition).

In Section 1 we define the thin subsets of a ballean and, for every ordinal ballean, characterize the ideal generated by the thin subsets.

The group and metric spaces have the natural ballean structures. In Section 2 we apply the result from Section 1 to justify the following two tests.

A subset $A$ of a metric space $X$ can be partitioned in $\leq m$ thin subsets if and only if, for every $r \in \mathbb{R}^+$, there exists a bounded subset $Y \subseteq X$ such that $|A \cap B(x, r)| \leq m$ for every $x \in A \setminus Y$.

A subset $A$ of a countable group $G$ can be partitioned in $\leq m$ thin subsets if and only if, for every finite subset $F$ of $G$, there exists a finite subset $Y$ of $G$ such that $|A \cap Fx| \leq m$ for every $x \in A \setminus Y$. We do not know whether this test is effective for an uncountable group.

1. Ballean context

A ball structure is a triple $\mathcal{B} = (X, P, B)$, where $X, P$ are not-empty sets and, for every $x \in X$ and $\alpha \in P$, $B(x, \alpha)$ is a subset of $X$ which is called a ball of radius $\alpha$ around $x$. It is supposed that $x \in B(x, \alpha)$ for all $x \in X$ and $\alpha \in P$. The set $X$ is called the support of $\mathcal{B}$, $P$ is called the set of radii.

Given any $x \in X$, $A \subseteq X$, we put

$$B^*(x, \alpha) = \{y \in X : x \in B(y, \alpha)\}, \ B(A, \alpha) = \bigcup_{\alpha \in A} B(a, \alpha).$$

A ball structures $\mathcal{B}$ is called a ballean if

- for any $\alpha, \beta \in P$, there exist $\alpha', \beta' \in P$ such that, for every $x \in X$,

$$B(x, \alpha) \subseteq B^*(x, \alpha'), \ B^*(x, \beta) \subseteq B(x, \beta');$$

- for any $\alpha, \beta \in P$, there exist $\gamma \in P$ such that, for every $x \in X$,

$$B(B(x, \alpha), \beta) \subseteq B(x, \gamma).$$

We note that a ballean can also be defined in terms of entourages of diagonal in $X \times X$. In this case it is called a coarse structures 7.

A ballean $\mathcal{B}$ is called connected if, for any $x, y \in X$, there exists $\alpha \in P$ such that $y \in B(x, \alpha)$. All balleans under considerations are supposed to be connected. Replacing each ball $B(x, \alpha)$ to $B(x, \alpha) \cap B^*(x, \alpha)$, we may suppose that $B^*(x, \alpha) = B(x, \alpha)$ for all $x \in X$, $\alpha \in P$. A subset $Y \subseteq X$ is called bounded if there exist $x \in X$ and $\alpha \in P$ such that $Y \subseteq B(x, \alpha)$.

We use a preordering $\preceq$ on the support $X$ of $\mathcal{B}$ defined by the rule: $\alpha \preceq \beta$ if and only if $B(x, \alpha) \subseteq B(x, \beta)$ for every $x \in X$. A subset $P' \subseteq P$ is called cofinal if, for every $\alpha \in P$, there exists $\alpha' \in P'$ such that $\alpha \preceq \alpha'$. A ballean $\mathcal{B}$ is called ordinal if there exists a cofinal subset $P' \subseteq P$ well ordered by $\preceq$.

Let $\mathcal{B} = (X, P, B)$ be a ballean, $m \in \mathbb{N}$. We say that a subset $A \subseteq X$ is $m$-thin if, for every $\alpha \in P$, there exists a bounded subset $Y_\alpha \subseteq X$ such that $|B(x, \alpha) \cap A| \leq m$ for every $x \in A \setminus Y_\alpha$. A 1-thin subset is called thin. Thus,
A is thin if, for every $\alpha \in P$, there exists a bounded subset $Y_\alpha$ of $X$ such that $B(x, \alpha) \cap A = \{x\}$ for every $x \in A \setminus Y_\alpha$. In the terminology of [5], the thin subsets are called pseudodiscrete. For pseudodiscreteness see also [2], [6].

We use the following notation:

- $T(B)$ is the family of all thin subsets of $X$;
- $T_m(B)$ is the family of all $m$-thin subsets of $X$;
- $\bigcup_m T(B)$ is the family of all unions of $\leq m$ thin subsets of $X$;
- $T^*(B)$ is the ideal generated by $T(B)$.

Clearly, $T^*(B) = \bigcup_{m \in \mathbb{N}}(\bigcup_m T(B))$.

**Lemma 1.1.** For every ballean $B$, we have $\bigcup_m T(B) \subseteq T_m(B)$.

**Proof.** Let $A_1, \ldots, A_n$ be thin subsets of $X$. For every $\alpha \in P$, we pick $\gamma(\alpha) \in P$ such that $B(B(x, \alpha), \alpha) = B(x, \gamma(\alpha))$. For all $\alpha \in P$ and $i \in \{1, \ldots, m\}$, we choose a bounded subset $Y_\alpha(i)$ such that $B(x, \alpha) \cap A_i = \{x\}$ for every $x \in A_i \setminus Y_\alpha(i)$, and put $Y_\alpha = Y_\alpha(1) \cup \ldots \cup Y_\alpha(m)$. We take an arbitrary element $a \in (A_1 \cup \ldots \cup A_m) \setminus Y_\alpha$ and suppose that $|B(a, \alpha) \cap (A_1 \cup \ldots \cup A_m)| > m$. Then there exists $j \in \{1, \ldots, m\}$ such that $|A_j \cap B(a, \alpha)| > 2$. Let $b, c \in A_j \cap B(a, \alpha)$, $b \neq c$. Then $c \in B(b, \gamma(\alpha))$ contradicting the choice of $Y_\alpha(j)$. \qed

The following theorem gives a characterization of $T^*(B)$ in the case of an ordinal ballean $B$.

**Theorem 1.2.** For every ordinal ballean $B$ and $m \in \mathbb{N}$, we have $T_m(B) = \bigcup_m T(B)$.

**Proof.** In view of Lemma 1.1, it suffices to show that $T_m(B) \subseteq \bigcup_m T(B)$. Let $A \in T_m(B)$. We may suppose that $P$ is well ordered by $\leq$. We construct inductively a family $\{Y_\alpha : \alpha \in P\}$ of bounded subsets of $X$ such that $|B(y, \alpha) \cap A| = 1$ for every $x \in A \setminus Y_\alpha$ and $y \subseteq Y_\alpha \subseteq Y_{\beta}$ for all $\alpha \leq \beta$. Then we consider a graph $\Gamma$ with the set of vertices $A$ and the set of edges $E$ defined as follows: $(x, y) \in E$ if and only if $x \neq y$ and there exists $\alpha \in P$ such that $x, y \in A \setminus Y_\alpha$ and $y \in B(x, \alpha)$. We show that $deg(x) \leq m - 1$ for every $x \in A$, where $deg(x) = |\{y \in A : (x, y) \in E\}|$. We suppose the contrary and choose $x \in A$ and distinct vertices $y_1, \ldots, y_m$ such that $(x, y_i) \in E$ for every $i \in \{1, \ldots, m\}$. By the definition of $E$, for every $i \in \{1, \ldots, m\}$, there exists $\alpha_i \in P$ and a bounded subset $Y_{\alpha_i}$ of $X$ such that $y_i \in B(x, \alpha_i)$ and $y_i \in A \setminus Y_{\alpha_i}$. Let $\alpha = max\{\alpha_1, \ldots, \alpha_m\}$ and $\alpha = \alpha_j$. Then $y_1, \ldots, y_m \in B(x, \alpha_j)$ and $y_1, \ldots, y_m \in A \setminus Y_{\alpha_j}$ because $Y_{\alpha_j} \subseteq Y_{\alpha_j}$ for all $i \in \{1, \ldots, m\}$, so we get a contradiction with the choice of $\alpha_j$ because $|B(x, \alpha_j) \cap A| \leq m$.

By [3] Corollary 12.2, the chromatic number of $\Gamma$ does not exceed $m$. Hence $A$ can be partitioned $A = A_1 \cup \ldots \cup A_k$, $k \leq m$ so that, for every $i \in \{1, \ldots, k\}$ and $x, y \in A_i$, we have $(x, y) \notin E$.

We show that each subset $A_i$ is thin. For every $\alpha \in P$, we put $Z_\alpha = B(Y_\alpha, \alpha)$. Suppose that there exists $x \in A_1 \setminus Z_\alpha$ such that $|B(x, \alpha) \cap A_1| > 1$. Let $y \in B(x, \alpha) \cap A_i$ and $y \neq x$. Since $x \notin Z_\alpha$ then $y \notin Y_\alpha$. Thus $x, y \in A \setminus Y_\alpha$ and $y \in B(x, \alpha)$, so $(x, y) \in E$ contradicting the choice of $A_i$. \qed
2. Applications

**Theorem 2.1.** Let \((X, d)\) be a metric space, \(m \in \mathbb{N}\). A subset \(A \subseteq X\) can be partitioned in \(\leq m\) thin subsets if and only if, for every \(r \in \mathbb{R}^+\), there exists a bounded subset \(Y\) of \(X\) such that \(|B(x, r) \cap A| \leq m\) for every \(x \in A \setminus Y\).

**Proof.** We consider \((X, d)\) as the ballean \(B(X, d) = (X, \mathbb{R}^+, B_d)\). Clearly, \(B(X, d)\) is ordinal so we can apply Theorem 1.2. □

Let \(G\) be a group, \(\kappa\) be an infinite cardinal, \(F_\kappa(G) = \{F \subseteq G : |F| < \kappa, e \in F\}\). We consider the ballean

\[B_\kappa(G) = (G, F_\kappa(G), B),\]

where \(B(g, F) = Fg\) for all \(g \in G, F \in F_\kappa(G)\). If \(\kappa > |G|, B_\kappa(G)\) is bounded. For \(\kappa = |G|, B_\kappa(G)\) is ordinal. Indeed, let \(g_0 = e, \{g_\alpha : \alpha < \kappa\}\) be a numeration of \(G, F_\alpha = \{g_\beta : \beta \leq \alpha\}\). Then the well ordered by \(\subseteq\) family \(F = \{F_\alpha : \alpha < \kappa\}\) is cofinal in \(F\).

We say that a subset \(A \subseteq G\) is \(\kappa\)-thin if \(|gA \cap A| < \kappa\) for every \(g \in G, g \neq e\). In the case \(\kappa = \aleph_0\), we get the thin subsets defined in the very beginning of the paper.

**Lemma 2.2.** Let \(A\) be a subset of a group \(G\). If \(A\) is thin in the ballean \(B_\kappa(G)\) then \(A\) is \(\kappa\)-thin. If \(A\) is \(\kappa\)-thin and \(\kappa\) is regular then \(A\) is thin in the ballean \(B_\kappa(G)\).

**Proof.** Let \(A\) be thin in \(B_\kappa(G)\). For every \(g \in G, g \neq e\), we put \(F_g = \{e, g\}\) and choose a bounded subset \(Y_g\) in \(B_\kappa(G)\) such that \(B(x, F) \cap A = \{x\}\) for every \(x \in A \setminus Y_g\). Then \(gx \notin A\) for every \(x \in A \setminus Y_g\) so \(gA \cap A \subseteq Y_g\). Since \(Y_g\) is bounded in \(B_\kappa(G)\) then \(|Y_g| < \kappa\) and \(A\) is \(\kappa\)-thin.

Let \(A\) be \(\kappa\)-thin, \(F \in F_\kappa(G)\). We put \(Y = \bigcup\{gA \cap A : g \in F \setminus \{e\}\}\). Since \(|gA \cap A| < \kappa, |F| < \kappa\) and \(\kappa\) is regular, \(|Y| < \kappa\) so \(Y\) is bounded in \(B_\kappa(G)\). For every \(x \in A \setminus Y\), we have \(Fx \cap A = \{x\}\) hence \(A\) is thin in \(B_\kappa(G)\). □

**Theorem 2.3.** Let \(G\) be a group of regular cardinality \(\kappa, m \in \mathbb{N}\). A subset \(A \subseteq G\) can be partitioned in \(\leq m\) \(\kappa\)-thin subsets if and only if, for every \(F \subseteq G, |F| < \kappa\), there exists a subset \(Y \subseteq G\) such that \(|Y| < \kappa\) and \(|Fx \cap A| \leq m\) for every \(x \in A \setminus Y\).

**Proof.** Since the ballean \(B_\kappa(G)\) is ordinal, in view of Lemma 2.2, we can apply Theorem 1.2. □

**Remark 2.4.** A subset \(A\) of a group \(G\) is called almost thin if the set \(\Delta(A) = \{g \in G : gA \cap A\text{ is infinite}\}\) is finite. By Theorem 3.1, every almost thin subset of a group \(G\) can be partitioned in \(3^{|\Delta(A)|-1}\) thin subsets, but the union of two thin subsets needs not to be almost thin [4, Theorem 3.2].
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References


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