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The structure of the poset of regular topologies on a set

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Abstract

We study the subposet $\Sigma_3(X)$ of the lattice $\mathcal{L}_1(X)$ of all T_1-topologies on a set X, being the collections of all T_3 topologies on X, with a view to deciding which elements of this partially ordered set have and which do not have immediate predecessors. We show that each regular topology which is not R-closed does have such a predecessor and as a corollary we obtain a result of Costantini that each non-compact Tychonoff space has an immediate predecessor in Σ_3. We also consider the problem of when an R-closed topology is maximal R-closed.

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1. Introduction

In a previous paper [3], we studied the problem of when a jump can occur in the order of the lattice $\mathcal{L}_1(X)$; that is to say, when there exist T_1-topologies τ and τ^+ on a set X such that whenever μ is a topology on X such that $\tau \subseteq \mu \subseteq \tau^+$ then $\mu = \tau$ or $\mu = \tau^+$. The existence of jumps in $\mathcal{L}_1(X)$ and in the subposet of Hausdorff topologies, has been studied in [5], [2], [10] and [16]; in the last two articles an immediate successor τ^+ was said to be *a cover of* (or simply *to cover*) τ. In the above cited paper [3], when a topology τ has a cover τ^+ we have called τ a *lower topology* and τ^+ an *upper topology* and we continue to use this terminology here.

In the present work we study the structure of the subposet $\Sigma_3(X)$ of all T_3-topologies of the lattice $\mathcal{L}_1(X)$, on a set X with a view to deciding which elements of this partially ordered sets have and which do not have covers.

In [1] it was shown that a T_3-topology on X which is not feebly compact is an upper topology in $\Sigma_3(X)$ and in [6], Costantini showed that every non-compact Tychonoff topology on X is upper in $\Sigma_3(X)$. In Section 2 of this paper we generalize both these results by showing that every T_3-topology which is not R-closed is upper in $\Sigma_3(X)$. (A T_3-space is *R-closed* if it is closed in every embedding in a T_3-space.) In Section 3 we consider the problem of the existence of spaces which are maximal with respect to being R-closed and in Section 4 we study lower topologies in Σ_3. In the final section we pose a number of open problems.

A set X with a topology ξ will be denoted by (X, ξ) and if $p \in X$, then $\xi(p, X)$ denotes the collection of all open sets in X which contain p. The closure (respectively, interior) of a set A in a topological space (X, τ) will be denoted by $\mathrm{cl}_\tau(A)$ (respectively, $\mathrm{int}_\tau(A)$) or simply by $\mathrm{cl}(A)$ (respectively $\mathrm{int}(A)$) when no confusion is possible. All undefined terms can be found in [7] or [13] and all spaces in this article are (at least) T_3. A comprehensive survey of results on R-closed spaces and many open questions can be found in [8]. We make the following formal definitions.

Definition 1.1. Say that two (distinct) T_3-topologies τ_1 and τ_2 on a set X are *adjacent in* $\Sigma_3(X)$ if whenever $\sigma \in \Sigma_3(X)$ and $\tau_1 \subseteq \sigma \subseteq \tau_2$, then either $\sigma = \tau_1$ or $\sigma = \tau_2$. We say that τ_1 is a *lower topology in* $\Sigma_3(X)$, τ_2 is an *upper topology in* Σ_3 and τ_2 is an *immediate successor* of τ_1. For a topology τ, τ^+ will always denote an immediate successor of τ. A T_3-topology on X is *R-minimal* if there is no weaker T_3-topology on X; it is well known that an R-minimal topology is R-closed. Clearly an R-minimal topology is not upper in $\Sigma_3(X)$. In the sequel, whenever the space X is understood, we will write Σ_3 instead of $\Sigma_3(X)$.

In [11], it was shown that the structure of basic intervals in Σ_3 is essentially different from those of the poset Σ_t of Tychonoff spaces in that not every finite interval is isomorphic to the power set of a finite ordinal. The following result is Lemma 22 of [11].

Lemma 1.2. *If σ is an immediate successor of τ in Σ_3, then τ and σ differ at precisely one point.*

An open filter (that is, a filter with a base of open sets) $\mathcal{F}$ is a *regular filter* if for each $U \in \mathcal{F}$ there is $V \in \mathcal{F}$ such that $\mathrm{cl}(V) \subseteq U$. A simple application of Zorn's Lemma shows that every regular filter can be embedded in a maximal regular filter and furthermore, in a regular space, if a maximal regular filter has an accumulation point, then it must converge to that point.

By Theorem 4.14 of [4], a T_3-space is R-closed if and only if every regular filter has an accumulation point or equivalently, if and only if every maximal regular filter converges.

2. Upper topologies in Σ_3

The next result generalizes Theorem 2.14 of [1].

Theorem 2.1. *Each T_3-topology which is not R-closed is upper in Σ_3.*

Proof. Suppose that (X,σ) is a T_3-space which is not R-closed. Then there is some maximal regular filter $\mathcal{F}$ in (X,σ) which is not fixed. Pick $p \in X$ and define a new topology τ on X as follows:

$$\tau = \{U \in \sigma : p \notin U\} \cup \{U \in \sigma : p \in U \in \mathcal{F}\}.$$

The topologies τ and σ differ only at the point p and hence for each $A \subseteq X$, $\mathrm{cl}_\tau(A) \subseteq \mathrm{cl}_\sigma(A) \cup \{p\}$.

We first show that (X,τ) is a T_3-space; suppose that $C \subseteq X$ is τ-closed and $q \notin C$. There are three cases to consider.
1) If $p \notin C \cup \{q\}$, then there are σ-open sets U, V separating C and q in $X \setminus \{p\}$ and U, V are τ-open.
2) If $p \in C$, then C is σ-closed and hence there are disjoint σ-open sets U and V such that $C \subseteq U$ and $q \in V$. Furthermore, since $\mathcal{F}$ is a free regular filter, there is $W \in \mathcal{F}$ such that $q \notin \mathrm{cl}_\sigma(W)$ and hence $q \notin \mathrm{cl}_\tau(W) = \mathrm{cl}_\sigma(W) \cup \{p\}$. It is now clear that $U \cup W$ and $U \setminus \mathrm{cl}_\tau(W)$ are disjoint τ-open sets containing C and q respectively.
3) If $p = q$, then since C is τ-closed and $p \notin C$, it follows that there is some element $W \in \mathcal{F}$ such that $W \cap C = \varnothing$. Furthermore, since C is σ-closed, there are disjoint sets $U, V \in \sigma$ such that $C \subseteq U$ and $p \in V$. Since $\mathcal{F}$ is a regular filter, there is some $T \in \mathcal{F}$ such that $\mathrm{cl}_\sigma(T) \subseteq W$. Since $\mathrm{cl}_\tau(T) = \mathrm{cl}_\sigma(T) \cup \{p\}$, it is now clear that $U \setminus \mathrm{cl}_\tau(T)$ and $V \cup T$ are disjoint τ-open sets containing C and p respectively.

We claim that τ is the immediate predecessor of σ in Σ_3. To see this, suppose that μ is a T_3-topology on X such that $\tau \subsetneq \mu \subsetneq \sigma$; note that μ differs from σ and τ only at the point p. If there is some μ-neighbourhood U of p which misses some element $F \in \mathcal{F}$, then if W is a σ-open neighbourhood of p, it follows that $W \cup F$ is a τ-open, hence μ-open neighbourhood of p. But then $(W \cup F) \cap U = W \cap U \subseteq W$ is a μ-open neighbourhood of p, implying that $\mu = \sigma$. Hence every μ-neighbourhood of p must meet every element of $\mathcal{F}$; we claim that this implies that $\mu = \tau$. To prove our claim, let $\mathcal{V}_p$ be the filter of μ-open neighbourhoods of p and let $\mathcal{G}$ be the open filter generated by $\{F \cap V : F \in \mathcal{F} \text{ and } V \in \mathcal{V}_p\}$. We will show that $\mathcal{G}$ is a regular filter in (X,σ), thus contradicting the maximality of $\mathcal{F}$. However, if $F \in \mathcal{F}$ and $V \in \mathcal{V}_p$, then there is $W \in \mathcal{V}_p$ and $H \in \mathcal{F}$ such that $V \supseteq \mathrm{cl}_\mu(W) \supseteq \mathrm{cl}_\sigma(W)$ and $\mathrm{cl}_\sigma(H) \subseteq F$. Hence $W \cap H \in \mathcal{G}$ and $\mathrm{cl}_\sigma(W \cap H) \subseteq F \cap V$. □

In [6], the concept of a *strongly upper topology* was defined. (A topology τ is strongly upper if whenever $\mu \subsetneq \tau$, there is an immediate predecessor τ^- of τ such that $\mu \subseteq \tau^- \subsetneq \tau$.) A simple modification of the above proof shows that every regular topology which is not R-closed is in fact strongly upper.

Clearly every R-closed Tychonoff space is compact and hence the following result of [6] is an immediate corollary.

Corollary 2.2. *Every Tychonoff topology which is not compact is (strongly) upper in Σ_3.*

Every completely Hausdorff topology possesses a weaker Tychonoff topology (the weak topology induced by the continuous real-valued functions). Thus every completely Hausdorff R-minimal topology is compact. The following question then arises:

Question 2.3. *Is every completely Hausdorff T_3-topology which is not compact an upper topology in Σ_3?*

Question 2.4. *Is every regular topology which has a compact Hausdorff subtopology an upper topology in Σ_3?*

3. Maximal R-closed topologies

Recall that a space is *submaximal* if every dense set is open (we do not assume that a submaximal space must be dense-in-itself). It follows from 7M of [13] that each H-closed topology is contained in a maximal H-closed topology and that a space is maximal H-closed if and only if it is H-closed and submaximal. However, as we show below, the class of submaximal R-closed spaces is much more restricted. Recall that a space is *feebly compact* if every locally finite family of open sets is finite.

Theorem 3.1. *Each submaximal regular, feebly compact topology has an isolated point.*

Proof. Suppose that (X,τ) is feebly compact, submaximal and has no isolated points. Fix $p \in X$ and let $\mathcal{C}$ be a maximal cellular family of open sets in X so that for each $C \in \mathcal{C}$, we have $p \notin \mathrm{cl}_\tau(C)$. The subset $\bigcup \mathcal{C}$ is dense in X and hence $F = X \setminus \bigcup \mathcal{C}$ is closed and discrete. Since $p \in F$, there are disjoint open sets U and V so that $p \in U$ and $F \setminus \{p\} \subseteq V$. Let $\mathcal{S} = \{U \cap C : C \in \mathcal{C}$ and $U \cap C \neq \varnothing\}$; since p is not isolated, it follows that $\mathcal{S}$ is an infinite cellular family of open sets. Since X is feebly compact, this family must have an accumulation point in F, and hence its only accumulation point is p. For each $U \cap C \in \mathcal{S}$, pick $x_C \in U \cap C$; since X has no isolated points, the set $\{x_C : U \cap C \in \mathcal{S}\} \cup \{p\}$ is closed and discrete and hence there are disjoint opens sets U' and V' such that $p \in U'$ and $\{x_C : U \cap C \in \mathcal{S}\} \subseteq V'$. It follows immediately that the infinite family of non-empty open sets $\{C \cap U \cap V' : U \cap C \in \mathcal{S}\}$ has no accumulation point in X, contradicting the fact that X is feebly compact. □

The following theorem is a result of Scarborough and Stone [14]. For completeness we include the simple proof.

Theorem 3.2. *An R-closed topology is feebly compact.*

Proof. Suppose to the contrary that $\mathcal{U} = \{U_n : n \in \omega\}$ is an infinite discrete family of open subsets of (X, τ). For each $n \in \omega$, pick $x_n \in U_n$. It is then straightforward to check that the family

$$\mathcal{B} = \{U \in \tau : U \supseteq \{x_n : n \geq k\} \text{ for some } k \in \omega\}$$

is a regular filter base on X with no accumulation point, contradicting the fact that (X, τ) is R-closed. □

Corollary 3.3. *Each submaximal R-closed space has an isolated point.*

Lemma 3.4. *An R-closed space which is scattered and of dispersion order 2 is compact.*

Proof. Suppose $X = X_0 \cup X_1$ where X_0 is the set of isolated points and X_1 is the set of accumulation points of X. For each $p \in X_1$, there is a closed neighbourhood U of p such that $U \subseteq X_0 \cup \{p\}$. It is clear that U is clopen and so X is 0-dimensional and hence Tychonoff. Thus X is compact. □

Stephenson's examples (see [15] and [9]) show that the previous result is false for R-closed scattered spaces of dispersion order 3.

Since a subspace of a submaximal space is submaximal, the closure of the set of isolated points of an R-closed submaximal space is scattered of dispersion order 2. Thus:

Corollary 3.5. *Each submaximal R-closed space is a compact scattered space of dispersion order 2.*

Proof. Suppose that (X, τ) is an R-closed submaximal space and let X_0 denote the set of isolated points of X; by Corollary 3.3, $X_0 \neq \varnothing$. Let $C = \mathrm{cl}(X \setminus \mathrm{cl}(X_0))$; If $C = \varnothing$ then we are done, so suppose to the contrary. Then C is a submaximal space without isolated points and so again by Corollary 3.3, $(C, \tau|C)$ is not feebly compact. Thus there is an infinite locally finite family $\mathcal{F}$ of open sets in $(C, \tau|C)$. But then, $\{F \cap (X \setminus \mathrm{cl}(X_0)) : F \in \mathcal{F}\}$ is an infinite locally finite family of open sets in X, implying that X is not feebly compact, which is a contradiction. □

Theorem 3.6. *A submaximal R-closed space is maximal R-closed.*

Proof. Suppose that (X, τ) is a submaximal R-closed space. By the previous corollary, X is compact scattered of dispersion order 2; let X_0 denote the set of isolated points of X and $X_1 = X \setminus X_0$. Suppose that $\sigma \supsetneq \tau$ is a regular topology on X which differs from τ at a point p. Then there is some σ-open neighbourhood U of p which is not τ-open and hence does not contain any τ-neighbourhood of p; there is also a compact τ-neighbourhood V of p such that $V \subseteq X_0 \cup \{p\}$. It is then clear that $V \setminus U$ is an infinite σ-closed subset of X_0, implying that (X, σ) is not feebly compact. □

Lemma 3.7. *A feebly compact regular space of countable pseudocharacter is first countable.*

Proof. Suppose that (X,τ) is feebly compact regular space and $\psi(X,p)=\omega$. There is a family $\mathcal{B}=\{B_n : n\in\omega\}$ of open sets such that $\bigcap\{B_n : n\in\omega\}=\{p\}$ and for each $n\in\omega$, $\mathrm{cl}(B_{n+1})\subseteq B_n$. If $\mathcal{B}$ is not a local base at p, then there is some open neighbourhood U of p such that for each $n\in\omega$, $B_n\nsubseteq U$. It is straightforward to check that the family of open sets $\{B_n\setminus(\mathrm{cl}_\tau(B_{n+1}\cup U)) : n\in\omega\}$ is an infinite locally finite family of open sets, contradicting the fact that X is feebly compact. □

The next theorem should be compared with Theorem 2.20 of [12].

Theorem 3.8. *A regular feebly compact first countable topology is maximal among regular feebly compact topologies.*

Proof. Suppose that (X,τ) is a regular feebly compact first countable space and $\sigma\supsetneq\tau$ is a regular topology on X; we will show that (X,σ) is not feebly compact.

To this end, suppose that $U\in\sigma\setminus\tau$; then $X\setminus U$ is σ-closed but not τ-closed and so since (X,τ) is first countable, there is some sequence $\{p_n\}$ in $X\setminus U$ convergent (in (X,τ)) to $p\in U$. By Lemma 4.1 of [2], there is a family of disjoint τ-open sets $\{U_n : n\in\omega\}$ whose only accumulation point (in (X,τ)) is p and such that $p_n\in U_n$ for each $n\in\omega$. Now by regularity of (X,σ) there is $W\in\sigma$ such that $p\in W\subseteq \mathrm{cl}_\sigma(W)\subseteq U$; then, the collection of sets $\mathcal{U}=\{U_n\setminus\mathrm{cl}_\sigma(W) : n\in\omega\}$ is a locally finite collection of open subsets of (X,σ) and so if an infinite number of elements of $\mathcal{U}$ are non-empty, then (X,σ) is not feebly compact. However, if for some $n_0\in\omega$, $U_n\setminus\mathrm{cl}_\sigma(W)=\varnothing$ for all $n\ge n_0$, then $p_n\in U_n\subseteq\mathrm{cl}_\sigma(W)$ for all $n\ge n_0$ contradicting the fact that $p_n\in X\setminus U\subseteq X\setminus\mathrm{cl}_\sigma(W)$. □

The following result is now an immediate consequence of Theorems 3.2 and 3.8 and Lemma 3.7.

Corollary 3.9. *An R-closed space of countable pseudocharacter is maximal R-closed.*

Remark 3.10. Note that we have proved something a little stronger: If (X,τ) is R-closed and $\sigma\supseteq\tau$ differs from τ at a point of countable pseudocharacter, then (X,σ) is not R-closed.

Corollary 3.11. *A regular space with a strictly weaker R-closed first countable topology is upper in Σ_3.*

Corollary 3.12. *A first countable compact Hausdorff space is maximal R-closed.*

Question 3.13. *Is a Fréchet compact Hausdorff space maximal R-closed?*

4. Lower topologies

A point p is a *maximal regular point* of a regular space (X,τ) if the trace of the regular filter $\mathcal{V}_p^\tau$ generated by $\tau(p,X)$ on $X\setminus\{p\}$ is a maximal regular filter.

Lemma 4.1. *A point p in a regular topological space (X,τ) is a maximal regular point of X if and only if whenever $\tau \subsetneq \sigma$ is a regular topology on X such that $\sigma|(X\setminus\{p\}) = \tau|(X\setminus\{p\})$ then p is an isolated point of (X,σ).*

Proof. For the sufficiency suppose that the regular filter $\mathcal{V}_p^\tau$ generated by $\tau(p,X)$ when restricted to $X\setminus\{p\}$ is not maximal. Then there is some regular filter $\mathcal{F} \supsetneq \mathcal{V}_p^\tau|(X\setminus\{p\})$. Define σ to be that topology on X generated by the subbase

$$\tau \cup \{V\cup\{p\} : V\in\mathcal{F}\};$$

it is straightforward to show that σ is a regular topology on X strictly finer that τ in which p is not an isolated point.

To show the necessity, suppose that p is a maximal regular point of (X,τ). Then if $\sigma \supsetneq \tau$ and $\sigma|(X\setminus\{p\}) = \tau|(X\setminus\{p\})$, it follows that the trace of the neighbourhood filter $\mathcal{V}_p^\sigma$ at p on $X\setminus\{p\}$ is strictly larger than the trace of the neighbourhood filter $\mathcal{V}_p^\tau$ at p on $X\setminus\{p\}$ and since $\sigma|(X\setminus\{p\}) = \tau|(X\setminus\{p\})$, $\mathcal{V}_p^\sigma|(X\setminus\{p\})$ is a τ-open collection strictly larger than the maximal regular filter $\mathcal{V}_p^\tau|(X\setminus\{p\})$. It follows that p is an isolated point of (X,σ). □

It was essentially shown in Theorem 2.13 of [1] that a point of first countability in a space is not a maximal regular point.

Corollary 4.2. *If (X,τ) has a maximal regular point then τ is a lower topology in Σ_3.*

In [3] we characterized lower topologies in the poset of Hausdorff spaces as those having a closed subspace with a maximal point. Example 4.10 below shows that having a closed subspace with a maximal regular point does not guarantee that a topology is lower in Σ_3. However, we have the following result:

Lemma 4.3. *If $\sigma\in\Sigma_3(X)$ is a simple extension of $\tau\in\Sigma_3(X)$ which differs from τ at precisely one point $p\in X$, then σ is upper and each lower topology μ corresponding to σ has a closed subspace with a maximal regular point.*

Proof. It was shown in [6] that if a T_3-topology σ is a simple extension of a T_3-topology τ that differs from τ at precisely one point p, then σ is upper in $\Sigma_3(X)$ and is generated by the subbase $\tau\cup\{U\cup\{p\}\}$ for some $U\in\tau$. Clearly $\mu\cup\{U\cup\{p\}\}$ is also a subbase for σ and hence p is an isolated point of $A=(X\setminus U)\cup\{p\}$ in the topology σ but not in μ. Thus p is a maximal regular point of $(A,\mu|A)$. □

Remark 4.4. If τ is a lower topology in Σ_3 and τ and τ^+ differ at $p\in X$ then there is some $U_0\in\tau$ such that $U_0\cup\{p\}\in\tau^+\setminus\tau$. Then since τ^+ is regular, for each $n \geq 1$ there is $U_n \in \tau$ such that $U_n \cup \{p\} \in \tau^+ \setminus \tau$ and $U_n \cup \{p\} \subseteq \mathrm{cl}_{\tau^+}(U_n) \cup \{p\} \subseteq U_{n-1} \cup \{p\}$. It is clear that τ^+ is generated by the subbase $\tau \cup \{U_n \cup \{p\} : n \in \omega\}$ and hence the character of p in (X, τ^+) is no greater than its character in (X, τ).

A family $\mathcal{S} = \{S_n : n \in \omega\}$ is said to be *strongly decreasing at p* if for each $n \in \omega$, $\mathrm{cl}(S_{n+1}) \cup \{p\} \subseteq S_n \cup \{p\}$. We now formulate the above Remark as a lemma:

Lemma 4.5. *Let (X, τ) be a T_3-space; if τ has an immediate successor $\tau^+ \in \Sigma_3$, then there is $p \in X$ and a family $\mathcal{U} = \{U_n : n \in \omega\} \subseteq \tau$ which is strongly decreasing at p, such that for each $n \in \omega$, $U_n \cup \{p\} \notin \tau$ and τ^+ is generated by the subbase $\tau \cup \{U_n \cup \{p\} : n \in \omega\}$.*

This result allows us to characterize (rather abstractly it must be said) lower topologies in Σ_3 in the next theorem. In order to simplify the notation somewhat, when $\mathcal{W} = \{W_n : n \in \omega\} \subseteq \tau$ and $\mathcal{V} = \{V_n : n \in \omega\} \in \tau$ are strongly decreasing families at (a fixed) $p \in X$, $\tau_{\mathcal{W}}$ will denote the topology generated by $\tau \cup \{W_n \cup \{p\} : n \in \omega\}$ and $\mathcal{W} \cap \mathcal{V}$ will denote the family $\{W_n \cap V_n : n \in \omega\}$ which is also strongly decreasing at p.

Theorem 4.6. *A topology τ on X is lower in Σ_3 if and only if there is $p \in X$ and a strongly decreasing family $\mathcal{U} = \{U_n : n \in \omega\} \subseteq \tau$ at p such that whenever $\mathcal{V} = \{V_n : n \in \omega\} \subseteq \tau$ is strongly decreasing at p and $\tau_{\mathcal{U}} = \tau_{\mathcal{U} \cap \mathcal{V}}$, then either $\tau_{\mathcal{V}} = \tau_{\mathcal{U}}$ or $\tau_{\mathcal{V}} = \tau$.*

Proof. Suppose that τ is not lower and fix $p \in X$; if $\mathcal{U} = \{U_n : n \in \omega\} \subseteq \tau$ is strongly decreasing at p, then there is $\sigma \in \Sigma_3$ such that $\tau \subsetneq \sigma \subsetneq \tau_{\mathcal{U}}$. We may then choose a strongly decreasing family (at p) $\mathcal{V} = \{V_n : n \in \omega\} \subseteq \sigma$, such that for each $n \in \omega$, $V_n \cup \{p\} \in \sigma \setminus \tau$ and so $\tau \subsetneq \tau_{\mathcal{V}} \subsetneq \tau_{\mathcal{U}}$. However, since for each $n \in \omega$, $V_n \cup \{p\} \in \tau_{\mathcal{U}}$, we have that $(U_n \cap V_n) \cup \{p\} \in \tau_{\mathcal{U}}$ which implies that $\tau_{\mathcal{U}} = \tau_{\mathcal{U} \cap \mathcal{V}}$, giving a contradiction.

Conversely, suppose that τ is lower in Σ_3; by Lemma 4.5, there is $p \in X$ and a strongly decreasing family $\mathcal{U}$ at p such that $\tau^+ = \tau_{\mathcal{U}}$. Then, if $\mathcal{V} = \{V_n : n \in \omega\} \subseteq \tau$ is a strongly decreasing family at p such that $\tau_{\mathcal{U}} = \tau_{\mathcal{U} \cap \mathcal{V}}$ it follows that for each $n \in \omega$, $V_n \cup \{p\} \in \tau_{\mathcal{U}}$ and so $\tau_{\mathcal{V}} \subseteq \tau_{\mathcal{U} \cap \mathcal{V}} = \tau_{\mathcal{U}}$. □

Theorem 4.7. *A compact LOTS is maximal R-closed.*

Proof. Suppose that $(X, \tau, <)$ is a compact LOTS and $\sigma \supsetneq \tau$. Then there is some $U \in \sigma \setminus \tau$ and $p \in U$ such that U is not a τ-neighbourhood of p, and hence $L_p \setminus U$ is cofinal in $L_p \setminus \{p\}$ or $R_p \setminus U$ is cofinal in $R_p \setminus \{p\}$, where $L_p = \{x \in X : x \leq p\}$ and $R_p = \{x \in X : x \geq p\}$. It is easy to see that (X, τ) is maximal R-closed if and only if both of the compact subspaces (L_p, τ) and (R_p, τ) are maximal R-closed. Thus, if p is a point of first countability of (X, τ), then it is also of first countability in both (L_p, τ) and (R_p, τ) and so the result is an immediate consequence of Remark 3.10.

Suppose then that $\chi(p, X) > \omega$, say $\chi(p, L_p) = \kappa > \omega$ (where κ is a regular uncountable cardinal); in the sequel we consider only the subspace L_p. Let

$V \in \sigma$ be such that $p \in V \subseteq \mathrm{cl}_\sigma(V) \subseteq U$, then clearly, either, $V = \{p\}$ or $V \setminus \{p\}$ is a cofinal σ-closed subset of $L_p \setminus \{p\}$. If the former occurs, then clearly $L_p \setminus \{p\}$ is open and closed in (L_p, σ) which then cannot be R-closed.

If $V \setminus \{p\}$ is cofinal in $L_p \setminus \{p\}$ then, inductively we may construct interpolating sequences $\{v_n : n \in \omega\} \subseteq V \setminus \{p\}$ and $\{w_n : n \in \omega\} \subseteq L_p \setminus U$ such that $w_n < v_n < w_{n+1}$ for all $n \in \omega$. Since $(X, <)$ is complete, $q = \sup\{v_n : n \in \omega\} = \sup\{w_n : n \in \omega\}$ exists. Now for each $n \in \omega$, let $O_n = V \cap (w_n, w_{n+1})$. The sets $\{O_n : n \in \omega\}$ are σ-open and their only possible accumulation point in (X, σ) is q. There are now two possibilities:

1) If $q \in \mathrm{cl}_\sigma(\{w_n : n \in \omega\})$, then $q \in L_p \setminus U$ and so q is not an accumulation point in (X, σ) of the family $\{O_n : n \in \omega\}$, showing that (X, σ) is not feebly compact and hence not R-closed.

2) If on the other hand, $q \notin \mathrm{cl}_\sigma(\{w_n : n \in \omega\})$, then $\{w_n : n \in \omega\})$ is closed and discrete in (X, σ). Since σ is regular, we may construct a discrete family of σ-open sets $\{W_n : n \in \omega\}$ such that $w_n \in W_n$, again showing that (X, σ) is not feebly compact. □

The same proof essentially shows that:

Theorem 4.8. *If $(X, \tau, <)$ is a LOTS and $\chi(p, L_p) > \omega$, then p is a maximal regular point of L_p.*

Corollary 4.9. *A compact LOTS is lower in Σ_3 if and only if it is not first countable.*

Proof. The sufficiency follows from Theorem 4.8 and Corollary 4.2. The necessity was proved in Theorem 2.13 of [1]. □

Compactness is essential in the previous theorem. It is straightforward to show that the one-point Lindelofication of a discrete space of cardinality ω_1 is a LOTS but is neither first countable nor lower in Σ_3.

From Theorem 4.8 we see that if κ is an uncountable regular cardinal, then κ is a maximal regular point of $\kappa + 1$ (with the order topology).

Example 4.10. *Let κ denote the first ordinal of cardinality $\mathfrak{c}^+$ and let X denote the set $(\kappa + 1) \times [0, 1]$, τ the product topology on X and σ the topology generated by $\tau \cup \{(\kappa, 1)\}$. We will show that $\sigma = \tau^+$. To this end, suppose that μ is a regular topology such that $\tau \subsetneq \mu \subseteq \sigma$; clearly μ differs from τ and σ only at the point $(\kappa, 1)$ and hence there is some open μ-neighbourhood V which is not a τ-neighbourhood of $(\kappa, 1)$ and some μ-neighbourhood U of $(\kappa, 1)$ such that $cl_\mu(U) \subseteq V$. Since $\kappa > \mathfrak{c}$, there are a number of possibilities:*

1) There is an infinite set $J = \{r_n : n \in \omega\} \subseteq [0, 1)$ with $1 \in \mathrm{cl}(J)$ and for each $n \in \omega$ a set $S_n \subseteq \kappa$ such that either,

a) S_n is cofinal in κ or

b) $\kappa \in S_n$

and $\bigcup\{S_n \times \{r_n\} : n \in J\} \cap V = \varnothing$. Or,

2) There is a cofinal set $S_\omega \subset \kappa$ such that $(S_\omega \times \{1\}) \cap V = \varnothing$; furthermore, since $V \setminus \{(\kappa, 1)\}$ is τ-open, we may assume that S_ω is τ-closed in κ.

If 1a) occurs, then $\{\kappa\} \times J \subseteq X \setminus V \subseteq X \setminus \mathrm{cl}_\mu(U)$*; and if 1b) occurs, then since* $V \setminus \{(\kappa, 1)\}$ *is* τ*-open, it follows that* $\{\kappa\} \times J \subseteq X \setminus V \subseteq X \setminus \mathrm{cl}_\mu(U)$*.*

Thus in either case 1a) or 1b), there is an infinite subset $J \subseteq [0, 1)$ *with* $1 \in \mathrm{cl}(J)$ *such that* $\{\kappa\} \times J \subseteq X \setminus V \subseteq X \setminus \mathrm{cl}_\mu(U)$*. It then follows that for each* $r_n \in J$ *there is* $\alpha_n \in \kappa$ *such that* $\bigcup\{(\alpha_n, \kappa] \times \{r_n\} : n \in J\} \subseteq X \setminus \mathrm{cl}_\mu(U)$*. Letting* $\alpha = \sup\{\alpha_n : n \in J\} \in \kappa$ *we have that* $(\alpha, \kappa] \times J \subseteq X \setminus \mathrm{cl}_\mu(U)$ *and so* $(\alpha, \kappa) \times \{1\} \subseteq X \setminus U$*. Again using regularity of* (X, μ)*, there is some* μ*-open neighbourhood* W *of* $(\kappa, 1)$ *such that* $\mathrm{cl}_\mu(W) \subseteq U$ *and hence* $\mathrm{cl}_\mu(W) \cap ((\alpha, \kappa) \times \{1\}) = \varnothing$*.*

If on the other hand, 2) occurs, then since $\mathrm{cl}_\mu(U)$ *is also* τ*-closed and it follows that* $\mathrm{cl}_\mu(U) \cap (\kappa \times \{1\})$ *is a* τ*-closed subset of* $\kappa \times \{1\})$*. Thus, since* κ *is a regular cardinal with uncountable cofinality and* $\mathrm{cl}_\mu(U) \cap S_\omega = \varnothing$*, it follows that there is some* $\alpha \in \kappa$ *such that* $\mathrm{cl}_\mu(U) \cap ((\alpha, \kappa) \times \{1\}) = \varnothing$*.*

Thus in both cases 1) and 2) we have shown that there is a μ*-open neighbourhood* O *of* $(\kappa, 1)$ *and* $\alpha \in \kappa$ *such that* $\mathrm{cl}_\mu(O) \cap ((\alpha, \kappa) \times \{1\}) = \varnothing$*.*

Now, since $1 \in \mathrm{cl}(J)$*, it follows that* $\{(r_n, 1] : n \in J\}$ *is a local base at 1 and so for each* $\alpha < \gamma \in \kappa$*, there is* $r_{n_\gamma} \in J$ *and* O_γ *open in* κ *such that* $O_\gamma \times (r_{n_\gamma}, 1] \subseteq X \setminus \mathrm{cl}_\mu(O)$*. Now denoting by* L_n *the set* $\{\gamma : n_\gamma = n \in J\}$ *and by* M_n *the set* $\bigcup\{O_\gamma : \gamma \in L_n\}$ *we have that for each* $n \in J$*,* $M_n \times (r_n, 1] \subseteq X \setminus \mathrm{cl}_\mu(O)$*. However,* $\bigcup\{M_n : n \in J\} \supseteq (\alpha, \kappa)$ *and hence there is a finite subset* $\{M_{n_1}, \ldots, M_{n_k}\}$ *which covers* (α, κ)*. Letting* $r = \max\{r_{n_1}, \ldots, r_{n_k}\}$*, we have that* $(\alpha, \kappa) \times (r, 1] \subseteq S \setminus \mathrm{cl}_\mu(O)$ *and hence* $O \cap ((\alpha, \kappa+1] \times (r, 1]) \subseteq \{\kappa\} \times [0, 1]$*. Since* $O \cap (X \setminus \{(\kappa, 1)\})$ *is* τ*-open this shows that* $O \cap ((\alpha, \kappa+1] \times (r, 1]) = \{(\kappa, 1)\}$*, that is to say,* $(\kappa, 1)$ *is an isolated point of* (X, μ)*.*

Of course, for each $r \in [0, 1]$*, the same argument applies to the point* $(\kappa, r) \in X$*. Thus each point of* X *is either a maximal regular point or a point of first countability; it follows that* $(\kappa + 1) \times [0, 1]$ *is maximal R-closed and is lower in* Σ_3*.*

Now let L *denote the ordered set* $(\kappa+1) \bigoplus \omega^{-1}$ *(that is to say,* $\kappa+1$ *with its usual ordering followed by* ω *with its reverse ordering, with the order topology) and* $Y = L \times [0, 1]$ *with the product topology* τ*. The space* Y *is the product of two LOTS, is not first countable and contains* X *as a closed subspace. Nonetheless, we claim that* Y *is not lower in* Σ_3*. To see this suppose that* $\tau \subsetneq \sigma$ *and that* τ *and* σ *differ at precisely one point* $p \in Y$*. By Theorem 2.13 of [1],* p *is not a point of first countability, hence* $p = (\kappa, r) \in \{\kappa\} \times [0, 1]$*. Clearly the neighbourhood filter* $\mathcal{V}_p^\sigma$ *of* p *in* (Y, σ) *must differ from that in* (Y, τ)*,* $\mathcal{V}_p^\tau$*, either on the subset* $(\kappa + 1) \times [0, 1]$ *or on* $Y \setminus (\kappa \times [0, 1])$*. Suppose then that the traces of* $\mathcal{V}_p^\sigma$ *and* $\mathcal{V}_p^\tau$ *on* $(\kappa + 1) \times [0, 1]$ *are the same; then* $\mathcal{V}_p^\sigma$ *and* $\mathcal{V}_p^\tau$ *differ on* $Z = Y \setminus (\kappa \times [0, 1])$*, however,* (Z, τ) *is first countable and hence again by Theorem 2.13 of [1] there are* T_3*-topologies on it lying strictly between* τ *and* σ*. Thus* τ *and* σ *differ on* $(\kappa + 1) \times [0, 1]$ *and so by what we showed above,* p *must be an isolated point of* $((\kappa + 1) \times [0, 1], \sigma)$ *and hence also of* $(\{\kappa\} \times [0, 1], \sigma)$*. However, the topology on* $Y \setminus (\kappa \times [0, 1])$ *obtained by declaring* $\{\kappa\} \times ([0, 1] \setminus \{r\})$ *to be closed is not regular, and an argument similar to that* *employed in Theorem 2.13 of* [1] *shows that there is no topology, minimal in the class of regular topologies larger than it.*

With a little more work, using the fact that $[0,1]$ is second countable, it is possible to substitute ω_1 instead of κ in the previous example.

However the following questions remain open.

Question 4.11. *If a regular topology is lower does some closed subspace have a maximal regular point?*

Question 4.12. *Is there an internal concrete characterization of lower topologies in* Σ_3*?*

5. First countable regular topologies

Denote by $\Sigma_3'(X)$ the partially ordered set of first countable T_3-topologies on a set X.

Theorem 5.1. *There are no jumps in* $\Sigma_3'(X)$*; between any two first countable* T_3*-topologies on* X *there are at least* $\mathfrak{c}$ *incomparable first countable* T_3*-topologies.*

Proof. Suppose that ξ and τ are two first countable T_3-topologies on X which differ precisely at the point $x \in X$, Let $\{V_n : n \in \omega\}$ and $\{W_n : n \in \omega\}$ be nested local bases at x in the topologies ξ and τ respectively. We may now choose a sequence $\{x_m\}_{m\in\omega}$ which converges to x in (X,ξ) but not in (X,τ) and by passing to a subsequence if necessary, we may assume that $x_m \in V_m$ and $\{x_m : m \in \omega\}$ is a closed, discrete subset of (X,τ). For each $m \in \omega$, let $\{U_m^n : n \in \omega\}$ be a local base of τ-open sets at x_m such that $x \notin \mathrm{cl}_\tau(U_m^{n+1}) \subseteq U_m^n \subseteq V_m$ for each $m,n \in \omega$; since (X,τ) is regular, we may assume that $\{U_m^1 : m \in \omega\}$ is a discrete family of τ-open sets. Note that each set U_m^n is ξ-open and for each $n \in \omega$, the family $\{U_m^n : m \in \omega\}$ has x as its unique accumulation point in (X,ξ). Now let $\mathcal{A}$ be an almost disjoint family of subsets of ω of size $\mathfrak{c}$ and for each $A \in \mathcal{A}$ we define

$$\begin{aligned}\mathcal{F}_A = \{U \in \tau : {}& \text{if } x \in U \text{ then there is } n \in \omega \text{ and some}\\ & \text{finite } F \subseteq \omega \text{ such that } U \supseteq \textstyle\bigcup\{U_m^n : m \in A \setminus F\}\}.\end{aligned}$$

It is clear that this is a sub-base for a first countable topology $\mu_A \subseteq \tau$ on X and since $\{x_m\}_{m\in A}$ converges to x in (X,μ_A) it follows that $\mu_A \neq \tau$. Furthermore, since $U_m^n \subseteq V_m$ for each $m,n \in \omega$, it follows that $\xi \subseteq \mu_A$ and since $\{x_m\}_{m\in\omega\setminus A}$ does not converge to x in (X,μ_A) it follows that $\mu_A \neq \xi$. Finally, note that if $A, B \in \mathcal{A}$ are distinct, then μ_A and μ_B are incomparable topologies. Finally, we need to show that each topology μ_A is regular. To this end, suppose that $x \in U \in \mu_A$; then there is some finite set $F \subseteq \omega$ such that $U \supseteq \bigcup\{U_m^n : m \in A\setminus F\}$. It follows that $\bigcup\{\mathrm{cl}_\tau(U_m^{n+1}) : m \in A\setminus F\}$ is a μ_A-closed neighbourhood of x which is contained in U. If $x \neq z \in U \in \tau$, then there is some τ-closed neighbourhood $W \subseteq U$ of z and some $n \in \omega$ such that $W \cap \bigcup\{U_m^n : m \in \omega\} = \varnothing$ and hence W is a μ_A-closed neighbourhood of z contained in U. Thus (X,μ_A) is regular. ☐

In Theorem 2.13 of [1] it was shown that a sequential T_3-topology of countable pseudocharacter is not a lower topology in Σ_3. However, we do not know the answer to the following question:

Question 5.2. *Is every first countable T_3-topology which is not R-minimal, upper in Σ_3?*

6. Some more open problems

The supremum of a chain of regular topologies is regular. Thus a positive answer to the first question would imply a positive answer to the second.

Question 6.1. *Is the supremum of a chain of R-closed topologies R-closed?*

Question 6.2. *Is every R-closed topology contained in a maximal R-closed topology ?*

Note: There are maximal R-closed topologies which are not compact. In [15], Stephenson gave an example under CH of a first countable non-compact R-closed topology - by Corollary 3.9, this must be maximal R-closed. In [9] it was shown that the same construction can be done in ZFC. This space is scattered and has dispersion order 3. The topology contains a weaker compact Hausdorff topology of dispersion order 3 (which is clearly not maximal R-closed).

Question 6.3. *Is a maximal R-closed topology which is not R-minimal, upper in Σ_3?*

Stephenson's examples show that maximal R-closed topologies need not be lower. Finally, the most general question of all:

Question 6.4. *Is every regular topology which is not R-minimal an upper topology in Σ_3?*

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