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The equality of the Patch topology and the Ultrafilter topology: A shortcut

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Abstract

In this work R denotes a commutative ring with non-zero identity and we prove that the patch topology and the ultrafilter topology defined on the prime spectrum of R are equal, in a different way as the given by Marco Fontana and K. Alan Loper in ([2]).

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1. Terminology and basic definitions

Let R be a commutative ring with non-zero identity. $Spec(R)$ denotes the set of all prime ideals of R. For every proper subset I of R, we denote by $V(I)$ the set of all prime ideals of R containing I, and $D_0(I) = Spec(R) - V(I)$. $V(a)$ will denote the set $V(aR)$ and $D_0(a)$ the set $D_0(aR)$. The Zariski topology t_z on $Spec(R)$ is the one that has as its closed sets those of the form $V(I)$ ([1]). The patch topology on $Spec(R)$ is defined as the smallest topology having the collections $V(I)$ and $D_0(a)$ as closed sets. Let C be a subset of $Spec(R)$, and let Ω be an ultrafilter on C. It was shown in ([2]) that the set $P_\Omega = \{a \in R : V(a) \cap C \in \Omega\}$ is a prime ideal of R. The set C is said to be *ultrafilter-closed* if for every ultrafilter Ω on C, $P_\Omega \in C$. The ultrafilter-closed sets define a topology on $Spec(R)$ called the Ultrafilter topology ([2]), and is denoted by τ_U. In this work we prove that the patch topology and the ultrafilter topology are equal, in a different way as the given by Fontana and Loper in ([2]).

2. The shortcut

Theorem 2.1. *The Ultrafilter topology τ_U is compact.*

Proof. Let $\mathcal{U}$ be a non principal ultrafilter in $Spec(R)$. We want to prove that $\mathcal{U}$ τ_U-converge to $P_{\mathcal{U}}$. Let θ be a τ_U-open set containing $P_{\mathcal{U}}$. Suppose that $\mathcal{A} = \theta^C$ belongs to $\mathcal{U}$ and let $\mathcal{U}_{\mathcal{A}} = \{U \cap \mathcal{A} : U \in \mathcal{U}\}$ be the ultrafilter on $\mathcal{A}$ induced by $\mathcal{U}$. Since $\mathcal{A}$ is τ_U-closed, then $P_{\mathcal{U}_{\mathcal{A}}} \in \mathcal{A}$. If $a \in P_{\mathcal{U}}$, $V(a) \in \mathcal{U}$, then $V(a) \cap \mathcal{A} \in \mathcal{U}_{\mathcal{A}}$ and therefore $a \in P_{\mathcal{U}_{\mathcal{A}}}$. Also, if $b \in P_{\mathcal{U}_{\mathcal{A}}}$ it follows that $V(b) \cap \mathcal{A} \in \mathcal{U}_{\mathcal{A}}$ and there exists $U \in \mathcal{U}$ such that $V(b) \cap \mathcal{A} = U \cap \mathcal{A}$. Since $U \cap \mathcal{A} \in \mathcal{U}$, then $V(b) \cap \mathcal{A} \in \mathcal{U}$ which implies that $V(b) \in \mathcal{U}$ and so $b \in P_{\mathcal{U}}$. Therefore $P_{\mathcal{U}} = P_{\mathcal{U}_{\mathcal{A}}} \in \mathcal{A}$ which is a contradiction. □

Corollary 2.2. *The Ultrafilter topology and the patch topology are equal.*

Proof. Since the patch topology is Hausdorff ([3]), weaker than the ultrafilter topology ([2]) and the well known fact that any compact topology does not admit a weaker Hausdorff topology unless they are equal, the result follows. □

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