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Some remarks on chaos in topological dynamics

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Abstract

Bau-Sen Du introduced a notion of chaos which is stronger than Li-Yorke sensitivity. A TDS (X, f) is called chaotic if there is a positive ε such that for any x and any nonempty open set V of X there is a point y in V such that the pair (x, y) is proximal but not ε-asymptotic. In this article, we show that a TDS (T, f) is transitive but not mixing if and only if (T, f) is Li-Yorke sensitive but not chaotic, where T is a tree. Moreover, we compare such chaos with other notions of chaos.

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1. Introduction

Throughout this paper a topological dynamical system is a pair (X, f) (TDS for short), where X is a compact metric space with a metric d and $f : X \to X$ is a continuous surjective map. A TDS (X, f) is **nontrivial** if X contains at least two points. Chaotic behavior is a manifestation of the complexity of the dynamical system. Now we recall some concepts of complexity.

A TDS (X, f) is **sensitive** [3], if there exists a positive ε such that for any x in X and any open neighborhood U of x, there exist $y \in U$ and a positive integer n with $d(f^n(x), f^n(y)) > \varepsilon$.

Let ε be a positive number. A subset C in X is a **Li-Yorke ε-scrambled set** of a TDS (X, f), if any pair (x, y) of distinct points x and y in C is proximal but not ε-asymptotic, that is,

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0 \quad \text{and} \quad \limsup_{n\to\infty} d(f^n(x), f^n(y)) > \varepsilon.$$

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A TDS (X, f) is **Li-Yorke ε-chaotic** [5], if it has an uncountable Li-Yorke ε-scrambled set.

In 2003, Akin and Kolyada [1] introduced the notion of Li-Yorke sensitivity which links the Li-Yorke version of chaos with sensitivity. A TDS (X, f) is called **Li-Yorke sensitive** [1] if there is a positive ε such that every x in X is a limit of points y in X such that the pair (x, y) is proximal but not ε-asymptotic.

What is the nature of chaos? Various people have various understandings. In [4], Bau-Sen Du believed that chaos should involve not only nearby points could diverge apart but also faraway points could get close to each other. Therefore in 2006, he [4] proposed a new definition of chaos as follows, which is stronger than Li-Yorke sensitivity. A TDS (X, f) is called **chaotic** [4] if there is a positive ε such that for any x and any nonempty open set V of X there is a point y in V such that the pair (x, y) is proximal but not ε-asymptotic. There is a TDS (X, f) which is Li-Yorke sensitive but not chaotic (see [4, Theorem 4]).

The present article goes on studying the nature of chaos, and is written on basis of the preprint [4]. This article is organized as follows. In Section 2, we investigate the chaos of transitive maps on trees. We show that a TDS (T, f) is transitive but not mixing if and only if (T, f) is Li-Yorke sensitive but not chaotic, where T is a tree. Finally, we compare the chaos with other notion of the same.

2. The chaos of transitive maps on trees

In this section, the chaos of transitive maps on trees are investigated. By a **tree** we mean a connected compact one-dimensional polyhedron, which does not contain any subset homeomorphic to a circle and which contains a subset homeomorphic to an interval. Let T be a tree. Given point $x \in T$, we define the valence of x, $\text{val}(x)$, as the number of connected components of $T - \{x\}$. Each point of valence 1 is an endpoint of T. A subtree of a tree T is a subset of T, which is a tree itself.

A TDS (X, f) is (topologically) **transitive** if for any two nonempty open sets U and V there exists a positive integer n such that $f^n(U) \cap V \neq \varnothing$. A TDS (X, f) is (topologically) **weakly mixing** if $(X \times X, f \times f)$ is (topologically) transitive. A TDS (X, f) is (topologically) **totally transitive** if (X, f^n) is transitive for any positive integer n. A TDS (X, f) is (topologically) **mixing** if for any two nonempty open sets U and V there exists a positive integer N such that $f^n(U) \cap V \neq \varnothing$ for any $n \geq N$. A TDS (X, f) is **minimal** if the set $orb(x, f) = \{f^n(x) : n = 0, 1, 2, \cdots\}$ is dense in X for any x of X.

The main aim of this section is to prove the following result.

Theorem 2.1. *Let T be a tree and let (T, f) be transitive. Then the following results hold.*

(1) *(T, f) is mixing if and only if it is chaotic;*
(2) *(T, f) is not mixing if and only if it is Li-Yorke sensitive but not chaotic.*

Remark 2.2. Theorem 2.1 may not be true for a general TDS. Let g be an irrational rotation of the unit circle S^1, then (S^1, g) is minimal and totally transitive but not Li-Yorke sensitive.

We need the following lemmas which come from [8] and [2] respectively.

Lemma 2.3. *Let T be a tree and let (T, f) be transitive. Then $\overline{P(f)} = T$, where $\overline{P(f)}$ denotes the closure of the set of all periodic points of f.*

Lemma 2.4. *Let T be a tree and let (T, f) be transitive. Then exactly one of the following alternatives holds.*

(1) *(T, f) is totally transitive.*
(2) *There is a positive integer n_0 such that there are an interior fixed point y and subtrees $T_1, T_2, \cdots, T_{n_0}$ of T with $\cup_{i=1}^{n_0} T_i = T$, $T_i \cap T_j = \{y\}$ for $i \neq j$ and $f(T_i) = T_{i+1 (mod\ n_0)}$ for $1 \leq i \leq n_0$. Moreover, $(T_i, f^{n_0}|_{T_i})$ is transitive, $1 \leq i \leq n_0$.*

Proposition 2.5. *Let $f : T \to T$ be a tree map. (T, f) is mixing if and only if it is totally transitive.*

Proof. Let us denote by $E(T)$ the set of endpoints of the tree T and suppose (T, f) is totally transitive.

Suppose that U and V are nonempty open connected subsets of T. We may assume that U is contained in $T - E(T)$. Since (T, f) is transitive, then $\overline{P(f)} = T$ by Lemma 2.3. Let y be any periodic point in U which orbit $Orb(y, f)$ is contained in $T - E(T)$. Let x be any periodic point in V. Let m be a common multiple of the periods of x and y, and set $g = f^m$. Then every point of $Orb(y, f) \cup \{x\}$ is a fixed point of g. Let $K = \cup_{n=0}^{\infty} g^n(V)$. Then K is a connected subset of T, since x is a fixed point of g. Since (T, g) is transitive, then K is a dense connected subset of T. This implies K contains $T - E(T)$.

For any $u \in Orb(y, f)$, there is an integer $k_u \geq 0$ such that $u \in g^{k_u}(V)$. Let $k = \max\{k_u : u \in Orb(y, f)\}$. Since every point u of $Orb(y, f)$ is a fixed point of g, then $f^{km}(V) = g^k(V)$ which contains $Orb(y, f)$. Thus $f^n(V)$ contains point y for any $n \geq km$. This implies $f^n(V) \cap U \neq \varnothing$, hence (T, f) is mixing.

Conversely, it is obvious. □

Proposition 2.6. *Let T be a tree and let (T, f) be transitive. Then exactly one of the following alternatives holds.*

(1) *(T, f) is mixing.*
(2) *There is a positive integer n_0 such that there are an interior fixed point y and subtrees $T_1, T_2, \cdots, T_{n_0}$ of T with $\cup_{i=1}^{n_0} T_i = T$, $T_i \cap T_j = \{y\}$ for*

$i \neq j$ and $f(T_i) = T_{i+1(mod\ n_0)}$ for $1 \leq i \leq n_0$. Moreover, $(T_i, f^{n_0}|_{T_i})$ is mixing, $1 \leq i \leq n_0$.

Proof. Suppose that (T, f) is transitive. If (T, f) is totally transitive, then (T, f) is mixing by Proposition 2.5. Otherwise, there is a positive integer n_0 such that there are an interior fixed point y and subtrees $T_1, T_2, \cdots, T_{n_0}$ of T with $\cup_{i=1}^{n_0} T_i = T$, $T_i \cap T_j = \{y\}$ for $i \neq j$ and $f(T_i) = T_{i+1(mod\ n_0)}$ for $1 \leq i \leq n_0$. Moreover, $(T_i, f^{n_0}|_{T_i})$ is transitive, $1 \leq i \leq n_0$. Then $f^{n_0}|_{T_i}$ satisfies the condition (1) of Lemma 2.4, since y is a fixed point of $f^{n_0}|_{T_i}$, and y is the endpoint of T_i. Hence $(T_i, f^{n_0}|_{T_i})$ is mixing by Lemma 2.4 and Proposition 2.5. □

By [1, Theorem 3.4 and Lemma 3.8], the following lemma holds.

Lemma 2.7. *If a nontrivial TDS (X, f) is weakly mixing, then it is chaotic.*

Proof of Theorem 2.1. If (T, f) is mixing, then (T, f) is chaotic by Lemma 2.7.

Otherwise, by Proposition 2.6 there are a positive integer n_0 such that there is an interior fixed point y and subtrees $T_1, T_2, \cdots, T_{n_0}$ of T with $\cup_{i=1}^{n_0} T_i = T$, $T_i \cap T_j = \{y\}$ for $i \neq j$ and $f(T_i) = T_{i+1(mod\ n_0)}$ for $1 \leq i \leq n_0$. Moreover, $(T_i, f^{n_0}|_{T_i})$ is mixing, $1 \leq i \leq n_0$. Since every $(T_i, f^{n_0}|_{T_i})$ is chaotic, hence (T, f) is Li-Yorke sensitive.

Next we show that (T, f) is not chaotic. Let x be a periodic point of f in the interior of T_1. Let V be an open subset of T which is contained in T_2. Since x and V are jumping alternatively and never get close to each other, then (T, f) is not chaotic. Hence, Theorem 2.1 holds. □

3. Comparison of various notions of chaos

In this section X will denote a general compact metric space. Below, we discuss the interrelations between the notions of chaos.

A TDS (X, f) is **Devaney's chaotic** [3] if it is transitive and the set of periodic points of f is dense in X. Recall that a **Mycielski set** is a countable union of Cantor sets, while a Cantor set is a set homeomorphic to the standard middle-third Cantor set on the real line.

The following lemmas come from [1] and [6] respectively.

Lemma 3.1. *For a TDS (X, f) the following conditions are equivalent.*

1. *(X, f) is sensitive.*
2. *There exists a positive ε such that $\{(x, y) \in X \times X : \limsup_{n\to\infty} d(f^n(x), f^n(y)) > \varepsilon\}$ is a dense G_δ set of $X \times X$.*

Lemma 3.2 (Mycielski). *Let X be a separable complete metric space without isolated point. If for each natural number n, R_n is a residual set of X^n, then there is a Mycielski set K of X such that (1) for any nonempty open set U of X, $K \cap U$ contains a nonempty perfect set; (2) for each natural number n, for all $x_1, x_2, \cdots, x_n$ mutually distinct points in K, $(x_1, x_2, \cdots, x_n) \in R_n$.*

Theorem 3.3. *If a TDS (X, f) is chaotic, then there is a dense Mycielski set K in X such that K is a Li-Yorke ε-scrambled set for some positive ε. Hence, (X, f) is Li-Yorke ε-chaotic.*

Proof. Since (X, f) is chaotic, then (X, f) is sensitive, this implies X without isolated point. There is a positive ε such that $C_1(\varepsilon) := \{(x, y) \in X \times X : \limsup_{n\to\infty} d(f^n(x), f^n(y)) > \varepsilon\}$ is a dense G_δ set of $X \times X$, by Lemma 3.1. Moreover, $C_2 := \{(x, y) \in X \times X : \liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0\}$ is a dense G_δ set of $X \times X$ because (X, f) is chaotic.

Put $R_2 = C_1(\varepsilon) \cap C_2$. So Theorem 3.3 holds by Lemma 3.2. □

Remark 3.4. There is a TDS with positive topological entropy, which is Li-Yorke ε-chaotic and Devaney's chaotic but not chaotic.

Let $f(x) = \frac{1}{2} + 2x$ if $0 \le x \le \frac{1}{4}$; $f(x) = \frac{3}{2} - 2x$ if $\frac{1}{4} \le x \le \frac{1}{2}$; $f(x) = 1 - x$ if $\frac{1}{2} \le x \le 1$. Then $f : [0, 1] \to [0, 1]$ is transitive but not mixing. Since f is transitive, then the set of periodic points of f is dense in X. This implies that $([0, 1], f)$ is Devaney's chaotic. Because f has positive topological entropy, then $([0, 1], f)$ is Li-Yorke ε-chaotic for some positive ε but not chaotic.

Remark 3.5. There is a TDS which is chaotic with zero topological entropy.

A minimal system (X, f) which is weakly mixing and has zero topological entropy has been built, such as in [7]. Then the minimal system (X, f) is chaotic, and has zero topological entropy.

Remark 3.6. If a TDS (X, f) is Li-Yorke sensitive, the product system $(X \times Y, f \times g)$ is Li-Yorke sensitive for any TDS (Y, g) (see [1, Theorem 3.11]). However, for any TDS (X, f) which is chaotic, there is a TDS (Y, g) such that the product system $(X \times Y, f \times g)$ is not chaotic.

Actually, let g be an irrational rotation of the unit circle S^1. The product system $(X \times S^1, f \times g)$ is Li-Yorke sensitive but not chaotic for any chaotic TDS (X, f).

Remark 3.7. There is a TDS (X, σ) which is chaotic but not Devaney's chaotic.

Let $(\sum_2, \sigma)$ be the full shift over two letters $\{0, 1\}$. Let $A_1 = 1, A_2 = 101, \cdots, A_{n+1} = A_n 0^n A_n$. Then $x = \lim_{n\to\infty} A_n A_n \cdots \in \sum_2$. Put $X = \overline{Orb(x, \sigma)}$. Then (X, σ) is mixing and has a fixed point $00\cdots$ which is the unique minimal subset of X (see [9]). (X, σ) is chaotic but not Devaney's chaotic since the set of periodic points of σ is not dense in X.

There exists a TDS which is chaotic but not transitive, a fortiori not weakly mixing(see [4, Theorem 5]). However we have

Remark 3.8. If a TDS (X, f) is minimal, then (X, f) is chaotic if and only if it is weakly mixing.

If (X, f) is chaotic, then the proximal cell $P(f)(x)$ is dense in X for any x of X, where $P(f)(x) = \{y \in X : \liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0\}$. Hence (X, f) is weakly mixing by [1, Theorem 3.7]. Conversely, it is clear.

Below, we summarize the interrelations between the notions of chaos, see Figure 1.

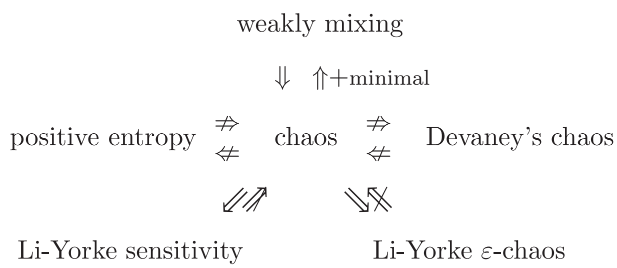

Figure 1. Relations of various notions of chaos

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