

## Core compactness and diagonality in spaces of open sets

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### ABSTRACT

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We investigate when the space  $\mathcal{O}_X$  of open subsets of a topological space  $X$  endowed with the Scott topology is core compact. Such conditions turn out to be related to infraconsonance of  $X$ , which in turn is characterized in terms of coincidence of the Scott topology of  $\mathcal{O}_X \times \mathcal{O}_X$  with the product of the Scott topologies of  $\mathcal{O}_X$  at  $(X, X)$ . On the other hand, we characterize diagonality of  $\mathcal{O}_X$  endowed with the Scott convergence and show that this space can be diagonal without being pretopological. New examples are provided to clarify the relationship between pretopologicity, topologicity and diagonality of this important convergence space.

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2010 MSC: 54A20, 54A10, 54B20, 54D45.

KEYWORDS: Scott convergence, Scott topology, upper Kuratowski convergence, upper Kuratowski topology, core compact, diagonal convergence, pretopology, consonance, infraconsonance.

### 1. INTRODUCTION

Definitions and notations concerning convergence structures follow [3] and are gathered as an appendix at the end of these notes (<sup>1</sup>). In particular, if  $X$  and  $Y$  are two convergence spaces, the *continuous convergence*  $[X, Y]$  on the set  $C(X, Y)$  of continuous maps from  $X$  to  $Y$  is the coarsest convergence making the evaluation jointly continuous. This is the canonical function space structure in the cartesian closed category of convergence spaces and continuous maps. This paper is concerned with certain properties of this canonical convergence on functions valued into the Sierpiński space:

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<sup>1</sup>Terms and notations that are not defined in the text can be found in the appendix.

Let  $\mathbb{S}_0$  and  $\mathbb{S}_1$  denote two versions of the Sierpiński space on  $\{0, 1\}$ :  $\{0\}$  is the only non-trivial open subset of  $\mathbb{S}_0$ , and  $\{1\}$  is the only non-trivial open subset of  $\mathbb{S}_1$ . Let  $1_A : X \rightarrow \{0, 1\}$  denote the indicator function of a subset  $A$  of a convergence space  $X$  defined by  $1_A(x) = 1$  if and only if  $x \in A$ . With those conventions,  $A$  is an open subset of  $X$  if and only if  $1_A : X \rightarrow \mathbb{S}_1$  is continuous and closed if and only if  $1_A : X \rightarrow \mathbb{S}_0$  is continuous. Therefore,  $C(X, \mathbb{S}_1)$  can be identified with the set  $\mathcal{O}_X$  of open subsets of  $X$ , and  $C(X, \mathbb{S}_0)$  can be identified with the set  $\mathcal{C}_X$  of closed subsets of  $X$ .

If  $X$  is a topological space, the continuous convergences  $[X, \mathbb{S}_1]$  and  $[X, \mathbb{S}_0]$  turn out to be familiar convergences, on  $\mathcal{O}_X$  and  $\mathcal{C}_X$  respectively (see, e.g., [5]):

$$(1.1) \quad U \in \lim_{[X, \mathbb{S}_1]} \mathcal{F} \iff U \subseteq \bigcup_{F \in \mathcal{F}} \text{int} \left( \bigcap_{O \in F} O \right)$$

$$(1.2) \quad C \in \lim_{[X, \mathbb{S}_0]} \mathcal{F} \iff \bigcap_{F \in \mathcal{F}} \text{cl} \left( \bigcup_{A \in F} A \right) \subseteq C.$$

Both are instances of *Scott convergence* (in the sense of, for instance, [11]), i.e.,

$$(1.3) \quad x \in \lim \mathcal{F} \iff \bigvee_{F \in \mathcal{F}} \bigwedge F \geq x,$$

in the complete lattices  $(\mathcal{O}_X, \subseteq)$  and  $(\mathcal{C}_X, \supseteq)$  respectively. However, (1.2) is usually called *upper Kuratowski convergence*. The topological modification  $T[X, \mathbb{S}_0]$  is called *upper Kuratowski topology*. The topological modification  $T[X, \mathbb{S}_1]$  is the *Scott topology*, whose open sets are exactly *compact families*: families  $\mathcal{A}$  of open subsets of  $X$  that are closed under open supersets and satisfy

$$\bigcup_{i \in I} O_i \in \mathcal{A} \implies \exists F \in [I]^{<\infty} : \bigcup_{i \in F} O_i \in \mathcal{A}$$

for any collection  $\{O_i : i \in I\}$  of open subsets of  $X$ , where  $[I]^{<\infty}$  denotes the set of finite subsets of  $I$ . For instance, if  $K$  is a compact subset of  $X$  then

$$\mathcal{O}(K) := \{O \in \mathcal{O}_X : K \subseteq O\}$$

is a compact family. The family of all sets  $\mathcal{O}(K)$  where  $K$  ranges over compact subsets of  $X$  is a basis for a topology on  $\mathcal{O}_X = C(X, \mathbb{S}_1)$ . We write  $C_k(X, \mathbb{S}_1)$  for the corresponding topological space.

Of course, complementation  $c : \mathcal{O}_X \rightarrow \mathcal{C}_X$  defined by  $c(U) = X \setminus U$  is an homeomorphism between  $[X, \mathbb{S}_1]$  and  $[X, \mathbb{S}_0]$  and we only need study one of these two convergences. We choose to focus on  $\mathcal{O}_X$ . Hence, from now on,  $\mathbb{S}$  means  $\mathbb{S}_1$  and we only formulate results on  $\mathcal{O}_X$  but they have counterparts for the upper Kuratowski convergence and upper Kuratowski topology on  $\mathcal{C}_X$ , which the interested reader can easily write out (See e.g., [8] for a study of  $[X, \mathbb{S}_0]$  and  $T[X, \mathbb{S}_0]$  on  $\mathcal{C}_X$ ).

It is clear from the definitions that

$$[X, \mathbb{S}] \geq T[X, \mathbb{S}] \geq C_k(X, \mathbb{S}).$$

A convergence space  $X$  is called  $T$ -dual if  $[X, \$] = T[X, \$]$  and *consonant* [5] if  $T[X, \$] = C_k(X, \$)$ .

It is easily seen that

$$\begin{aligned} \cap : [X, \$] \times [X, \$] &\rightarrow [X, \$] \\ (U, V) &\mapsto U \cap V \end{aligned}$$

is continuous for any convergence space  $X$ .

To simplify the discussion, let us momentarily assume that  $X$  is a completely regular topological space. In this case,  $X$  is  $T$ -dual if and only if  $X$  is locally compact. Moreover,

$$C_k(X, \$) = [KX, \$],$$

where  $KX$  is the locally compact modification of  $X$  <sup>(2)</sup> [21, Proposition 4.3]. Therefore intersection is also jointly continuous for  $C_k(X, \$)$ . Additionally, infraconsonance in the sense of [9] can then be characterized in similar terms:  $X$  is *infraconsonant* if  $\cap : T[X, \$] \times T[X, \$] \rightarrow T[X, \$]$  is continuous. Thus

$$T\text{-dual} \implies \text{consonant} \implies \text{infraconsonant}.$$

The problem of characterizing  $T$ -dual topological spaces has long been settled (e.g., [13], [23]): a topological space  $X$  is  $T$ -dual if and only if it is *core compact*. Recall that a topological space  $X$  is *core compact* if for every  $x$  and  $O \in \mathcal{O}(x)$ , there is  $U \in \mathcal{O}(x)$  such that every open cover of  $O$  has a finite subfamily that covers  $U$ .

In the case of a general convergence space  $X$ , the situation is more complicated. It is known (e.g., [24], [7]) that the following are equivalent:

$$(1.4) \quad \begin{aligned} \forall Y, T(X \times Y) &\leq X \times TY; \\ \forall Y = TY, [X, Y] &= T[X, Y] \\ T(X \times [X, \$]) &\leq X \times T[X, \$]; \\ X &\text{ is } T\text{-dual.} \end{aligned}$$

Moreover, it was shown in [7] that

$$(1.5) \quad X \text{ is core compact} \implies X \text{ is } T\text{-dual} \implies X \text{ is } T\text{-core compact},$$

where a *convergence space* is called *core compact* if whenever  $x \in \lim \mathcal{F}$ , there is  $\mathcal{G} \leq \mathcal{F}$  with  $x \in \lim \mathcal{G}$  and for every  $G \in \mathcal{G}$  there is  $G' \in \mathcal{G}$  such that  $G'$  is compact at  $G$ ; and a convergence space is called  *$T$ -core compact* if whenever  $x \in \lim \mathcal{F}$  and  $U \in \mathcal{O}_X(x)$ , there is  $F \in \mathcal{F}$  that is compact at  $U$ .

The three notions clearly coincide if  $X$  is topological. However, so far, it was not known whether they do in general. At the end of the paper, we provide an example (Example 5.8) of a  $T$ -dual convergence that is not core-compact.

Section 2 examines when  $[X, \$]$  and  $T[X, \$]$  are  $T$ -dual. The latter question, while natural in itself, is motivated by its connection (established in Section 3) with the (now recently solved [18]) problem [9, Problem 1.2] of finding a

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<sup>2</sup>with the abuse of notation that  $[KX, \$]$  is identified with the convergence it induces on the subset  $C(X, \$)$  of  $C(KX, \$)$ .

completely regular infraconsonant topological space that is not consonant. We obtain that  $X$  is infraconsonant whenever  $T[X, \$]$  is  $T$ -dual, and we prove more generally that  $X$  is infraconsonant if and only if the Scott topology on  $\mathcal{O}_X \times \mathcal{O}_X$  for the product order coincides with the product of the Scott topologies at the point  $(X, X)$  (Theorem 4.2).

Infraconsonance was introduced while studying the Isbell topology on the set of real-valued continuous functions over a topological space. In fact a completely regular space  $X$  is infraconsonant if and only if the Isbell topology on the set of real-valued continuous functions on  $X$  is a group topology [6, Corollary 4.6]. On the other hand, the fact that the Scott topology on the product does not coincide in general with the product of the Scott topologies has been at the origin of a number of errors, as pointed out for instance in [11, p.197]. Therefore, Theorem 4.2 provides new motivations to investigate infraconsonance.

In [7], it is shown that a convergence space  $X$  is  $T$ -core compact if and only if  $[X, \$]$  is pretopological. Therefore, if  $X$  is topological,  $[X, \$]$  is topological whenever it is pretopological. As topologies are exactly the diagonal <sup>(3)</sup> pretopologies, it raises the question of whether  $[X, \$]$  is diagonal whenever  $X$  is topological. In Section 5, diagonality of  $[X, \$]$  is characterized in terms of a variant of core-compactness that do not need to coincide with core-compactness. As a result  $[X, \$]$  does not need to be diagonal even if  $X$  is topological.

## 2. CORE-COMPACTNESS OF $\mathcal{O}_X$

For a general convergence space  $X$ , the underlying set of  $[X, \$]$  can still be identified with the collection  $\mathcal{O}_X$  of open subsets of  $X$  (or  $TX$ ), but the characterization (1.1) of convergence in  $[X, \$]$  needs to be modified. A family  $\mathcal{S}$  of subsets of a convergence space  $Y$  is a *cover of*  $A \subseteq Y$  if every filter on  $Y$  converging to a point of  $A$  contains an element of the family  $\mathcal{S}$ . Then we have:

$$U \in \lim_{[X, \$]} \mathcal{F} \iff \left\{ \bigcap_{O \in \mathcal{F}} O : \mathcal{F} \text{ is a cover of } U \right\}$$

The space  $[[X, \$], \$]$  has as underlying set the set of Scott-open subsets of  $\mathcal{O}_X$ , that is, if  $X$  is topological, the set  $\kappa(X)$  of openly isotone compact families on  $X$ . Note that the family

$$\{U^\sim := \{\mathcal{A} \in \kappa(X) : U \in \mathcal{A}\} : U \in \mathcal{O}_X\}$$

forms a subbase for a topology on  $\kappa(X)$ , called *Stone topology*. It is the analog on  $\kappa(X)$  of the standard topology on the set  $\beta X$  of ultrafilters on  $X$ .

As observed in [10, Proposition 5.2], when  $X$  is topological, the convergence  $[X, \$]$  is based in filters of the form

$$(2.1) \quad \mathcal{O}^{\natural}(\mathcal{P}) := \{\mathcal{O}(P) : P \in \mathcal{P}\},$$

where  $\mathcal{P}$  is an *ideal subbase* of open subsets of  $X$ , that is, such that there is  $P \in \mathcal{P}$  with  $\bigcup_{Q \in \mathcal{P}_0} Q \subseteq P$  whenever  $\mathcal{P}_0$  is a finite subfamily of  $\mathcal{P}$ . More

<sup>3</sup>in the sense of e.g., [4]. See Definition 5.1.

precisely, for every filter  $\mathcal{F}$  on  $[X, \$]$  with  $U \in \lim_{[X, \$]} \mathcal{F}$  there is an open cover  $\mathcal{P}$  of  $U$  that forms an ideal subbase, such that  $U \in \lim_{[X, \$]} \mathcal{O}^{\natural}(\mathcal{P})$  and  $\mathcal{O}^{\natural}(\mathcal{P}) \leq \mathcal{F}$ .

Note also that

$$(2.2) \quad \mathcal{A} \subseteq \mathcal{B} \implies \mathcal{A} \in \lim_{[[X, \$], \$]} \{\mathcal{B}\}^{\uparrow},$$

for every  $\mathcal{A}$  and  $\mathcal{B}$  in  $\kappa(X)$ . In particular if  $O$  is  $[[X, \$], \$]$ -open,  $\mathcal{A} \in O$  and  $\mathcal{A} \subseteq \mathcal{B} \in \kappa(X)$  then  $\mathcal{B} \in O$ .

It was observed in [12], as a consequence of a general theory, that if  $X$  is topological, then so is  $[[X, \$], \$]$ . We provide here an independent proof, that shows that  $[X, \$]$  is then a core compact convergence.

**Proposition 2.1.** *If  $X$  is topological, then  $[X, \$]$  is core compact, so that  $[[X, \$], \$]$  is topological. More precisely, it is homeomorphic to  $\kappa(X)$  with the Stone topology.*

*Proof.* Let  $U \in \lim_{[X, \$]} \mathcal{O}^{\natural}(\mathcal{P})$  for an ideal subbase  $\mathcal{P}$  of open subsets of  $X$ . Then for each  $P \in \mathcal{P}$ , the set  $\mathcal{O}(P)$  is a compact subset of  $[X, \$]$  because  $P \in \lim_{[X, \$]} \mathcal{O}(P)$ . Indeed,  $P = \text{int} \left( \bigcap_{O \in \mathcal{O}(P)} O \right)$ .

$U^{\sim}$  is  $[[X, \$], \$]$ -open for each  $U \in \mathcal{O}_X$ . Indeed, if  $\mathcal{A} \in U^{\sim} \cap \lim_{[[X, \$], \$]} \mathcal{F}$  then  $\{\bigcap_{\mathcal{B} \in \mathcal{F}} \mathcal{B} : \mathcal{B} \in \mathcal{F}\}$  is a cover of  $\mathcal{A}$  (in the sense of convergence) so that there is  $F \in \mathcal{F}$  with  $\bigcap_{\mathcal{B} \in F} \mathcal{B} \in \{U\}^{\uparrow}$  because  $U \in \lim_{[X, \$]} \{U\}^{\uparrow} \cap \mathcal{A}$ . In other words,  $F \subseteq U^{\sim}$ , so that  $U^{\sim} \in \mathcal{F}$ .

Conversely, if  $O$  is  $[[X, \$], \$]$ -open and  $\mathcal{A} \in O$ , there is  $U \in \mathcal{A}$  such that  $U^{\sim} \subseteq O$ . Otherwise, for each  $U \in \mathcal{A}$ , there is  $\mathcal{B} \in \kappa(X)$  with  $U \in \mathcal{B}$  and  $\mathcal{B} \notin O$ . In that case,  $\widehat{U} := \{\mathcal{B} \in \kappa(X) : U \in \mathcal{B}, \mathcal{B} \notin O\} \neq \emptyset$  for all  $U \in \mathcal{A}$ . Note also that in view of (2.2),  $\mathcal{B}_U \cap \mathcal{B}_V \in \widehat{U} \cap \widehat{V}$  whenever  $\mathcal{B}_U \in \widehat{U}$  and  $\mathcal{B}_V \in \widehat{V}$ . Therefore  $\{\bigcap_{i \in I} \widehat{U}_i : U_i \in \mathcal{A} : \text{card } I < \infty\}$  is a filter-base generating a filter  $\mathcal{F}$ . This filter converges to  $\mathcal{A}$  in  $[[X, \$], \$]$ . To show that, we need to see that  $\{\bigcap_{\mathcal{B} \in \widehat{U}} \mathcal{B} : U \in \mathcal{A}\}$  is a cover of  $\mathcal{A}$  for  $[X, \$]$ . In view of the form (2.1) of a base for  $[X, \$]$ , it is enough to show that if  $U_0 \in \mathcal{A}$  and  $\mathcal{P}$  is an ideal subbase of open subsets of  $X$  covering  $U_0$ , then there is  $A \in \mathcal{A}$  with  $\bigcap_{\mathcal{B} \in \widehat{A}} \mathcal{B} \in \mathcal{O}^{\natural}(\mathcal{P})$ . Because  $U_0 \subseteq \bigcup_{P \in \mathcal{P}} P$  and  $\mathcal{A}$  is a compact family, there is a finite subfamily  $\mathcal{P}_0$  of  $\mathcal{P}$  such that  $\bigcup_{P \in \mathcal{P}_0} P \in \mathcal{A}$ . Since  $\mathcal{P}$  is an ideal subbase, there is  $P \in \mathcal{P} \cap \mathcal{A}$ . Then  $\mathcal{O}(P) \subseteq \bigcap_{\mathcal{B} \in \widehat{P}} \mathcal{B}$ , which concludes the proof that  $\mathcal{A} \in \lim_{[[X, \$], \$]} \mathcal{F}$ . On the other hand,  $O \notin \mathcal{F}$ , which contradicts the fact that  $O$  is open for  $[[X, \$], \$]$ .  $\square$

**Remark 2.2.** If  $X$  is a non topological convergence space, then by [8, Corollary 16.3], the open subsets of  $[X, \$]$  are the *rigidly compact families*: families  $\mathcal{A}$  of open subsets of  $X$ , closed under open supersets, such that  $\text{adh}_{\xi} \mathcal{H} \# \mathcal{A}$  whenever  $\mathcal{H}$  is a filter such that for every  $H \in \mathcal{H}$  there is a closed subset  $B$  of  $H$  with  $B \in \mathcal{A}^{\#}$ . Hence the underlying set of  $[[X, \$], \$]$  is no longer  $\kappa(X)$  but the larger set of rigidly compact families on  $X$ . We will see below (Proposition 2.3) that the convergence  $[[X, \$], \$]$  fails to be topological in this case. We do not know

whether  $T[[X, \$], \$]$  can be expressed as an analog of the Stone topology on the set of rigidly compact families on  $X$ .

In order to investigate when  $T[X, \$]$  is core compact, we will need notions and results from [7]. The concrete endofunctor  $\text{Epi}_T$  of the category of convergence spaces (and continuous maps) is defined (on objects) by

$$\text{Epi}_T X = i^- [T[X, \$], \$]$$

where  $i : X \rightarrow [[X, \$], \$]$  is defined by  $i(x)(f) = f(x)$ . In view of [7, Theorem 3.1]

$$(2.3) \quad W \geq \text{Epi}_T X \iff T[X, \$] \geq [W, \$],$$

where  $X \geq W$  have the same underlying set. In particular,  $X$  is  $T$ -dual if and only if  $X \geq \text{Epi}_T X$ . A convergence space  $X$  is called *epitopological* if  $i : X \rightarrow [[X, \$], \$]$  is initial (in the category **Conv** of convergence spaces and continuous maps). Epitopologies form a reflective subcategory **Epi** of **Conv** and the (concrete) reflector is given (on objects) by  $\text{Epi} X = i^- [[X, \$], \$]$ . Because  $[\text{Epi} X, \$] = [X, \$]$ , it is enough to consider epitopologies in the study of dual convergences. Observe that a topological space is epitopological. Note that if  $[X, \$]$  is  $T$ -dual, then  $\text{Epi} X = X$  is topological. Therefore, in contrast to Proposition 2.1,  $[X, \$]$  is not  $T$ -dual if  $X$  is not topological.

**Proposition 2.3.** *Let  $X$  be an epitopological space. Then  $X$  is topological if and only if  $[X, \$]$  is  $T$ -dual.*

Note also that  $\text{Epi} X \leq \text{Epi}_T X$  and that  $\text{Epi}_T \circ \text{Epi} = \text{Epi}_T$ , so that  $\text{Epi}_T$  restricts to an expansive endofunctor of **Epi**. By iterating this functor, we obtain the coreflector on  $T$ -dual epitopologies. More precisely, if  $F$  is an expansive concrete endofunctor of **C**, we define the transfinite sequence of functors  $F^\alpha$  by  $F^1 = F$  and  $F^\alpha X = F \left( \bigvee_{\beta < \alpha} F^\beta X \right)$ . For each epitopological space  $X$ , there is an ordinal  $\alpha(X)$  such that

$$\text{Epi}_T^{\alpha(X)} X = \text{Epi}_T^{\alpha(X)+1} X := D_T X.$$

**Proposition 2.4.** *The class of  $T$ -dual epitopologies is concretely coreflective in **Epi** and the coreflector is  $D_T$ .*

While this proposition easily follows from general results in [7] or [20], and Galois connections, we provide a self-contained proof.

*Proof.* The class of  $T$ -dual convergences is closed under infima because

$$\left[ \bigwedge_{i \in I} X_i, Z \right] = \bigvee_{i \in I} [X_i, Z].$$

Indeed, if each  $X_i$  is  $T$ -dual, then

$$\left[ \bigwedge_{i \in I} X_i, \$ \right] = \bigvee_{i \in I} [X_i, \$] = \bigvee_{i \in I} T[X_i, \$] \leq T \left( \bigvee_{i \in I} [X_i, \$] \right) = T \left( \left[ \bigwedge_{i \in I} X_i, \$ \right] \right),$$

and  $\bigwedge_{i \in I} X_i$  is  $T$ -dual. The functor  $\text{Epi}_T$  is expansive on **Epi** and therefore, so is  $D_T$ . Moreover,  $D_T X$  is  $T$ -dual for each epitopological space  $X$  because

$$[D_T X, \$] = [\text{Epi}_T^{\alpha(X)+1} X, \$] \leq T[\text{Epi}_T^{\alpha(X)} X, \$] = T[D_T X, \$].$$

Therefore, for each epitopological space  $X$ , there exists the coarsest  $T$ -dual convergence  $\overline{X}$  finer than  $X$ . By definition  $X \leq \overline{X} \leq D_T X$ . Then  $[\overline{X}, \$] \leq [X, \$]$  and  $[\overline{X}, \$]$  is topological, so that  $[\overline{X}, \$] \leq T[X, \$]$ . But  $\text{Epi}_T X$  is the coarsest convergence with this property. Therefore  $\text{Epi}_T X \leq \overline{X} = \text{Epi}_T \overline{X}$  and  $D_T X \leq \overline{X}$ .  $\square$

**Proposition 2.5.** *If  $X$  is a core compact topological space, then  $[X, \$]$  is also a core compact topological space.*

*Proof.*  $[X, \$] = T[X, \$]$  because  $X$  is core compact, and  $[X, \$]$  is  $T$ -dual by Proposition 2.1, because  $X$  is topological. Therefore  $T[X, \$]$  is a core compact topology.  $\square$

However, if  $X$  is a non-topological  $T$ -dual convergence space (4), then  $[X, \$] = T[X, \$]$  is *not* core compact, by Proposition 2.3. In other words, we have:

**Proposition 2.6.** *If  $[X, \$]$  is topological then  $X$  is topological if and only if  $[X, \$]$  is core compact.*

In particular,  $D_T X$  is topological if and only if  $[D_T X, \$]$  is core compact.

**Theorem 2.7.** *If  $X \geq TD_T X$  then  $T[X, \$]$  is core compact if and only if  $X$  is a core compact topological space.*

*Proof.* We already know that if  $X$  is a core compact topological space then  $[X, \$] = T[X, \$]$  and that  $[X, \$]$  is core compact by Proposition 2.1. Conversely, if  $T[X, \$]$  is core compact then  $[T[X, \$], \$]$  is topological, so that  $\text{Epi}_T X$  is topological. Under our assumptions,

$$X \geq TD_T X \geq T \text{Epi}_T X = \text{Epi}_T X,$$

hence by (2.3),  $X$  is  $T$ -dual. Therefore  $[X, \$] = T[X, \$]$  is core compact and, in view of Proposition 2.3,  $X$  is topological, and  $T$ -dual, hence a core compact topological space.  $\square$

**Remark 2.8.** Note that, at least among Hausdorff topological spaces, Theorem 2.7 generalizes [19, Corollary 3.6] that states that if  $X$  is first countable, then  $X$  is core compact if and only if  $T[X, \$]$  is core compact. Indeed, the locally compact coreflection  $KX$  of a Hausdorff topological space is  $T$ -dual so that  $D_T X \leq KX$ . Moreover, [1] characterizes a number of topological properties in terms of functorial inequalities of the form

$$X \geq JE(X),$$

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<sup>4</sup>Such convergences exist: take for a instance a non-locally compact Hausdorff regular topological  $k$ -space. Then  $X = TK_h X$  but  $X < K_h X$  so that  $K_h X$  is non-topological.

where  $J$  is a concrete reflector and  $E$  a concrete coreflector in the category of convergence spaces. For instance, it is observed that a (Hausdorff) topological space  $X$  is a  $k$ -space if and only if

$$(2.4) \quad X \geq TKX,$$

so that (2.4) can be taken as a definition of a  $k$ -convergence. Hence if  $X$  is a Hausdorff topological  $k$ -space (in particular a first-countable space) then  $X \geq TD_T X$ .

On the other hand, in view of the results of [1], if  $f : X \rightarrow Y$  is a quotient map (in the topological sense) and  $X$  is core compact (so that  $X = D_T X$ ) then  $Y \geq TD_T Y$ .

We will see in the next section that similarly, if  $X$  is a consonant topological space, then  $T[X, \$]$  is core compact if and only if  $X$  is locally compact.

**Problem 2.9.** *Are there completely regular non locally compact topological spaces  $X$  such that  $T[X, \$]$  is core compact?*

Of course, in view of Remark 2.8, such a space cannot be a  $k$ -space or consonant.

### 3. CORE COMPACT DUAL, CONSONANCE, AND INFRACONSONANCE

A topological space is *consonant* if  $T[X, \$] = C_k(X, \$)$ , that is, if every Scott open subset  $\mathcal{A}$  of  $\mathcal{O}_X$  is compactly generated, that is, there are compact subsets  $(K_i)_{i \in I}$  of  $X$  such that  $\mathcal{A} = \bigcup_{i \in I} \mathcal{O}(K_i)$  [5]. A space is *infraconsonant* [9] if for every Scott open subset  $\mathcal{A}$  of  $\mathcal{O}_X$  there is a Scott open set  $\mathcal{C}$  such that  $\mathcal{C} \vee \mathcal{C} \subseteq \mathcal{A}$ , where  $\mathcal{C} \vee \mathcal{C} := \{C \cap D : C, D \in \mathcal{C}\}$ .

The notion's importance stems from Theorem 3.1 below. If the set  $C(X, Y)$  of continuous functions from  $X$  to  $Y$  is equipped with the Isbell topology (<sup>5</sup>), we denote it  $C_\kappa(X, Y)$ , while  $C_k(X, Y)$  denotes  $C(X, Y)$  endowed with the compact-open topology. Note that  $C_\kappa(X, \$) = T[X, \$]$ .

**Theorem 3.1** ([6]). *Let  $X$  be a completely regular topological space. The following are equivalent:*

- (1)  $X$  is infraconsonant;
- (2) addition is jointly continuous at the zero function in  $C_\kappa(X, \mathbb{R})$ ;
- (3)  $C_\kappa(X, \mathbb{R})$  is a topological vector space;
- (4)  $\cap : T[X, \$] \times T[X, \$] \rightarrow T[X, \$]$  is jointly continuous.

On the other hand, if  $X$  is consonant then  $C_\kappa(X, \mathbb{R}) = C_k(X, \mathbb{R})$  so that consonance provides an obvious sufficient condition for  $C_\kappa(X, \mathbb{R})$  to be a topological vector space.

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<sup>5</sup>whose sub-basic open sets are given by

$$[\mathcal{A}, U] := \{f \in C(X, Y) : \exists A \in \mathcal{A}, f(A) \subseteq U\},$$

where  $\mathcal{A}$  ranges over openly isotone compact families on  $X$  and  $U$  ranges over open subsets of  $Y$ .



Hence Theorem 3.1 becomes truly interesting if completely regular examples of infraconsonant non consonant spaces can be provided [9, Problem 1.2]. The first author recently obtained the first example of this kind [18]. The following results show that a space answering positively Problem 2.9 would necessarily be infraconsonant and non-consonant and might provide an avenue to construct new examples.

**Theorem 3.2.** *If  $X$  is topological and  $T[X, \$]$  is core compact then  $X$  is infraconsonant.*

*Proof.* [9, Lemma 3.3] shows the equivalence between (1) and (4) in Theorem 3.1, and that the implication (4) $\implies$ (1) does not require any separation. Therefore, it is enough to show that  $\cap : T[X, \$] \times T[X, \$] \rightarrow T[X, \$]$  is continuous. Since  $X$  is topological,  $[X, \$]$  is  $T$ -dual by Proposition 2.1. In view of (1.4)

$$T([X, \$] \times [X, \$]) \leq [X, \$] \times T[X, \$]$$

so that  $T([X, \$] \times [X, \$]) \leq T([X, \$] \times T[X, \$])$ . If  $T[X, \$]$  is core compact, hence  $T$ -dual then  $T([X, \$] \times T[X, \$]) \leq T[X, \$] \times T[X, \$]$  so that

$$(3.1) \quad T([X, \$] \times [X, \$]) \leq T[X, \$] \times T[X, \$].$$

Therefore the continuity of  $\cap : [X, \$] \times [X, \$] \rightarrow [X, \$]$  implies that of  $\cap : T([X, \$] \times [X, \$]) \rightarrow T[X, \$]$  because  $T$  is a functor, and in view of (3.1), that of  $\cap : T[X, \$] \times T[X, \$] \rightarrow T[X, \$]$ .  $\square$

Recall that a basis for the topology of  $C_k(X, \$)$  is given by sets of the form  $\mathcal{O}(K)$  where  $K$  ranges over compact subsets of  $X$ .

**Theorem 3.3.** *Let  $X$  be a topological space. If  $C_k(X, \$)$  is core compact then  $X$  is locally compact.*

*Proof.* If  $X$  is not locally compact, then  $C_k(X, \$) \not\cong [X, \$]$  (e.g., [23, 2.19]) so that there is  $U_0 \in \mathcal{O}_X$  with  $U_0 \notin \lim_{[X, \$]} \mathcal{N}_k(U_0)$ . Therefore, there is  $x_0 \in U_0$  such that  $x_0 \notin \text{int}(\bigcap_{V \in \mathcal{O}(K)} V)$  whenever  $K$  is a compact subset of  $X$  with  $K \subseteq U_0$ . In other words, for each such  $K$  and for each  $U \in \mathcal{O}(x_0)$  there is  $V_U \in \mathcal{O}(K)$  and  $x_U \in U \setminus V_U$ . Then  $C_k(X, \$)$  is not core compact at  $U_0$ . Indeed, there is  $U_0 \in \mathcal{O}(x_0)$  such that for every compact set  $K$  with  $K \subseteq U_0$ , the  $k$ -open set  $\mathcal{O}(K)$  is not relatively compact in  $\mathcal{O}(x_0)$ . To see that, consider the cover  $\mathcal{S} := \{\mathcal{O}(x_U) : U \in \mathcal{O}(x_0)\}$  of  $\mathcal{O}(x_0)$ . No finite subfamily of  $\mathcal{S}$  covers  $\mathcal{O}(K)$  because for any finite choice of  $U_1, \dots, U_n$  in  $\mathcal{O}(x_0)$ , we have  $W := \bigcap_{i=1}^n V_{U_i} \in \mathcal{O}(K)$  but  $W \notin \bigcup_{i=1}^n \mathcal{O}(x_{U_i})$ .  $\square$

Note that a Hausdorff topological space  $X$  is locally compact if and only if it is core compact, and that the Scott open filter topology on  $\mathcal{O}_X$  then coincides with  $C_k(X, \$)$  (e.g., [11, Lemma II.1.19]). Hence Theorem 3.3 could also be deduced (for the Hausdorff case) from [19, Corollary 3.6].

**Corollary 3.4.** *If  $X$  is a consonant topological space such that  $T[X, \$]$  is core compact, then  $X$  is locally compact.*

4. SCOTT TOPOLOGY OF THE PRODUCT VERSUS PRODUCT OF SCOTT TOPOLOGIES

We now turn to a new characterization of infraconsonance, which motivates further the systematic investigation of the notion.

Recall that in a complete lattice  $(L, \leq)$  the Scott convergence is given by (1.3), and the *Scott topology* is its topological modification. A subset  $A$  of  $L$  is Scott-open if and only if it is upper-closed and satisfies

$$\bigvee D \in A \implies \exists d \in D \cap A,$$

for every directed supset  $D$  of  $L$  (e.g., [11]). A product of complete lattices is a complete lattice for the coordinatewise order, and we can therefore consider the Scott topology on the product for the coordinatewise order, and compare it with the product of the Scott topologies.

**Proposition 4.1.**  *$T([X, \$]^2)$  is the Scott topology on  $\mathcal{O}_X \times \mathcal{O}_X$ .*

**Theorem 4.2.** *A space  $X$  is infraconsonant if and only if the product  $T[X, \$] \times T[X, \$]$  of the Scott topologies and the Scott topology  $T([X, \$] \times [X, \$])$  on the product coincide at  $(X, X)$ .*

**Lemma 4.3.** *A subset  $\mathcal{S}$  of  $\mathcal{O}_X \times \mathcal{O}_X$  is  $[X, \$]^2$ -open if and only if*

- (1)  $\mathcal{S} = \mathcal{S}^\uparrow$ , that is, if  $(U, V) \in \mathcal{S}$  and  $U \subseteq U'$ ,  $V \subseteq V'$  then  $(U', V') \in \mathcal{S}$ ;
- (2)  $\mathcal{S}$  is coordinatewise compact, that is,

$$\left( \bigcup_{i \in I} O_i, \bigcup_{j \in J} V_j \right) \in \mathcal{S} \implies \exists I_0 \in [I]^{<\omega}, J_0 \in [J]^{<\omega} : \left( \bigcup_{i \in I_0} O_i, \bigcup_{j \in J_0} V_j \right) \in \mathcal{S}$$

*Proof.* Assume  $\mathcal{S}$  is  $[X, \$]^2$ -open and let  $(U, V) \in \mathcal{S}$  and  $U \subseteq U'$ ,  $V \subseteq V'$ . Then  $(U, V) \in \lim_{[X, \$]^2} \{(U', V')\}^\uparrow$  so that  $(U', V') \in \mathcal{S}$ . Assume now that  $(\bigcup_{i \in I} O_i, \bigcup_{j \in J} V_j) \in \mathcal{S}$ . Then  $\{\mathcal{O}(\bigcup_{i \in F} O_i) : F \in [I]^{<\infty}\}$  is a filter-base for a filter  $\gamma$  on  $\mathcal{O}_X$  such that  $\bigcup_{i \in I} O_i \in \lim_{[X, \$]} \gamma$  and  $\{\mathcal{O}(\bigcup_{j \in D} V_j) : D \in [J]^{<\infty}\}$  is a filter-base for a filter  $\eta$  on  $\mathcal{O}_X$  such that  $\bigcup_{j \in J} V_j \in \lim_{[X, \$]} \eta$ . Hence  $\mathcal{S} \in \gamma \times \eta$  because  $\mathcal{S}$  is  $[X, \$]^2$ -open. Therefore, there are finite subsets  $I_0$  of  $I$  and  $J_0$  of  $J$  such that  $\mathcal{O}(\bigcup_{i \in I_0} O_i) \times \mathcal{O}(\bigcup_{j \in J_0} V_j) \subseteq \mathcal{S}$ , so that  $(\bigcup_{i \in I_0} O_i, \bigcup_{j \in J_0} V_j) \in \mathcal{S}$ .

Conversely, assume that  $\mathcal{S}$  satisfies the two conditions of the Lemma and  $(U, V) \in \mathcal{S} \cap \lim_{[X, \$]^2} (\gamma \times \eta)$ . Since  $U \subseteq \bigcup_{\mathcal{G} \in \gamma} \text{int}(\bigcap \mathcal{G})$  and  $V \subseteq \bigcup_{\mathcal{H} \in \eta} \text{int}(\bigcap_{H \in \mathcal{H}} H)$ , we have, by the first condition, that

$$\left( \bigcup_{\mathcal{G} \in \gamma} \text{int}(\bigcap_{G \in \mathcal{G}} G), \bigcup_{\mathcal{H} \in \eta} \text{int}(\bigcap_{H \in \mathcal{H}} H) \right) \in \mathcal{S}.$$

By the second condition, there are  $\mathcal{G}_1, \dots, \mathcal{G}_k \in \gamma$  and  $\mathcal{H}_1, \dots, \mathcal{H}_n \in \eta$  such that

$$\left( \bigcup_{i=1}^k \text{int}(\bigcap_{G \in \mathcal{G}_i} G), \bigcup_{j=1}^n \text{int}(\bigcap_{H \in \mathcal{H}_j} H) \right) \in \mathcal{S}.$$

Therefore  $(\text{int}(\bigcap_{G \in \bigcap_{i=1}^k \mathcal{G}_i} G), \text{int}(\bigcap_{H \in \bigcap_{j=1}^n \mathcal{H}_j} H)) \in \mathcal{S}$  so that

$$\left( \bigcap_{i=1}^k \mathcal{G}_i, \bigcap_{j=1}^n \mathcal{H}_j \right) \subseteq \mathcal{S},$$

and  $\mathcal{S} \in \gamma \times \eta$ .  $\square$

*Proof of Proposition 4.1.* In view of Lemma 4.3, every  $[X, \$]^2$ -open subset of  $\mathcal{O}_X \times \mathcal{O}_X$  is Scott open. Conversely, consider a Scott open subset  $\mathcal{S}$  of  $\mathcal{O}_X \times \mathcal{O}_X$ . We only have to check that  $\mathcal{S}$  satisfies the second condition in Lemma 4.3. Let  $(\bigcup_{i \in I} O_i, \bigcup_{j \in J} V_j) \in \mathcal{S}$ . The set  $D := \{(\bigcup_{i \in I_0} O_i, \bigcup_{j \in J_0} V_j) : I_0 \in [I]^{<\omega}, J_0 \in [J]^{<\omega}\}$  is a directed subset of  $\mathcal{O}_X \times \mathcal{O}_X$  (for the coordinatewise inclusion order) whose supremum is  $(\bigcup_{i \in I} O_i, \bigcup_{j \in J} V_j)$ . As  $\mathcal{S}$  is Scott-open, there are finite subsets  $I_0$  of  $I$  and  $J_0$  of  $J$  such that  $(\bigcup_{i \in I_0} O_i, \bigcup_{j \in J_0} V_j) \in \mathcal{S}$ .  $\square$

**Lemma 4.4.** *If  $\mathcal{A} \in \kappa(X)$  then  $\mathcal{S}_{\mathcal{A}} := \{(U, V) \in \mathcal{O}_X \times \mathcal{O}_X : U \cap V \in \mathcal{A}\}^\uparrow$  is  $[X, \$]^2$ -open.*

*Proof.* Let  $(\bigcup_{i \in I} O_i, \bigcup_{j \in J} V_j) \in \mathcal{S}_{\mathcal{A}}$ . Then

$$\left( \bigcup_{i \in I} O_i \right) \cap \left( \bigcup_{j \in J} V_j \right) = \bigcup_{(i,j) \in I \times J} O_i \cap V_j \in \mathcal{A}.$$

By compactness of  $\mathcal{A}$ , there is a finite subset  $I_0$  of  $I$  and a finite subset  $J_0$  of  $J$  such that  $\bigcup_{(i,j) \in I_0 \times J_0} O_i \cap V_j \in \mathcal{A}$ , so that  $(\bigcup_{i \in I_0} O_i, \bigcup_{j \in J_0} V_j) \in \mathcal{S}_{\mathcal{A}}$ . In view of Lemma 4.3,  $\mathcal{S}_{\mathcal{A}}$  is  $[X, \$]^2$ -open.  $\square$

**Lemma 4.5.** *If  $\mathcal{S}$  is  $[X, \$]^2$ -open, then*

$$\downarrow \mathcal{S} := \mathcal{O}_X(\{U \cup V : (U, V) \in \mathcal{S}\})$$

*is a compact family on  $X$ .*

*Proof.* If  $U \cup V \subseteq \bigcup_{i \in I} O_i$  for some  $(U, V) \in \mathcal{S}$  then  $(\bigcup_{i \in I} O_i, \bigcup_{i \in I} O_i) \in \mathcal{S}$  so that, in view of Lemma 4.3, there is a finite subset  $I_0$  of  $I$  such that  $(\bigcup_{i \in I_0} O_i, \bigcup_{i \in I_0} O_i) \in \mathcal{S}$ . Hence  $\bigcup_{i \in I_0} O_i \in \downarrow \mathcal{S}$ .  $\square$

*Proof of Theorem 4.2.* Suppose that  $X$  is infraconsonant. Note that  $(T[X, \$])^2 \leq T([X, \$]^2)$  is always true, so that we only have to prove the reverse inequality at  $(X, X)$ . Consider an  $[X, \$]^2$ -open neighborhood  $\mathcal{S}$  of  $(X, X)$ . By Lemma 4.5, the family  $\downarrow \mathcal{S}$  is compact. By infraconsonance, there is  $\mathcal{C} \in \kappa(X)$  with  $\mathcal{C} \vee \mathcal{C} \subseteq \downarrow \mathcal{S}$ . Note that

$$\mathcal{C} \times \mathcal{C} \subseteq \mathcal{S},$$

because if  $(C_1, C_2) \in \mathcal{C} \times \mathcal{C}$  then  $C_1 \cap C_2 \in \downarrow \mathcal{S}$  so that  $C_1 \cap C_2 \supseteq U \cup V$  for some  $(U, V) \in \mathcal{S}$ , and therefore  $(C_1, C_2) \in \mathcal{S}$ .

Conversely, assume that  $\mathcal{N}_{[X, \$]^2}(X, X) = \mathcal{N}_{T[X, \$]^2}(X, X)$  and let  $\mathcal{A} \in \kappa(X)$ . By Lemma 4.4,  $\mathcal{S}_{\mathcal{A}} \in \mathcal{N}_{[X, \$]^2}(X, X)$  so that  $\mathcal{S}_{\mathcal{A}} \in \mathcal{N}_{T[X, \$]^2}(X, X)$ . In other words, there are families  $\mathcal{B}$  and  $\mathcal{C}$  in  $\kappa(X)$  such that  $\mathcal{B} \times \mathcal{C} \subseteq \mathcal{S}_{\mathcal{A}}$ . In particular

$\mathcal{D} := \mathcal{B} \cap \mathcal{C}$  belongs to  $\kappa(X)$  and satisfies  $\mathcal{D} \times \mathcal{D} \subseteq \mathcal{S}_A$ . By definition of  $\mathcal{S}_A$ , we have that  $\mathcal{D} \vee \mathcal{D} \subseteq \mathcal{A}$  and  $X$  is infraconsonant.  $\square$

### 5. TOPOLOGICITY, PRETOPOLOGICITY AND DIAGONALITY OF $[X, \$]$

A *selection* for a convergence space  $X$  is a map  $\mathcal{S}[\cdot] : X \rightarrow \mathbb{F}X$  such that  $x \in \lim_X \mathcal{S}[x]$  for all  $x \in X$ .

**Definition 5.1.** A convergence space  $X$  is *diagonal* if for every selection  $\mathcal{S}[\cdot]$  and every filter  $\mathcal{F}$  with  $x_0 \in \lim_X \mathcal{F}$  the filter

$$(5.1) \quad \mathcal{S}[\mathcal{F}] := \bigcup_{F \in \mathcal{F}} \bigcap_{x \in F} \mathcal{S}[x]$$

converges to  $x_0$ . If this property only holds when  $\mathcal{F}$  is additionally principal, we say that  $X$  is  $\mathbb{F}_0$ -*diagonal*.

Of course, every topology is diagonal. In fact a convergence is topological if and only if it is both pretopological and diagonal (e.g., [4]).

In order to compare our condition for diagonality of  $[X, \$]$  with core-compactness, we first rephrase the latter.

**Lemma 5.2.** *A topological space is core compact if and only if for every  $x \in X$ , every  $U \in \mathcal{O}(x)$  and every family  $\mathbb{H}$  of filters on  $X$ , we have*

$$(5.2) \quad \forall \mathcal{H} \in \mathbb{H} : \text{adh } \mathcal{H} \cap U = \emptyset \implies x \notin \text{adh } \bigwedge_{\mathcal{H} \in \mathbb{H}} \mathcal{H}.$$

*Proof.* If  $X$  is core compact, then there is  $V \in \mathcal{O}(x)$  which is relatively compact in  $U$ . If  $\text{adh } \mathcal{H} \cap U = \emptyset$ , then  $U \subseteq \bigcup_{H \in \mathcal{H}} (\text{cl } H)^c$  so that, by relative compactness of  $V$  in  $U$  there is, for each  $\mathcal{H} \in \mathbb{H}$ , a set  $H_{\mathcal{H}} \in \mathcal{H}$  with  $V \cap \text{cl } H_{\mathcal{H}} = \emptyset$ . Then  $\bigcup_{\mathcal{H} \in \mathbb{H}} H_{\mathcal{H}} \in \bigwedge_{\mathcal{H} \in \mathbb{H}} \mathcal{H}$  but  $\bigcup_{\mathcal{H} \in \mathbb{H}} H_{\mathcal{H}} \cap V = \emptyset$  so that  $x \notin \text{adh } \bigwedge_{\mathcal{H} \in \mathbb{H}} \mathcal{H}$ .

Conversely, if (5.2) is true, consider the family  $\mathbb{H} := \{\mathcal{H} \in \mathbb{F}X : \text{adh } \mathcal{H} \cap U = \emptyset\}$ . In view of (5.2),  $x \notin \text{adh } \bigwedge_{\mathcal{H} \in \mathbb{H}} \mathcal{H}$  so that there is  $V \in \mathcal{O}(x)$  such that  $V \notin (\bigwedge_{\mathcal{H} \in \mathbb{H}} \mathcal{H})^\#$ . Now  $V$  is relatively compact in  $U$  because any filter than meshes with  $V$  cannot be in  $\mathbb{H}$  and has therefore an adherence point in  $U$ .  $\square$

Recall that  $[X, \$] = P[X, \$]$  if and only if  $X$  is  $T$ -core compact, and that, if  $X$  is topological,  $[X, \$]$  is topological whenever it is pretopological. While the latest is well-known, and follows for instance from the results of [7], it seems difficult to find an elementary argument in the literature, which is why we include the following proposition, which also illustrates the usefulness of Lemma 5.2.

**Proposition 5.3.** *If  $X$  is topological and  $[X, \$]$  is pretopological, then  $[X, \$]$  is topological.*

*Proof.* We will show that under these assumptions,  $X$  satisfies (5.2). Let  $x \in X$  and  $U \in \mathcal{O}(x)$ . Let  $\mathbb{H}$  be a family of filters satisfying the hypothesis of (5.2). Let  $\mathcal{H} \in \mathbb{H}$ . Consider the filter base  $\mathcal{H}^* := \{\mathcal{O}(X \setminus \text{cl}(H)) : H \in \mathcal{H}\}$  on  $[X, \$]$ . Since  $\text{adh}(\mathcal{H}) \cap U = \emptyset$ , it follows that  $U \in \lim \mathcal{H}^*$ . Since  $[X, \$]$  is pretopological,

$U \in \lim \bigwedge_{\mathcal{H} \in \mathbb{H}} \mathcal{H}^*$ . In particular, there exist, for each  $\mathcal{H} \in \mathbb{H}$ , a  $H_{\mathcal{H}} \in \mathcal{H}$  such that

$$\begin{aligned} x \in \text{int}\left(\bigcap_{\mathcal{H} \in \mathbb{H}} \bigcup \mathcal{O}(X \setminus \text{cl } H_{\mathcal{H}})\right) &= \text{int}\left(\bigcap_{\mathcal{H} \in \mathbb{H}} (X \setminus \text{cl } H_{\mathcal{H}})\right) \\ &= \text{int}\left(X \setminus \left(\bigcup_{\mathcal{H} \in \mathbb{H}} \text{cl } H_{\mathcal{H}}\right)\right) \\ &\subseteq X \setminus \text{cl}\left(\bigcup_{\mathcal{H} \in \mathbb{H}} H_{\mathcal{H}}\right). \end{aligned}$$

Thus,  $x \notin \text{adh}(\bigwedge_{\mathcal{H} \in \mathbb{H}} \mathcal{H})$ .  $\square$

In other words, if  $[X, \$]$  is pretopological it is also diagonal, provided that  $X$  is topological. We will see that even if  $X$  is topological,  $[X, \$]$  is not always diagonal. Moreover it can be diagonal without being pretopological (examples 5.5 and 5.7).

We call a topological space *injectively core compact* if for every  $x \in X$  and  $U \in \mathcal{O}(x)$  the conclusion of (5.2) holds for every family  $\mathbb{H}$  of filters such that there is an injection  $\theta : \mathbb{H} \rightarrow \mathcal{O}(U)$  satisfying  $\text{adh } \mathcal{H} \cap \theta(\mathcal{H}) = \emptyset$  for each  $\mathcal{H} \in \mathbb{H}$ . As such a family  $\mathbb{H}$  clearly satisfies the premise of (5.2), every core compact space is in particular injectively core compact.

**Theorem 5.4.** *Let  $X$  be a topological space. The following are equivalent:*

- (1)  $X$  is injectively core compact;
- (2)  $[X, \$]$  is diagonal;
- (3)  $[X, \$]$  is  $\mathbb{F}_0$ -diagonal.

*Proof.* (1) $\implies$ (2): Let  $\mathcal{S}[\cdot] : \mathcal{O}_X \rightarrow \mathbb{F}\mathcal{O}_X$  be a selection for  $[X, \$]$  and let  $U \in \lim_{[X, \$]} \mathcal{F}$ . If  $x \in U$ , there is  $F \in \mathcal{F}$  such that  $x \in \text{int}(\bigcap_{O \in F} O) := V$ . Note that  $F \subseteq \mathcal{O}(V)$ . For each  $O \in F$ , consider the filter  $\mathcal{H}_O$  on  $X$  generated by  $\{\text{cl}_X(\bigcup_{W \in S} W^c) : S \in \mathcal{S}[O]\}$ . Because  $O \in \lim_{[X, \$]} \mathcal{S}[O]$ , we have that  $\text{adh}_X \mathcal{H}_O \cap O = \emptyset$ . Choose  $G \subseteq F$  so that  $\{\mathcal{H}_O : O \in G\} = \{\mathcal{H}_O : O \in F\}$  and  $\mathcal{H}_O \neq \mathcal{H}_P$  for every two distinct  $O, P \in G$ . Because  $X$  is injectively core compact and  $\mathbb{H} := \{\mathcal{H}_O : O \in G\}$  satisfies the required condition (with  $\theta(\mathcal{H}_O) = O$ ), we conclude that  $x \notin \text{adh}_X \bigwedge_{O \in G} \mathcal{H}_O$ . By the way we chose  $G$ , we have  $\bigwedge_{O \in G} \mathcal{H}_O = \bigwedge_{O \in F} \mathcal{H}_O$ . So,  $x \notin \text{adh}_X \bigwedge_{O \in F} \mathcal{H}_O$ . In other words, there is an  $H \in \bigwedge_{O \in F} \mathcal{H}_O$  such that  $x \notin \text{cl}_X H$ , that is,  $x \in \text{int}_X H^c$ . Therefore, for each  $O \in F$  there is  $S_O \in \mathcal{S}[O]$  such that

$$x \in \text{int}\left(\bigcap_{O \in F} \text{int}\left(\bigcap_{W \in S_O} W\right)\right) \subseteq \text{int}\left(\bigcap_{W \in \bigcup_{O \in F} S_O} W\right).$$

In other words, there is  $F \in \mathcal{F}$  and  $M \in \bigwedge_{O \in F} \mathcal{S}[O]$  such that  $x \in \text{int}_X(\bigcap_{W \in M} W)$ , that is,  $U \in \lim_{[X, \$]} \mathcal{S}[\mathcal{F}]$ .

(2) $\implies$ (3) is clear. (3) $\implies$ (1): Suppose  $X$  is not injectively core compact. Then there is  $x \in X$ ,  $U \in \mathcal{O}(x)$  and a family  $\mathbb{H}$  of filters on  $X$  with an injective map  $\theta : \mathbb{H} \rightarrow \mathcal{O}(U)$  such that  $\theta(\mathcal{H}) \cap \text{adh}_X \mathcal{H} = \emptyset$  for each  $\mathcal{H} \in \mathbb{H}$  but  $x \in \text{adh}_X \bigwedge_{\mathcal{H} \in \mathbb{H}} \mathcal{H}$ . Define a relation  $\sim$  on  $\mathbb{H}$  by  $\mathcal{H}_1 \sim \mathcal{H}_2$  provided that

the collections  $\{\text{cl}(H): H \in \mathcal{H}_1\}$  and  $\{\text{cl}(H): H \in \mathcal{H}_2\}$  both generate the same filter. Clearly,  $\sim$  is an equivalence relation. Let  $\mathbb{H}^* \subseteq \mathbb{H}$  be such that  $\mathbb{H}^*$  contains exactly one element of each equivalence class of  $\sim$ . For each  $\mathcal{H} \in \mathbb{H}^*$  let  $\mathcal{H}^*$  be the filter with base  $\{\text{cl}(H): H \in \mathcal{H}\}$ . Let  $\mathbb{J} = \{\mathcal{H}^*: \mathcal{H} \in \mathbb{H}^*\}$ .

Define  $\theta^*: \mathbb{J} \rightarrow \mathcal{O}(U)$  so that  $\theta^*(\mathcal{J}) = \theta(\mathcal{H})$ , where  $\mathcal{H} \in \mathbb{H}^*$  is such that  $\mathcal{J} = \mathcal{H}^*$ . It is easily checked that  $\theta^*$  is injective. Since  $\text{adh}(\mathcal{H}^*) = \text{adh}(\mathcal{H})$  for every  $\mathcal{H} \in \mathbb{H}^*$ , we have  $\theta^*(\mathcal{J}) \cap \text{adh}(\mathcal{J}) = \emptyset$ . It is also easy to check that  $x \in \text{adh}\left(\bigwedge_{\mathcal{J} \in \mathbb{J}} \mathcal{J}\right)$ .

For each  $\mathcal{J} \in \mathbb{J}$ , the filter  $\tilde{\mathcal{J}}$  generated on  $\mathcal{O}_X$  by the filter-base  $\{O_X(X \setminus J) : J \in \mathcal{J}\}$  converges to  $\theta^*(\mathcal{J})$ . Consider now the subset  $\theta^*(\mathbb{J})$  of  $\mathcal{O}(U) \subseteq \mathcal{O}_X$  and the selection  $\mathcal{S}[\cdot] : \mathcal{O}_X \rightarrow \mathbb{F}\mathcal{O}_X$  defined by  $\mathcal{S}[\theta(\mathcal{J})] = \tilde{\mathcal{J}}$  for each  $\mathcal{J} \in \mathbb{J}$  and  $\mathcal{S}[O] = \{O\}^\uparrow$  for  $O \notin \theta^*(\mathbb{J})$ . This is indeed a well-defined selection because  $\theta^*$  is injective.

Notice that  $U \in \lim_{[X, \mathbb{S}]} \theta^*(\mathbb{J})$  because  $\theta^*(\mathbb{J}) \subseteq \mathcal{O}(U)$ . Let  $L \in \mathcal{S}[\theta^*(\mathbb{J})]$ . We may pick from each  $\mathcal{J} \in \mathbb{J}$  a closed set  $J_{\mathcal{J}} \in \mathcal{J}$  such that  $\bigcup_{\mathcal{J} \in \mathbb{J}} O_x(X \setminus J_{\mathcal{J}}) \subseteq L$ . Let  $V$  be an open neighborhood of  $x$ . Since  $x \in \text{adh}_X \bigwedge_{\mathcal{J} \in \mathbb{J}} \mathcal{J}$  and  $\bigcup_{\mathcal{J} \in \mathbb{J}} J_{\mathcal{J}} \in \bigwedge_{\mathcal{J} \in \mathbb{J}} \mathcal{J}$ , there is an  $\mathcal{J}_0 \in \mathbb{J}$  such that  $V \cap J_{\mathcal{J}_0} \neq \emptyset$ . Since  $V \not\subseteq X \setminus J_{\mathcal{J}_0}$  and  $X \setminus J_{\mathcal{J}_0} \in \mathcal{O}_X(X \setminus J_{\mathcal{J}_0})$ ,  $V \not\subseteq \bigcap \mathcal{O}_X(X \setminus J_{\mathcal{J}_0})$ . Since  $\mathcal{O}_X(X \setminus J_{\mathcal{J}_0}) \subseteq L$ ,  $V \not\subseteq \bigcap L$ . Since  $V$  was an arbitrary neighborhood of  $x$ ,  $x \notin \text{int}(\bigcap L)$ . Thus,  $U \notin \mathcal{S}[\theta^*(\mathbb{J})]$ . Therefore,  $[X, \mathbb{S}]$  is not  $\mathbb{F}_0$ -diagonal.  $\square$

A cardinal number  $\kappa$  is *regular* if a union of less than  $\kappa$ -many sets of cardinality less than  $\kappa$  has cardinality less than  $\kappa$ . A strong limit cardinal  $\kappa$  is a cardinal for which  $\text{card}(2^A) < \kappa$  whenever  $\text{card}(A) < \kappa$ . A *strongly inaccessible cardinal* is a regular strong limit cardinal. Uncountable strongly inaccessible cardinals cannot be proved to exist within ZFC, though their existence is not known to be inconsistent with ZFC. Let us denote by  $(*)$  the assumption that such a cardinal exist.

**Example 5.5** (A Hausdorff space  $X$  such that  $[X, \mathbb{S}]$  is diagonal but not pretopological under  $(*)$ ). Assume that  $\kappa$  is a (uncountable) strong limit cardinal. Let  $X$  be the subspace of  $\kappa \cup \{\kappa\}$  endowed with the order topology, obtained by removing all the limit ordinals but  $\kappa$ . Since  $X$  is a non locally compact Hausdorff topological space,  $[X, \mathbb{S}]$  is not pretopological. To show that  $X$  is injectively core compact, we only need to consider  $x = \kappa$  and  $U \in \mathcal{O}(\kappa)$  in the definition, because  $\kappa$  is the only non-isolated point of  $X$ . Let  $\mathbb{H}$  be a family of filters on  $X$  admitting an injective map  $\theta : \mathbb{H} \rightarrow \mathcal{O}(U)$  such that  $\text{adh} \mathcal{H} \cap \theta(\mathcal{H}) = \emptyset$  for each  $\mathcal{H} \in \mathbb{H}$ . For each  $\mathcal{H} \in \mathbb{H}$  there is  $H_{\mathcal{H}} \in \mathcal{H}$  such that  $\kappa \notin \text{cl}(H_{\mathcal{H}})$  so that  $\text{card}(H_{\mathcal{H}}) < \kappa$ . Since  $U$  is a neighborhood of  $\kappa$ , there is a  $\beta < \kappa$  such that  $\{\xi \in X : \beta \leq \xi\} \subseteq U$ . Since  $V \setminus U \subseteq \{\xi \in X : \xi < \beta\}$  for every  $V \in \mathcal{O}(U)$ , we have  $\text{card} \mathbb{H} \leq \text{card} \mathcal{O}(U) \leq 2^\beta$ . Since  $\kappa$  is a strong limit cardinal,  $\text{card} \mathbb{H} < \kappa$ . Since  $\kappa$  is regular,  $\text{card} \bigcup_{\mathcal{H} \in \mathbb{H}} H_{\mathcal{H}} < \kappa$  so that  $\kappa \notin \text{adh} \bigwedge_{\mathcal{H} \in \mathbb{H}} \mathcal{H}$ .

We do not know if the existence of large cardinals is necessary for the construction of a Hausdorff space  $X$  such that  $[X, \$]$  is diagonal and not pretopological, but, as the next proposition shows, such a space cannot be too small. Let  $\mathfrak{c}$  denote the cardinality of the real numbers.

**Proposition 5.6.** *Let  $X$  be a Hausdorff topological space. If  $X$  is a non locally compact space of character not exceeding  $\mathfrak{c}$ , then  $[X, \$]$  is not diagonal.*

*Proof.* Let  $p \in X$  be such that  $X$  is not locally compact at  $p$ . Since  $X$  is not compact, there is a neighborhood  $U$  of  $p$  such that  $X \setminus U$  is infinite. Since  $X$  is Hausdorff, there exists a countably infinite  $A \subseteq X \setminus U$  and mutually disjoint open sets  $\{W_a : a \in A\}$  such that  $a \in W_a$  for every  $a \in A$ . It follows that the collection  $\{U \cup \bigcup_{a \in E} W_a : E \subseteq A\}$  is a collection of  $\mathfrak{c}$ -many distinct elements of  $\mathcal{O}(U)$ . Since the character of  $X$  is at most  $\mathfrak{c}$ , there is a neighborhood base  $\mathcal{B}$  at  $p$  with at most  $\mathfrak{c}$ -many elements. Since  $X$  is not locally compact at  $p$ , there is for each  $B \in \mathcal{B}$  a filter  $\mathcal{H}_B$  on  $B$  such that  $\text{adh}(\mathcal{H}_B) = \emptyset$ . Let  $\mathbb{H} = \{\mathcal{H}_B : B \in \mathcal{B}\}$ . Since  $\text{card } \mathcal{B} \leq \text{card } \mathcal{O}(U)$ , there is an injection  $\theta : \mathbb{H} \rightarrow \mathcal{O}(U)$ . Clearly,  $\text{adh}(\mathcal{H}_B) \cap \theta(\mathcal{H}_B) = \emptyset$  for every  $B \in \mathcal{B}$ . However,  $p \in \text{adh}(\bigwedge_{B \in \mathcal{B}} \mathcal{H}_B)$ . Hence,  $X$  is not injectively core compact at  $p$ . Thus,  $[X, \$]$  is not diagonal.  $\square$

On the other hand, we can construct in ZFC a  $T_0$  space  $X$  such that  $[X, \$]$  is diagonal and not pretopological.

**Example 5.7** (A  $T_0$  space  $X$  such that  $[X, \$]$  is diagonal but not pretopological). *Let  $\mathbb{Z}$  stand for integers and  $\mathfrak{c}^+$  be the cardinal successor of  $\mathfrak{c}$ . Let  $\infty$  be a point that is not in  $\mathfrak{c}^+ \times \mathbb{Z}$  and  $X = \{\infty\} \cup (\mathfrak{c}^+ \times \mathbb{Z})$ . For each  $(\alpha, n) \in \mathfrak{c}^+ \times \mathbb{Z}$  define  $S_{\alpha, n} = \{(\beta, k) : \alpha \leq \beta \text{ and } n \leq k\}$ . For each  $\alpha \in \mathfrak{c}^+$ , let  $T_\alpha = \{(\beta, k) : \alpha \leq \beta \text{ and } k \in \mathbb{Z}\} \cup \{\infty\}$ . Topologize  $X$  by declaring all sets of the form  $T_\alpha$  and  $S_{\alpha, n}$  to be sub-basic open sets.*

*We show that  $X$  is not core compact at  $\infty$ . Let  $U$  be a neighborhood of  $\infty$ . There is an  $\alpha$  such that  $T_\alpha \subseteq U$ . Notice that  $T_{\alpha+1} \cup \{S_{0, n} : n \in \mathbb{Z}\}$  is a cover of  $X$  but no finite subcollection covers  $T_\alpha$ . Thus,  $X$  is not core compact at  $\infty$ . In particular,  $[X, \$]$  is not pretopological.*

*Let  $(\alpha, n) \in X \setminus \{\infty\}$ . Let  $U$  be an open neighborhood of  $(\alpha, n)$ . Since  $(\alpha, n) \in U$  it follows from the way we chose our sub-base that  $S_{\alpha, n} \subseteq U$ . Since  $(\alpha, n)$  has a minimal open neighborhood,  $X$  is core compact at  $(\alpha, n)$ .*

*Let  $V$  be an open neighborhood of  $\infty$ . There is an  $\alpha$  such that  $T_\alpha \subseteq V$ . Let  $U \subseteq X$  be an open superset of  $V$ . For every  $n \in \mathbb{Z}$ ,  $U \cap (\mathfrak{c}^+ \times \{n\}) \neq \emptyset$ . For each  $n \in \mathbb{Z}$  define  $\alpha_n = \min\{\beta : (\beta, n) \in U\}$ . Notice that  $\{\beta : \alpha_n \leq \beta\} \times \{n\} = U \cap (\mathfrak{c}^+ \times \{n\})$  and  $\alpha_n \leq \alpha$ . Since each open superset of  $V$  will determine a unique sequence  $(\alpha_n)_{n \in \mathbb{Z}}$ , it follows that the open supersets of  $V$  can injectively be mapped into the countable sequences on  $\{\beta : \beta \leq \alpha\} \times \mathbb{Z}$ . Since  $\{\beta : \beta \leq \alpha\} \times \mathbb{Z}$  has cardinality at most  $\mathfrak{c}$ ,  $\{\beta : \beta \leq \alpha\} \times \mathbb{Z}$  has at most  $\mathfrak{c}$ -many countable sequences. Thus,  $V$  has at most  $\mathfrak{c}$ -many supersets.*

*Let  $V$  be an open neighborhood of  $\infty$ ,  $\mathbb{H}$  be a collection of filters, and  $\theta : \mathbb{H} \rightarrow \mathcal{O}_X(V)$  be an injection such that  $\text{adh}(\mathcal{H}) \cap \theta(\mathcal{H}) = \emptyset$  for every  $\mathcal{H} \in \mathbb{H}$ . Since  $V$*



has at most  $\mathfrak{c}$ -many open supersets,  $\text{card } \mathbb{H} \leq \mathfrak{c}$ . Let  $\mathcal{H} \in \mathbb{H}$ . Since  $\infty \notin \text{adh } \mathcal{H}$ , there is an  $\alpha_{\mathcal{H}} \in \mathfrak{c}^+$  such that  $\text{adh}(\mathcal{H}) \cap T_{\alpha_{\mathcal{H}}} = \emptyset$ . Let  $\alpha = (\sup_{\mathcal{H} \in \mathbb{H}} \alpha_{\mathcal{H}}) + 1 < \mathfrak{c}^+$ . It is easy to check that,  $\text{adh}(\bigwedge_{\mathcal{H} \in \mathbb{H}} \mathcal{H}) \cap T_{\alpha} = \emptyset$ . Thus,  $X$  is injectively core compact at  $\infty$ .

Since  $X$  is injectively core compact at each point,  $[X, \$]$  is diagonal, by Theorem 5.4.

**Example 5.8** (A  $T$ -dual convergence space that is not core compact). Consider a partition  $\{\mathbb{A}_n : n \in \omega\}$  of the set  $\omega^*$  of free ultrafilters on  $\omega$  satisfying the condition that for every infinite subset  $S$  of  $\omega$  and every  $n \in \omega$ , there is  $\mathcal{U} \in \mathbb{A}_n$  with  $S \in \mathcal{U}$ . Let  $M := \{m_n : n \in \omega\}$  be disjoint from  $\omega$  and let  $X := \omega \cup M$ . Define on  $X$  the finest convergence in which  $\lim\{m_n\}^\uparrow = M$  for all  $n \in \omega$ , and each free ultrafilter  $\mathcal{U}$  on  $\omega$  converges to  $m_n$  (and  $m_n$  only), where  $n$  is defined by  $\mathcal{U} \in \mathbb{A}_n$ .

*Claim.*  $X$  is not core compact.

*Proof.* Let  $m_n \in M$  and  $\mathcal{U} \in \mathbb{A}_n$ . Pick  $S \subseteq \omega$ ,  $S \in \mathcal{U}$ , and  $k \neq n$ . For every  $U \in \mathcal{U}$  there is  $\mathcal{W} \in \mathbb{A}_k$  such that  $U \in \mathcal{W}$ . But  $\lim \mathcal{W} = \{m_k\}$  is disjoint from  $S$ .  $\square$

*Claim.*  $X$  is  $T$ -core compact, and therefore  $[X, \$]$  is pretopological.

*Proof.* For each  $m_n \in M$ , the set  $M$  is included in every open set containing  $m_n$  because  $m_n \in \bigcap_{k \in \omega} \lim\{m_k\}^\uparrow$ . If  $\mathcal{U}$  is a non-trivial convergent ultrafilter in  $X$  then  $\lim \mathcal{U} = \{m_n\}$  for some  $n \in \omega$ . For any  $S \in \mathcal{U}$ ,  $S \cap \omega$  is infinite and any free ultrafilter  $\mathcal{W}$  on  $S \cap \omega$  belongs to one of the element  $\mathbb{A}_k$  of the partition, so that  $\lim \mathcal{W} = \{m_k\}$  intersects  $M$ , and therefore any open set containing  $m_n$ .  $\square$

*Claim.*  $[X, \$]$  is diagonal.

*Proof.* Let  $\mathcal{S}[\cdot] : \mathcal{O}_X \rightarrow \mathbb{F}\mathcal{O}_X$  be a selection for  $[X, \$]$  and let  $U \in \lim_{[X, \$]} \mathcal{F}$ . Now,  $\{\bigcap F : F \in \mathcal{F}\}$  is a (convergence) cover of  $U$ .

Let  $x \in U$  and  $\mathcal{D}$  be a filter on  $X$  such that  $x \in \lim \mathcal{D}$ . There is an  $F \in \mathcal{F}$  and a  $D \in \mathcal{D}$  such that  $D \subseteq \bigcap F := V$ .

Assume  $x \in \omega$ , in which case  $\mathcal{D} = \{x\}^\uparrow$ . In particular,  $x \in O$  for every  $O \in F$ . For every  $O \in F$  there is a  $T_O \in \mathcal{S}[O]$  such that  $x \in \bigcap T_O$ . Now,  $x \in \bigcap \bigcap_{O \in F} T_O \in \mathcal{S}[F]$ . So,  $\bigcap \bigcap_{O \in F} T_O \in \{x\}^\uparrow = \mathcal{D}$ .

Assume  $x \in M$ . In this case,  $M \cap O \neq \emptyset$  for all  $O \in F$  and, by definition of the convergence on  $X$ ,  $M \subseteq O$  for all  $O \in F$ . Since  $O \in \lim_{[X, \$]} \mathcal{S}[O]$  and  $M \subseteq O$ , there is  $S \in \mathcal{S}[O]$  such that  $x \in \bigcap S$ , and, since each element of  $S$  is open,  $M \subseteq \bigcap S$ . If there is no  $S \in \mathcal{S}[O]$  such that  $O \subseteq \bigcap S$  then the filter  $\mathcal{H}$  generated by  $\{(O \cap \omega) \setminus \bigcap S : S \in \mathcal{S}[O]\}$  is non degenerate. Notice that it is not free, for otherwise there would be an  $n \in \omega$  and  $\mathcal{U} \in \mathbb{A}_n$  with  $\mathcal{U} \geq \mathcal{H}$ . But  $m_n \in \lim \mathcal{U} \cap O$ , and there would be  $S \in \mathcal{S}[O]$  such that  $\bigcap S \in \mathcal{U}$ , which is not possible. Therefore, there is  $y \in \bigcap_{S \in \mathcal{S}[O]} ((O \cap \omega) \setminus \bigcap S)$  which



contradicts  $O \in \lim_{[X, \$]} \mathcal{S}[O]$ . Hence, there is  $S_O \in \mathcal{S}[O]$  such that  $O \subseteq \bigcap S_O$ . Now,  $D \subseteq \bigcap F \subseteq \bigcap_{O \in F} \bigcap S_O$ . In particular,  $\bigcap_{O \in F} \bigcap S_O \in \mathcal{D}$ .

Thus,  $\{\bigcap J : J \in \mathcal{S}[\mathcal{F}]\}$  is a cover of  $U$ , and  $[X, \$]$  is diagonal.  $\square$

Therefore  $[X, \$]$  is pretopological and diagonal, hence topological, and  $X$  is  $T$ -dual.

## 6. APPENDIX: CONVERGENCE SPACES

A family  $\mathcal{A}$  of subsets of a set  $X$  is called *isotone* if  $B \in \mathcal{A}$  whenever  $A \in \mathcal{A}$  and  $A \subseteq B$ . We denote by  $\mathcal{A}^\uparrow$  the smallest isotone family containing  $\mathcal{A}$ , that is, the collection of subsets of  $X$  that contain an element of  $\mathcal{A}$ . If  $\mathcal{A}$  and  $\mathcal{B}$  are two families of subsets of  $X$  we say that  $\mathcal{B}$  is *finer than*  $\mathcal{A}$ , in symbols  $\mathcal{A} \leq \mathcal{B}$ , if for every  $A \in \mathcal{A}$  there is  $B \in \mathcal{B}$  such that  $B \subseteq A$ . Of course, if  $\mathcal{A}$  and  $\mathcal{B}$  are isotone, then  $\mathcal{A} \leq \mathcal{B} \iff \mathcal{A} \subseteq \mathcal{B}$ . This defines a partial order on isotone families, in particular on the set  $\mathbb{F}X$  of filters on  $X$ . Every family  $(\mathcal{F}_\alpha)_{\alpha \in I}$  of filters on  $X$  admits an infimum

$$\bigwedge_{\alpha \in I} \mathcal{F}_\alpha := \bigcap_{\alpha \in I} \mathcal{F}_\alpha = \left\{ \bigcup_{\alpha \in I} F_\alpha : F_\alpha \in \mathcal{F}_\alpha \right\}^\uparrow.$$

On the other hand the supremum even of a pair of filters may fail to exist. We call *grill* of  $\mathcal{A}$  the collection  $\mathcal{A}^\# := \{H \subseteq X : \forall A \in \mathcal{A}, H \cap A \neq \emptyset\}$ . It is easy to see that  $\mathcal{A} = \mathcal{A}^{\#\#}$  if and only if  $\mathcal{A}$  is isotone. In particular  $\mathcal{F} = \mathcal{F}^{\#\#} \subseteq \mathcal{F}^\#$  if  $\mathcal{F}$  is a filter. We say that two families  $\mathcal{A}$  and  $\mathcal{B}$  of subsets of  $X$  *mesh*, in symbols  $\mathcal{A} \# \mathcal{B}$ , if  $\mathcal{A} \subseteq \mathcal{B}^\#$ , equivalently if  $\mathcal{B}^\# \subseteq \mathcal{A}$ . The supremum of two filters  $\mathcal{F}$  and  $\mathcal{G}$  exists if and only if they mesh, in which case  $\mathcal{F} \vee \mathcal{G} = \{F \cap G : F \in \mathcal{F}, G \in \mathcal{G}\}^\uparrow$ . An infinite family  $(\mathcal{F}_\alpha)_{\alpha \in I}$  of filters has a supremum  $\bigvee_{\alpha \in I} \mathcal{F}_\alpha$  if pairwise suprema exist and for every  $\alpha, \beta \in I$  there is  $\gamma \in I$  with  $\mathcal{F}_\gamma \geq \mathcal{F}_\alpha \vee \mathcal{F}_\beta$ .

A *convergence*  $\xi$  on a set  $X$  is a relation between  $X$  and the set  $\mathbb{F}X$  of filters on  $X$ , denoted  $x \in \lim_\xi \mathcal{F}$  whenever  $x$  and  $\mathcal{F}$  are in relation, satisfying that  $x \in \lim_\xi \{x\}^\uparrow$  for every  $x \in X$ , and  $\lim_\xi \mathcal{F} \subseteq \lim_\xi \mathcal{G}$  whenever  $\mathcal{F} \leq \mathcal{G}$ . The pair  $(X, \xi)$  is called a *convergence space*. A function  $f : (X, \xi) \rightarrow (Y, \sigma)$  between two convergence space is *continuous* if

$$x \in \lim_\xi \mathcal{F} \implies f(x) \in \lim_\sigma f(\mathcal{F}),$$

where  $f(\mathcal{F})$  is the filter  $\{f(F) : F \in \mathcal{F}\}^\uparrow$  on  $Y$ . If  $\xi$  and  $\tau$  are two convergences on the same set  $X$ , we say that  $\xi$  is *finer than*  $\tau$ , in symbols  $\xi \geq \tau$ , if  $\lim_\xi \mathcal{F} \subseteq \lim_\tau \mathcal{F}$  for every  $\mathcal{F} \in \mathbb{F}X$ . This defines a partial order on the set of convergence structures on  $X$ , which defines a complete lattice for which supremum  $\bigvee_{i \in I} \xi_i$  and infimum  $\bigwedge_{i \in I} \xi_i$  of a family  $\{\xi_i : i \in I\}$  of convergences are defined by

$$\lim_{\bigvee_{i \in I} \xi_i} \mathcal{F} = \bigcap_{i \in I} \lim_{\xi_i} \mathcal{F},$$

$$\lim_{\bigwedge_{i \in I} \xi_i} \mathcal{F} = \bigcup_{i \in I} \lim_{\xi_i} \mathcal{F}.$$

Every topology can be identified with a convergence, in which  $x \in \lim \mathcal{F}$  if  $\mathcal{F} \geq \mathcal{N}(x)$ , where  $\mathcal{N}(x)$  is the neighborhood filter of  $x$  for this topology. A convergence obtained this way is called *topological*. Moreover, a function  $f : X \rightarrow Y$  between two topological spaces is continuous in the usual topological sense if and only if it is continuous in the sense of convergence. On the other hand, every convergence determines a topology in the following way: A subset  $C$  of a convergence space  $(X, \xi)$  is *closed* if  $\lim_{\xi} \mathcal{F} \subseteq C$  for every filter  $\mathcal{F}$  on  $X$  with  $C \in \mathcal{F}$ . A subset  $O$  is *open* if its complement is closed, that is, if  $O \in \mathcal{F}$  whenever  $\lim_{\xi} \mathcal{F} \cap O \neq \emptyset$ . The collection of open subsets for a convergence  $\xi$  is a topology  $T\xi$  on  $X$ , called *topological modification of  $\xi$* . The topology  $T\xi$  is the finest topological convergence coarser than  $\xi$ . If  $f : (X, \xi) \rightarrow (Y, \tau)$  is continuous, so is  $f : (X, T\xi) \rightarrow (Y, T\tau)$ . In other words,  $T$  is a concrete endofunctor of the category **Conv** of convergence spaces and continuous maps.

Continuity induces canonical notions of subspace convergence, product convergence, and quotient convergence. Namely, if  $f : X \rightarrow Y$  and  $Y$  carries a convergence  $\tau$ , there is the coarsest convergence on  $X$  making  $f$  continuous (to  $(Y, \tau)$ ). It is denoted  $f^{-}\tau$  and called *initial convergence for  $f$  and  $\tau$* . For instance if  $S \subseteq X$  and  $(X, \xi)$  is a convergence space, the *induced convergence by  $\xi$  on  $S$*  is by definition  $i^{-}\xi$  where  $i$  is the inclusion map of  $S$  into  $X$ . Similarly, if  $\{(X_i, \xi_i) : i \in I\}$  is a family of convergence space, then the product convergence  $\prod_{i \in I} \xi_i$  on the cartesian product  $\prod_{i \in I} X_i$  is the coarsest convergence making each projection  $p_j : \prod_{i \in I} X_i \rightarrow X_j$  continuous. In other words,  $\prod_{i \in I} \xi_i = \vee_{i \in I} p_i^{-}\xi_i$ . In the case of a product of two factors  $(X, \xi)$  and  $(Y, \tau)$ , we write  $\xi \times \tau$  for the product convergence on  $X \times Y$ .

Dually, if  $f : X \rightarrow Y$  and  $(X, \xi)$  is a convergence space, there is the finest convergence on  $Y$  making  $f$  continuous (from  $(X, \xi)$ ). It is denoted  $f\xi$  and called *final convergence for  $f$  and  $\xi$* . If  $f : (X, \xi) \rightarrow Y$  is a surjection, the associated *quotient convergence on  $Y$*  is  $f\xi$ . Note that if  $\xi$  is a topology, the quotient topology is not  $f\xi$  but  $Tf\xi$ .

The functor  $T$  is a reflector. In other words, the subcategory **Top** of **Conv** formed by topological spaces and continuous maps is closed under initial constructions. Note however that the functor  $T$  does not commute with initial constructions. In particular  $T\xi \times T\tau \leq T(\xi \times \tau)$  but the reverse inequality is generally not true. Similarly, if  $i : S \rightarrow (X, \xi)$  is an inclusion map,  $i^{-}(T\xi) \leq T(i^{-}\xi)$  but the reverse inequality may not hold. A convergence  $\xi$  is *pretopological* or a *pretopology* if  $\lim_{\xi} \bigwedge_{\alpha \in I} \mathcal{F}_{\alpha} = \bigcap_{\alpha \in I} \lim_{\xi} \mathcal{F}_{\alpha}$ . Of course, every topology is a pretopology, but not conversely. For any convergence  $\xi$  there is the finest pretopology  $P\xi$  coarser than  $\xi$ . Moreover,  $x \in \lim_{P\xi} \mathcal{F}$  if and only if  $\mathcal{F} \geq \mathcal{V}_{\xi}(x)$  where  $\mathcal{V}_{\xi}(x) := \bigwedge_{x \in \lim_{\xi} \mathcal{F}} \mathcal{F}$  is called *vicinity filter of  $x$* . The subcategory **PrTop** of **Conv** formed by pretopological spaces and continuous maps is reflective (closed under initial constructions). Moreover, in contrast with topologies, the reflector  $P$  commutes with subspaces. However, like  $T$ , it does not commute with products.

The *adherence*  $\text{adh}_\xi \mathcal{F}$  of a filter  $\mathcal{F}$  on a convergence space  $(X, \xi)$  is by definition

$$\text{adh}_\xi \mathcal{F} := \bigcup_{\mathcal{H} \# \mathcal{F}} \lim_\xi \mathcal{H} = \bigcup_{\mathcal{U} \in \mathbb{U}(\mathcal{F})} \lim_\xi \mathcal{U},$$

where  $\mathbb{U}X$  denotes the set of ultrafilters on  $X$  and  $\mathbb{U}(\mathcal{F})$  denotes the set of ultrafilters on  $X$  finer than the filter  $\mathcal{F}$ . We write  $\text{adh}_\xi A$  for  $\text{adh}_\xi \{A\}^\uparrow$ . Note that in a convergence space  $X$ ,  $\text{adh}_\xi$  may not be idempotent on subsets of  $X$ . In fact a pretopology is a topology if and only if  $\text{adh}$  is idempotent on subsets. We reserve the notations  $\text{cl}$  and  $\text{int}$  to topological closure and interior operators.

A family  $\mathcal{A}$  of subsets of  $X$  is *compact at a family*  $\mathcal{B}$  for  $\xi$  if

$$\mathcal{F} \# \mathcal{A} \implies \text{adh}_\xi \mathcal{F} \# \mathcal{B}.$$

We call a family *compact* if it is compact at itself. In particular, a subset  $A$  of  $X$  is *compact* if  $\{A\}$  is compact, and *compact at*  $B \subseteq X$  if  $\{A\}$  is compact at  $\{B\}$ .

Given a class  $\mathbb{D}$  of filters, a convergence is called *based in*  $\mathbb{D}$  or  $\mathbb{D}$ -*based* if for every convergent filter  $\mathcal{F}$ , say  $x \in \lim \mathcal{F}$ , there is a filter  $\mathcal{D} \in \mathbb{D}$  with  $\mathcal{D} \leq \mathcal{F}$  and  $x \in \lim \mathcal{D}$ . A convergence is called *locally compact* if every convergent filter contains a compact set, and *hereditarily locally compact* if it is based in filters with a filter-base composed of compact sets. For every convergence, there is the coarsest locally compact convergence  $K\xi$  that is finer than  $\xi$  and the coarsest hereditarily locally compact convergence  $K_h\xi$  that is finer than  $\xi$ . Both  $K$  and  $K_h$  are concrete endofunctors of **Conv** that are also coreflectors.

If  $A \subseteq X$  and  $(X, \xi)$  is a convergence space, then  $\mathcal{O}(A)$  denotes the collection of open subsets of  $X$  that contain  $A$  and if  $\mathcal{A}$  is a family of subsets of  $X$  then  $\mathcal{O}(\mathcal{A}) := \bigcup_{A \in \mathcal{A}} \mathcal{O}(A)$ . A family is called *openly isotone* if  $\mathcal{A} = \mathcal{O}(\mathcal{A})$ . Note that in a topological space  $X$ , an openly isotone family  $\mathcal{A}$  of open subsets of  $X$  is compact if and only if, whenever  $\bigcup_{i \in I} O_i \in \mathcal{A}$  and each  $O_i$  is open, there is a finite subset  $J$  of  $I$  such that  $\bigcup_{i \in J} O_i \in \mathcal{A}$ .

If  $(X, \xi)$  and  $(Y, \sigma)$  are two convergence spaces,  $C(X, Y)$  or  $C(\xi, \sigma)$  denote the set of continuous maps from  $X$  to  $Y$ . The coarsest convergence on  $C(X, Y)$  making the evaluation map  $e : X \times C(X, Y) \rightarrow Y$ ,  $e(x, f) = f(x)$ , jointly continuous is called *continuous convergence* and denoted  $[X, Y]$  or  $[\xi, \sigma]$ . Explicitly,

$$f \in \lim_{[X, Y]} \mathcal{F} \iff \forall x \in X \forall \mathcal{G} \in \mathbb{F}X : x \in \lim_\xi \mathcal{G} \ f(x) \in \lim_\sigma e(\mathcal{G} \times \mathcal{F}).$$

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(Received December 2010 – Accepted June 2011)

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