On a type of generalized open sets

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Abstract

In this paper, a new class of sets called $\mu$-generalized closed (briefly $\mu g$-closed) sets in generalized topological spaces are introduced and studied. The class of all $\mu g$-closed sets is strictly larger than the class of all $\mu$-closed sets (in the sense of Á. CsáSZár). Furthermore, $g$-closed sets (in the sense of N. Levine) is a special type of $\mu g$-closed sets in a topological space. Some of their properties are investigated here. Finally, some characterizations of $\mu$-regular and $\mu$-normal spaces have been given.

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1. Introduction

In the past few years, different forms of open sets have been studied. Recently, a significant contribution to the theory of generalized open sets, was extended by A. CsáSZár. Especially, the author defined some basic operators on generalized topological spaces.

It is observed that a large number of papers is devoted to the study of generalized open like sets of a topological space, containing the class of open sets and possessing properties more or less similar to those of open sets. For example, [22] has introduced $g$-open sets, [4, 30, 2] $sg$-open sets, [25] $pg$-open sets, [27, 28] $g\alpha$-open sets, [13] $g\delta^*$-open sets, [21, 17] $bg$-open sets.

Owing to the fact that corresponding definitions have many features in common, it is quite natural to conjecture that they can be obtained and a considerable part of the properties of generalized open sets can be deduced from suitable more general definitions. The purpose of this paper is to point

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out extremely elementary character of the proofs and to get many unknown results by special choice of the generalized topology.

We recall some notions defined in [9]. Let $X$ be a non-empty set, $\exp X$ denotes the power set of $X$. We call a class $\mu \subseteq \exp X$ a generalized topology [9], (briefly, GT) if $\emptyset \in \mu$ and union of elements of $\mu$ belongs to $\mu$. A set $X$, with a GT $\mu$ on it is said to be a generalized topological space (briefly, GTs) and is denoted by $(X, \mu)$. The $\theta$-closure [35] (resp. $\delta$-closure [35]) of a subset $A$ of a topological space $(X, \tau)$ is defined by $\{x \in X : cU \cap A \neq \emptyset \text{ for all } U \in \tau \text{ with } x \in U\}$ (resp. $\{x \in X : A \cap U \neq \emptyset \text{ for all regular open sets } U \text{ containing } x\}$, where a subset $A$ is called regular open if $A = \text{int}(cl(A)))$. A is called $\delta$-closed [35] (resp. $\theta$-closed [35]) if $A = cl_\delta A$ (resp. $A = cl_\theta A$) and the complement of a $\delta$-closed set (resp. $\theta$-closed) set is known as a $\delta$-open (resp. $\theta$-open) set. A subset $A$ of a topological space $(X, \tau)$ is called preopen [29] (resp. semiopen [23], $\delta$-preopen [33], $\delta$-semiopen [32], $\alpha$-open [27], $\beta$-open [1], $b$-open [21]) if $A \subseteq \text{int}(cl(A))$ (resp. $A \subseteq cl(\text{int}(A)), A \subseteq cl(cl_\delta A), A \subseteq cl(\text{int}(A)), A \subseteq cl(\text{int}(A)) \cup cl(cl(A))$). We note that for any topological space $(X, \tau)$, the collection of all open sets denoted by $\tau$ (preopen sets denoted by $PO(X)$, semi-open sets denoted by $SO(X)$, $\delta$-open sets denoted by $\delta O(X)$, $\delta$-preopen sets denoted by $\delta PO(X)$, $\delta$-semiopen sets denoted by $\delta SO(X)$, $\alpha$-open sets denoted by $\alpha O(X)$, $\beta$-open sets denoted by $\beta O(X)$, $\theta$-open sets denoted by $\theta O(X)$, $b$-open sets denoted by $BO(X)$ or $\gamma O(X)$) forms a GT.

For a GTS $(X, \mu)$, the elements of $\mu$ are called $\mu$-open sets and the complement of $\mu$-open sets are called $\mu$-closed sets. For $A \subseteq X$, we denote by $c_\mu(A)$ the intersection of all $\mu$-closed sets containing $A$, i.e., the smallest $\mu$-closed set containing $A$; and by $i_\mu(A)$ the union of all $\mu$-open sets contained in $A$, i.e., the largest $\mu$-open set contained in $A$ (see [9, 10]). Obviously in a topological space $(X, \tau)$, if one takes $\tau$ as the GT, then $c_\mu$ becomes equivalent to the usual closure operator. Similarly, $c_\mu$ becomes $pcl, scl, cl_\delta, pcl_\delta, scl_\delta, cl_\alpha, cl_\beta, bcl$ if $\mu$ stands for $PO(X)$ (resp. $SO(X)$, $\delta O(X)$, $\delta PO(X)$, $\delta SO(X)$, $\alpha O(X)$, $\beta O(X)$, $BO(X)$ or $\gamma O(X)$).

It is easy to observe that $i_\mu$ and $c_\mu$ are idempotent and monotonic, where $\gamma : \exp X \rightarrow \exp X$ is said to be idempotent iff $A \subseteq B \subseteq X$ implies $\gamma(\gamma(A)) = \gamma(A)$ and monotonic iff $\gamma(A) \subseteq \gamma(B)$. It is also well known from [10, 11] that if $g$ is a GT on $X$ and $A \subseteq X$, $x \in X$, then $x \in c_\mu(A)$ iff $x \in M \in \mu \Rightarrow M \cap A \neq \emptyset$ and $c_\mu(X \setminus A) = X \setminus i_\mu(A)$.

In this paper we introduce the concepts of $\mu g$-closed sets and $\mu g$-open sets. It is shown that many results in previous papers can be considered as special cases of our results.

2. Properties of $\mu g$-closed sets

**Definition 2.1.** Let $(X, \mu)$ be a GTS. Then a subset $A$ of $X$ is called a $\mu$-generalized closed set (or in short, $\mu g$-closed set) iff $c_\mu(A) \subseteq U$ whenever $A \subseteq U$ where $U$ is $\mu$-open in $X$. The complement of a $\mu g$-closed set is called a $\mu g$-open set.
Remark 2.2.

(i) If \((X, \tau)\) is a topological space, the definition of \(q\)-open set [22] (resp. 
\(sg\)-open set [4, 2], \(pg\)-open set [25], \(go\)-open set [27], \(\delta g^*\)-open set [13], 
\(bg\)-open set [21] or \(\gamma g\)-open set [17]) can be obtained by taking \(\mu = \tau\) 
(resp. \(SO(X), PO(X), \alpha O(X), \delta O(X), \gamma O(X))\).

(ii) Every \(\mu\)-open set in a GTS \((X, \mu)\) is \(\mu g\)-open. In fact, if \(A\) is a \(\mu\)-open 
set in \((X, \mu)\), then \(X \setminus A\) is a \(\mu\)-closed set. Let \(X \setminus A \subseteq U \in \mu\). Then 
\(c_{\mu}(X \setminus A) = X \setminus A \subseteq U\). Thus \(X \setminus A\) is a \(\mu g\)-closed set and hence \(A\) is 
a \(\mu g\)-open set.

The converse of Remark 2.2(ii) is not true as seen from the next example :

Example 2.3. Let \(X = \{a, b, c\}\) and \(\mu = \{\emptyset, X, \{a\}, \{b, c\}, \{a, c\}\}\). Then 
\((X, \mu)\) is a GTS. It is easy to verify that \(\{c\}\) is \(\mu g\)-open in \((X, \mu)\) but not 
\(\mu\)-open.

The next two examples show that the union (intersection) of two \(\mu g\)-open 
sets is not in general \(\mu g\)-open.

Example 2.4.

(a) Let \(X = \{a, b, c\}\) and \(\mu = \{\emptyset, X, \{a\}\}\). Then \((X, \mu)\) is a GTS. It can 
be shown that if \(A = \{b\}\) and \(B = \{c\}\), then \(A\) and \(B\) are two \(\mu g\)-open 
sets but \(A \cup B = \{b, c\}\) is not a \(\mu g\)-open set.

(b) Let \(X = \{a, b, c, d\}\) and \(\mu = \{\emptyset, X, \{a, b\}, \{a, c, d\}, \{a, b, d\}, \{b, c, d\}\}\). 
Then \((X, \mu)\) is a GTS. It follows from Remark 2.2(ii) that \(\{a, b\}\) and 
\(\{a, c, d\}\) are two \(\mu g\)-open sets but it is easy to check that their intersec-
tion \(\{a\}\) is not \(\mu g\)-open.

Theorem 2.5. A subset \(A\) of a GTS \((X, \mu)\) is \(\mu g\)-closed iff \(c_{\mu}(A) \setminus A\) contains 
no non-empty \(\mu\)-closed set.

Proof. Let \(F\) be a \(\mu\)-closed subset of \(c_{\mu}(A) \setminus A\). Then \(A \subseteq F^c\) (where \(F^c\) 
denotes as usual the complement of \(F\)). Hence by \(\mu g\)-closedness of \(A\), we have 
\(c_{\mu}(A) \subseteq F^c\) or \(F \subseteq (c_{\mu}(A))^c\). Thus \(F \subseteq c_{\mu}(A) \cap (c_{\mu}(A))^c = \emptyset\), i.e., \(F = \emptyset\).

Conversely, suppose that \(A \subseteq U\) where \(U\) is \(\mu\)-open. If \(c_{\mu}(A) \not\subseteq U\), then 
\(c_{\mu}(A) \cap U^c (\neq \emptyset)\) is a \(\mu\)-closed subset of \(c_{\mu}(A) \setminus A\), a contradiction. Hence 
\(c_{\mu}(A) \subseteq U\).

Theorem 2.6. If a \(\mu g\)-closed subset \(A\) of a GTS \((X, \mu)\) be such that \(c_{\mu}(A) \setminus A\) 
is \(\mu\)-closed, then \(A\) is \(\mu\)-closed.

Proof. Let \(A\) be a \(\mu g\)-closed subset such that \(c_{\mu}(A) \setminus A\) is \(\mu\)-closed. Then 
\(c_{\mu}(A) \setminus A\) is a \(\mu\)-closed subset of itself. Then by Theorem 2.5, \(c_{\mu}(A) \setminus A = \emptyset\) 
and hence \(c_{\mu}(A) = A\), showing \(A\) to be a \(\mu\)-closed set.

That the converse is false follows from the following example.

Example 2.7. Let \(X = \{a, b, c\}\) and \(\mu = \{\emptyset, \{a\}, \{a, b\}\}\). Then \((X, \mu)\) is a 
GTS. It is easy to observe that \(\{b, c\}\) is \(\mu\)-closed and hence a \(\mu g\)-closed set (by 
Remark 2.2), but \(c_{\mu}(A) \setminus A = \emptyset\), which is not \(\mu\)-closed.
Theorem 2.8. Let $A$ be a $\mu$-closed set in a GTS $(X, \mu)$ and $A \subseteq B \subseteq c_\mu(A)$. Then $B$ is $\mu$-closed.

Proof. Let $B \subseteq U$, where $U$ is $\mu$-open in $(X, \mu)$. Since $A$ is $\mu$-closed and $A \subseteq U$, $c_\mu(A) \subseteq U$. Now, $B \subseteq c_\mu(A) \Rightarrow c_\mu(B) \subseteq c_\mu(A)$. So $c_\mu(B) \subseteq U$. \hfill \Box

Theorem 2.9. In a GTS $(X, \mu)$, $\mu = \Omega$ (the collection of all $\mu$-closed sets) iff every subset of $X$ is $\mu$-closed.

Proof. Suppose $\mu = \Omega$ and $A \subseteq X$ be such that $A \subseteq U \subseteq \mu$. Then $c_\mu(A) \subseteq c_\mu(U) = U$ and hence $A$ is $\mu$-closed.

Conversely, suppose that every subset of $X$ is $\mu$-closed. Let $U \subseteq \mu$. Then $U \subseteq \mu$ and by $\mu$-closedness of $U$, we have $c_\mu(U) \subseteq U$, i.e., $U \in \Omega$. Thus $\mu \subseteq \Omega$.

Now, if $F \subseteq \Omega$ then $F^c \subseteq \mu$, so $F^c \subseteq \Omega$ (as $\mu \subseteq \Omega$), i.e., $F \in \mu$. \hfill \Box

Theorem 2.10. A subset $A$ of a GTS $(X, \mu)$ is $\mu$-open iff $F \subseteq i_\mu(A)$, whenever $F$ is $\mu$-closed and $F \subseteq A$.

Proof. Obvious and hence omitted. \hfill \Box

Theorem 2.11. A set $A$ is $\mu$-open in a GTS $(X, \mu)$ iff $U = X$ whenever $U$ is $\mu$-open and $i_\mu(A) \cup A^c \subseteq U$.

Proof. Suppose $U$ is $\mu$-open and $i_\mu(A) \cup A^c \subseteq U$. Now, $U^c \subseteq (i_\mu(A))^c \cap A = c_\mu(X \setminus A) \setminus (X \setminus A)$. Since $U^c$ is $\mu$-closed and $X \setminus A$ is $\mu$-closed, by Theorem 2.5, $U^c = \emptyset$, i.e., $U = X$.

Conversely, let $F$ be a $\mu$-closed set and $F \subseteq A$. Then by Theorem 2.10, it is enough to show that $F \subseteq i_\mu(A)$. Now, $i_\mu(A) \cup A^c \subseteq i_\mu(A) \cup F^c$, where $i_\mu(A) \cup F^c$ is $\mu$-open. Hence by the given condition, $i_\mu(A) \cup F^c = X$, i.e., $F \subseteq i_\mu(A)$. \hfill \Box

Theorem 2.12. A subset $A$ of a GTS $(X, \mu)$ is $\mu$-closed iff $c_\mu(A) \setminus A$ is $\mu$-open.

Proof. Suppose $A$ is $\mu$-closed and $F \subseteq c_\mu(A) \setminus A$, where $F$ is a $\mu$-closed subset of $X$. Then by Theorem 2.5, $F = \emptyset$ and hence $F \subseteq i_\mu[c_\mu(A) \setminus A]$. Then by Theorem 2.10, $c_\mu(A) \setminus A$ is $\mu$-open.

Conversely, suppose that $A \subseteq U$ where $U$ is $\mu$-open. Now, $c_\mu(A) \cap U^c \subseteq c_\mu(A) \cap A^c = c_\mu(A) \setminus A$. Since $c_\mu(A) \cap U^c$ is $\mu$-closed and $c_\mu(A) \setminus A$ is $\mu$-open, $c_\mu(A) \cap U^c = \emptyset$ (by Theorem 2.5). Thus $c_\mu(A) \subseteq U$, i.e., $A$ is $\mu$-closed. \hfill \Box

Definition 2.13. A GTS $(X, \mu)$ is said to be

(i) $\mu$-$T_0$ [34] iff $x, y \in X$, $x \neq y$ implies the existence of $K \subseteq \mu$ containing precisely one of $x$ and $y$.

(ii) $\mu$-$T_1$ [34] iff $x, y \in X$, $x \neq y$ implies the existence of $K, K^1 \subseteq \mu$ such that $x \in K$, $y \notin K$ and $x \notin K^1$, $y \in K^1$.

(iii) $\mu$-$T_{1/2}$ iff every $\mu$-closed set is $\mu$-closed.
Remark 2.14. A topological space \((X, \tau)\) is \(T_i\) \([16]\) (resp. \(semi-T_i\) \([4]\), \(pre-T_i\) \([25]\), \(\alpha-T_i\) \([28]\), \(\delta-T_i\) \([13]\), \(b-T_i\) \([21]\)) for \(i = 0, 1/2, 1\) by taking \(\mu = \tau\) (resp. \(SO(X), PO(X), \alpha O(X), \delta O(X), BO(X)\) or \(\gamma O(X)\)).

Theorem 2.15. If a GTS \((X, \mu)\) is \(\mu-T_{1/2}\) then it is \(\mu-T_0\).

Proof. Suppose that \((X, \mu)\) is not a \(\mu-T_0\) space. Then there exist distinct points \(x\) and \(y\) in \(X\) such that \(c_\mu(\{x\}) = c_\mu(\{y\})\). Let \(A = c_\mu(\{x\}) \cap \{x\}^c\). We shall show that \(A\) is \(\mu\)-closed but not \(\mu\)-closed. Suppose that \(A \subseteq V \subseteq \mu\). We have to show that \(c_\mu(A) \subseteq V\). Thus it is enough to show that \(c_\mu(\{x\}) \subseteq V\) (as \(\mu \subseteq c_\mu(\{x\})\)). Again, since \(c_\mu(\{x\}) \cap \{x\}^c = A \subseteq V\), we need only to show that \(x \in V\). In fact, if \(x \notin V\), then \(y \in c_\mu(\{x\}) \subseteq V^c\) (as \(V^c\) is \(\mu\)-closed). So \(y \in A \subseteq V^c\) and hence \(y \in V \cap V^c\) - a contradiction.

If \(x \in U \subseteq \mu\), then \(U \cap A \subseteq \{y\} \neq \emptyset\), and hence \(x \in c_\mu(A)\). Clearly, \(x \notin A\) and thus \(A\) is not \(\mu\)-closed.

Example 2.16. Let \(X = \{a, b, c, d\}\) and \(\mu = \{\emptyset, X, \{a, b\}, \{a, c, d\}, \{a, b, d\}, \{b, c, d\}\}\). Then \((X, \mu)\) is a GTS. Clearly, this GTS is \(\mu-T_0\) and it can be shown that the collection of all \(\mu\)-open sets are \(\{\emptyset, X, \{a, b\}, \{a, c, d\}, \{a, b, d\}, \{b, c, d\}\}\). Thus this space is not \(\mu-T_{1/2}\).

Theorem 2.17. If a GTS \((X, \mu)\) is \(\mu-T_1\) then it is \(\mu-T_{1/2}\).

Proof. Suppose that \(A\) is a subset of \(X\) which is not \(\mu\)-closed. Take \(x \in c_\mu(A) \setminus A\). Then \(\{x\} \subseteq c_\mu(A) \setminus A\) and \(x\) is \(\mu\)-closed (as \((X, \mu)\) is \(\mu-T_1\)). Thus by Theorem 2.5, \(A\) is not \(\mu\)-closed.

Example 2.18. Let \(X = \{a, b, c, d\}\) and \(\mu = \{\emptyset, X, \{a, b\}, \{b, c, d\}, \{a, c, d\}, \{a, b, d\}\}\). Then \((X, \mu)\) is a GTS. It is easy to verify that \((X, \mu)\) is \(\mu-T_{1/2}\) but not \(\mu-T_1\).

Definition 2.19. A GTS \((X, \mu)\) is said to be \(\mu\)-symmetric iff for each \(x, y \in X\), \(x \in c_\mu(\{y\}) \Rightarrow y \in c_\mu(\{x\})\).

Remark 2.20. It is easy to check that the above definition of a \(\mu\)-symmetric space GT unifies the existing definitions of \(\delta\)-symmetric space \([8]\), \(\delta(p)\)-symmetric space \([5]\), \(\alpha\)-symmetric \([6]\), \(\delta\)-semi symmetric space \([7]\) if \((X, \tau)\) is a topological space and \(\mu = \delta O(X), \delta-PO(X), \alpha O(X), \delta-SO(X)\) respectively.

Theorem 2.21. A GTS \((X, \mu)\) is \(\mu\)-symmetric iff \(\{x\}\) is \(\mu\)-closed for each \(x \in X\).

Proof. Let \(\{x\} \subseteq U \subseteq \mu\) and \((X, \mu)\) be \(\mu\)-symmetric but \(c_\mu(\{x\}) \not\subseteq U\). Then \(c_\mu(\{x\}) \cap U^c \neq \emptyset\). Let \(y \in c_\mu(\{x\}) \cap U^c\). Then \(x \in c_\mu(\{y\}) \subseteq U^c\) ⇒ \(x \notin U\) - a contradiction.

Conversely, let for each \(x \in X\), \(\{x\}\) is \(\mu\)-closed and \(x \in c_\mu(\{y\}) \subseteq (c_\mu(\{x\}))^c\) (as \(\{y\}\) is \(\mu\)-closed). Thus \(x \in (c_\mu(\{x\}))^c\) - a contradiction.

Corollary 2.22. If a GTS \((X, \mu)\) is \(\mu-T_1\) then it is \(\mu\)-symmetric.

Example 2.23. Let \(X = \{a, b\}\) and \(\mu = \{\emptyset, X\}\). Then \((X, \mu)\) is a \(\mu\)-symmetric space which is not \(\mu-T_1\).
Theorem 2.24. A GTS \((X, \mu)\) is \(\mu\)-symmetric and \(\mu\)-\(T_0\) iff \((X, \mu)\) is \(\mu\)-\(T_1\).

Proof. If \((X, \mu)\) is \(\mu\)-\(T_1\) then it is \(\mu\)-symmetric (by Corollary 2.22) and \(\mu\)-\(T_0\) (by Definition 2.13).

Conversely, let \((X, \mu)\) be \(\mu\)-symmetric and \(\mu\)-\(T_0\). We shall show that \((X, \mu)\) is \(\mu\)-\(T_1\). Let \(x, y \in X\) and \(x \neq y\). Then by \(\mu\)-\(T_0\)-ness of \((X, \mu)\), there exists \(U \in \mu\) such that \(x \in U \subseteq \{y\}^c\). Then \(x \not\in c_\mu(\{y\})\) and hence \(y \not\in c_\mu(\{x\})\).

Thus there exists \(V \in \mu\) such that \(y \in V\) and \(x \not\in V\). Thus \((X, \mu)\) is \(\mu\)-\(T_1\). \(\square\)

Theorem 2.25. If \((X, \mu)\) is \(\mu\)-symmetric, then \((X, \mu)\) is \(\mu\)-\(T_0\) iff \((X, \mu)\) is \(\mu\)-\(T_{1/2}\) iff \((X, \mu)\) is \(\mu\)-\(T_1\).

Proof. Follows from Theorem 2.24 and the fact that \(\mu\)-\(T_1 \Rightarrow \mu\)-\(T_{1/2} \Rightarrow \mu\)-\(T_0\). \(\square\)

3. Preservation of \(\mu\)-\(g\)-closed sets

Definition 3.1. Let \((X, \mu_1)\) and \((Y, \mu_2)\) be two GTS’s. A mapping \(f : (X, \mu_1) \to (Y, \mu_2)\) is said to be

(i) \((\mu_1, \mu_2)\) continuous [9] iff \(f^{-1}(G_2) \in \mu_1\) for each \(G_2 \in \mu_2\);

(ii) \((\mu_1, \mu_2)\)-closed iff for any \(\mu_1\)-closed subset \(A\) of \(X\), \(f(A)\) is \(\mu_2\)-closed in \(Y\).

Theorem 3.2. Let \((X, \mu_1)\) and \((Y, \mu_2)\) be two GTS’s and \(f : (X, \mu_1) \to (Y, \mu_2)\) be \((\mu_1, \mu_2)\)-continuous and \((\mu_1, \mu_2)\)-closed mapping. If \(A\) is \(\mu_1\)-\(g\)-closed in \(X\) then \(f(A)\) is \(\mu_2\)-\(g\)-closed in \(Y\).

Proof. Let \(f(A) \subseteq G_2\), where \(G_2\) is a \(\mu_2\)-open set in \(Y\). Then \(A \subseteq f^{-1}(G_2)\), where \(f^{-1}(G_2)\) is a \(\mu_1\)-open set in \(X\). Thus by \(\mu_1\)-\(g\)-closedness of \(A\), \(c_{\mu_1}(A) \subseteq f^{-1}(G_2)\). Thus \(f(c_{\mu_1}(A)) \subseteq G_2\) and \(f(c_{\mu_1}(A))\) is \(\mu_2\)-closed in \(Y\). It thus follows that \(c_{\mu_2}(f(A)) \subseteq c_{\mu_2}(f(c_{\mu_1}(A))) = f(c_{\mu_1}(A)) \subseteq G_2\). Thus \(f(A)\) is \(\mu_2\)-\(g\)-closed in \(Y\). \(\square\)

Theorem 3.3. Let \((X, \mu_1)\) and \((Y, \mu_2)\) be two GTS’s and \(f : (X, \mu_1) \to (Y, \mu_2)\) be a \((\mu_1, \mu_2)\)-continuous and \((\mu_1, \mu_2)\)-closed mapping. If \(B\) is a \(\mu_2\)-\(g\)-closed set in \(Y\), then \(f^{-1}(B)\) is \(\mu_1\)-\(g\)-closed in \(X\).

Proof. Suppose that \(B\) is a \(\mu_2\)-\(g\)-closed set in \(Y\) and \(f^{-1}(B) \subseteq G_1\), where \(G_1\) is \(\mu_1\)-open in \(X\). We shall show that \(c_{\mu_1}(f^{-1}(B)) \subseteq G_1\). Now \(f(c_{\mu_1}(f^{-1}(B))) \cap G_1^c \subseteq c_{\mu_2}(B) \setminus B\) and by Theorem 2.5, \(f(c_{\mu_1}(f^{-1}(B))) \setminus G_1^c = \emptyset\). Thus \(c_{\mu_2}(f^{-1}(B)) \cap G_1 = \emptyset\). Thus \(c_{\mu_1}(f^{-1}(B)) \subseteq G_1\) and hence \(f^{-1}(B)\) is \(\mu_1\)-\(g\)-closed in \(X\). \(\square\)

Next two examples show that \((\mu_1, \mu_2)\)-continuity and \((\mu_1, \mu_2)\)-closedness in both of the above theorems are essential.

Example 3.4. Let \(X = \{a, b, c, d\}\), \(\mu_1 = \{\emptyset, X, \{a, b\}, \{c, d\}, \{a, c, d\}, \{a, b, d\}\}\) and \(\mu_2 = \{\emptyset, X, \{a, b\}, \{c, d\}, \{a, c, d\}\}\). Then \((X, \mu_1)\) and \((X, \mu_2)\) are two GTS’s. Consider the identity mapping \(f : (X, \mu_1) \to (X, \mu_2)\). It is easy to see
that $f$ is a $(\mu_1, \mu_2)$-continuous mapping which is not $(\mu_1, \mu_2)$-closed. The families of $\mu_1$-$g$-open and $\mu_2$-$g$-open sets are respectively $\{\emptyset, X, \{a\}, \{d\}, \{a, d\}, \{a, b\}, \{a, c\}, \{a, c, d\}, \{a, b, d\}, \{a, b, c\}\}$ and $\{\emptyset, X, \{a\}, \{c\}, \{d\}, \{a, d\}, \{a, b\}, \{a, c\}, \{a, c, d\}, \{a, b, d\}, \{a, b, c\}\}$. We note that $\{d\}$ is $g\mu_2$-closed but $f^{-1}(\{d\})$ is not $g\mu_1$-closed.

Again, the identity map $h$ defined by $h : (X, \mu_2) \rightarrow (X, \mu_1)$ is not a $(\mu_2, \mu_1)$-continuous mapping but it is $(\mu_2, \mu_1)$-closed. Clearly, $\{d\}$ is a $\mu_2$-$g$-closed set but $h(\{d\})$ is not a $\mu_1$-$g$-closed set.

Example 3.5. Let $X = \{a, b, c, d\}$, $\mu_1 = \{\emptyset, X, \{a\}, \{c\}, \{a, c\}, \{a, b\}, \{a, d\}, \{a, c, d\}, \{a, b, d\}, \{a, b, c\}\}$ and $\mu_2 = \{\emptyset, X, \{a, b\}, \{a, c\}, \{a, d\}, \{a, b, c\}\}$. Then $(X, \mu_1)$ and $(X, \mu_2)$ are GTS’s. Now, consider the identity map $f : (X, \mu_1) \rightarrow (X, \mu_2)$. It is easy to verify that $f$ is a $(\mu_1, \mu_2)$-continuous mapping which is not $(\mu_1, \mu_2)$-closed. The family of $g\mu_1$-$open$ and $g\mu_2$-$open$ sets are respectively $\{\emptyset, X, \{a\}, \{d\}, \{a, d\}, \{a, b\}, \{a, c\}, \{a, c, d\}, \{a, b, d\}, \{a, b, c\}\}$ and $\{\emptyset, X, \{a\}, \{d\}, \{a, d\}, \{a, b\}, \{a, c\}, \{a, c, d\}\}$. We note that $\{a, b\}$ is $\mu_2$-$g$-closed but $f(\{a, b\})$ is not $\mu_2$-$g$-closed.

Again, consider the identity map $h : (X, \mu_2) \rightarrow (X, \mu_1)$. Then, clearly $h$ is a $(\mu_2, \mu_1)$-closed map which is not $(\mu_2, \mu_1)$-continuous. Clearly, $\{a, b\}$ is $\mu_1$-$g$-closed but $h^{-1}(\{a, b\})$ is not a $\mu_2$-$g$-closed set.

4. Properties of $\mu$-regular and $\mu$-normal spaces

Definition 4.1. A GTS $(X, \mu)$ is said to be $\mu$-regular if for each $\mu$-$closed$ set $F$ of $X$ not containing $x$, there exist disjoint $\mu$-$open$ set $U$ and $V$ such that $x \in U$ and $F \subseteq V$.

Remark 4.2. Regular space, pre-regular space, semi-regular space, $\beta$-regular space, $\alpha$-regular space are defined and studied in [16, 31, 15, 19, 20] respectively. The above definition gives a unified version of all these definitions if $\mu$ takes the role of $\tau$, $PO(X)$, $SO(X)$, $\beta O(X)$, $\alpha O(X)$ respectively.

Theorem 4.3. For a GTS $(X, \mu)$ the followings are equivalent:

(a) $X$ is $\mu$-regular.
(b) For each $x \in X$ and each $U \in \mu$ containing $x$, there exists $V \in \mu$ such that $x \in V \subseteq c_\mu(V) \subseteq U$.
(c) For each $\mu$-$closed$ set $F$ of $X$, $\cap\{c_\mu(V) : F \subseteq V \subseteq \mu\} = F$.
(d) For each subset $A$ of $X$ and each $U \in \mu$ with $A \cap U \neq \emptyset$, there exists a $V \in \mu$ such that $A \cap V \neq \emptyset$ and $c_\mu(V) \subseteq U$.
(e) For each non-empty subset $A$ of $X$ and each $\mu$-$closed$ subset $F$ of $X$ with $A \cap F = \emptyset$, there exist $U, V \in \mu$ such that $A \cap V \neq \emptyset$, $F \subseteq W$ and $W \cap V = \emptyset$.
(f) For each $\mu$-$closed$ set $F$ with $x \not\in F$ there exist $U \in \mu$ and a $\mu$-$g$-$open$ set $V$ such that $x \in U$, $F \subseteq V$ and $U \cap V = \emptyset$.
(g) For each $A \subseteq X$ and each $\mu$-$closed$ set $F$ with $A \cap F = \emptyset$ there exist a $U \in \mu$ and a $\mu$-$g$-$open$ set $V$ such that $A \cap U \neq \emptyset$, $F \subseteq V$ and $U \cap V = \emptyset$.
(h) For each $\mu$-$closed$ set $F$ of $X$, $F = \cap\{c_\mu(V) : F \subseteq V, V \text{ is } \mu$-$open\}$. 
Then there exists a $\subseteq F U$ exist a $\subseteq \mu g V$ V. Now put $A$ of $\subseteq x \in W, V W \cap \mu W$ such that $x \in c_\mu(V)$, where $A$ is closed subsets $\subseteq c_\mu(W)$. We put $V = X \cap_\mu(W)$, which is a $\mu$-open set containing $x$ and hence $A \cap V \neq \emptyset$ (as $x \in A \cap V$). Now $V \subseteq X \cap W$ and so $c_\mu(V) \subseteq X \cap W \subseteq U$.

(b) $\Rightarrow$ (c): Let $X \cap F \in \mu$ be such that $x \notin F$. Then by (b) there exists $U \in \mu$ such that $x \in U \subseteq c_\mu(U) \subseteq X \cap F$. So, $F \subseteq X \cap c_\mu(U) = V \in \mu$ and $U \cap V = \emptyset$. Thus $x \notin c_\mu(V)$. Hence $F \supseteq \cap \{c_\mu(V) : F \subseteq V \in \mu\}$.

(c) $\Rightarrow$ (d): Let $U \in \mu$ with $x \in U \cap A$. Then $x \notin X \cap U$ and hence by (c) there exists a $\mu$-open set $W$ such that $X \cap U \subseteq W$ and $x \notin c_\mu(W)$. We put $V = X \cap c_\mu(W)$, which is a $\mu$-open set containing $x$ and hence $A \cap V \neq \emptyset$ (as $x \in A \cap V$). Now $V \subseteq X \cap W$ and so $c_\mu(V) \subseteq X \cap W \subseteq U$.

(d) $\Rightarrow$ (e): Let $F$ be a $\mu$-closed set as in the hypothesis of (e). Then $X \cap F$ is a $\mu$-open set and $(X \cap F) \cap A \neq \emptyset$. Then there exists $V \in \mu$ such that $A \cap V \neq \emptyset$ and $c_\mu(V) \subseteq X \cap F$. If we put $W = X \cap c_\mu(V)$, then $F \subseteq W$ and $W \cap V = \emptyset$.

(e) $\Rightarrow$ (a): Let $F$ be a $\mu$-closed set not containing $x$. Then by (e), there exist $W, V \in \mu$ such that $F \subseteq W$ and $x \in V$ and $W \cap V = \emptyset$.

(a) $\Rightarrow$ (f): Obvious as every $\mu$-open set is $\mu g$-open (by Remark 2.2).

(f) $\Rightarrow$ (g): Let $F$ be a $\mu$-closed set such that $A \cap F = \emptyset$ for any subset $A$ of $X$. Thus for $a \in A$, $a \notin F$ and hence by (f), there exist a $U \in \mu$ and a $\mu g$-open set $V$ such that $a \in U$, $F \subseteq V$ and $U \cap V = \emptyset$. So $A \cap U \neq \emptyset$.

(g) $\Rightarrow$ (a): Let $x \notin F$, where $F$ is $\mu$-closed. Since $\{x\} \cap F = \emptyset$, by (g) there exist a $U \in \mu$ and a $\mu g$-open set $W$ such that $x \in U$, $F \subseteq W$ and $U \cap W = \emptyset$. Now put $V = i_\mu(W)$. Then $F \subseteq V$ (by Theorem 2.10) and $U \cap V = \emptyset$.

(c) $\Rightarrow$ (h): We have $F \subseteq \cap \{c_\mu(V) : F \subseteq V \cap V$ is $\mu g$-open $\} \subseteq \cap \{c_\mu(V) : F \subseteq V \cap V$ is $\mu$-open $\} = F$.

(h) $\Rightarrow$ (a): Let $F$ be a $\mu$-closed set in $X$ not containing $x$. Then by (h) there exists a $\mu g$-open set $W$ such that $F \subseteq W$ and $x \in X \cap c_\mu(W)$. Since $F$ is $\mu$-closed and $W$ is $\mu g$-open, $F \subseteq i_\mu(W)$ (by Theorem 2.10). Take $V = i_\mu(W)$. Then $F \subseteq V$, $x \in X \cap c_\mu(V) = U$ (say) (as $(X \cap F) \cap V = \emptyset$) and $U \cap V = \emptyset$. \[ \square \]

**Definition 4.4.** A GTS $(X, \mu)$ is $\mu$-normal [12] if for any pair of disjoint $\mu$-closed subsets $A$ and $B$ of $X$, there exist disjoint $\mu$-open sets $U$ and $V$ such that $A \subseteq U$ and $B \subseteq V$.

**Remark 4.5.** Normal space, pre-normal space, semi-normal space, $\alpha$-normal space, $\beta$-normal space, $\gamma$-normal space are defined and studied in [16, 31, 2, 19].
Remark 4.7. \( \subseteq \) \( B \) \( \mu \) \( g \) \( \text{families are obtained from} \) \( \alpha O \) \( \mu \) \( a \) \( \text{by} \) \( \mu g \) \( \text{Proof.} \) \( (a) \) \( X \) \( \text{is} \) \( \mu \) \( \text{-normal}; \)

(b) For any pair of disjoint \( \mu \) -closed sets \( A \) and \( B \), there exist disjoint \( \mu g \) -open sets \( U \) and \( V \) such that \( A \subseteq U \) and \( B \subseteq V \);

(c) For every \( \mu \) -closed set \( A \) and \( \mu \) -open set \( B \) containing \( A \), there exists a \( \mu g \) -open set \( U \) such that \( A \subseteq U \subseteq c_{\mu}(U) \subseteq B \);

(d) For every \( \mu \) -closed set \( A \) and every \( \mu g \) -open set \( B \) containing \( A \), there exists a \( \mu \) -open set \( U \) such that \( A \subseteq U \subseteq c_{\mu}(U) \subseteq i_{\mu}(B) \);

(e) For every \( \mu g \) -closed set \( A \) and every \( \mu \) -open set \( B \) containing \( A \), there exists a \( \mu g \) -open set \( U \) such that \( A \subseteq c_{\mu}(A) \subseteq U \subseteq c_{\mu}(U) \subseteq B \).

Proof. \( (a) \Rightarrow (b) \) : Let \( A \) and \( B \) be two disjoint \( \mu g \) -closed subsets of \( X \). Then by \( \mu \) -normality of \( X \), there exist disjoint \( \mu \) -open sets \( U \) and \( V \) such that \( A \subseteq U \) and \( B \subseteq V \). Then \( U \) and \( V \) are \( \mu g \) -open by Remark 2.2.

\( (b) \Rightarrow (c) \) : Let \( A \) be a \( \mu \) -closed set and \( B \) be a \( \mu \) -open set containing \( A \). Then \( A \) and \( B^{c} \) are two disjoint \( \mu \) -closed sets in \( X \). Then by (b), there exist disjoint \( \mu g \) -open sets \( U \) and \( V \) such that \( A \subseteq U \) and \( B^{c} \subseteq V \). Thus \( A \subseteq U \subseteq X \setminus V \subseteq B \). Again, since \( B \) is \( \mu g \) -open and \( X \setminus V \) is \( \mu g \) -closed, \( c_{\mu}(X \setminus V) \subseteq B \). Hence \( A \subseteq U \subseteq c_{\mu}(U) \subseteq B \).

\( (c) \Rightarrow (d) \) : Let \( A \) be a \( \mu g \) -closed subset of \( X \) and \( B \) be a \( \mu g \) -open set containing \( A \). Since \( B \) is a \( \mu g \) -open set containing \( A \) and \( A \) is \( \mu g \) -closed, by Theorem 2.10, \( A \subseteq i_{\mu}(B) \). Thus by (c) there exists a \( \mu g \) -open set \( U \) such that \( A \subseteq U \subseteq c_{\mu}(U) \subseteq i_{\mu}(B) \).

\( (d) \Rightarrow (e) \) : Let \( A \) be a \( \mu g \) -closed set and \( B \) be a \( \mu \) -open set in \( X \) containing \( A \). \( A \subseteq B \) implies \( c_{\mu}(A) \subseteq B \), where \( c_{\mu}(A) \) is \( \mu \) -closed and \( B \) is \( \mu g \) -open (as \( B \) is \( \mu \) -open). Then by (d), there exists a \( \mu \) -open set \( U \) such that \( A \subseteq c_{\mu}(A) \subseteq U \subseteq c_{\mu}(U) \subseteq i_{\mu}(B) \). Thus \( A \subseteq c_{\mu}(A) \subseteq U \subseteq c_{\mu}(U) \subseteq B \).

\( (e) \Rightarrow (a) \) : Let \( A \) and \( B \) be two disjoint \( \mu \) -closed subsets of \( X \). Then \( A \) is \( \mu g \) -closed and \( A \subseteq X \setminus B \), where \( X \setminus B \) is \( \mu \) -open. Thus by (e), there exists a \( \mu \) -open set \( U \) such that \( A \subseteq c_{\mu}(A) \subseteq U \subseteq c_{\mu}(U) \subseteq X \setminus B \). Thus \( A \subseteq U \), \( B \subseteq X \setminus c_{\mu}(U) \) and \( U \cap (X \setminus c_{\mu}(U)) = \emptyset \). Hence \( X \) is \( \mu \) -normal. \( \square \)

Remark 4.7. (a) By using \( \mu = \tau \) [22] (resp. \( PO(X) \) [25], \( SO(X) \) [4], \( \alpha O(X) \) [27], \( \delta O(X) \) [13], \( BO(X) \) [17, 21]) on a topological space \((X, \tau)\) several modifications of \( g \)-closed sets (resp. \( sg \)-closed sets, \( ga \)-closed sets, \( \delta g^{*} \)-closed sets, \( bg \)-closed sets) are introduced and investigated. Since each of \( \tau \), \( PO(X) \), \( SO(X) \), \( \alpha O(X) \), \( \delta O(X) \), \( BO(X) \) forms a GT on \( X \), the characterizations of each of the families are obtained from \( \mu g \) -open set.
(b) The definition of many other similar types of generalized closed sets can be defined on a topological space \((X, \tau)\) from the definition of \(\mu g\)-closed set by replacing \(\mu\) by the corresponding GT on \(X\).

References


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