Some fixed point theorems on the class of comparable partial metric spaces

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ABSTRACT

In this paper we present existence and uniqueness criteria of a fixed point for a self mapping on a non-empty set endowed with two comparable partial metrics.


KEYWORDS: partial metric space, fixed point theory, comparable metrics.

1. Introduction and Preliminaries

In 1992, Matthews [10, 11] introduced the notion of a partial metric space which is a generalization of usual metric spaces in which \( d(x, x) \) are no longer necessarily zero. After this remarkable contribution, many authors focused on partial metric spaces and its topological properties (see e.g. [15, 16, 2, 1, 3, 4, 5, 6]). Partial metric spaces have extensive potential applications in the research area of computer domains and semantics (see e.g. [7, 12, 8, 13, 14]). Consequently, the attention paid to such spaces rapidly increases.

A partial metric space (see e.g. [10, 11]) is a pair \((X, p)\) such that  \( X \) is non-empty set and \( p: X \times X \to \mathbb{R}^+ \) (where \( \mathbb{R}^+ \) denotes the set of all non negative real numbers) satisfies:

(\text{PM1}) \quad p(x, y) = p(y, x) \quad \text{(symmetry)}

(\text{PM2}) \quad p(x, x) = p(x, y) = p(y, y) \quad \text{then} \quad x = y \quad \text{(equality)}

(\text{PM3}) \quad p(x, x) \leq p(x, y) \quad \text{(small self-distances)}

(\text{PM4}) \quad p(x, z) + p(y, y) \leq p(x, y) + p(y, z) \quad \text{(triangle inequality)}

for all \( x, y, z \in X \). We use the abbreviation PMS for the partial metric space \((X, p)\).
Notice that for a partial metric $p$ on $X$, the function $d_p : X \times X \to \mathbb{R}^+$ given by
\begin{equation}
(1.1) \quad d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)
\end{equation}
is a (usual) metric on $X$. Observe that each partial metric $p$ on $X$ generates a $T_0$ topology $\tau_p$ on $X$ with a base the family open $p$-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. Similarly, closed $p$-ball is defined as $B_p[x, \varepsilon] = \{y \in X : p(x, y) \leq p(x, x) + \varepsilon\}$

**Definition 1.1** (see e.g.[10, 11, 1]).

(i) A sequence $\{x_n\}$ in a PMS $(X, p)$ converges to $x \in X$ if $p(x, x) = \lim_{n \to \infty} p(x, x_n)$,

(ii) a sequence $\{x_n\}$ in a PMS $(X, p)$ is called a Cauchy sequence if $\lim_{n,m \to \infty} p(x_n, x_m)$ exists (and finite),

(iii) A PMS $(X, p)$ is said to be complete if every Cauchy sequence $\{x_n\}$ in $X$ converges, with respect to $\tau_p$, to a point $x \in X$ such that $p(x, x) = \lim_{n,m \to \infty} p(x_n, x_m)$.

**Lemma 1.2** (see e.g.[10, 11, 1]).

(A) A sequence $\{x_n\}$ in a PMS $(X, p)$ is Cauchy if and only if $\{x_n\}$ is Cauchy in a metric space $(X, d_p)$.

(B) A PMS $(X, p)$ is complete if and only if a metric space $(X, d_p)$ is complete. Moreover,

\begin{equation}
(1.2) \quad \lim_{n \to \infty} d_p(x, x_n) = 0 \Leftrightarrow p(x, x) = \lim_{n \to \infty} p(x, x_n) = \lim_{n,m \to \infty} p(x_n, x_m)
\end{equation}

In this manuscript, we present some new fixed point theorems on a non-empty set on which there exists two partial metrics with certain conditions.

2. **Main Results**

The following two lemmas will be used in the proof of the main theorem.

**Lemma 2.1** (see e.g. [3]). Let $(X, p)$ be a complete PMS. Then

(A) If $p(x, y) = 0$ then $x = y$,

(B) If $x \neq y$, then $p(x, y) > 0$.

**Lemma 2.2** (see e.g. [1, 3]). Assume $x_n \to z$ as $n \to \infty$ in a PMS $(X, p)$ such that $p(z, z) = 0$. Then $\lim_{n \to \infty} p(x_n, y) = p(z, y)$ for every $y \in X$.

The following theorem is an extension of the result of Maia [9].

**Theorem 2.3.** Let $X$ be a non-empty set endowed with two partial metrics $p_1, p_2$, and let $T$ be a mapping of $X$ into itself. Suppose that

(i) $(X, p_1)$ is complete,

(ii) $p_1(x, y) \leq p_2(x, y)$ for all $x, y \in X$,

(iii) $T$ is continuous with respect to $\tau_{p_1}$,

(iv) $T$ is a contraction with respect to $p_2$, that is, $p_2(Tx, Ty) \leq kp_2(x, y)$ for all $x, y \in X$, where $0 \leq k < 1$. 
Then $T$ has a unique fixed point in $X$.

Proof. Fix $x \in X$. We construct a sequence $\{x_n\}$ in the following way:

(S1) $x_0 = x$,
(S2) $x_n = Tx_{n-1} = T^n x_0$ for each $n \in \mathbb{N}$.

Then, by assumption (iv) we have

$$p_2(x_{n+1}, x_n) = p_2(Tx_n, Tx_{n-1}) \leq kp_2(x_n, x_{n-1}) \leq \cdots \leq k^n p_2(Tx_0, x_0).$$

Hence, by standard calculations, we get that $\lim_{n,m \to \infty} p_2(x_n, x_m) = 0$, and by assumption (ii), $\lim_{n,m \to \infty} p_1(x_n, x_m) = 0$, i.e., $\{x_n\}$ is a Cauchy sequence in $(X, p_1)$. So, by assumption (i) and Lemma 1.2, it converges in $(X, d_{p_1})$ to a point $z \in X$. Again by Lemma 1.2,

$$p_1(z, z) = \lim_{n \to \infty} p_1(x_n, z) = \lim_{n,m \to \infty} p_1(x_n, x_m)$$

Since $\lim_{n,m \to \infty} p_1(x_n, x_m) = 0$, then by (2.1) we have $p_1(z, z) = 0$.

By the continuity of $T$ and also Lemma 2.2, one can get

$$p_1(z, z) = \lim_{n \to \infty} p_1(z, x_{n+1}) = \lim_{n \to \infty} p_1(z, T^n x_0) = p_1(z, T \lim_{n \to \infty} T^n x_0) = p_1(z, Tz).$$

Hence $P(Tz, z) = p(z, z) = 0$. Due to Lemma 2.1 the point $z$ is a unique fixed point of $T$. Suppose not, that is, there exist $z, y \in X$ such that $Tz = z$ and $Ty = y$. Then, $p_2(z, y) = p_2(Tz, Ty) \leq kp_2(z, y)$. Thus, $p_2(z, y) = 0$. Regarding Lemma 2.1, $z = y$.

**Theorem 2.4.** Let $(X, p_1)$ be a PMS and $T : X \to X$ a mapping. Consider the series:

$$\sum_{n=0}^{\infty} t^n p_1(T^n x, T^n y)$$

Suppose that for some $t > 1$, the series (2.2) converges for every $x, y \in X$. Then, for such a point $t$, the function $p_2 : X \times X \to \mathbb{R}^+$ defined by

$$p_2(x, y) = \sum_{n=0}^{\infty} t^n p_1(T^n x, T^n y)$$

is a partial metric on $X$, moreover,

(i) $p_2$ is an upper bound partial metric for $p_1$,
(ii) $T$ is a contraction with respect to $p_2$.

Proof. Since $t > 1$ and $p_1(T^n x, T^n y) \geq 0$ for all $x, y \in X$ and $n \in \mathbb{N}$, then $p_2(x, y) \geq 0$. It is clear that $p_2$ satisfies (PM1). For the proof of (PM2), assume $p_2(x, x) = p_2(x, y) = p_2(x, y)$ which is equivalent to

$$\sum_{n=0}^{\infty} t^n p_1(T^n x, T^n x) = \sum_{n=0}^{\infty} t^n p_1(T^n x, T^n y) = \sum_{n=0}^{\infty} t^n p_1(T^n y, T^n y)$$
Thus, and (PM4) are obtained by definition. In particular, $p_1(x, y) = p_1(x, x) = p_1(y, y)$, and hence, $x = y$. Moreover, (PM3) and (PM4) are obtained by definition.

Let us prove (i) and (ii).

$p_2(x, y) = \sum_{n=0}^{\infty} t^n p_1(T^n x, T^n y) = p_1(x, y) + \sum_{n=1}^{\infty} t^n p_1(T^n x, T^n y) = p_1(x, y) + t \sum_{n=0}^{\infty} t^n p_1(T^{n+1} x, T^{n+1} y) = p_1(x, y) + tp_2(Tx, Ty)$

Thus,

$p_2(Tx, Ty) = \frac{1}{t}(p_2(x, y) - p_1(x, y)) \leq \frac{1}{t}p_2(x, y)$.

**Theorem 2.5.** Suppose $(X, p_1)$ is a PMS and $T : X \to X$ is a mapping such that $p_1(T^m x, T^m y) \leq kp_1(x, y)$ for some $m \in \mathbb{N}$, where $0 \leq k < 1$. Then the series $p_2(x, y) = \sum_{n=0}^{\infty} t^n p_1(T^n x, T^n y)$ converges for $t > 1$, whatever the points $x, y \in X$.

**Proof.** By assumption, $p_1(T^m x, T^m y) \leq kp_1(x, y)$ for some $m \in \mathbb{N}$, and $0 \leq k < 1$. It yields that $p_1(T^{mn} x, T^{mn} y) \leq k^n p_1(x, y)$ for every $n$ integer. Then,

$p_2(x, y) = \sum_{n=0}^{\infty} t^n p_1(T^n x, T^n y) = \sum_{n=0}^{\infty} t^{mn} p_1(T^{mn} x, T^{mn} y) + \sum_{n=0}^{\infty} t^{mn+1} p_1(T^{mn+1} x, T^{mn+1} y) + \cdots + \sum_{n=0}^{\infty} t^{mn+n-1} p_1(T^{mn+n-1} x, T^{mn+n-1} y) \leq \sum_{n=0}^{\infty} t^{mn} k^n p_1(x, y) + \sum_{n=0}^{\infty} t^{mn} k^n p_1(Tx, Ty) + \cdots + t^{n-1} \sum_{n=0}^{\infty} t^{mn} k^n p_1(T^{n-1} x, T^{n-1} y)$

Just then take $t$ such that: $1 < t^n < \frac{1}{k}$, because the series converges regardless of the points $x, y \in X$.

**Theorem 2.6.** Let $X$ be a non-empty set endowed with two partial metrics $p_1, p_2$, and let $T$ be a mapping of $X$ into itself. Suppose that

(i) There exists a point $x_0 \in X$ such that the sequence of iterates $\{T^n(x_0)\}$ has a subsequence $\{T^{n_i}(x_0)\}$ converging to a point $z \in X$ for $\tau_{p_1}$,

(ii) $p_1(x, y) \leq p_2(x, y)$ for all $x, y \in X$,

(iii) $T$ is continuous at $z$ with respect to $p_1$,

(iv) $T$ is contraction with respect to $p_2$, that is, $p_2(Tx, Ty) \leq kp_2(x, y)$ for all $x, y \in X$, where $0 \leq k < 1$.

Then $T$ has a unique fixed point in $X$. 
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Proof. Fix $x_0 \in X$ and define $x_{n+1} = Tx_n$ for $n \in \mathbb{N} \cup \{0\}$. As it shown in the proof of Theorem 2.3, this sequence $\{x_n\}$ is Cauchy with respect to $p_2$. By $(ii)$, the sequence $\{x_n\}$ is also Cauchy with respect to $p_1$. By $(i)$, Cauchy sequence $\{x_n\}$ has a subsequence $\{x_{n_i}\}$ which converges $z \in X$ for $\tau_{p_1}$. Thus, $\{x_n\}$ converges to $z$ for $\tau_{p_1}$. By the continuity of $T$ and also Lemma 2.2 one can get

$$p_1(z, z) = \lim_{n \to \infty} p_1(z, x_{n+1}) = \lim_{n \to \infty} p(z, T^{n+1}x_0) = p_1(z, T(\lim_{n \to \infty} T^n x_0)) = p_1(z, Tz).$$

Hence $P(Tz, z) = p(z, z) = 0$. Due to Lemma 2.1 the point $z$ is a unique fixed point of $T$. To show uniqueness, assume the contrary. Let $z$ and $w$ be two different fixed points. Then, by $(iv)$,

$$p_2(z, w) = p_2(Tz, Tw) \leq kp_2(z, w)$$

Since $0 \leq k < 1$, one can get a contradiction. Thus, $T$ has a unique fixed point. \qed

Remark 2.7. Consider the following condition:

$(iv)^*$ There is a point $x_0 \in X$ such that the iterated sequence $\{T^n(x_0)\}$ is a Cauchy sequence with respect to $p_2$.

If the condition $(iv)$ is replaced by $(iv)^*$ in Theorem 2.6, the theorem will still guarantee the existence of fixed point.

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REFERENCES


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