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On density and π-weight of $L^p(\beta\mathbb{N},\mathbb{R},\mu)$

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Abstract

The new topological concept of selective separability is able to give some estimates for density and π-weight of the Lebesgue space $L^p(\beta\mathbb{N},\mathbb{R},\mu)$ with $1 \leq p < +\infty$. In particular, we deduce a purely topological proof of the non-separability of such a space.

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In Integration Theory, it is important to establish the separability or not of Lebesgue spaces of the type L^p, with $1 \leq p < +\infty$. In general, the usual proof of this type of results for certain Lebesgue spaces, is conducted through methods of Real Analysis.

In this work, we use some concepts and methods of pure General Topology in proving the non-separability of a particular Lebesgue space. Further, we provide some estimates for density and π-weight of such a space.

1. Introduction

In the context of infinite-combinatorial topology, M. Scheepers (see [14]) has introduced and studied a particular selection principle $\mathcal{S}_{fin}(\mathcal{D},\mathcal{D})$ connected with some problems of topological diagonalization of covers of a given completely regular topological space X, whose dense set family is $\mathcal{D}$. Subsequently, A. Bella and coworkers (see [3]) have extended and generalized some formal properties introduced by M. Scheepers in [14], into a pure topological context, proposing the notion of *selective separability,* defined as follows.

A completely regular topological space (or a $T_{3\frac{1}{2}}$-space), say (X,τ), is said to be *selectively separable* (or *M-separable*) if, for any sequence $\{D_n; n \in \mathbb{N}\}$ of dense sets of X, there exists a sequence $\{F_n; n \in \mathbb{N}\}$ of finite subsets of X such that $F_n \subseteq D_n$ for each $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} F_n$ is dense in X.

A selectively separable space is a separable space too. In fact, if we put $D_n = X$ for any $n \in \mathbb{N}$, then there exists a finite subset $F_n \subseteq X$ for each $n \in \mathbb{N}$ such that $\overline{\bigcup_{n\in\mathbb{N}} F_n} = X$, with $\bigcup_{n\in\mathbb{N}} F_n$ countable.
There exist however separable spaces that are not selectively separable, as, for example, the Tychonov cube $\mathbb{I}^{\mathfrak{c}}$ of weight $\mathfrak{c} = 2^{\aleph_0}$, when $\mathbb{I} = [0,1]$ is equipped with the usual Euclidean topology.
One of the purposes of this note is that of discussing some elementary properties of selectively separable spaces in view of their applications.

2. Preliminary concepts

Given an arbitrary topological space (X,τ), we put $\tau^* = \tau \setminus \{\varnothing\}$.
A *base* of X is a family $(\varnothing \neq)\mathcal{B} \subseteq \tau^*$ such that every element of τ^* is the union of elements of $\mathcal{B}$. A π*-base* of X is a family $(\varnothing \neq)\mathcal{B} \subseteq \tau^*$ such that, for every $A \in \tau^*$, there is a $B \in \mathcal{B}$ such that $B \subseteq A$. Any base of X is also a π-base, but the inverse, in general, is not true. The minimal cardinality of a base [π-base] of X, is called the *weight* [π*-weight*] of X, say $w(X)$ [$\pi w(X)$]. The property $\pi w(X) \leq w(X)$ holds true; moreover, if $\beta\mathbb{N}$ is the Čech-Stone compactification of the discrete space $\mathbb{N}$, then it is possible to prove that $\pi w(\beta\mathbb{N}) = \aleph_0 < \mathfrak{c} = w(\beta\mathbb{N})$.
If $d(X)$ is the density of X, it is immediate to prove that $d(X) \leq \pi w(X)$. In fact, if $\mathcal{B}$ is a π-base of X such that *card* $\mathcal{B} = \pi w(X)$, and A is a dense subset of X, then we have $A \cap B \neq \varnothing$ for any $B \in \mathcal{B}$. In addition, if we choose a $x_B \in A \cap B$ for every $B \in \mathcal{B}$, then the set $\{x_B; B \in \mathcal{B}\}$ is dense in X, so that $d(X) \leq \mathit{card}\ \{x_B; B \in \mathcal{B}\} \leq \mathit{card}\ \mathcal{B} = \pi w(X)$.
Therefore, we have the following relation between cardinal functions

$$d(X) \leq \pi w(X) \leq w(X). \tag{1}$$

If (X,τ) is a Hausdorff topological space, then it is known that it has an infinite disjoint base, say $\mathcal{B}$. It follows that $\aleph_0 \leq \pi(X)$, because, if $\tilde{\mathcal{B}}$ is any π-base of X, then, by the definition of π-base, there exists a $A_B \in \tilde{\mathcal{B}}$, for each $B \in \mathcal{B}$, such that $A_B \subseteq B$, for which card $\tilde{\mathcal{B}} \geq \aleph_0$ being $\mathcal{B}$ an infinite disjoint family. Since $\tilde{\mathcal{B}}$ is an arbitrary π-base of X, it follows that $\pi w(X) \geq \aleph_0$. In particular, if (X,τ) is a metrizable space, then it follows that $d(X) = \pi w(X) = w(X) \geq \aleph_0$ (see [11], Section 6, Remark 6.2, and [7]).
The space (X,τ) has *countable fan tightness* if, for any $x \in X$ and any sequence $\{A_n; n \in \mathbb{N}\}$ of subsets of X such that $x \in \bigcap_{n\in\mathbb{N}} \overline{A_n}$, there exists a finite set $B_n \subseteq A_n$, for each $n \in \mathbb{N}$, such that $x \in \overline{\bigcup_{n\in\mathbb{N}} B_n}$. In this case, we write concisely *vet* $(X) \leq \aleph_0$.
The space (X,τ) has *countable tightness* if, for any $x \in X$ and $A \subseteq X$ with $x \in \overline{A}$, there exists a countable set $B \subseteq A$ such that $x \in \overline{B}$. In this case, we write concisely $t(X) \leq \aleph_0$. We have $(vet(X) \leq \aleph_0) \Rightarrow (t(X) \leq \aleph_0)$, but the inverse is, in general, not true.

3. Selective separability: basic properties

We here recall some properties of selectively separable spaces.

Theorem 3.1. *Every dense subspace of a selectively separable space is also selectively separable.*

Proof. If S is a dense subspace of a selectively separable space (X, τ) and $\{D_n; n \in \mathbb{N}\}$ is a sequence of dense subsets of (S, τ_S), it follows that each D_n is also dense in X (as a consequence of the density of S in X), so that, for every $n \in \mathbb{N}$, there exists a finite set $F_n \subseteq D_n$ such that $\overline{\bigcup_{n\in\mathbb{N}} F_n}^X = X$. Hence, we have

$$\overline{\bigcup_{n\in\mathbb{N}} F_n}^S = \overline{\bigcup_{n\in\mathbb{N}} F_n}^X \cap S = X \cap S = S.$$

□

Theorem 3.2. *A topological space (X, τ) has the property that*

(1) *if $\pi w(X) = \aleph_0$, then X is selectively separable,*
(2) *if X is separable and vet $(X) \leq \aleph_0$, then X is selectively separable.*

Proof. If $\pi w(X) = \aleph_0$, let $\mathcal{B} = \{B_n; n \in \mathbb{N}\}$ be a π-base of X. If $\{D_n; n \in \mathbb{N}\}$ is a sequence of dense subsets of X, then we have $D_n \cap B_m \neq \varnothing$ for all $n, m \in \mathbb{N}$, so that we can choose a point $x_n \in D_n \cap B_n$ for each $n \in \mathbb{N}$. Hence, $\{x_n; n \in \mathbb{N}\}$ is dense in X, and thus X is selectively separable. This proves (1).

If *vet* $(X) \leq \aleph_0$, and X is separable, let $\{a_n; n \in \mathbb{N}\}$ be a countable dense subset of X, and let $\{D_n; n \in \mathbb{N}\}$ be a sequence of dense subsets of X. Since $\overline{D_k} = X = \overline{\{a_n; n \in \mathbb{N}_0\}}$ for each $k \in \mathbb{N}$, we have, for each $n \in \mathbb{N}$, $a_n \in \overline{D_k}$ for infinitely many $k \in \mathbb{N}$, so that $a_n \in \overline{D_k}\ \ \forall k \in L_n$, with L_n infinite subset of $\mathbb{N}$. Therefore, we may always choose a disjoint family $\{L_n; n \in \mathbb{N}\}$ of infinite subsets of $\mathbb{N}$ in such a way that[1] $\mathbb{N} = \bigcup_{n\in\mathbb{N}} L_n$. Hence $a_n \in \bigcap_{k\in L_n} \overline{D_k}$ for every $n \in \mathbb{N}$, and since X has countable fan tightness, it follows that there exists a finite set $F_k \subseteq D_k$, for each $k \in L_n$, such that $a_n \in \overline{\bigcup_{k\in L_n} F_k}$, whence we have

$$\{a_n; n \in \mathbb{N}\} \subseteq \bigcup_{n\in\mathbb{N}} \big(\overline{\bigcup_{k\in L_n} F_k}\big) \subseteq \overline{\bigcup_{n\in\mathbb{N}} \bigcup_{k\in L_n} F_k},$$

so that

$$\left(X = \overline{\{a_n; n \in \mathbb{N}\}} \subseteq \overline{\bigcup_{n\in\mathbb{N}} \bigcup_{k\in L_n} F_k} \subseteq X\right) \Rightarrow \left(X = \overline{\bigcup_{n\in\mathbb{N}} \bigcup_{k\in L_n} F_k}\right).$$

The countability of $\bigcup_{n\in\mathbb{N}} \bigcup_{k\in L_n} F_k$ proves the selective separability of X. This proves (2). □

Remark 3.3. In general, the proposition (1) of Theorem 3.2, is not valid if we suppose $\pi w(X) < \aleph_0$. Moreover, since $w(X) = \aleph_0$ for every separable

[1]For instance, if $\{p_m; m \in \mathbb{N}\}$ is the infinite sequence of the prime numbers of $\mathbb{N}$, then $L_n = \{p_m^n; m \in \mathbb{N}\}\ \ \forall n \in \mathbb{N}$, verifies as required.

metrizable spaces X, from what has been said in Section 2, it follows that $w(X) = \pi w(X) = d(X) = \aleph_0$ for such spaces[2].

If X is a completely regular topological space, and $C_p(X)$ is the space of continuous functions $f : X \to \mathbb{R}$, equipped with the pointwise topology (see [6], Section 2.6.), then it is possible to prove the following

Theorem 3.4. *$C_p(X)$ is selectively separable if and only if X is separable and vet $(X) = \aleph_0$.*

The proof is given in [3].

Nevertheless, the study of $C(X)$ from this viewpoint has been only done with respect to the pointwise topology (see [2]).

4. The Lebesgue space $L^p(\beta\mathbb{N}, \mathbb{R}, \mu)$

Following [15], Counterexamples 110 and 111, if we endow $\mathbb{N}$ with the discrete topology, said $\beta\mathbb{N}$ the Čech-Stone compactification of the discrete space $\mathbb{N}$, then $\beta\mathbb{N}$ is a compact Hausdorff space. Let us recall that $\aleph_0 = \pi w(\beta\mathbb{N}) < w(\beta\mathbb{N}) = \mathfrak{c}$.

The definition of $L^p(\beta\mathbb{N}, \mathbb{R}, \mu)$ can be given in three ways[3] as follows.

1. Following [10], Chap. 7, and[4] [4], Chapter IX, Section 5, let μ be a prescribed Radon measure on $\beta\mathbb{N}$, so that we can consider the Lebesgue space $L^p(\beta\mathbb{N}, \mathbb{R}, \mu)$ of the L^p-sommable functions $f : \beta\mathbb{N} \to \mathbb{R}$, with $1 \leq p < +\infty$.
   It is known that the space of functions with compact support $C_0(\beta\mathbb{N}, \mathbb{R})$ is dense in $L^p(\beta\mathbb{N}, \mathbb{R}, \mu)$, and since $\beta\mathbb{N}$ is compact, we have that $C_0(\beta\mathbb{N}, \mathbb{R}) = C(\beta\mathbb{N}, \mathbb{R})$ is dense too in $L^p(\beta\mathbb{N}, \mathbb{R}, \mu)$.
2. Taking into account that $\beta\mathbb{N}$ is a Hausdorff compact space, let μ be a positive Borel measure on $\beta\mathbb{N}$ (see [13], Chap. 3, and [9], Chap. 10), defined over the Borel σ-algebra generated by the topology of $\beta\mathbb{N}$, and having a countable base (see [9], Section 37.1, and [8], Chap. III, Section 3, Problems 418 - 420).
   Hence, reasoning as in *1.,* it follows that $C(\beta\mathbb{N}, \mathbb{R})$ is dense in $L^p(\beta\mathbb{N}, \mathbb{R}, \mu)$.
3. Let X be a compact space with a countable base ($w(X) = \aleph_0$), and let μ be a Baire measure on X. Then $L^p(X, \mathbb{R}, \mu)$ is a separable space for each $1 \leq p < +\infty$. Indeed (see [12], Chap. IV, Section 4, and Problem 43 (a)), if $\{B_n; n \in \mathbb{N}\}$ is a countable base of X, for all $n, m \in \mathbb{N},\ n \neq m$,

[2] From here, it follows that the concepts of separability and selective separability are the same, in the case of metric spaces.

[3] There exists a fourth way, based on integration theory over separable measure spaces (see [5], and [16], Chap. 7, Section 5), leading to the same results.

[4] Section 5 of N. Bourbaki's Chapter IX, deals with measures on a completely regular space. These results can be applied to $\beta\mathbb{N}$ since this is a Hausdorff compact space, and hence a completely regular space (see [6], Section 3.3.).

with $\overline{B}_n \cap \overline{B}_m = \varnothing$, it is possible to define $f_{n,m} \in C(X, \mathbb{R})$ in such a way that $f_{n,m} = 0$ on B_n and $f_{n,m} = 1$ on B_m. Hence, the $f_{n,m}$ can be used to define a countable dense set in $C(X, \mathbb{R})$, whence a countable dense set in $L^p(X, \mathbb{R}, \mu)$, being $C(X, \mathbb{R})$ dense in $L^p(X, \mathbb{R}, \mu)$.
Nevertheless, this proof is no longer valid when $\pi w(X) = \aleph_0$, hence for $X = \beta\mathbb{N}$ since $\aleph_0 = \pi w(\beta\mathbb{N}) < w(\beta\mathbb{N}) = \mathfrak{c}$, so that we cannot say that such a Lebesgue space is separable, but only that it has a dense subset (namely $C(X, \mathbb{R})$).

On the other hand, it is well-known too as the Banach space (respect to the supremum norm) of all real bounded sequences, namely $l^\infty(\mathbb{N}, \mathbb{R})$, is not separable as metric space. Since each $f \in l^\infty(\mathbb{N}, \mathbb{R})$ is continuous and bounded, by the universal properties of $\beta\mathbb{N}$ (see [6], Section 3.6.), it is possible, in a unique manner[5], to extend it to a continuous function $\psi(f) : \beta\mathbb{N} \to \mathbb{R}$. It is possible to prove (see [1], Sections 2.17 and 2.18) that the correspondence $f \to \psi(f)$ is an isometric isomorphism from $l^\infty(\mathbb{N}, \mathbb{R})$ to $C(\beta\mathbb{N}, \mathbb{R})$, hence a homeomorphism. It follows that $C(\beta\mathbb{N}, \mathbb{R})$ is not separable, but dense in $L^p(\beta\mathbb{N}, \mathbb{R}, \mu)$, with $1 \leq p < +\infty$.

Finally, it also follows that the space $L^p(\beta\mathbb{N}, \mathbb{R}, \mu)$ cannot be separable: indeed, as metric space, taking into account what has been said in Section 2 and in Remark 3.3, we have the following density and π-weight estimates for such a space

$$\aleph_0 < d(L^p(\beta\mathbb{N}, \mathbb{R}, \mu)) = \pi w(L^p(\beta\mathbb{N}, \mathbb{R}, \mu)) = w(L^p(\beta\mathbb{N}, \mathbb{R}, \mu)). \tag{2}$$

Hence, if $L^p(\beta\mathbb{N}, \mathbb{R}, \mu)$ were separable, then we would have $w(L^p(\beta\mathbb{N}, \mathbb{R}, \mu)) = \aleph_0$ (as separable metric space), hence $\pi w(L^p(\beta\mathbb{N}, \mathbb{R}, \mu)) = \aleph_0$, so that, by (1) of Theorem 3.2, $L^p(\beta\mathbb{N}, \mathbb{R}, \mu)$ would be selectively separable, whence it would follow that every dense subspace of it would be selectively separable too (by Theorem 3.1), and this is impossible since $C(\beta\mathbb{N}, \mathbb{R})$ is a no separable dense subspace of $L^p(\beta\mathbb{N}, \mathbb{R}, \mu)$, with $1 \leq p < +\infty$.

Following the lines of this paper, it would be of a certain interest to analyze the possible role played by the various notions of tightness, fan tightness (see [2] for the topological function spaces case) and others properties of selectively separable spaces, in the case of a generic Lebesgue space $L^p(X, \mathbb{R}, \mu)$, with $1 \leq p < +\infty$.

[5]Recalling, on the other hand, that $\mathbb{N}$ is dense in $\beta\mathbb{N}$.

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