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Relative dimension r-dim and finite spaces

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Abstract

In [4] a relative covering dimension is defined and studied which is denoted by r-dim. *In this paper we give an algorithm of polynomial order for computing the dimension* r-dim *of a pair* (Q, X), *where* Q *is a subset of a finite space* X, *using matrix algebra.*

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1. Introduction and preliminaries

The "relative dimensions" or "positional dimensions" are functions whose domains are classes of subsets. By a class of subsets we mean a class consisting of pairs (Q, X), where Q is a subset of a space X.

The class of finite topological spaces was first studied by P.A. Alexandroff in 1937 in [1]. A topological space X is *finite* if the set X is finite. In what follows we denote by $X = \{x_1, \ldots, x_n\}$ a finite space of n elements and by $\mathbf{U}_i$ the smallest open set of X containing the point x_i, $i = 1, \ldots, n$. The cardinality of a set X is denoted by $|X|$ and the first infinite cardinal is denoted by ω.

Let $X = \{x_1, \ldots, x_n\}$ be a finite space of n elements. The $n \times n$ matrix $T = (t_{ij})$, where

$$t_{ij} = \begin{cases} 1, & \text{if } x_i \in \mathbf{U}_j \\ 0, & \text{otherwise} \end{cases}$$

is called the *incidence matrix* of X. We observe that

$$\mathbf{U}_j = \{x_i : t_{ij} = 1\}, \ j = 1, \ldots, n.$$

We denote by $c_1, \ldots, c_n$ the n columns of the matrix T. Let

$$c_i = \begin{pmatrix} c_{1i} \\ c_{2i} \\ \vdots \\ c_{ni} \end{pmatrix} \text{ and } \mathrm{c_j} = \begin{pmatrix} c_{1j} \\ c_{2j} \\ \vdots \\ c_{nj} \end{pmatrix}$$

be two $n \times 1$ matrices. Then, by $\max c_i$ we denote the maximum

$$\max\{c_{1i}, c_{2i}, \ldots, c_{ni}\}$$

and by $c_i + c_j$ the $n \times 1$ matrix

$$c_i + c_j = \begin{pmatrix} c_{1i} + c_{1j} \\ c_{2i} + c_{2j} \\ \vdots \\ c_{ni} + c_{nj} \end{pmatrix}.$$

Also, we write $c_i \leq c_j$ if only if $c_{ki} \leq c_{kj}$ for each $k = 1, \ldots, n$.

For the following notions see for example [2].

Let X be a space. A *cover* of X is a non-empty set of subsets of X, whose union is X. A cover c of X is said to be open (closed) if all elements of c is open (closed). A family r of subsets of X is said to be a *refinement* of a family c of subsets of X if each element of r is contained in an element of c.

Define the *order* of a family r of subsets of a space X as follows:

(a) $\mathrm{ord}(r) = -1$ if and only if r consists of only the empty set.
(b) $\mathrm{ord}(r) = n$, where $n \in \omega$, if and only if the intersection of any $n + 2$ distinct elements of r is empty and there exist $n + 1$ distinct elements of r, whose intersection is not empty.
(c) $\mathrm{ord}(r) = \infty$, if and only if for every $n \in \omega$ there exist n distinct elements of r, whose intersection is not empty.

Definition 1.1 (see [4])**.** We denote by r-dim the (unique) function that has as domain the class of all subsets and as range the set $\omega \cup \{-1, \infty\}$ satisfying the following condition r-dim$(Q, X) \leq n$, where $n \in \{-1\} \cup \omega$ if and only if for every finite family c of open subsets of X such that $Q \subseteq \cup\{U : U \in c\}$ there exists a finite family r of open subsets of X refinement of c such that $Q \subseteq \cup\{V : V \in r\}$ and $\mathrm{ord}(r) \leq n$.

Finite topological spaces and the notion of dimension play an important role in digital spaces, computer graphics, and image analysis. In [5] the authors gave an algorithm for computing the covering dimension of a finite topological space using matrix algebra. In this paper we give an algorithm of polynomial order for computing the dimension r-dim of a pair (Q, X), where Q is a subset of a finite space X, using matrix algebra.

2. Finite spaces and dimension r-dim

In this section we present some propositions concerning the dimension r-dim of a pair (Q, X), where Q is a subset of a finite space X.

Proposition 2.1. *Let $X = \{x_1, \ldots, x_n\}$ be a finite space and $Q \subseteq X$. Then, $\text{r-dim}(Q, X) \leq k$, where $k \in \omega$, if and only if there exists a family $\{\mathbf{U}_{j_1}, \ldots, \mathbf{U}_{j_m}\}$ such that $\{x_{j_1}, \ldots, x_{j_m}\} \subseteq Q \subseteq \mathbf{U}_{j_1} \cup \ldots \cup \mathbf{U}_{j_m}$ and $\text{ord}(\{\mathbf{U}_{j_1}, \ldots, \mathbf{U}_{j_m}) \leq k$.*

Proof. Let $\text{r-dim}(Q, X) \leq k$, where $k \in \omega$. We prove that there exists a family $\{\mathbf{U}_{j_1}, \ldots, \mathbf{U}_{j_m}\}$ such that $\{x_{j_1}, \ldots, x_{j_m}\} \subseteq Q \subseteq \mathbf{U}_{j_1} \cup \ldots \cup \mathbf{U}_{j_m}$ and $\text{ord}(\{\mathbf{U}_{j_1}, \ldots, \mathbf{U}_{j_m}) \leq k$.

Let

$$\nu = \min\{m \in \omega : \text{there exist } j_1, \ldots, j_m \in \{1, \ldots, n\} \text{ such that } \{x_{j_1}, \ldots, x_{j_m}\} \subseteq Q \subseteq \mathbf{U}_{j_1} \cup \cdots \cup \mathbf{U}_{j_m}\}$$

and $c = \{\mathbf{U}_{j_1}, \ldots, \mathbf{U}_{j_\nu}\}$ be a family such that

$$\{x_{j_1}, \ldots, x_{j_\nu}\} \subseteq Q \subseteq \mathbf{U}_{j_1} \cup \ldots \cup \mathbf{U}_{j_\nu}.$$

Since $\text{r-dim}(Q, X) \leq k$, there exists a family $r = \{V_1, \ldots, V_\mu\}$ of open subsets of X refinement of c such that $Q \subseteq V_1 \cup \ldots \cup V_\mu$ and $\text{ord}(r) \leq k$. Clearly, it suffices to prove that $\{\mathbf{U}_{j_1}, \ldots, \mathbf{U}_{j_\nu}\} \subseteq r$. Indeed, we suppose that there exists $\alpha \in \{1, \ldots, \nu\}$ such that $\mathbf{U}_{j_\alpha} \notin r$. Since $x_{j_\alpha} \in Q$, there exists $\beta \in \{1, \ldots, \mu\}$ such that $x_{j_\alpha} \in V_\beta$. By the fact that $\mathbf{U}_{j_\alpha}$ is the smallest open set of X containing the point x_{j_α} we have that $\mathbf{U}_{j_\alpha} \subseteq V_\beta$. Also, since $\mathbf{U}_{j_\alpha} \notin r$, we have $\mathbf{U}_{j_\alpha} \neq V_\beta$. Therefore, $\mathbf{U}_{j_\alpha} \subset V_\beta$. Since r is a refinement of c, there exists $\gamma \in \{1, \ldots, \nu\}$ such that $V_\beta \subseteq \mathbf{U}_{j_\gamma}$. Hence,

$$\mathbf{U}_{j_\alpha} \subset \mathbf{U}_{j_\gamma}.$$

We observe that $Q \subseteq (\mathbf{U}_{j_1} \cup \ldots \cup \mathbf{U}_{j_\nu}) \setminus \mathbf{U}_{j_\alpha}$, which is a contradiction by the choice of ν. Thus, $c \subseteq r$.

Conversely, we suppose that there exists a family $\{\mathbf{U}_{j_1}, \ldots, \mathbf{U}_{j_m}\}$ such that $\{x_{j_1}, \ldots, x_{j_m}\} \subseteq Q \subseteq \mathbf{U}_{j_1} \cup \ldots \cup \mathbf{U}_{j_m}$ and $\text{ord}(\{\mathbf{U}_{j_1}, \ldots, \mathbf{U}_{j_m}) \leq k$. We prove that $\text{r-dim}(Q, X) \leq k$.

Indeed, let c be a finite family of open subsets of X such that $Q \subseteq \cup\{U : U \in c\}$. It suffices to prove that the family $\{\mathbf{U}_{j_1}, \ldots, \mathbf{U}_{j_m}\}$ is a refinement of c. For every $i \in \{1, \ldots, m\}$ there exists $V_i \in c$ such that $x_{j_i} \in \mathbf{U}_{j_i} \subseteq V_i$. This means that the family $\{\mathbf{U}_{j_1}, \ldots, \mathbf{U}_{j_m}\}$ is a refinement of c. □

Proposition 2.2. *Let $X = \{x_1, \ldots, x_n\}$ be a finite space, where $n > 1$, and $Q \subseteq X$. Then,*

$$\text{r-dim}(Q, X) \leq |Q| - 1.$$

Proof. Let $Q = \{x_{j_1}, \ldots, x_{j_m}\}$. The family $\{\mathbf{U}_{j_1}, \ldots, \mathbf{U}_{j_m}\}$ has m elements and, therefore, $\text{ord}(\{\mathbf{U}_{j_1}, \ldots, \mathbf{U}_{j_m}\}) \leq m - 1$. Thus, by Proposition 2.1, $\text{r-dim}(Q, X) \leq m - 1 = |Q| - 1$. □

Note 1. In the following propositions we suppose that $X = \{x_1, \ldots, x_n\}$ is a finite space with n elements, $Q \subseteq X$, $T = (t_{ij})$, $i = 1, \ldots, n$, $j = 1, \ldots, n$, the incidence matrix of X, and $c_1, \ldots, c_n$ the n columns of the matrix T. We denote by $\mathbf{1}_Q$ the $n \times 1$ matrix

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix},$$

where

$$a_i = \begin{cases} 1, & \text{if } x_i \in Q \\ 0, & \text{otherwise.} \end{cases}$$

Example 2.3. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ and $Q = \{x_1, x_3, x_4\}$. Then,

$$\mathbf{1}_Q = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

Proposition 2.4. *If $c_j = \mathbf{1}_Q$ and $x_j \in Q$ for some $j \in \{1, \ldots, n\}$, then* r-dim$(Q, X) = 0$.

Proof. Since $c_j = \mathbf{1}_Q$, we have $t_{ij} = 1$ for every $x_i \in Q$ and, therefore, $Q \subseteq \mathbf{U}_j$. Since $\text{ord}(\{\mathbf{U}_j\}) = 0$, by Proposition 2.1, we have r-dim$(Q, X) = 0$. □

Proposition 2.5. *Let c_{j_i}, $i = 1, \ldots, m$, be m columns of the matrix T. Then, $c_{j_1} + \ldots + c_{j_m} \geq \mathbf{1}_Q$ if and only if $Q \subseteq \mathbf{U}_{j_1} \cup \ldots \cup \mathbf{U}_{j_m}$.*

Proof. Let $c_{j_1} + \ldots + c_{j_m} \geq \mathbf{1}_Q$. We prove that $Q \subseteq \mathbf{U}_{j_1} \cup \ldots \cup \mathbf{U}_{j_m}$. Let $x_{i_0} \in Q$. By the definition of the matrix T and by the assumption $c_{j_1} + \ldots + c_{j_m} \geq \mathbf{1}_Q$, there exists $\kappa \in \{1, \ldots, m\}$ such that $t_{i_0 j_\kappa} = 1$. Since $\mathbf{U}_{j_\kappa} = \{x_i : t_{ij_\kappa} = 1\}$, we have $x_{i_0} \in \mathbf{U}_{j_\kappa}$. Thus, $Q \subseteq \mathbf{U}_{j_1} \cup \ldots \cup \mathbf{U}_{j_m}$.

Conversely, we suppose that $Q \subseteq \mathbf{U}_{j_1} \cup \ldots \cup \mathbf{U}_{j_m}$. Then, for every $x_i \in Q$ there exists $\kappa(i) \in \{1, \ldots, m\}$ such that $x_i \in \mathbf{U}_{j_{\kappa(i)}}$. Therefore, by the definition of the matrix T, $t_{ij_{\kappa(i)}} = 1$. Thus, $c_{j_1} + \ldots + c_{j_m} \geq \mathbf{1}_Q$. □

Proposition 2.6 (see Proposition 2.6 of [5])**.** *Let c_{j_i}, $i = 1, \ldots, m$, be m columns of the matrix T and $k = \max(c_{j_1} + \ldots + c_{j_m})$, that is k is the maximum element of the $n \times 1$ matrix $c_{j_1} + \ldots + c_{j_m}$. Then,*

$$\text{ord}(\{\mathbf{U}_{j_1}, \ldots, \mathbf{U}_{j_m}\}) = k - 1.$$

Definition 2.7. We define a preorder $\leqslant$ on the set of all families $\{x_{j_1}, \ldots, x_{j_m}\}$ with $\{x_{j_1}, \ldots, x_{j_m}\} \subseteq Q \subseteq \mathbf{U}_{j_1} \cup \cdots \cup \mathbf{U}_{j_m}$ by

$$\{x_{j_1}, \ldots, x_{j_{m_1}}\} \leqslant \{x_{j'_1}, \ldots, x_{j'_{m_2}}\}$$

if and only if

$$\{\mathbf{U}_{j_1}, \dots, \mathbf{U}_{j_{m_1}}\} \subseteq \{\mathbf{U}_{j'_1}, \dots, \mathbf{U}_{j'_{m_2}}\}.$$

Remark 2.8. The space X is T_0 if and only if $\mathbf{U}_i = \mathbf{U}_j$ implies $x_i = x_j$ for every i, j (see [1]). Therefore, if the space X is T_0, then the relation $\leqslant$ is an order. We note that if the space X is T_0, then there exists exactly one minimal family on the set of all families $\{x_{j_1}, \dots, x_{j_m}\}$ with $\{x_{j_1}, \dots, x_{j_m}\} \subseteq Q \subseteq \mathbf{U}_{j_1} \cup \dots \cup \mathbf{U}_{j_m}$.

Proposition 2.9. *Let* $\{x_{i_1}, \dots, x_{i_\mu}\} \subseteq Q \subseteq \{\mathbf{U}_{i_1}, \dots, \mathbf{U}_{i_\mu}\}$,

$$\nu = \min\{m \in \omega : \text{there exist } j_1, \dots, j_m \in \{1, \dots, n\} \text{ such that } \{x_{j_1}, \dots, x_{j_m}\} \subseteq Q \subseteq \mathbf{U}_{j_1} \cup \dots \cup \mathbf{U}_{j_m}\},$$

and $\{x_{j_1}, \dots, x_{j_\nu}\} \subseteq Q \subseteq \{\mathbf{U}_{j_1}, \dots, \mathbf{U}_{j_\nu}\}$. *Then,*

$$\{x_{j_1}, \dots, x_{j_\nu}\} \leqslant \{x_{i_1}, \dots, x_{i_\mu}\}.$$

Proof. The proof is similar to that of Proposition 2.1. □

Proposition 2.10. *Let* $\{x_{j_1}, \dots, x_{j_\nu}\}$ *be a minimal family on the set of all families* $\{x_{j_1}, \dots, x_{j_m}\}$ *with* $\{x_{j_1}, \dots, x_{j_m}\} \subseteq Q \subseteq \mathbf{U}_{j_1} \cup \dots \cup \mathbf{U}_{j_m}$. *If*

$$\operatorname{ord}(\{\mathbf{U}_{j_1}, \dots, \mathbf{U}_{j_\nu}\}) = k \geq 0,$$

then for every family $\{x_{r_1}, \dots, x_{r_\mu}\}$ *with* $\{x_{r_1}, \dots, x_{r_\mu}\} \subseteq Q \subseteq \mathbf{U}_{r_1} \cup \dots \cup \mathbf{U}_{r_\mu}$ *we have* $\operatorname{ord}(\{\mathbf{U}_{r_1}, \dots, \mathbf{U}_{r_\mu}\} \geq k$.

Proof. Let $\{\mathbf{U}_{r_1}, \dots, \mathbf{U}_{r_\mu}\}$ be a family such that

$$\{x_{r_1}, \dots, x_{r_\mu}\} \subseteq Q \subseteq \mathbf{U}_{r_1} \cup \dots \cup \mathbf{U}_{r_\mu}.$$

Then,

$$\{x_{j_1}, \dots, x_{j_\nu}\} \leqslant \{x_{r_1}, \dots, x_{r_\mu}\}$$

and, therefore,

$$\{\mathbf{U}_{j_1}, \dots, \mathbf{U}_{j_\nu}\} \subseteq \{\mathbf{U}_{r_1}, \dots, \mathbf{U}_{r_\mu}\}.$$

Since $\operatorname{ord}(\{\mathbf{U}_{j_1}, \dots, \mathbf{U}_{j_\nu}\}) = k$, we have $\operatorname{ord}(\{\mathbf{U}_{r_1}, \dots, \mathbf{U}_{r_\mu}\} \geq k$. □

Proposition 2.11. *Let* $\{x_{j_1}, \dots, x_{j_\nu}\}$ *be a minimal family on the set of all families* $\{x_{j_1}, \dots, x_{j_m}\}$ *with* $\{x_{j_1}, \dots, x_{j_m}\} \subseteq Q \subseteq \mathbf{U}_{j_1} \cup \dots \cup \mathbf{U}_{j_m}$. *Then,*

$$\text{r-}\dim(Q, X) = \max(c_{j_1} + \ldots + c_{j_\nu}) - 1.$$

Proof. Let $k = \max(c_{j_1} + \ldots + c_{j_\nu})$. Then, by Proposition 2.6, we have

$$\operatorname{ord}(\{\mathbf{U}_{j_1}, \dots, \mathbf{U}_{j_\nu}\}) = k - 1$$

and, therefore, by Proposition 2.1, $\text{r-}\dim(Q, X) \leq k - 1$. We prove that $\text{r-}\dim(Q, X) = k - 1$. We suppose that $\text{r-}\dim(Q, X) < k - 1$. Then, by Proposition 2.1, there exists a family $\{\mathbf{U}_{r_1}, \dots, \mathbf{U}_{r_\mu}\}$ such that

$$\{x_{r_1}, \dots, x_{r_\mu}\} \subseteq Q \subseteq \mathbf{U}_{r_1} \cup \dots \cup \mathbf{U}_{r_\mu}$$

and

$$\operatorname{ord}(\{\mathbf{U}_{r_1}, \dots, \mathbf{U}_{r_\mu}\}) < k - 1.$$

Since $\operatorname{ord}(\{\mathbf{U}_{j_1},\ldots,\mathbf{U}_{j_\nu}\}) = k-1$, by Proposition 2.10, we have

$$\operatorname{ord}(\{\mathbf{U}_{r_1},\ldots,\mathbf{U}_{r_\mu}\}) \geq k-1$$

which is a contradiction. Thus, $\text{r-}\dim(Q,X) = k-1$. □

Proposition 2.12. *Let c_{j_i}, $i = 1,\ldots,\nu$, be ν columns of the matrix T such that $c_{j_1}+\ldots+c_{j_\nu} \geq \mathbf{1}_Q$ and $\{x_{j_1},\ldots,x_{j_\nu}\} \subseteq Q$. If $c_{r_1}+\ldots+c_{r_q} \ngeq \mathbf{1}_Q$ for every $\{x_{r_1},\ldots,x_{r_q}\} \subseteq Q$ and $q<\nu$, then $\{x_{j_1},\ldots,x_{j_\nu}\}$ is a minimal family on the set of all families $\{x_{j_1},\ldots,x_{j_m}\}$ with $\{x_{j_1},\ldots,x_{j_m}\} \subseteq Q \subseteq \mathbf{U}_{j_1}\cup\cdots\cup\mathbf{U}_{j_m}$.*

Proof. Since $c_{j_1}+\ldots+c_{j_\nu} \geq \mathbf{1}_Q$ and $c_{r_1}+\ldots+c_{r_q} \ngeq \mathbf{1}_Q$ for every $\{x_{r_1},\ldots,x_{r_q}\} \subseteq Q$ and $q<m$, by Proposition 2.5, we have

$$\nu = \min\{m\in\omega : \text{there exist } j_1,\ldots,j_m \in \{1,\ldots,n\} \text{ such that } \{x_{j_1},\ldots,x_{j_m}\} \subseteq Q \subseteq \mathbf{U}_{j_1}\cup\cdots\cup\mathbf{U}_{j_m}\}.$$

Thus, by Proposition 2.9, $\{x_{j_1},\ldots,x_{j_\nu}\}$ is a minimal family on the set of all families $\{x_{j_1},\ldots,x_{j_m}\}$ with $\{x_{j_1},\ldots,x_{j_m}\} \subseteq Q \subseteq \mathbf{U}_{j_1}\cup\cdots\cup\mathbf{U}_{j_m}$. □

By Propositions 2.11 and 2.12 we have the following corollary.

Corollary 2.13. *Let c_{j_i}, $i = 1,\ldots,\nu$, be ν columns of the matrix T such that $c_{j_1}+\ldots+c_{j_\nu} \geq \mathbf{1}_Q$ and $\{x_{j_1},\ldots,x_{j_\nu}\} \subseteq Q$. If $c_{r_1}+\ldots+c_{r_q} \ngeq \mathbf{1}_Q$ for every $\{x_{r_1},\ldots,x_{r_q}\} \subseteq Q$ and $q<\nu$, then*

$$\text{r-}\dim(Q,X) = \max(c_{j_1}+\ldots+c_{j_\nu}) - 1.$$

3. An algorithm for computing the covering dimension

In this section we give an algorithm of polynomial order for computing the dimension $\text{r-dim}(Q,X)$, where Q is a subset of a finite space X, using the Propositions 2.11 and 2.5.

Algorithm 3.1. *Let $X = \{x_1,\ldots,x_n\}$ be a finite space of n elements, $Q = \{x_{\lambda_1},\ldots,x_{\lambda_l}\} \subseteq X$, and $T = (t_{ij})$ the $n\times n$ incidence matrix of X. Our intended algorithm contains $l-1$ steps:*

Step 1. *Read the l columns $c_{\lambda_1},\ldots,c_{\lambda_l}$ of the matrix T. If some column is equal to $\mathbf{1}_Q$, then print*

$$\text{r-dim}(Q,X) = 0.$$

Otherwise go to the Step 2.

Step 2. *Find the sums*

$$c_{\lambda_{j_{11}}} + c_{\lambda_{j_{21}}} + \ldots + c_{\lambda_{j_{(l-1)1}}}$$

for each $\{j_{11}, j_{21},\ldots,j_{(l-1)1}\} \subseteq \{1,\ldots,l\}$.

If there exists $\{j^0_{11}, j^0_{21},\ldots,j^0_{(l-1)1}\} \subseteq \{1,\ldots,l\}$ such that

$$c_{\lambda_{j^0_{11}}} + c_{\lambda_{j^0_{21}}} + \ldots + c_{\lambda_{j^0_{(l-1)1}}} \geq \mathbf{1}_Q,$$

then go to the Step 3.

Otherwise print

$$\text{r-dim}(Q,X)=\max(c_{\lambda_1}+c_{\lambda_2}+\ldots+c_{\lambda_l})-1.$$

Step 3. *Find the sums*

$$c_{\lambda_{j_{12}}}+c_{\lambda_{j_{22}}}+\ldots+c_{\lambda_{j_{(l-2)2}}}$$

for each $\{j_{12},j_{22},\ldots,j_{(l-2)2}\}\subseteq\{j^0_{11},j^0_{21},\ldots,j^0_{(l-1)1}\}$.

If there exists $\{j^0_{12},j^0_{22},\ldots,j^0_{(l-2)2}\}\subseteq\{j^0_{11},j^0_{21},\ldots,j^0_{(l-1)1}\}$ *such that*

$$c_{\lambda_{j^0_{12}}}+c_{\lambda_{j^0_{22}}}+\ldots+c_{\lambda_{j^0_{(l-2)2}}}\geq \mathbf{1}_Q,$$

then go to the Step 4.

Otherwise print

$$\text{r-dim}(Q,X)=\max(c_{\lambda_{j^0_{11}}}+c_{\lambda_{j^0_{21}}}+\ldots+c_{\lambda_{j^0_{(l-1)1}}})-1.$$

. . . .
. . . .
. . . .

Step $l-2$. *Find the sums*

$$c_{\lambda_{j_{1(l-3)}}}+c_{\lambda_{j_{2(l-3)}}}+c_{\lambda_{j_{3(l-3)}}}$$

for each $\{j_{1(l-3)},j_{2(l-3)},j_{3(l-3)}\}\subseteq\{j^0_{1(l-4)},j^0_{2(l-4)},j^0_{3(l-4)},j^0_{4(l-4)}\}$.

If there exists $\{j^0_{1(l-3)},j^0_{2(l-3)},j^0_{3(l-3)}\}\subseteq\{j^0_{1(l-4)},j^0_{2(l-4)},j^0_{3(l-4)},j^0_{4(l-4)}\}$ *such that*

$$c_{\lambda_{j^0_{1(l-3)}}}+c_{\lambda_{j^0_{2(l-3)}}}+c_{\lambda_{j^0_{3(l-3)}}}\geq \mathbf{1}_Q,$$

then go to the Step $l-1$.

Otherwise print

$$\text{r-dim}(Q,X)=\max(c_{\lambda_{j^0_{1(l-4)}}}+c_{\lambda_{j^0_{2(l-4)}}}+c_{\lambda_{j^0_{3(l-4)}}}+c_{\lambda_{j^0_{4(l-4)}}})-1.$$

Step $l-1$. *Find the sums*

$$c_{\lambda_{j_{1(l-2)}}}+c_{\lambda_{j_{2(l-2)}}}$$

for each $\{j_{1(l-2)},j_{2(l-2)}\}\subseteq\{j^0_{1(l-3)},j^0_{2(l-3)},j^0_{3(l-3)}\}$.

If there exists $\{j^0_{1(l-2)},j^0_{2(l-2)}\}\subseteq\{j^0_{1(l-3)},j^0_{2(l-3)},j^0_{3(l-3)}\}$ *such that*

$$c_{\lambda_{j^0_{1(l-2)}}}+c_{\lambda_{j^0_{2(l-2)}}}\geq \mathbf{1},$$

then print

$$\text{r-dim}(Q,X)=\max(c_{\lambda_{j^0_{1(l-2)}}}+c_{\lambda_{j^0_{2(l-2)}}})-1.$$

Otherwise print

$$\text{r-dim}(Q,X)=\max(c_{\lambda_{j^0_{1(l-3)}}}+c_{\lambda_{j^0_{2(l-3)}}}+c_{\lambda_{j^0_{3(l-3)}}})-1.$$

Example 3.2. Let $X = \{x_1, x_2, x_3, x_4\}$ with the topology

$$\tau = \{\varnothing, \{x_2\}, \{x_1, x_2\}, \{x_2, x_3\}, \{x_1, x_2, x_3\}, X\}$$

and $Q = \{x_1, x_3\}$. Then,

$$\mathbf{1}_Q = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

We observe that $\mathbf{U}_1 = \{x_1, x_2\}$, $\mathbf{U}_2 = \{x_2\}$, $\mathbf{U}_3 = \{x_2, x_3\}$, $\mathbf{U}_4 = X$. Therefore,

$$T = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ c_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \ c_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

Moreover,

$$c_1 + c_3 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} \geq \mathbf{1}_Q$$

and

$$\max(c_1 + c_3) = 2.$$

Thus, $\text{r-dim}(Q, X) = \max(c_1 + c_3) - 1 = 1$.

4. Remarks on the algorithm for computing the covering dimension of finite topological spaces

Remark 4.1. Let $A = (\alpha_{ij})$ be a $n \times n$ matrix and $B = (\beta_{ij})$ a $m \times m$ matrix. The *Kronecker product* of A and B (see [3]) is the $mn \times mn$ block matrix

$$A \otimes B = \begin{pmatrix} \alpha_{11}B & \dots & \alpha_{1n}B \\ \vdots & \ddots & \vdots \\ \alpha_{n1}B & \dots & \alpha_{nn}B \end{pmatrix}.$$

More explicitly, the Kronecker product of A and B is the matrix

$$\begin{pmatrix} \alpha_{11}\beta_{11} & \dots & \alpha_{11}\beta_{1m} & \dots & \alpha_{1n}\beta_{11} & \dots & \alpha_{1n}\beta_{1m} \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ \alpha_{11}\beta_{m1} & \dots & \alpha_{11}\beta_{mm} & \dots & \alpha_{1n}\beta_{m1} & \dots & \alpha_{1n}\beta_{mm} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \alpha_{n1}\beta_{11} & \dots & \alpha_{n1}\beta_{1m} & \dots & \alpha_{nn}\beta_{11} & \dots & \alpha_{nn}\beta_{1m} \\ \vdots & \ddots & \vdots & & \vdots & \ddots & \vdots \\ \alpha_{n1}\beta_{m1} & \dots & \alpha_{n1}\beta_{mm} & \dots & \alpha_{nn}\beta_{m1} & \dots & \alpha_{nn}\beta_{mm} \end{pmatrix}.$$

Let $X = \{x_1, \dots, x_n\}$ be a finite space of n elements and $Y = \{y_1, \dots, y_m\}$ a finite space of m elements. It is known that if T_X is the incidence matrix of X and T_Y is the incidence matrix of Y, then the incidence matrix of

$$X \times Y = \{(x_1, y_1), \dots, (x_1, y_m), \dots, (x_n, y_1), \dots, (x_n, y_m)\}$$

is the Kronecker product $T_X \otimes T_Y$ of T_X and T_Y (see [8]).

Example 4.2. Let $X = \{x_1, x_2, x_3\}$ with the topology

$$\tau_X = \{\varnothing, \{x_2\}, \{x_1, x_2\}, \{x_2, x_3\}, X\}$$

and $Y = \{y_1, y_2, y_3, y_4\}$ with the topology

$$\tau_Y = \{\varnothing, \{y_3\}, \{y_1, y_3\}, \{y_2, y_3\}, \{y_1, y_2, y_3\}, Y\}.$$

Also, let $Q^X = \{x_1, x_3\}$ and $Q^Y = \{y_1, y_2, y_3\}$. Then,

$$Q^X \times Q^Y = \{(x_1, y_1), (x_1, y_2), (x_1, y_3), (x_3, y_1), (x_3, y_2), (x_3, y_3)\}$$

and

$$\mathbf{1}_{Q^X} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{1}_{Q^Y} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{1}_{Q^X \times Q^Y} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

The incidence matrix T_X of X is

$$T_X = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and the incidence matrix T_Y of Y is

$$T_Y = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore, the incidence matrix $T_{X\times Y}$ of the product space $X \times Y$ is

$$T_{X\times Y} = T_X \otimes T_Y = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We observe that

$$c_1 + c_2 + c_9 + c_{10} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 2 \\ 2 \\ 4 \\ 0 \\ 1 \\ 1 \\ 2 \\ 0 \end{pmatrix} > \mathbf{1}_{Q^X \times Q^Y},$$

$c_{r_1} + c_{r_2} + c_{r_3} \ngeq \mathbf{1}_{Q^X \times Q^Y}$ for every $\{r_1, r_2, r_3\} \subseteq \{1, 2, 9, 10\}$, and

$$\max(c_1 + c_2 + c_9 + c_{10}) = 4.$$

Thus,

$$\text{r-dim}(Q^X \times Q^Y, X \times Y) = \max(c_1 + c_2 + c_9 + c_{10}) - 1 = 3.$$

Also, we observe that $\text{r-dim}(Q^X, X) = 1$ and $\text{r-dim}(Q^Y, Y) = 1$.

Remark 4.3. Let $X = \{x_1, \ldots, x_n\}$ be a finite T_0-space and $Q \subseteq X$. Then, there exists a finite space Y homeomorphic to X such that the incidence matrix T_Y of Y is an upper triangular matrix. Let h a homeomorphism from X to Y such that the incidence matrix T_Y of Y is an upper triangular matrix. In order to calculate the $\text{r-dim}(Q, X)$ it suffices to calculate $\text{r-dim}(h(Q), Y)$.

Example 4.4. Let $X = \{x_1, x_2, x_3\}$ with the topology

$$\tau_X = \{\varnothing, \{x_2\}, \{x_1, x_2\}, \{x_2, x_3\}, X\}$$

and $Q = \{x_2, x_3\}$. We consider the space $Y = \{y_1, y_2, y_3\}$ with the topology

$$\tau_Y = \{\varnothing, \{y_1\}, \{y_1, y_2\}, \{y_1, y_3\}, Y\}.$$

We observe that the map $h : X \to Y$ defined by $h(x_1) = y_2$, $h(x_2) = y_1$, and $h(x_3) = y_3$ is a homeomorphism from X to Y with $h(Q) = \{y_1, y_3\}$. The incidence matrix T_Y of Y is

$$T_Y = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Since

$$c_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \mathbf{1}_{h(Q)},$$

we have r-dim$(h(Q), Y) = 0$. Therefore, r-dim$(Q, X) = 0$.

Proposition 4.5. *An upper bound on the number of iterations of the algorithm for computation of the dimension* r-dim *of a pair* (Q, X)*, where* Q *is a subset of a finite space* X*, is the number* $\frac{1}{2}|Q|^2 + \frac{3}{2}|Q| - 3$.

Proof. Let $|Q| = l$. We observe that the number of iterations the algorithm performs in Steps

$$1,\ 2,\ 3,\ 4,\ \ldots,\ l-2,\ l-1$$

is

$$l,\ l,\ l-1,\ l-2,\ \ldots,\ 4,\ 3$$

respectively. Thus, the number of iterations the algorithm performs is

$$\begin{aligned} l + l + (l-1) + (l-2) + \ldots + 4 + 3 &= l + \frac{(l-2)(l+3)}{2} = \frac{1}{2}l^2 + \frac{3}{2}l - 3 \\ &= \frac{1}{2}|Q|^2 + \frac{3}{2}|Q| - 3. \end{aligned}$$

□

5. Problems

In [9] (see also [6] and [7]) two relative covering dimensions are defined and studied which are denoted by dim and dim*. The given two definitions below are actually the definitions of dimensions dim and dim* given in [9] for regular spaces.

Definition 5.1. We denote by dim the (unique) function with domain the class of all subsets and range the set $\omega \cup \{-1, \infty\}$, satisfying the following condition dim$(Q, X) \leq n$, where $n \in \{-1\} \cup \omega$ if and only if for every finite open cover c of the space X there exists a finite open cover r_Q of Q such that r_Q is a refinement of c and ord$(r_Q) \leq n$.

Definition 5.2. We denote by $\dim^*$ the (unique) function with domain the class of all subsets and range the set $\omega \cup \{-1, \infty\}$, satisfying the following condition $\dim^*(Q, X) \leq n$, where $n \in \{-1\} \cup \omega$ if and only if for every finite open cover c of the space X there exists a finite family r of open subsets of X refinement of c such that $Q \subseteq \cup\{V : V \in r\}$ and $\mathrm{ord}(r) \leq n$.

Problem 5.3. *Find an algorithm for computing the dimension* dim *of a pair* (Q, X)*, where* Q *is a subset of a finite space* X*, using matrix algebra.*

Problem 5.4. *Find an algorithm for computing the dimension* $\dim^*$ *of a pair* (Q, X)*, where* Q *is a subset of a finite space* X*, using matrix algebra.*

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