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Tripled coincidence and fixed point results in partial metric spaces

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Abstract

In this paper, we introduce the concept of W-compatiblity of mappings $F : X \times X \times X \to X$ and $g : X \to X$ and based on this notion, we obtain tripled coincidence and common tripled fixed point results in the setting of partial metric spaces. The presented results generalize and extend several well known comparable results in the existing literature. We also provide an example to support our results.

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1. Introduction

Matthews [19, 20] introduced the notion of a partial metric space which is a generalization of usual metric spaces in which the distance of a point to itself is no longer necessarily zero.

A partial metric space (see [19, 20]) is a pair (X, p) such that X is a (non-empty) set and $p : X \times X \to \mathbb{R}^+$ (where $\mathbb{R}^+$ denotes the set of all non negative real numbers) satisfies

(p1) $p(x, y) = p(y, x)$ (symmetry)
(p2) If $p(x, x) = p(x, y) = p(y, y)$ then $x = y$ (equality)
(p3) $p(x, x) \le p(x, y)$ (small self-distances)
(p4) $p(x, z) + p(y, y) \le p(x, y) + p(y, z)$ (triangularity)

for all $x, y, z \in X$. In this case we say that p is a partial metric on X.

Each partial metric p on X generates a T_0 topology τ_p on X with a base of the family open of p-balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in$

$X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$. Similarly, a closed p-ball is defined as $B_p[x, \varepsilon] = \{y \in X : p(x, y) \leq p(x, x) + \varepsilon\}$.

Definition 1.1 ([19, 20]). (*i*) A sequence $\{x_n\}$ in a partial metric space (X, p) is called Cauchy if $\lim_{n,m\to\infty} p(x_n, x_m)$ exists (and finite),

(*ii*) A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to τ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to\infty} p(x_n, x_m)$.

Notice that for a partial metric p on X, the function $p^s : X \times X \to \mathbb{R}^+$ given by

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y) \tag{1.1}$$

is a (usual) metric on X.

It is well known and easy to see that

$$\lim_{n\to\infty} p^s(x, x_n) = 0 \Leftrightarrow p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n,m\to\infty} p(x_n, x_m). \tag{1.2}$$

Lemma 1.2 ([19, 20]). (*A*) *A sequence $\{x_n\}$ is Cauchy in a partial metric space (X, p) if and only if $\{x_n\}$ is Cauchy in the metric space (X, p^s).*

(*B*) *A partial metric space (X, p) is complete if and only if the metric space (X, p^s) is complete.*

For simplicity, we denote from now on $\underbrace{X \times X \cdots X \times X}_{k \text{ terms}}$ by X^k where $k \in \mathbb{N}$ and X a non-empty set. We start by recalling some definitions where X is a non-empty set.

Definition 1.3 (Bhashkar and Lakshmikantham [13]). An element $(x, y) \in X^2$ is called a coupled fixed point of a mapping $F : X^2 \to X$ if $x = F(x, y)$ and $y = F(y, x)$.

Definition 1.4 (Lakshmikantham and Ćirić [18]). An element $(x, y) \in X^2$ is called

(i) a coupled coincidence point of mappings $F : X^2 \to X$ and $g : X \to X$ if $gx = F(x, y)$ and $gy = F(y, x)$, and (gx, gy) is called coupled point of coincidence;

(ii) a common coupled fixed point of mappings $F : X^2 \to X$ and $g : X \to X$ if $x = gx = F(x, y)$ and $y = gy = F(y, x)$.

Note that if g is the identity mapping, then Definition 1.4 reduces to Definition 1.3.

In 2011, Samet and Vetro [21] introduced a fixed point of order $N \geq 3$. In particular, for $N = 3$ we have following definition.

Definition 1.5 (Samet and Vetro [21]). An element $(x, y, z) \in X^3$ is called a tripled fixed point of a given mapping $F : X^3 \to X$ if $x = F(x, y, z)$, $y = F(y, z, x)$ and $z = F(z, x, y)$.

Note that, Berinde and Borcut [12] defined differently the notion of a tripled fixed point in the case of ordered sets in order to keep true the mixed monotone property. For more details, see [12].

Now, we give the following definitions.

Definition 1.6. An element $(x, y, z) \in X^3$ is called

(i) a tripled coincidence point of mappings $F : X^3 \to X$ and $g : X \to X$ if $gx = F(x, y, z)$, $gy = F(y, x, z)$ and $gz = F(z, x, y)$. In this case (gx, gy, gz) is called tripled point of coincidence;

(ii) a common tripled fixed point of mappings $F : X^3 \to X$ and $g : X \to X$ if $x = gx = F(x, y, z)$, $y = gy = F(y, z, x)$ and $z = gz = F(z, x, y)$.

Fixed point theorems on partial metric spaces have received a lot of attention in the last years (see, for instance, [2, 3, 4, 5, 6, 7, 8, 9, 10, 14, 15, 16, 17, 22, 23, 24] and their references). Abbas et al. [1] introduced the concept of w-compatible mappings and obtained coupled coincidence point and coupled point of coincidence for mappings satisfying a contractive condition in cone metric spaces. Very recently, Aydi et al. [11] introduced the concepts of $\widetilde{w}$-compatible mappings and generalized the results in [1].

The aim of this paper is to introduce the concepts of W-compatible mappings. Based on this notion, tripled coincidence point and common tripled fixed point for mappings $F : X \times X \times X \to X$ and $g : X \to X$ are obtained in partial metric space. The presented theorems generalize and extend several well known comparable results in the literature. An example is also given in support of our results.

Definition 1.7 (Abbas, Khan and Radenović [1])**.** The mappings $F : X^2 \to X$ and $g : X \to X$ are called w-compatible if $g(F(x, y)) = F(gx, gy)$ whenever $gx = F(x, y)$ and $gy = F(y, x)$.

Definition 1.8. Mappings $F : X^3 \to X$ and $g : X \to X$ are called W-compatible if

$$F(gx, gy, gz) = g(F(x, y, z))$$

whenever $F(x, y, z) = gx$, $F(y, z, x) = gy$ and $F(z, y, x) = gz$.

2. Main results

We present our first result as follows.

Theorem 2.1. *Let (X, p) be a partial metric space and $F : X^3 \longrightarrow X$ and $g : X \to X$ be mappings such that $F(X^3) \subseteq g(X)$ and $g(X)$ is a complete subspace of X. Suppose that for any $x, y, z, u, v, w \in X$, the following condition*

$$\begin{aligned}
p(F(x,y,z),F(u,v,w)) &\le a_1p(F(x,y,z),gx) + a_2p(F(y,z,x),gy) + a_3p(F(z,x,y),gz) \\
&+ a_4p(F(u,v,w),gu) + a_5p(F(v,w,u),gv) + a_6p(F(w,u,v),gw) + a_7p(F(u,v,w),gx) \\
&+ a_8p(F(v,w,u),gy) + a_9p(F(w,u,v),gz) + a_{10}p(F(x,y,z),gu) + a_{11}p(F(y,z,x),gv) \\
&+ a_{12}p(F(z,x,y),gw) + a_{13}p(gx,gu) + a_{14}p(gy,gv) + a_{15}p(gz,gw),
\end{aligned}$$

holds, where a_i, $i = 1, \cdots, 15$ *are nonnegative real numbers. If* $\sum_{i=7}^{9} a_i + \sum_{i=1}^{15} a_i < 1$ *or* $\sum_{i=10}^{12} a_i + \sum_{i=1}^{15} a_i < 1$, *then* F *and* g *have a tripled coincidence point.*

Proof. Let x_0, y_0 and z_0 be three arbitrary points in X. By given assumptions, there exists (x_1, y_1, z_1) such that

$$F(x_0, y_0, z_0) = gx_1, \quad F(y_0, z_0, x_0) = gy_1 \quad \text{and} \quad F(z_0, x_0, y_0) = gz_1.$$

Continuing this process, we construct three sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ in X such that

$$F(x_n, y_n, z_n) = gx_{n+1}, \quad F(y_n, z_n, x_n) = gy_{n+1} \quad \text{and} \quad F(z_n, x_n, y_n) = gz_{n+1} \quad \forall\, n \in \mathbb{N}. \tag{2.1}$$

Denote

$$\delta_n = p(gx_{n+1}, gx_n) + p(gy_{n+1}, gy_n) + p(gz_{n+1}, gz_n).$$

We claim that

$$\delta_{n+1} \leq \kappa \delta_n \quad \forall\, n \in \mathbb{N}, \tag{2.2}$$

where $\kappa \in [0, 1)$ will be chosen conveniently.

First, taking $(x, y, z) = (x_n, y_n, z_n)$ and $(u, v, w) = (x_{n+1}, y_{n+1}, z_{n+1})$ in the considered contractive condition and using (2.1), we have

$$\begin{aligned}
&p(gx_{n+1}, gx_{n+2}) = p(F(x_n, y_n, z_n), F(x_{n+1}, y_{n+1}, z_{n+1}))\\
&\leq a_1 p(F(x_n, y_n, z_n), gx_n) + a_2 p(F(y_n, z_n, x_n), gy_n) + a_3 p(F(z_n, x_n, y_n), gz_n)\\
&+ a_4 p(F(x_{n+1}, y_{n+1}, z_{n+1}), gx_{n+1}) + a_5 p(F(y_{n+1}, z_{n+1}, x_{n+1}), gy_{n+1})\\
&+ a_6 p(F(z_{n+1}, x_{n+1}, y_{n+1}), gz_{n+1}) + a_7 p(F(x_{n+1}, y_{n+1}, z_{n+1}), gx_n)\\
&+ a_8 p(F(y_{n+1}, z_{n+1}, x_{n+1}), gy_n) + a_9 p(F(z_{n+1}, x_{n+1}, y_{n+1}), gz_n) + a_{10} p(F(x_n, y_n, z_n), gx_{n+1})\\
&+ a_{11} p(F(y_n, z_n, x_n), gy_{n+1}) + a_{12} p(F(z_n, x_n, y_n), gz_{n+1})\\
&+ a_{13} p(gx_n, gx_{n+1}) + a_{14} p(gy_n, gy_{n+1}) + a_{15} p(gz_n, gz_{n+1})\\
&= a_1 p(gx_{n+1}, gx_n) + a_2 p(gy_{n+1}, gy_n) + a_3 p(gz_{n+1}, gz_n) + a_4 p(gx_{n+2}, gx_{n+1})\\
&+ a_5 p(gy_{n+2}, gy_{n+1}) + a_6 p(gz_{n+2}, gz_{n+1}) + a_7 p(gx_{n+2}, gx_n) + a_8 p(gy_{n+2}, gy_n)\\
&+ a_9 p(gz_{n+2}, gz_n) + a_{10} p(gx_{n+1}, gx_{n+1}) + a_{11} p(gy_{n+1}, gy_{n+1}) + a_{12} p(gz_{n+1}, gz_{n+1})\\
&+ a_{13} p(gx_n, gx_{n+1}) + a_{14} p(gy_n, gy_{n+1}) + a_{15} p(gz_n, gz_{n+1}).
\end{aligned}$$

Then, using $(p2)$ and the triangular inequality (which holds even for partial metrics), one can write for any $n \in \mathbb{N}$

$$\begin{aligned}
&(1 - a_4 - a_7 - a_{10}) p(gx_{n+2}, gx_{n+1}) \leq (a_1 + a_7 + a_{13}) p(gx_{n+1}, gx_n) + (a_2 + a_8 + a_{14}) p(gy_n, gy_{n+1})\\
&+ (a_3 + a_9 + a_{15}) p(gz_n, gz_{n+1}) + (a_5 + a_8 + a_{11}) p(gy_{n+2}, gy_{n+1}) + (a_6 + a_9 + a_{12}) p(gz_{n+2}, gz_{n+1}).
\end{aligned} \tag{2.3}$$

Similarly, following similar arguments to those given above, we obtain

$$(1-a_4-a_7-a_{10})p(gy_{n+2},gy_{n+1}) \le (a_1+a_7+a_{13})p(gy_{n+1},gy_n)+(a_2+a_8+a_{14})p(gz_n,gz_{n+1}) \tag{2.4}$$
$$+(a_3+a_9+a_{15})p(gx_n,gx_{n+1})+(a_5+a_8+a_{11})p(gz_{n+2},gz_{n+1})+(a_6+a_9+a_{12})p(gx_{n+2},gx_{n+1}),$$

and

$$(1-a_4-a_7-a_{10})p(gz_{n+2},gz_{n+1}) \preceq (a_1+a_7+a_{13})p(gz_{n+1},gz_n)+(a_2+a_8+a_{14})p(gx_n,gx_{n+1}) \tag{2.5}$$
$$+(a_3+a_9+a_{15})p(gy_n,gy_{n+1})+(a_5+a_8+a_{11})p(gx_{n+2},gx_{n+1})+(a_6+a_9+a_{12})p(gy_{n+2},gy_{n+1}).$$

Adding (2.3) to (2.5) we have

$$(1-a_4-a_5-a_6-a_7-a_8-a_9-a_{10}-a_{11}-a_{12})\delta_{n+1} \le (a_1+a_2+a_3+a_7+a_8+a_9+a_{13}+a_{14}+a_{15})\delta_n, \tag{2.6}$$

that is

$$\delta_{n+1} \le \kappa_1\, \delta_n \quad \forall\, n \in \mathbb{N}, \tag{2.7}$$

where

$$\kappa_1 = \frac{a_1+a_2+a_3+a_7+a_8+a_9+a_{13}+a_{14}+a_{15}}{1-\sum_{i=4}^{12} a_i}.$$

As $\sum_{i=7}^{9} a_i + \sum_{i=1}^{15} a_i < 1$, so $0 \le \kappa_1 < 1$. Hence (2.2) holds for $\kappa = \kappa_1$.

On the other hand, we have

$$\begin{aligned}
&p(gx_{n+2},gx_{n+1}) = p(F(x_{n+1},y_{n+1},z_{n+1}),F(x_n,y_n,z_n))\\
&\le a_1p(F(x_{n+1},y_{n+1},z_{n+1}),gx_{n+1})+a_2p(F(y_{n+1},z_{n+1},x_{n+1}),gy_{n+1})\\
&+a_3p(F(z_{n+1},x_{n+1},y_{n+1}),gz_{n+1})+a_4p(F(x_n,y_n,z_n),gx_n)\\
&+a_5p(F(y_n,z_n,x_n),gy_n)+a_6p(F(z_n,x_n,y_n),gz_n)+a_7p(F(x_n,y_n,z_n),gx_{n+1})\\
&+a_8p(F(y_n,z_n,x_n),gy_{n+1})+a_9p(F(z_n,x_n,y_n),gz_{n+1})+a_{10}p(F(x_{n+1},y_{n+1},z_{n+1}),gx_n)\\
&+a_{11}p(F(y_{n+1},z_{n+1},x_{n+1}),gy_n)+a_{12}p(F(z_{n+1},x_{n+1},y_{n+1}),gz_n)+a_{13}p(gx_{n+1},gx_n)\\
&+a_{14}p(gy_{n+1},gy_n)+a_{15}p(gz_{n+1},gz_n)\\
&=a_1p(gx_{n+2},gx_{n+1})+a_2p(gy_{n+2},gy_{n+1})+a_3p(gz_{n+2},gz_{n+1})+a_4p(gx_{n+1},gx_n)\\
&+a_5p(gy_{n+1},gy_n)+a_6p(gz_{n+1},gz_n)+a_7p(gx_{n+1},gx_{n+1})+a_8p(gy_{n+1},gy_{n+1})\\
&+a_9p(gz_{n+1},gz_{n+1})+a_{10}p(gx_{n+2},gx_n)+a_{11}p(gy_{n+2},gy_n)+a_{12}p(gz_{n+2},gz_n)\\
&+a_{13}p(gx_n,gx_{n+1})+a_{14}p(gy_n,gy_{n+1})+a_{15}p(gz_n,gz_{n+1}).
\end{aligned}$$

Thus, using again $(p2)$ and the triangular inequality

$$(1-a_1-a_7-a_{10})p(gx_{n+2},gx_{n+1}) \le (a_4+a_{10}+a_{13})p(gx_{n+1},gx_n)+(a_5+a_{11}+a_{14})p(gy_n,gy_{n+1}) \tag{2.8}$$
$$+(a_6+a_{12}+a_{15})p(gz_n,gz_{n+1})+(a_2+a_8+a_{11})p(gy_{n+2},gy_{n+1})+(a_3+a_9+a_{12})p(gz_{n+2},gz_{n+1}).$$

Similarly,

$$(1-a_1-a_7-a_{10})p(gy_{n+2},gy_{n+1}) \le (a_4+a_{10}+a_{13})p(gy_{n+1},gy_n)+(a_5+a_{11}+a_{14})p(gz_n,gz_{n+1})$$
(2.9)
$$+(a_6+a_{12}+a_{15})p(gx_n,gx_{n+1})+(a_2+a_8+a_{11})p(gz_{n+2},gz_{n+1})+(a_3+a_9+a_{12})p(gx_{n+2},gx_{n+1})$$

and

$$(1-a_1-a_7-a_{10})p(gz_{n+2},gz_{n+1}) \le (a_4+a_{10}+a_{13})p(gz_{n+1},gz_n)+(a_5+a_{11}+a_{14})p(gx_n,gx_{n+1})$$
(2.10)
$$+(a_6+a_{12}+a_{15})p(gy_n,gy_{n+1})+(a_2+a_8+a_{11})p(gx_{n+2},gx_{n+1})+(a_3+a_9+a_{12})p(gy_{n+2},gy_{n+1}).$$

Adding (2.8) to (2.10), we obtain that
(2.11)
$$(1-a_1-a_2-a_3-a_7-a_8-a_9-a_{10}-a_{11}-a_{12})\delta_{n+1} \le (a_4+a_5+a_6+a_{10}+a_{11}+a_{12}+a_{13}+a_{14}+a_{15})\delta_n.$$

From (2.11), one can write

$$\delta_{n+1} \le \kappa_2\delta_n \quad \forall\ n\in\mathbb{N}$$

where

$$\kappa_2 = \frac{a_4+a_5+a_6+a_{10}+a_{11}+a_{12}+a_{13}+a_{14}+a_{15}}{1-a_1-a_2-a_3-\sum_{i=7}^{12}a_i}.$$

Since $\sum_{i=10}^{12}a_i+\sum_{i=1}^{15}a_i<1$, so $0\le\kappa_2<1$. Thus, (2.2) holds for $\kappa=\kappa_2$.

By (2.2), we have

$$\delta_n \le \kappa\delta_{n-1} \le \cdots \le \kappa^n\delta_0. \tag{2.12}$$

If $\delta_0=0$, we get $p(gx_0,gx_1)+p(gy_0,gy_1)=p(gz_0,gz_1)=0$, that is, $gx_0=gx_1$, $gy_0=gy_1$ and $gz_0=gz_1$. Therefore, from (2.1) we have

$$F(x_0,y_0,z_0)=gx_1=gx_0, \quad F(y_0,z_0,x_0)=gy_1=gy_0 \quad \text{and} \quad F(z_0,x_0,y_0)=gz_1=gz_0,$$

that is, (x_0,y_0,z_0) is a tripled coincidence point of F and g. Now, assume that $\delta_0\ne0$. If $m>n$, we have

$$p(gx_m,gx_n)\le p(gx_m,gx_{m-1})+p(gx_{m-1},gx_{m-2})+\cdots+p(gx_{n+1},gx_n),$$
$$p(gy_m,gy_n)\le p(gy_m,gy_{m-1})+p(gy_{m-1},gy_{m-2})+\cdots+p(gy_{n+1},gy_n),$$

and

$$p(gz_m,gz_n)\le p(gz_m,gz_{m-1})+p(gz_{m-1},gz_{m-2})+\cdots+p(gz_{n+1},gz_n).$$

Adding above inequalities and using (2.12), we obtain (for $m>n$)

$$\begin{aligned} p(gx_m,gx_n)+p(gy_m,gy_n)+p(gz_m,gz_n) &\le \delta_{m-1}+\delta_{m-2}+\cdots+\delta_n \\ &\le (\kappa^{m-1}+\kappa^{m-1}+\cdots+\kappa^n)\delta_0 \\ &\le \frac{\kappa^n}{1-\kappa}\delta_0 \to 0 \quad \text{since } \kappa\in[0,1). \end{aligned}$$

This implies that

$$\lim_{n,m\to\infty}p(gx_m,gx_n)=\lim_{n,m\to\infty}p(gy_m,gy_n)=\lim_{n,m\to\infty}p(gz_m,gz_n)=0. \tag{2.13}$$

We deduce that $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences in $(g(X), p)$ which is complete, then by Lemma 1.2, $\{gx_n\}$, $\{gy_n\}$ and $\{gz_n\}$ are Cauchy sequences in the metric subspace $(g(X), p^s)$. Since is also $(g(X), p^s)$ complete, so that there exist $x, y, z \in X$ such that

$$\lim_{n\to\infty} p^s(gx_n, gx) = \lim_{n\to\infty} p^s(gy_n, gy) = \lim_{n\to\infty} p^s(gz_n, gz) = 0. \tag{2.14}$$

Again, by Lemma 1.2 and (2.13), we get that

$$p(gx, gx) = \lim_{n\to\infty} p(gx_n, gx) = \lim_{n,m\to\infty} p(gx_m, gx_n) = 0, \tag{2.15}$$

$$p(gy, gy) = \lim_{n\to\infty} p(gy_n, gy) = \lim_{n,m\to\infty} p(gy_m, gy_n) = 0, \tag{2.16}$$

and

$$p(gz, gz) = \lim_{n\to\infty} p(gz_n, gz) = \lim_{n,m\to\infty} p(gz_m, gz_n) = 0. \tag{2.17}$$

Now, we prove that $F(x, y, z) = gx$, $F(y, z, x) = gy$ and $F(z, x, y) = gz$. Note that

$$\begin{aligned} p(F(x, y, z), gx) &\leq p(F(x, y, z), F(x_n, y_n, z_n)) + p(F(x_n, y_n, z_n), gx) \\ &= p(F(x, y, z), F(x_n, y_n, z_n)) + p(gx_{n+1}, gx). \end{aligned} \tag{2.18}$$

On the other hand, applying the given contractive condition, we obtain

$$\begin{aligned} &p(F(x, y, z), F(x_n, y_n, z_n)) \leq a_1 p(F(x, y, z), gx) + a_2 p(F(y, z, x), gy) + a_3 p(F(z, x, y), gz) \\ &+ a_4 p(F(x_n, y_n, z_n), gx_n) + a_5 p(F(y_n, z_n, x_n), gy_n) + a_6 p(F(z_n, x_n, y_n), gz_n) \\ &+ a_7 p(F(x_n, y_n, z_n), gx) + a_8 p(F(y_n, z_n, x_n), gy) + a_9 p(F(z_n, x_n, y_n), gz) + a_{10} p(F(x, y, z), gx_n) \\ &+ a_{11} p(F(y, z, x), gy_n) + a_{12} p(F(z, x, y), gz_n) + a_{13} p(gx, gx_n) + a_{14} p(gy, gy_n) + a_{15} p(gz, gz_n) \\ &= a_1 p(F(x, y, z), gx) + a_2 p(F(y, z, x), gy) + a_3 p(F(z, x, y), gz) + a_4 p(gx_{n+1}, gx_n) + a_5 p(gy_{n+1}, gy_n) \\ &+ a_6 p(gz_{n+1}, gz_n) + a_7 p(gx_{n+1}, gx) + a_8 p(gy_{n+1}, gy) + a_9 p(gz_{n+1}, gz) + a_{10} p(F(x, y, z), gx_n) \\ &+ a_{11} p(F(y, z, x), gy_n) + a_{12} p(F(z, x, y), gz_n) + a_{13} p(gx, gx_n) + a_{14} p(gy, gy_n) + a_{15} p(gz, gz_n). \end{aligned}$$

Combining above inequality with (2.18) and using a triangular inequality, we have

$$\begin{aligned} &p(F(x, y, z), gx) \preceq a_1 p(F(x, y, z), gx) + a_2 p(F(y, z, x), gy) + a_3 p(F(z, x, y), gz) + a_4 p(gx_{n+1}, gx_n) \\ &+ a_5 p(gy_{n+1}, gy_n) + a_6 p(gz_{n+1}, gz_n) + a_7 p(gx_{n+1}, gx) + a_8 p(gy_{n+1}, gy) + a_9 p(gz_{n+1}, gz) \\ &+ a_{10} p(F(x, y, z), gx) + a_{10} p(gx, gx_n) + a_{11} p(F(y, z, x), gy) + a_{11} p(gy, gy_n) + a_{12} p(F(z, x, y), gz) \\ &+ a_{12} p(gz, gz_n) + a_{13} p(gx, gx_n) + a_{14} p(gy, gy_n) + a_{15} p(gz, gz_n) + p(gx_{n+1}, gx). \end{aligned}$$

Therefore,

$$(1-a_1-a_{10})p(F(x,y,z),gx)-(a_2+a_{11})p(F(y,z,x),gy)-(a_3+a_{12})p(F(z,x,y),gz)$$
$$\leq a_4p(gx_{n+1},gx_n)+a_5p(gy_{n+1},gy_n)+a_6p(gz_{n+1},gz_n)+(1+a_7)p(gx_{n+1},gx)$$
$$+a_8p(gy_{n+1},gy)+a_9p(gz_{n+1},gz)+(a_{10}+a_{13})p(gx,gx_n)$$
(2.19)
$$+(a_{11}+a_{14})p(gy,gy_n)+(a_{12}+a_{15})p(gz,gz_n).$$

Similarly, we obtain

$$(1-a_1-a_{10})p(F(y,z,x),gy)-(a_2+a_{11})p(F(z,x,y),gz)-(a_3+a_{12})p(F(x,y,z),gx)$$
$$\leq a_4p(gy_{n+1},gy_n)+a_5p(gz_{n+1},gz_n)+a_6p(gx_{n+1},gx_n)+(1+a_7)p(gy_{n+1},gy)$$
$$+a_8p(gz_{n+1},gz)+a_9p(gx_{n+1},gx)+(a_{10}+a_{13})p(gy,gy_n)$$
(2.20)
$$+(a_{11}+a_{14})p(gz,gz_n)+(a_{12}+a_{15})p(gx,gx_n),$$

and

$$(1-a_1-a_{10})p(F(z,x,y),gz)-(a_2+a_{11})p(F(x,y,z),gx)-(a_3+a_{12})p(F(y,z,x),gy)$$
$$\leq a_4p(gz_{n+1},gz_n)+a_5p(gx_{n+1},gx_n)+a_6p(gy_{n+1},gy_n)+(1+a_7)p(gz_{n+1},gz)$$
$$+a_8p(gx_{n+1},gx)+a_9p(gy_{n+1},gy)+(a_{10}+a_{13})p(gz,gz_n)$$
(2.21)
$$+(a_{11}+a_{14})p(gx,gx_n)+(a_{12}+a_{15})p(gy,gy_n).$$

Letting in (2.19)-(2.21) and using (2.15)-(2.17), we get that

$$(1-a_1-a_2-a_3-a_{10}-a_{11}-a_{12})[p(F(x,y,z),gx)+p(F(y,z,x),gy)+p(F(z,x,y),gz)]=0.$$

It follows that $p(F(x,y,z),gx)=p(F(y,z,x),gy)=p(F(z,x,y),gz)=0$, that is $F(x,y,z)=gx$, $F(y,z,x)=gy$ and $F(z,x,y)=gz$. □

As consequences of Theorem 2.1, we give the following corollaries.

Corollary 2.2. *Let (X,p) be a partial metric space. Let $F:X^3\longrightarrow X$ and $g:X\to X$ be mappings such that $F(X^3)\subseteq g(X)$ and $g(X)$ is a complete subspace of X. Suppose that for any $x,y,z,u,v,w\in X$*

$$\begin{aligned}p(F(x,y,z),F(u,v,w))\leq\ &\alpha_1[p(F(x,y,z),gx)+p(F(y,z,x),gy)+p(F(z,x,y),gz)]\\&+\alpha_2[p(F(u,v,w),gu)+p(F(v,w,u),gv)+p(F(w,u,v),gw)]\\&+\alpha_3[p(F(u,v,w),gx)+p(F(v,w,u),gy)+p(F(w,u,v),gz)]\\&+\alpha_4[p(F(x,y,z),gu)+p(F(y,z,x),gv)+p(F(z,x,y),gw)]\\&+\alpha_5[p(gx,gu)+p(gy,gv)+p(gz,gw)],\end{aligned}$$

where α_i, $i=1,\cdots,5$ are nonnegative real numbers. If $\alpha_3+\sum_{i=1}^{5}\alpha_i<1/3$ or $\alpha_4+\sum_{i=1}^{5}\alpha_i<1/3$, then F and g have a tripled coincidence.

Proof. Take $a_1 = a_2 = a_3 = \alpha_1$, $a_4 = a_5 = a_6 = \alpha_2$, $a_7 = a_8 = a_9 = \alpha_3$, $a_{10} = a_{11} = a_{12} = \alpha_4$ and $a_{13} = a_{14} = a_{15} = \alpha_5$ in Theorem 2.1 with $\alpha_3 + \sum_{i=1}^{5} \alpha_i < 1/3$ or $\alpha_4 + \sum_{i=1}^{5} \alpha_i < 1/3$. The result follows. □

Corollary 2.3. *Let (X, p) be a partial metric space. Let $\tilde{F} : X^2 \longrightarrow X$ and $g : X \to X$ be mappings satisfying $\tilde{F}(X^2) \subseteq g(X)$, $(g(X), p)$ is a complete subspace of X and for any $x, y, u, v \in X$,*

$$\begin{aligned} p(\tilde{F}(x,y), \tilde{F}(u,v)) &\leq a_1 p(\tilde{F}(x,y), gx) + a_2 p(\tilde{F}(u,v), gu) + a_3 p(\tilde{F}(u,v), gx) \\ & a_4 p(\tilde{F}(x,y), gu) + a_5 p(gx, gu) + a_6 p(gy, gv), \end{aligned} \tag{2.22}$$

where a_i, $i = 1, \cdots, 6$ are nonnegative real numbers such that $a_3 + \sum_{i=1}^{6} a_i < 1$ or $a_4 + \sum_{i=1}^{6} a_i < 1$. Then $\tilde{F}$ and g have a coupled coincidence point $(x, y) \in X^2$, that is, $\tilde{F}(x, y) = gx$ and $\tilde{F}(y, x) = gy$.

Proof. Consider the mappings $F : X^3 \to X$ defined by $F(x, y, z) = \tilde{F}(x, y)$ for all $x, y, z \in X$. Then, the contractive condition (2.22) implies that, for all $x, y, z, u, v, w \in X$

$$\begin{aligned} p(F(x,y,z), F(u,v,w)) \leq\ & a_1 p(F(x,y,z), gx) + a_2 p(F(u,v,w), gu) + a_3 p(F(x,y,z), gu) \\ & + a_4 p(F(u,v,w), gx) + a_5 p(gx, gu) + a_6 p(gy, gv). \end{aligned}$$

Then F and g satisfy the contractive condition of Theorem 2.1. Clearly other conditions of Theorem 2.1 are also satisfied as $\tilde{F}(X^2) \subseteq g(X)$ and $g(X)$ is a complete subspace of X. Therefore, from Theorem 2.1, F and g have a tripled fixed point $(x, y, z) \in X^3$ such that $F(x, y, z) = gx$, $F(y, z, x) = gy$ and $F(z, x, y) = gz$, that is, $\tilde{F}(x, y) = gx$ and $\tilde{F}(y, x) = gy$. This makes end to the proof. □

Now, we are ready to state and prove a result of common tripled fixed point.

Theorem 2.4. *Let $F : X^3 \to X$ and $g : X \to X$ be two mappings which satisfy all the conditions of Theorem 2.1. If F and g are W-compatible, then F and g have a unique common tripled fixed point. Moreover, common tripled fixed point of F and g is of the form (u, u, u) for some $u \in X$.*

Proof. First, we'll show that the tripled point of coincidence is unique. Suppose that (x, y, z) and $(x^*, y^*, z^*) \in X^3$ with

$$\begin{cases} gx = F(x,y,z) \\ gy = F(y,z,x) \\ gz = F(z,x,y), \end{cases} \quad \text{and} \quad \begin{cases} gx^* = F(x^*,y^*,z^*) \\ gy^* = F(y^*,z^*,x^*) \\ gz^* = F(z^*,x^*,y^*). \end{cases}$$

Using contractive condition in Theorem 2.1 and $(p2)$, we obtain

$$p(gx,gx^*)=p(F(x,y,z),F(x^*,y^*,z^*))\leq a_1p(F(x,y,z),gx)+a_2p(F(y,z,x),gy)+a_3p(F(z,x,y),gz)$$
$$+a_4p(F(x^*,y^*,z^*),gx^*)+a_5p(F(y^*,z^*,x^*),gy^*)+a_6p(F(z^*,x^*,y^*),gz^*)$$
$$+a_7p(F(x^*,y^*,z^*),gx)+a_8p(F(y^*,z^*,x^*),gy)+a_9p(F(z^*,x^*,y^*),gz)+a_{10}p(F(x,y,z),gx^*)$$
$$+a_{11}p(F(y,z,x),gy^*)+a_{12}p(F(z,x,y),gz^*)+a_{13}p(gx,gx^*)+a_{14}p(gy,gy^*)+a_{15}p(gz,gz^*)$$
$$\leq(a_1+a_4+a_7+a_{10}+a_{13})p(gx^*,gx)+(a_2+a_5+a_8+a_{11}+a_{14})p(gy^*,gy)$$
$$+(a_3+a_6+a_9+a_{12}+a_{15})p(gz^*,gz).$$

Similarly, we have

$$p(gy,gy^*)=p(F(y,z,x),F(y^*,z^*,x^*))\leq(a_1+a_4+a_7+a_{10}+a_{13})p(gy^*,gy)$$
$$+(a_2+a_5+a_8+a_{11}+a_{14})p(gz^*,gz)+(a_3+a_6+a_9+a_{12}+a_{15})p(gx^*,gx),$$

and

$$p(gz,gz^*)=p(F(z,x,y),F(z^*,x^*,y^*))\leq(a_1+a_4+a_7+a_{10}+a_{13})p(gz^*,gz)+(a_2+a_5+a_8+a_{11}$$
$$+a_{14})p(gx^*,gx)+(a_3+a_6+a_9+a_{12}+a_{15})p(gy^*,gy).$$

Adding above three inequalities, we get

$$p(gx,gx^*)+p(gy,gy^*)+p(gz,gz^*)\leq(\sum_{i=1}^{15}a_i)[p(gx,gx^*)+p(gy,gy^*)+p(gz,gz^*)].$$

Since $\sum_{i=1}^{15}a_i<1$, we obtain

$$p(gx,gx^*)+p(gy,gy^*)+p(gz,gz^*)=0,$$

which implies that

$$gx=gx^*,\quad gy=gy^*\quad\text{and}\quad gz=gz^*,\tag{2.23}$$

which implies uniqueness of the tripled point of coincidence of F and g, that is, (gx,gy,gz). Note that

$$p(gx,gy^*)=p(F(x,y,z),F(y^*,z^*,x^*))\leq a_1p(F(x,y,z),gx)+a_2p(F(y,z,x),gy)+a_3p(F(z,x,y),gz)$$
$$+a_4p(F(y^*,z^*,x^*),gy^*)+a_5p(F(z^*,x^*,y^*),gz^*)+a_6p(F(x^*,y^*,z^*),gx^*)$$
$$+a_7p(F(y^*,z^*,x^*),gx)+a_8p(F(z^*,x^*,y^*),gy)+a_9p(F(x^*,y^*,z^*),gz)+a_{10}p(F(x,y,z),gy^*)$$
$$+a_{11}p(F(y,z,x),gz^*)+a_{12}p(F(z,x,y),gx^*)+a_{13}p(gx,gy^*)+a_{14}p(gy,gz^*)+a_{15}p(gz,gx^*)$$
$$\leq(a_1+a_4+a_7+a_{10}+a_{13})p(gy^*,gx)+(a_2+a_5+a_8+a_{11}+a_{14})p(gz^*,gy)$$
$$+(a_3+a_6+a_9+a_{12}+a_{15})p(gx^*,gz).$$

Similarly

$$p(gy,gz^*)\leq(a_1+a_4+a_7+a_{10}+a_{13})p(gz^*,gy)+(a_2+a_5+a_8+a_{11}+a_{14})p(gx^*,gz)$$
$$+(a_3+a_6+a_9+a_{12}+a_{15})p(gy^*,gx),$$

and

$$p(gz, gx^*) \leq (a_1 + a_4 + a_7 + a_{10} + a_{13})p(gx^*, gz) + (a_2 + a_5 + a_8 + a_{11} + a_{14})p(gy^*, gx) + (a_3 + a_6 + a_9 + a_{12} + a_{15})p(gz^*, gy).$$

Adding the above inequalities, we obtain

$$p(gx, gy^*)+p(gy, gz^*)+p(gz, gx^*) \leq (\sum_{i=1}^{15} a_i)(p(gx, gy^*)+p(gy, gz^*)+p(gz, gx^*)),$$

which again yields that

$$gx = gy^*, \quad gy = gz^* \quad \text{and} \quad gz = gx^*. \tag{2.24}$$

In view of (2.23) and (2.24), one can assert that

$$gx = gy = gz. \tag{2.25}$$

That is, the unique tripled point of coincidence of F and g is (gx, gy, gz).

Now, let $u = gx$, then we have $u = gx = F(x, y, z) = gy = F(y, z, x) = gz = F(z, x, y)$. Since F and g are W-compatible, we have

$$F(gx, gy, gz) = g(F(x, y, z)),$$

which due to (2.25) gives that

$$F(u, u, u) = gu.$$

Consequently, (u, u, u) is a tripled coincidence point of F and g, and so (gu, gu, gu) is a tripled point of coincidence of F and g, and by its uniqueness, we get $gu = gx$. Thus, we obtain

$$u = gx = gu = F(u, u, u).$$

Hence, (u, u, u) is the unique common tripled fixed point of F and g. This completes the proof. □

Corollary 2.5. *Let (X, p) be a cone partial metric space. Let $\tilde{F} : X^2 \longrightarrow X$ and $g : X \to X$ be mappings satisfying $\tilde{F}(X^2) \subseteq g(X)$, $(g(X), p)$ is a complete subspace of X and for any $x, y, u, v \in X$,*

$$p(\tilde{F}(x, y), \tilde{F}(u, v)) \leq a_1 p(\tilde{F}(x, y), gx) + a_2 p(\tilde{F}(u, v), gu) + a_3 p(\tilde{F}(u, v), gx) + a_4 p(\tilde{F}(x, y), gu) + a_5 p(gx, gu) + a_6 p(gy, gv),$$

where a_i, $i = 1, \cdots, 6$ are nonnegative real numbers such that $a_3 + \sum_{i=1}^{6} a_i < 1$ or $a_4 + \sum_{i=1}^{6} a_i < 1$. If $\tilde{F}$ and g are w-compatible, then $\tilde{F}$ and g have a unique common coupled fixed point. Moreover, the common fixed point of $\tilde{F}$ and g is of the form (u, u) for some $u \in X$.

Proof. Consider the mappings $F : X^3 \to X$ defined by $F(x,y,z) = \tilde{F}(x,y)$ for all $x,y,z \in X$. From the proof of Corollary 2.3 and the result given by Theorem 2.4, we have only to show that F and g are W-compatible. Let $(x,y,z) \in X^3$ such that $F(x,y,z) = gx$, $F(y,z,x) = gy$ and $F(z,x,y) = gz$. From the definition of F, we get $\tilde{F}(x,y) = gx$ and $\tilde{F}(y,x) = gy$. Since $\tilde{F}$ and g are w-compatible, this implies that

$$g(\tilde{F}(x,y)) = \tilde{F}(gx,gy). \tag{2.26}$$

Using (2.26), we have

$$F(gx,gy,gz) = \tilde{F}(gx,gy) = g(\tilde{F}(x,y)) = g(F(x,y,z)).$$

Thus, we proved that F and g are W-compatible mappings, and the desired result follows immediately from Theorem 2.4. □

Remark 2.6.
- Theorem 2.1 of Aydi [5] is a particular case of Corollary 2.5 by taking $a_1 = a_1 = a_3 = a_4 = 0$ and $g = I_X$, the identity on X.
- Theorem 2.4 of Aydi [5] is a particular case of Corollary 2.5 by taking $a_3 = a_4 = a_5 = a_6 = 0$ and $g = I_X$, the identity on X.
- Theorem 2.5 of Aydi [5] is a particular case of Corollary 2.5 by taking $a_1 = a_2 = a_5 = a_6 = 0$ and $g = I_X$, the identity on X.
- Corollary 2.2 extends Theorem 2.9 of Samet and Vetro [21] to partial metric spaces (corresponding to the case $N = 3$).
- Theorem 2.4 extends Theorem 2.10 of Samet and Vetro [21] to partial metric spaces (case $N = 3$).
- Theorem 2.4 extends Theorem 2.11 of Samet and Vetro [21] to partial metric spaces (case $N = 3$).

Similar to the Corollaries 2.3 and 2.5, by considering $F(x,y,z) = fx$ for all $x,y,z \in X$ where $f : X \to X$, we may state the following consequence of Theorem 2.4.

Corollary 2.7. *Let (X,p) be a partial metric space and $f,g : X \to X$ be mappings such that*

$$\begin{aligned} p(fx,fu) &\leq a_1 p(fx,gx) + a_2 p(fu,gu) + a_3 p(fu,gx) \\ &+ a_4 p(fx,gu) + a_5 p(gu,gx) \end{aligned} \tag{2.27}$$

for all $x,u \in X$, where $a_i \in [0,1), i = 1,\cdots,5$ and $a_3 + \sum_{i=1}^{5} a_i < 1$ or $a_4 + \sum_{i=1}^{5} a_i < 1$. Suppose that f and g are weakly compatible, $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of X. Then the mappings f and g have a unique common fixed point.

Now, we give an example to illustrate our obtained results.

Example 2.8. Let $X = \mathbb{R}^+$ endowed with the partial metric metric $p(x, y) = \max(x, y)$ for all $x, y \in X$. Define the mappings $F : X^3 \to X$ and $g : X \to X$ by

$$gx = \frac{x}{2} \quad \text{and} \quad F(x, y, z) = \frac{2x + 3y + 4z}{72}.$$

We will check that all the hypotheses of Theorem 2.1 are satisfied.

Note that $F(X^3) \subseteq g(X)$ with $g(X)$ is complete in X. Now, for all $x, y, z, u, v, w \in X$, we have

$$\begin{aligned} p(F(x,y,z), F(u,v,w)) &= \max(F(x,y,z), F(u,v,w)) \\ &\leq \max(\frac{2x+3y+4z}{72}, \frac{2u+3v+4w}{72}) \\ &\leq \frac{1}{4}[\max\{\frac{x}{2}, \frac{u}{2}\} + \max\{\frac{y}{2}, \frac{v}{2}\} + \max\{\frac{z}{2}, \frac{w}{2}\}] \\ &= \frac{1}{4}p(gx, gu) + \frac{1}{4}p(gy, gv) + \frac{1}{4}p(gz, gw). \end{aligned}$$

Then, the contractive condition is satisfied with $a_i = 0$ for all $i = 1, \cdots, 12$ and $a_{13} = a_{14} = a_{15} = 1/4$. All conditions of Theorem 2.1 are satisfied. Consequently, (x, y, z) is a tripled coincidence point of F and g if and only if $x = y = z = 0$. This implies that F and g are W-compatible. Applying our Theorem 2.4, we obtain the existence and uniqueness of a common tripled fixed point of F and g. In this example, $(0, 0, 0)$ is the unique common tripled fixed point.

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