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Compactification of closed preordered spaces

E. Minguzzi

Abstract

A topological preordered space admits a Hausdorff T_2-preorder compactification if and only if it is Tychonoff and the preorder is represented by the family of continuous isotone functions. We construct the largest Hausdorff T_2-preorder compactification for these spaces and clarify its relation with Nachbin's compactification. Under local compactness the problem of the existence and identification of the smallest Hausdorff T_2-preorder compactification is considered.

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1. Introduction

A *topological preordered space* is a triple $(E, \mathscr{T}, \leq)$ where $(E, \mathscr{T})$ is a topological space and $\leq$ is a *preorder* on E, namely a reflexive and transitive relation on E. The preorder is an *order* if it is antisymmetric. There are many possible compatibility conditions between topology and preorder that can be added to this basic structure. We shall mainly consider the T_2*-preordered spaces* (*closed preordered spaces*), namely those spaces for which the graph

$$G(\leq) = \{(x, y) : x \leq y\},$$

is closed in the product topology $\mathscr{T} \times \mathscr{T}$ of $E \times E$. In this work we shall follow Nachbin's terminology [22] but we remark that in computer science T_2-ordered spaces are very much studied and called *pospaces*.

A T_2-preordered space E is a T_1-preordered space in the sense that for every $x \in E$, $i(x)$ and $d(x)$ are closed where $i(x) = \{y \in E : x \leq y\}$ is the increasing hull and $d(x) = \{y \in E : y \leq x\}$ is the decreasing hull.

We recall that an *isotone* function $f : E \to \mathbb{R}$ is a function such that $x \le y \Rightarrow f(x) \le f(y)$. We shall mostly work with continuous isotone functions with value in [0,1], although we could equivalently work with bounded continuous isotone functions.

In this work we shall consider the problem of compactification for T_2-preordered spaces. It is understood here that the compactification cE must be endowed with a preorder $\le_c$ which induces $\le$ on E, namely if $x, y \in E$, then $x \le y$ if and only if $x \le_c y$. The extended preorder is also demanded to be closed.

In the ordered case this problem has been solved by Nachbin who proved [4,22,23] that a topological ordered space admits a T_2-order compactification if and only if it is a *completely regularly ordered space*, where a *completely regularly preordered space* is a topological preordered space for which the following two conditions hold

(i) $\mathscr{T}$ coincides with the initial topology generated by the set of continuous isotone functions $f : E \to [0, 1]$,

(ii) $x \le y$ if and only if for every continuous isotone function $f : E \to [0, 1]$, $f(x) \le f(y)$.

For future reference let us introduce the equivalence relation $x \sim y$ on E, given by "$x \le y$ and $y \le x$". Let $E/\!\sim$ be the quotient space, $\mathscr{T}/\!\sim$ the quotient topology, and let $\lesssim$ be defined by, $[x] \lesssim [y]$ if $x \le y$ for some representatives (with some abuse of notation we shall denote with $[x]$ both a subset of E and a point on $E/\!\sim$). The quotient preorder is by construction an order. The triple $(E/\!\sim, \mathscr{T}/\!\sim, \lesssim)$ is a topological ordered space and $\pi : E \to E/\!\sim$ is the continuous quotient projection.

Nachbin proves [22, Prop. 8] that the completely regularly preordered spaces can be characterized as those topological preordered spaces $(E, \mathscr{T}, \le)$ which come from a quasi-uniformity $\mathcal{U}$, in the sense that $\mathscr{T} = \mathscr{T}(\mathcal{U}^*)$ and $G(\le) = \bigcap \mathcal{U}$ (see [4,22] for details on quasi-uniformities). Note that for these spaces, by (i) above, $(E, \mathscr{T})$ is completely regular but not necessarily Hausdorff (equivalently T_1). Nevertheless, from (ii) it follows that E is a T_2-preordered space, hence T_1-preordered thus $[x] = d(x) \cap i(x)$ is closed. We conclude that in a completely regularly preordered space, $\mathscr{T}$ is T_1, and hence $(E, \mathscr{T})$ is a Tychonoff space, if and only if $\le$ is an order [22].

In this work we look for topological preordered spaces that admit a Hausdorff T_2-preordered compactification. Since the T_2-preorder property is hereditary, and every topological space that admits a Hausdorff compactification is Tychonoff, the class that we are considering is contained in the family of T_2-preordered Tychonoff spaces. In fact we shall see that all these spaces admit a T_2-preorder compactification provided the family of continuous isotone functions determines the preorder. We shall then look for the largest Hausdorff T_2-preorder compactification and we shall clarify its connection with Nachbin's T_2-order compactification. We will end the paper with a discussion of the smallest Hausdorff T_2-preorder compactification.

2. A motivation: the spacetime boundary

Since the next sections will be particularly abstract, it will be convenient to motivate this study mentioning an application. This author is particularly interested in general relativity, but the reader will easily find other applications in closely related fields, for instance, in dynamical systems theory.

This author's interest for the compactifications of closed preordered spaces comes from the well-known problem of attaching a boundary to a spacetime (physicists term *boundary* what is known as *remainder* in topology). We recall that a spacetime is a connected, Hausdorff, time oriented Lorentzian manifold and is denoted (M, g), where g is the Lorentzian metric. In relativity theory the concept of singularity has proved to be quite elusive. One would like to attach a boundary to spacetime so as to distinguish between points at infinity and singularities, where the distinction is made considering the behavior of the Riemann tensor near the boundary point (e.g. diverging or not).

There have been numerous attempts to construct such a boundary. We mention Penrose's conformal boundary [24], Geroch, Kronheimer and Penrose's causal boundary [6], Scott and Szekeres' abstract boundary [28], and various other proposals by Budic and Sachs [1], Racz [25,26], Szabados [30,31], Harris [7], Flores [5] and others. Apart for the case of Penrose's conformal boundary, which cannot be applied in general, one does not demand that spacetime plus the boundary be still a manifold. In general, one wishes just to preserve some notion of continuity and provide a way of extending the causal relation to the boundary.

The above constructions are often quite involved. I propose a strategy which takes advantage of the fact that any spacetime is a topological preordered space. Let us clarify this point. The *causal relation* J^+ on M is given by the pairs (x, y) of points of M for which there is a C^1 curve $\gamma : [0, 1] \to M$, $\gamma(0) = x$, $\gamma(1) = y$, which is *causal*, in the sense that its tangent vector at any point stays in the future causal cone of g. In general J^+ might be non-closed, however, there is another relation, intimately connected with J^+, which is always closed: the Seifert's relation J^+_S [18,29]. The Seifert relation turns spacetime into a topological space endowed with a closed relation and, provided some topological conditions are satisfied, it is indeed possible to compactify spacetime along the lines suggested in this work.

We do not claim that the compactification constructed in this way, denoted $\beta(E)$, will be the most physical. Indeed, it will add many more points than intuitively required. Nevertheless, it will provide an important step since it will dominate any other possible compactification which, therefore, will be obtainable from $\beta(E)$ through a suitable identification of the boundary points. The possibility of adding a boundary and extending the preorder so as to keep its closure is not known among physicists. It suffices to say that the boundary constructions mentioned above, either apply to very special spacetimes, or do not share this property.

We could also try a different approach by first showing that the spacetime is not only a topological preordered space, but in fact a quasi-pseudo-metric space, and then completing it with a preorder generalization of the Cauchy completion. Unfortunately, although we could prove, using the results of [20], that most interesting spacetimes are quasi-pseudo-metrizable, the completion would depend on the chosen quasi-pseudo-metric. Therefore, this strategy is not entirely viable unless we prove the existence of some natural spacetime quasi-pseudo-metric.

Let us end this section explaining why we have to generalize Nachbin's compactification to the preordered case, even in those cases in which E is ordered. A key example is provided by Misner's spacetime, a 2-dimensional spacetime which retains several features of the Taub-NUT spacetime [8]. This spacetime has topology $S^1 \times \mathbb{R}$ and metric $g = 2\mathrm{d}\theta \mathrm{d}t + t\mathrm{d}\theta^2$. The line $t = 0$ of topology S^1 is a closed lightlike geodesic. Through any point of the region $t \le 0$ passes a closed causal curve.

The topological space E given by the region $t \ge 0$ of Misner's spacetime can be endowed with a preorder given by the causal relation. This relation is closed, and the subset $t > 0$ with the induced topology and preorder is a completely regularly ordered space (indeed it can be shown to be convex and it is normally preordered due to the results of [19]). The set $t = 0$ represents a natural connected piece which bounds the region $t > 0$, but Nachbin's compactification cannot dominate a compactification with this piece of boundary since Nachbin's compactification would be ordered while the set $t = 0$ is a closed null geodesic, and hence any pair of points in this set violates antisymmetry. In summary, although the region $t > 0$ is ordered, its most natural compactifications are not ordered. Evidently, Nachbin's compactification is too restrictive for applications, and the order condition on the compactified space must be relaxed.

3. Hausdorff T_2-preorder compactifications

Given two topological preordered spaces $(E_1, \mathscr{T}_1, \le_1)$ and $(E_2, \mathscr{T}_2, \le_2)$ the function $H : E_1 \to E_2$ is a *preorder homeomorphism* if H is bijective, continuous and isotone and so is its inverse. We speak of *preorder embedding* if H is a preorder homeomorphism of E_1 on its image $H(E_1) \subset E_2$, where $H(E_1)$ is given the induced topology and induced preorder.

We are interested in establishing under which conditions a topological preordered space $(E, \mathscr{T}, \le)$ admits a preorder compactification, namely a preorder embedding $c : E \to cE$ into a compact topological preordered space $(cE, \mathscr{T}_c, \le_c)$ in such a way that $c(E)$ is a dense subset of cE. We shall often identify E with $c(E)$ because c is a preorder homeomorphism between E and $c(E)$. We shall be especially interested in Hausdorff T_2-preordered compactifications, that is, in those preorder compactifications for which $(cE, \mathscr{T}_c, \le_c)$ is also a Hausdorff T_2-preordered space. Sometimes we shall write that $(cE, \mathscr{T}_c, \le_c)$ is a preorder compactification by meaning with this that the map $c : E \to cE$ is a preorder compactification.

Definition 3.1. If c_1E, c_2E, are two preorder compactifications of E we write $c_1 \le c_2$ if there is a continuous isotone map $C : c_2E \to c_1E$ such that $C \circ c_2 = c_1$ ($c_1 \le c_2$ reads "c_2 *dominates* over c_1"). The map C is just an extension to c_2E of the preorder homeomorphism $c_1 \circ c_2^{-1} : c_2(E) \to c_1(E)$. Two preorder compactifications are *equivalent* if $c_1 \le c_2$ and $c_2 \le c_1$.

We remark that two compactifications may be such that $c_1E = c_2E$, $C = Id$, but correspond to different preorders on c_1E. In this case $c_1 \le c_2$ means that, because Id must be isotone, $G(\le_{c_2}) \subset G(\le_{c_1})$ (in our conventions the set inclusion is reflexive). Intuitively, to enlarge the compactification means to enlarge the domain cE or to narrow the preorder $\le_c$ or both. From the definition it follows that the relation of domination on the set of all the compactification is a preorder. The next result establishes that it is actually an order provided we pass to the quotient made by the classes of compactifications related by preorder homeomorphisms.

Proposition 3.2. *If two Hausdorff preorder compactifications c_1, c_2, are equivalent, then there is a preorder homeomorphism $H : c_2E \to c_1E$ such that $H \circ c_2 = c_1$.*

Proof. Since $c_1 \le c_2$ there is a continuous isotone map $C_{12} : c_2E \to c_1E$ such that $C_{12} \circ c_2 = c_1$ and since $c_2 \le c_1$ there is a continuous isotone map $C_{21} : c_1E \to c_2E$ such that $C_{21} \circ c_1 = c_2$. Applying C_{12} to the latter equation and using the former equation we get $C_{12} \circ C_{21} \circ c_1 = C_{12} \circ c_2 = c_1$ which implies that $C_{12} \circ C_{21}|_{c_1(E)} = Id_{c_1E}|_{c_1(E)}$. Since $c_1(E)$ is dense in c_1E and c_1E is a Hausdorff space we have that $C_{12} \circ C_{21} = Id_{c_1E}$ (e.g. [32, Cor. 13.14]). Arguing with the roles of 1 and 2 exchanged we get $C_{21} \circ C_{12} = Id_{c_2E}$ thus C_{12} and C_{21} are one the inverse of the other. But they are both isotone thus $H := C_{12}$ is a preorder homeomorphism. □

Proposition 3.3. *If c_1, c_2 are two Hausdorff preorder compactifications of E and $c_1 \le c_2$ then the continuous isotone map $C : c_2E \to c_1E$ such that $C \circ c_2 = c_1$ satisfies $C(c_2E) = c_1E$, $C(c_2(E)) = c_1(E)$ and $C(c_2E\backslash c_2(E)) = c_1E\backslash c_1(E)$.*

Proof. The map C is necessarily onto because $C(c_2E)$ is compact and hence closed and the image of C includes $C(c_2(E)) = c_1(E)$ which is dense in c_1E. The preorder compactifications are compactifications so that the last equation follows from [3, Theor. 3.5.7]. □

Let $f : E \to [0, 1]$ be a continuous function on a topological space $(E, \mathscr{T})$, we shall denote by $\le_f$ the total preorder given by "$x \le_f y$ if $f(x) \le f(y)$". Its graph will be denoted with G_f.

The next proposition establishes some necessary conditions for the existence of a Hausdorff T_2-preorder compactification.

Proposition 3.4. *If $(E, \mathscr{T}, \le)$ is a subspace of a Hausdorff T_2-preordered compact space, then E is a T_2-preordered Tychonoff space and the family of continuous isotone functions $\mathcal{F}$, $f : E \to [0, 1]$, is such that $x \le y$ if and only if for every $f \in \mathcal{F}$, $f(x) \le f(y)$ (equivalently $G(\le) = \bigcap_{f\in\mathcal{F}} G_f$).*

Proof. Let E be a subspace of a Hausdorff T_2-preordered compact space which we denote $(E', \mathscr{T}', \leq')$. Since every compact Hausdorff space is Tychonoff and this property is hereditary, we have that E is Tychonoff. The T_2-preorder space property is also hereditary thus E is T_2-preordered. Finally, since every T_2-preordered compact space is normally preordered [19], for $x', y' \in E'$, $x' \leq y'$ iff $F(x') \leq F(y')$ where $F : E' \to [0, 1]$ is any continuous and isotone function on E' (see e.g. [21, Prop. 1.1]). Let $\mathcal{G}$ be the family of continuous isotone functions, $f : E \to [0, 1]$, which come from the restriction of some continuous isotone function $F : E' \to [0, 1]$. Evidently, for $x, y \in E$, $x \leq y$ iff for every $f \in \mathcal{G}$, $f(x) \leq f(y)$. Since $\mathcal{F}$ includes $\mathcal{G}$ and is made of isotone functions the last claim follows. □

3.1. **The largest Hausdorff T_2-preorder compactification.** The next result establishes that the previous necessary conditions are actually sufficient and that there is a Hausdorff T_2-preordered compactification which dominates over all the other Hausdorff T_2-preordered compactifications. The locally compact σ-compact Hausdorff T_2-preordered spaces satisfy these necessary and sufficient conditions [19].

Theorem 3.5. *Let $(E, \mathscr{T}, \leq)$ be a T_2-preordered Tychonoff space, let $\mathcal{F}$ be the family of continuous isotone functions $f : E \to [0, 1]$, and assume that the preorder is represented by the continuous isotone functions i.e. $G(\leq) = \bigcap_{f \in \mathcal{F}} G_f$. Let $\beta : E \to \beta E$ be the Stone-Čech compactification and let $\tilde{\mathcal{F}}$ be the set of continuous functions over βE obtained from the (unique) extension*[1] *of the elements of $\mathcal{F}$. There is a largest Hausdorff T_2-preordered compactification of $(E, \mathscr{T}, \leq)$ given by $(\beta E, \mathscr{T}_\beta, \leq_\beta)$ where $G(\leq_\beta) = \bigcap_{\tilde{f} \in \tilde{\mathcal{F}}} G_{\tilde{f}}$. Every continuous isotone function on E extends to a continuous isotone function on βE.*

Proof. Each graph $G_{\tilde{f}}$ is closed because the functions $\tilde{f} : \beta E \to [0, 1]$ are continuous, thus $G(\leq_\beta)$ being the intersection of closed sets is closed. Further the graphs $G_{\tilde{f}}$ contain the diagonal of βE, thus $G(\leq_\beta)$ contains the diagonal. Moreover, $\leq_{\tilde{f}}$ is transitive which implies that $\leq_\beta$ is transitive and hence a closed preorder on βE. For every $f \in \mathcal{F}$, if $x, y \in E$ then $f(x) \leq f(y)$ iff $\tilde{f}(x) \leq \tilde{f}(y)$ thus $G(\leq) = G(\leq_\beta) \cap (E \times E)$ which proves that $(\beta E, \mathscr{T}_\beta, \leq_\beta)$ is a preorder compactification.

If $f : E \to [0, 1]$ is a continuous isotone function on E then its continuous extension to βE, $\tilde{f}$, is such that $\tilde{f} \in \tilde{F}$ and by definition of $\leq_\beta$, $G(\leq_\beta) \subset G_{\tilde{f}}$ which means that $\tilde{f}$ is isotone.

Let $(cE, \mathscr{T}_c, \leq_c)$ be another preorder compactification then, since $(\beta E, \mathscr{T}_\beta)$ is the largest Hausdorff compactification [32, Theor. 19.9] there is a continuous map $H : \beta E \to cE$ such that $H \circ \beta = c$. The relation on βE, $R := (H \times H)^{-1} G(\leq_c)$ which is clearly reflexive and transitive is also closed in $\beta E \times \beta E$ because H is continuous.

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[1]Note that the extension $\tilde{F}$ is really the extension of $f \circ \beta^{-1}$.

The map H extends into a continuous function on βE the preorder homeomorphism $c \circ \beta^{-1} : \beta(E) \to c(E)$ thus $R \cap (\beta(E) \times \beta(E)) = G(\le_\beta) \cap (\beta(E) \times \beta(E))$, that is, $(\beta \times \beta)^{-1} R = G(\le)$. If a function $g : \beta E \to [0,1]$ is continuous and R-isotone then $g \circ \beta : E \to [0,1]$ is continuous and isotone which means that $g \in \tilde{\mathcal{F}}$ (the extension of a continuous function to a continuous function on βE is unique because $\beta(E)$ is dense in βE), that is g is also G_β-isotone.

Since $(\beta E, \mathscr{T}_\beta, R)$ is a compact T_2-preordered space it is normally preordered [19, Theor. 2.4] thus $R = \bigcap_{g \in \mathcal{G}} G_g$ where the intersection is with respect to the family $\mathcal{G}$ of all the continuous R-isotone functions on βE. As we have just proved, this family is a subset of $\tilde{\mathcal{F}}$ thus $G(\le_\beta) \subset R$. Since $G(\le_\beta) \subset (H \times H)^{-1} G(\le_c)$ we conclude that H is isotone and hence that $c \le \beta$. □

Theorem 3.6. *A Hausdorff T_2-preorder compactification $(cE, \mathscr{T}_c, \le_c)$ which shares the properties*

(a) *every continuous function $f : E \to [0,1]$ can be extended to a continuous function on cE,*
(b) *every continuous isotone function $f : E \to [0,1]$ can be extended to a continuous isotone function on cE,*

is necessarily equivalent to $(\beta E, \mathscr{T}_\beta, \le_\beta)$.

Proof. We already know that $c \le \beta$ because βE is the largest Hausdorff T_2-preorder compactification. Since the compactification $(cE, \mathscr{T}_c)$ shares property (a) it is equivalent with the Stone-Čech compactification $(\beta E, \mathscr{T}_\beta)$, in particular there is a continuous map $D : cE \to \beta E$ such that $D \circ c = \beta$. The relation on cE, $R := (D \times D)^{-1} G(\le_\beta)$ which is clearly reflexive and transitive is also closed in $cE \times cE$ because D is continuous.

D extends into a continuous function on cE the preorder homeomorphism $\beta \circ c^{-1} : c(E) \to \beta(E)$ thus $R \cap (c(E) \times c(E)) = G(\le_c) \cap (c(E) \times c(E))$, that is, $(c \times c)^{-1} R = G(\le)$. If a function $g : cE \to [0,1]$ is continuous and R-isotone then $g \circ c : E \to [0,1]$ is continuous and isotone which means by property (b) that g is also G_c-isotone (the extension of a continuous function to a continuous function on cE is unique because $c(E)$ is dense in cE).

Since $(cE, \mathscr{T}_c, R)$ is a compact T_2-preordered space it is normally preordered [19, Theor. 2.4] thus $R = \bigcap_{g \in \mathcal{G}} G_g$ where the intersection is with respect to the family $\mathcal{G}$ of all the continuous R-isotone functions on cE. As we have just proved, this family is contained in the family of continuous G_c-isotone functions $\mathcal{C}$, $\bigcap_{g \in \mathcal{C}} G_g \subset R$. Finally, note that $(cE, \mathscr{T}_c, \le_c)$ is also a compact T_2-preordered space hence normally preordered and hence with a preorder represented by the continuous G_c-isotone functions, $G(\le_c) = \bigcap_{g \in \mathcal{C}} G_g$, which implies $G(\le_c) \subset R$. The inclusion $G(\le_c) \subset (D \times D)^{-1} G(\le_\beta)$ proves that D is isotone and hence that $\beta \le c$. □

Adapting the terminology of Fletcher and Lindgren [4] for ordered compactifications we can say that the next result proves that $(\beta E, \mathscr{T}_\beta, \le_\beta)$ is a *strict* preorder compactification.

Theorem 3.7. *On $(\beta E, \mathscr{T}_\beta)$ the closed preorder $\le_\beta$ is the smallest closed preorder inducing $\le$ on E.*

Proof. Let $\le_R$ be another closed preorder such that $R \cap (E \times E) = G(\le)$. The map $\beta' : E \to \beta E$, $\beta' = \beta$, where βE is regarded as the preordered space $(\beta E, \mathscr{T}_\beta, R)$ is a preorder compactification. Since β is the largest $\beta' \le \beta$, which means that there is a continuous isotone function $B : \beta E \to \beta' E$ such that $B \circ \beta = \beta'$. On $\beta(E)$ the map B coincides with $\beta' \circ \beta^{-1} = \beta \circ \beta^{-1} = Id$, thus B is the identity over βE. The fact that it is isotone means $G(\le_\beta) \subset R$ which is the thesis. □

Theorem 3.8. *If $(E, \mathscr{T}, \le)$ is a compact Hausdorff T_2-preordered space, then its Hausdorff T_2-preorder compactification $\beta : E \to \beta E$ constructed in Theorem 3.5 is equivalent with the identity $Id : E \to E$.*

Proof. The map $c : E \to E$ where $c = Id_E$ and $(cE, \mathscr{T}_c, \le_c) = (E, \mathscr{T}, \le)$ is a preorder compactification which satisfies both conditions (a) and (b) of Theorem 3.6, thus the preorder compactification Id is equivalent to β. □

The discrete preorder is that preorder for which the increasing hull of a point is made only by the point (thus it is actually an order). The indiscrete preorder is that preorder for which the increasing hull of a point is the whole space. The indiscrete preorder is closed while the discrete preorder requires the Hausdorffness of the space, which we assume.

Corollary 3.9. *If $\le$ is the discrete (indiscrete) preorder then $(\beta E, \mathscr{T}_\beta, \le_\beta)$ is the Stone-Čech compactification endowed with the discrete (resp. indiscrete) preorder.*

Proof. The discrete preorder $\le_d$ on βE is clearly the smallest closed preorder inducing the discrete preorder $\le$, thus $\le_d = \le_\beta$.

For the indiscrete case let $x, y \in \beta E$ and let O_x, O_y be neighborhoods of x and y respectively. Since $\beta(E)$ is dense there are points $x', y' \in E$ such that $x' \in \beta(E) \cap O_x$, $y' \in \beta(E) \cap O_y$, from $\beta^{-1}(x') \le \beta^{-1}(y')$ since β is isotone we get $x' \le_\beta y'$ and since $\le_\beta$ is closed we conclude $x \le y$. □

3.2. **The relation with Nachbin's T_2-order compactification.** In this section we wish to study the relation between the compactification $\beta : E \to \beta E$ and the Nachbin's compactification $n : E \to nE$ in those cases in which E is a completely regularly ordered space so that the latter compactification applies. In this case, although $\le$ is an order, $\le_\beta$ need not be an order. We want to prove that the Nachbin's compactification is obtained by taking the quotient with respect to $\sim_\beta$.

Let $(E/\!\sim, \mathscr{T}/\!\sim, \lesssim)$ be the quotient topological preordered space and let $\pi : E \to E/\!\sim$ be the continuous quotient projection. Every open (closed) increasing (decreasing) set on E projects to an open (resp. closed) increasing (resp. decreasing) set on $E/\!\sim$ and all the latter sets can be regarded as such projections. As a consequence, $(E, \mathscr{T}, \le)$ is a normally preordered space (T_1-preordered space) if and only if $(E/\!\sim, \mathscr{T}/\!\sim, \lesssim)$ is a normally ordered space (resp. T_1-ordered space). Using this fact it is easy to prove (see [19, Cor. 4.3])

Theorem 3.10. *If $(E, \mathscr{T}, \le)$ is a compact T_2-preordered space, then $(E/\!\sim, \mathscr{T}/\!\sim, \lesssim)$ is a compact T_2-ordered space.*

We are ready to establish the connection with the Nachbin T_2-order compactification.

Theorem 3.11. *Let $(E, \mathscr{T}, \le)$ be a T_2-preordered Tychonoff space such that $E/\!\sim$ is a completely regularly ordered space, then the preorder $\le$ is represented by the continuous isotone functions on E. Let $\beta : E \to \beta E$ be the Hausdorff T_2-preorder compactification constructed in Theorem 3.5 and let $\Pi : \beta E \to \beta E/\!\sim_\beta$ be the quotient projection on the T_2-ordered space $(\beta E/\!\sim_\beta, \mathscr{T}_\beta/\!\sim_\beta, \lesssim_\beta)$, then[2] $\Pi \circ \beta \circ \pi^{-1} : E/\!\sim \to \beta E/\!\sim_\beta$ is a T_2-order compactification equivalent to the Nachbin T_2-order compactification $n : E/\!\sim \to n(E/\!\sim)$. In particular, up to equivalences, the following diagram commutes*

$$\begin{array}{ccc} E & \xrightarrow{\beta} & \beta E \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\Pi} \\ E/\!\sim & \xrightarrow{n} & n(E/\!\sim) \end{array}$$

Proof. The order $\lesssim$ on $E/\!\sim$ is represented by the continuous isotone functions because $E/\!\sim$ is completely regularly ordered. Since for $x, y \in E$, $x \le y$ iff $\pi(x) \lesssim \pi(y)$, and the continuous isotone functions on E pass to the quotient, the continuous isotone functions on E represent $\le$.

The fact that $(\beta E/\!\sim_\beta, \mathscr{T}_\beta/\!\sim_\beta, \lesssim_\beta)$ is T_2-ordered follows from Theorem 3.10.

The expression $\varphi := \Pi \circ \beta \circ \pi^{-1}$ gives a well defined function, indeed suppose $x, y \in E$ project on the same element $[x] \in E/\!\sim$, then $x \sim y$ and since β is a preorder embedding $\beta(x) \sim_\beta \beta(y)$ which implies $\Pi(\beta(x)) = \Pi(\beta(y))$.

The function φ is continuous, indeed let $O \subset \beta E/\!\sim_\beta$ be an open subset then $\beta^{-1}(\Pi^{-1}(O))$ is open and if $x \in \beta^{-1}(\Pi^{-1}(O))$ and $y \sim x$ then as β is a preorder embedding $\beta(y) \sim_\beta \beta(x)$, $\beta(x) \in \Pi^{-1}(O)$ which implies $\beta(y) \in \Pi^{-1}(O)$ and hence $y \in \beta^{-1}(\Pi^{-1}(O))$. The open set $\beta^{-1}(\Pi^{-1}(O)) \subset E$, being projectable has an open projection by definition of quotient topology which implies that $\varphi^{-1}(O)$ is open.

Let us prove that φ is isotone. Let $[x] \lesssim [y]$, $x, y \in E$, then $x \le y$ and, since β is a preorder embedding, $\beta(x) \le_\beta \beta(y)$, and finally $\Pi(\beta(x)) \le_\beta \Pi(\beta(y))$ by definition of quotient order.

Let us prove that φ is injective. Let $[x], [y] \in E/\!\sim$ be such that $\varphi([x]) = \varphi([y])$, that is, $\Pi(\beta(x)) = \Pi(\beta(y))$. This equality implies $\beta(x) \sim_\beta \beta(y)$, and since β is a preorder embedding $x \sim y$, that is, $[x] = [y]$.

[2] The inverse π^{-1} is multivalued but the composition $\Pi \circ \beta \circ \pi^{-1}$ is a well defined function.

Let us prove that $\varphi^{-1}|_{\varphi(E/\sim)} : \varphi(E/\sim) \to E/\sim$ is isotone. Let $x, y \in E$ and $\Pi(\beta(x)) \lesssim_\beta \Pi(\beta(y))$ then $\beta(x) \le_\beta \beta(y)$ and, since β is a preorder embedding, $x \le y$ which implies $[x] \lesssim [y]$.

Let us prove that φ is an embedding. Since π is continuous, given an open set $N \subset E/\sim$ we have that $\pi^{-1}(N)$ is open, thus we have only to prove that $\Pi \circ \beta$ sends open sets on E of the form $\pi^{-1}(N)$ to open sets on $\Pi \circ \beta(E)$ with the topology induced from $\beta E/\sim_\beta$. Let $O \subset E$ be an open set of the form $O = \pi^{-1}(N)$ with N open set on $E/\sim$ and let $x \in O$ (thus $[x] \in N$). Since $E/\sim$ is completely regularly ordered space there are [22] a continuous isotone function $\hat{f} : E/\sim \to [0, 1]$ and a continuous anti-isotone function $\hat{g} : E/\sim \to [0, 1]$ such that $\hat{f}([x]) = \hat{g}([x]) = 1$ and $\min(\hat{f}([y]), \hat{g}([y])) = 0$ for $[y] \in E\backslash N$.

Let us define $f = \hat{f} \circ \pi$, $g = \hat{g} \circ \pi$, so that they are respectively continuous isotone and continuous anti-isotone and such that $f(x) = g(x) = 1$ and $\min(f(y), g(y)) = 0$ for $y \in E\backslash O$.

The functions $f, g(\circ\beta^{-1})$ extend to functions $\tilde{f}, \tilde{g} : \beta E \to [0, 1]$ respectively isotone and anti-isotone (extend $-g$ in place of g and take minus the extended function). Since they are isotone or anti-isotone there are continuous functions $F, G : \beta E/\sim_\beta \to [0, 1]$, respectively isotone and anti-isotone, such that $\tilde{f} = F \circ \Pi$, $\tilde{g} = G \circ \Pi$ (continuity follows from the universality property of the quotient map [32, Theor. 9.4]).

The function $h = \min(\tilde{f}, \tilde{g}) = \min(F, G) \circ \Pi$ is continuous and vanishes on $\beta(E\backslash O)$ and hence $\min(F, G)$ vanishes on $(\Pi \circ \beta)(E\backslash O) = \varphi((E/\sim)\backslash N)$ and equals 1 on $[\beta(x)]_\beta = \varphi(x)$. Since φ is injective the open set $Q = \{[w]_\beta \in \beta E/\sim_\beta : \min(F([w]_\beta), G([w]_\beta)) > 0\}$ contains $\varphi(x)$ and is such that $Q \cap \varphi(E/\sim) \subset \varphi(N)$ which proves, due to the arbitrariness of $[x]$, that $\varphi(N)$ is open in the topology induced on $\varphi(E/\sim)$ by $\beta E/\sim_\beta$. We infer that φ is an embedding and since it is isotone with its inverse it is a preorder embedding.

If $[z]_\beta \in (\beta E/\sim_\beta)\backslash\varphi(E/\sim)$ and W is an open set containing $[z]_\beta$ then $\Pi^{-1}(W)$ is open and since β is a dense embedding there is some $r \in E$ such that $\beta(r) \in \Pi^{-1}(W)$, thus $[r] \in E/\sim$ is such that $\varphi([r]) \in W$, that is, $\varphi(E/\sim)$ is dense in $\beta E/\sim_\beta$ and hence φ is a T_2-order compactification.

Now, let $\hat{f} : E/\sim \to [0, 1]$ be a continuous isotone function, and let $f = \hat{f} \circ \pi$. The function $f : E \to [0, 1]$ is a continuous isotone function and we know that there is a continuous isotone function $\tilde{f} : \beta E \to [0, 1]$ which extends $f \circ \beta^{-1} : \beta(E) \to [0, 1]$. Since $\tilde{f}$ is isotone there is some continuous isotone function $F : \beta E/\sim_\beta \to [0, 1]$ (continuity follows from the universality property of the quotient map) such that $\tilde{f} = F \circ \Pi$, thus F extends $\hat{f} \circ \varphi^{-1} : \varphi(E/\sim) \to [0, 1]$. Since the Nachbin T_2-order compactification is characterized by this extension property [4, 22] it follows that φ is equivalent to n.

Finally, $\varphi \circ \pi = (\Pi \circ \beta \circ \pi^{-1}) \circ \pi = \Pi \circ \beta$ which proves that, up to equivalences, the diagram commutes.

□

Corollary 3.12. *Let E be a completely regularly ordered space, let $\beta : E \to \beta E$ be the Hausdorff T_2-preorder compactification constructed in Theorem 3.5 and let $\Pi : \beta E \to \beta E/\!\sim_\beta$ be the quotient projection on the T_2-ordered space $(\beta E/\!\sim_\beta, \mathscr{T}_\beta/\!\sim_\beta, \lesssim_\beta)$, then $\Pi \circ \beta : E \to \beta E/\!\sim_\beta$ is a T_2-order compactification equivalent to the Nachbin T_2-order compactification $n : E \to nE$.*

Proof. It follows from the previous theorem noting that a completely regularly ordered space is a T_2-preordered Tychonoff space. □

If E is a completely regularly ordered space the preorder compactification β need not be equivalent with the Nachbin compactification. Consider for instance the interval $[0, 1)$ with the usual topology and order. The Nachbin compactification is given by $[0, 1]$ but $\beta([0, 1))$ includes many more points.

3.3. **The smallest Hausdorff T_2-preorder compactification.** In this section we make some progress in the problem of finding the smallest Hausdorff T_2-preorder compactification of a topological preordered space in those cases in which it exists. The problem of identifying and characterizing the smallest T_2-order compactification was considered in [13, 15–17, 27].

In this section $(E, \mathscr{T}, \le)$ is a locally compact T_2-preordered Tychonoff space and $\mathcal{F}$ is the family of continuous isotone functions $f : E \to [0, 1]$. Accordingly with the necessary conditions singled out in Prop. 3.4, we shall assume that the preorder is represented by the continuous isotone functions i.e. $G(\le) = \bigcap_{f\in\mathcal{F}} G_f$.

Let $\mathcal{C}$, $\mathcal{C}^-$ and $\mathcal{C}^+$ be the families of continuous functions in $[0, 1]$ which are constant outside a compact set, which have compact support and which have value 1 outside a compact set, respectively.

For every $\mathcal{H} \subset \mathcal{F}$ such that $G(\le) = \bigcap_{h\in\mathcal{H}} G_h$ we can construct a T_2-preorder compactification $(cE, \mathscr{T}_c, \le_c)$, which we call $\mathcal{H}$-compactification, through the embedding $c : E \to [0, 1]^{\mathcal{H}\cup\mathcal{C}}$ identifying cE with the closure of the image. Indeed, the family $\mathcal{H}\cup\mathcal{C}$ separates points and has an initial topology coincident with $\mathscr{T}$ (thanks to local compactness and the inclusion of $\mathcal{C}$ in the family) thus c is an embedding [32, Theor. 8.12]. The topology $\mathscr{T}_c$ is that induced from the product topology in $[0, 1]^{\mathcal{H}\cup\mathcal{C}}$ on cE.

We define the T_2-preorder $\preceq$ on $[0, 1]^{\mathcal{H}\cup\mathcal{C}}$ as that given by $x \preceq y$ iff $x_h \le_h y_h$ for every $h \in \mathcal{H}$, where $\le_h$ is the usual order on the h-th factor [0,1]. This preorder is closed because the projections $\pi_h : [0, 1]^{\mathcal{H}\cup\mathcal{C}} \to \mathbb{R}$ are continuous, and hence $G(\preceq) = \bigcap_{h\in\mathcal{H}} (\pi_h \times \pi_h)^{-1} G(\le_h)$ is closed. It is a preorder rather than an order because two points can have the same h-components while being different. The T_2-preorder $\le_c$ on cE is that induced by $\preceq$ and is again closed because of the hereditarity of the T_2-preorder property. Finally, $c : E \to c(E)$ is isotone with its inverse because $G(\le) = \bigcap_{h\in\mathcal{H}} G_h$.

Observe that $h \circ c^{-1} : c(E) \to [0, 1]$ extends to the continuous isotone function $\pi_h|_{cE}$, that is, the continuous isotone functions belonging to $\mathcal{H}$ are extendable to the $\mathcal{H}$-compactification cE keeping the same properties.

Remark 3.13. The just defined $\mathcal{H}$-compactification gives back the usual one-point compactification if the preorder $\le$ is indiscrete and $\mathcal{H}$ is chosen empty (the additional point is that of coordinates f_c, $c \in \mathcal{C}$, where f_c is the constant value taken by c outside a compact set).

If the preorder $\le$ is discrete and $\mathcal{H}$ is chosen to coincide with $\mathcal{C}$ then the compactified space is still the one-point compactification but endowed with the discrete preorder. If $\mathcal{H}$ is chosen equal to $\mathcal{C}^-$, then the added point is less than any other point. If $\mathcal{H}$ is chosen equal to $\mathcal{C}^+$, then the added point is greater than any other point.

In the next proofs we shall often identify $c(E)$ with E especially when referring to the extension of functions.

Proposition 3.14. *Let $c : E \to cE$ be a $\mathcal{H}$-compactification. The remainder $cE\backslash c(E)$ endowed with the preorder induced from $\le_c$ is a T_2-ordered space.*

Proof. Since the T_2-preorder property is hereditary the remainder is a T_2-preordered space. Let $x, y \in cE\backslash c(E)$ and suppose that $x \le_c y \le_c x$ then $x \preceq y \preceq x$, that is for the (necessarily unique as $c(E)$ is dense in cE) continuous isotone extension $H : cE \to [0,1]$, $H = \pi_h|_{cE}$, of $h \in \mathcal{H}$ we have $H(x) \le H(y) \le H(x)$, which reads $H(x) = H(y)$. We have only to prove that for every $f \in \mathcal{C}$, $\pi_f(x) = \pi_f(y)$ from which it follows $x = y$. But by local compactness $c(E)$ is open in cE thus $cE\backslash c(E)$ is compact and can be separated by open sets (as cE is Hausdorff and compact hence normal) from the compact set outside which f is constant. Thus the extension $\pi_f|_{cE}$ of $f \in \mathcal{C}$ takes a constant value on the whole remainder, which implies $\pi_f(x) = \pi_f(y)$. □

We remark that the previous result does not imply that if $\le$ is an order then $\le_c$ is an order, but only that if $x \le_c y \le_c x$, then one point among x and y belongs to $c(E)$ while the other belongs to $cE\backslash c(E)$.

Proposition 3.15. *Let $(E, \mathscr{T}, \le)$ be a locally compact T_2-preordered Tychonoff space then every T_2-preordered Hausdorff compactification $c : E \to cE$ dominates a $\mathcal{H}$-compactification for a family $\mathcal{H} \subset \mathcal{F}$ where $\mathcal{H}$ is such that $G(\le) = \bigcap_{h\in\mathcal{H}} G_h$. The family $\mathcal{H}$ is made by those continuous isotone function with value in [0,1] in E that extend with the same properties to cE.*

Proof. Let $c_1 : E \to c_1E$ be a T_2-preordered Hausdorff compactification. Since $(c_1E, \mathscr{T}_{c_1}, \le_{c_1})$ is a compact T_2-preordered space it is normally preordered, thus the family of continuous isotone functions with values in $[0,1]$, $\mathcal{H}_{c_1}$, is such that for $x, y \in c_1E$, $x \le_{c_1} y$ if and only if for every $F \in \mathcal{H}_{c_1}$ we have $F(x) \le F(y)$. Let $\mathcal{H}$ be made by those functions which are the restriction of the elements of $\mathcal{H}_{c_1}$ to E. With this definition $G(\le) = \bigcap_{h\in\mathcal{H}} G_h$. Let $c_2 : E \to c_2E \subset [0,1]^{\mathcal{H}\cup\mathcal{C}}$ be the $\mathcal{H}$-compactification and let us prove that c_1 dominates c_2.

A continuous isotone map $C : c_1E \to c_2E$ such that $C \circ c_1 = c_2$ can be constructed as follows. By local compactness $c_1(E)$ is open and $c_1E\backslash c_1(E)$ is closed and compact. We consider the family $\mathcal{H}_{c_1} \cup \mathcal{C}_{c_1}$ where $\mathcal{C}_{c_1}$ is the family of continuous functions with value in $[0,1]$ on c_1E which are constant outside a compact set disjoint from $c_1E\backslash c_1(E)$. The restriction of the elements of the family $\mathcal{C}_{c_1}$ to $c_1(E)$ gives back $\mathcal{C}$. By definition, the map C sends $x \in c_1E$ to the point of $[0,1]^{\mathcal{H}_{c_1}\cup\mathcal{C}_{c_1}}$ whose f coordinate is the value $f(x)$, $f \in \mathcal{H}_{c_1}\cup\mathcal{C}_{c_1}$. This map is continuous [32, Theor. 8.8] and isotone, where we define the preorder on $[0,1]^{\mathcal{H}_{c_1}\cup\mathcal{C}_{c_1}}$ as that determined by the family $\mathcal{H}_{c_1}$. Let us prove that its image is included in c_2E. From the definitions we have that if $x \in c_1(E)$ then $C(x)$ belongs to $c_2(E)$. As C is continuous, and $c_1(E)$ is dense in c_1E, if $x \in c_1E$ its image $C(x)$ belongs to the closure of $c_2(E)$ namely to c_2E. □

Proposition 3.16. *If $\mathcal{H}_2 \supset \mathcal{H}_1$ then the $\mathcal{H}_2$-compactification dominates over the $\mathcal{H}_1$-compactification.*

Proof. Indeed, if $c_2 : E \to c_2E \subset [0,1]^{\mathcal{H}_2\cup\mathcal{C}}$ is the former and $c_1 : E \to c_1E \subset [0,1]^{\mathcal{H}_1\cup\mathcal{C}}$ is the latter preorder compactification, then there is a continuous isotone map $C : c_2E \to c_1E$ such that $C \circ c_2 = c_1$. This map is the restriction to c_2E of $\Pi : [0,1]^{\mathcal{H}_2\cup\mathcal{C}} \to [0,1]^{\mathcal{H}_1\cup\mathcal{C}}$ where Π identifies points with the same coordinates belonging to the set $\mathcal{H}_1 \cup \mathcal{C}$. □

Once a $\mathcal{H}$-compactification is given it is well possible that some $f \in \mathcal{F}\backslash\mathcal{H}$ could be extendable as a continuous isotone function to the whole compactification. Let $i(\mathcal{H})$ be the subset of $\mathcal{F}$ of so extendable functions. This set being larger than $\mathcal{H}$ has again the property that it represents $\leq$.

Proposition 3.17. *The $\mathcal{H}$-compactification and the $i(\mathcal{H})$-compactification are equivalent.*

Proof. Since $\mathcal{H} \subset i(\mathcal{H})$ the $i(\mathcal{H})$-compactification dominates over the $\mathcal{H}$-compactification. For the converse let $c_2 : E \to c_2E \subset [0,1]^{\mathcal{H}\cup\mathcal{C}}$ be the $\mathcal{H}$-compactification and let $c_1 : E \to c_1E \subset [0,1]^{i(\mathcal{H})\cup\mathcal{C}}$ be the $i(\mathcal{H})$-compactification. A continuous isotone map $C : c_2E \to c_1E$ such that $C \circ c_2 = c_1$ can be constructed as follows. All the functions of $i(\mathcal{H}) \cup \mathcal{C}$ extend (uniquely because $c_2(E)$ is dense in c_2E) from E to c_2E thus to every $x \in c_2E$ we assign the image $C(x)$ given by the point of $[0,1]^{i(\mathcal{H})\cup\mathcal{C}}$ having as coordinates the values taken by the functions belonging to $i(\mathcal{H}) \cup \mathcal{C}$. By construction C is continuous [32, Theor. 8.8]. Let us prove that the image is included in c_1E. From the definitions we have that if $x \in c_2(E)$ then $C(x)$ belongs to $c_1(E)$. As C is continuous, and $c_2(E)$ is dense in c_2E, if $x \in c_2E$ its image $C(x)$ belongs to the closure of $c_1(E)$ namely to c_1E. The fact that C is isotone follows immediately from the definition of preorder in $[0,1]^{i(\mathcal{H})\cup\mathcal{C}}$ and from the fact that the extension of the function in $i(\mathcal{H})$ to c_2E are, by assumption, continuous and isotone. □

Corollary 3.18. *Let $P(\mathcal{F})$ denote the family of subsets of $\mathcal{F}$. The map $i : P(\mathcal{F}) \to P(\mathcal{F})$ is idempotent, namely $i(i(\mathcal{H})) = i(\mathcal{H})$. Furthermore, if $\mathcal{H}_1 \subset \mathcal{H}_2$ then $i(\mathcal{H}_1) \subset i(\mathcal{H}_2)$.*

Proof. If a continuous isotone function $f : E \to [0,1]$ can be extended as a continuous isotone function to the $i(\mathcal{H})$-compactified space, i.e. $f \in i(i(\mathcal{H}))$ then, as the $\mathcal{H}$-compactification and the $i(\mathcal{H})$-compactification are equivalent, it can be extended as a continuous isotone function to the $\mathcal{H}$-compactified space that is $f \in i(\mathcal{H})$.

For the last statement, let $f \in i(\mathcal{H}_1)$ that is $f : E \to [0, 1]$ can be extended as a continuous isotone function $f_1 : c_1E \to [0, 1]$ to the $\mathcal{H}_1$-compactified space. But the $\mathcal{H}_2$-compactification dominates over the $\mathcal{H}_1$-compactification, that is if $c_2 : E \to c_2E$ is the former and $c_1 : E \to c_1E$ is the latter, there is a continuous isotone function $C : c_2E \to c_1E$ such that $C \circ c_2 = c_1$. The pullback with C of the extension to c_1E, namely $f_2 = f_1 \circ C$, is a continuous isotone extension on c_2E of f thus $f \in i(\mathcal{H}_2)$. □

Theorem 3.19. *The $\mathcal{H}$-compactification is the smallest Hausdorff T_2-preordered compactification for which the function belonging to $\mathcal{H}$ are extendable as continuous isotone functions to the compactified space.*

Proof. Let $c : E \to cE$ be a Hausdorff T_2-preordered compactification for which the functions belonging to $\mathcal{H}$ are extendable. By Prop. 3.15 the compactification c dominates a $\mathcal{G}$-compactification where $\mathcal{G}$ is the set of continuous isotone functions on E with value in [0,1] which are extendable with these properties to cE. Thus $\mathcal{H} \subset \mathcal{G}$ and by Prop. 3.16 the $\mathcal{G}$-compactification dominates over the $\mathcal{H}$-compactification, thus c dominates the $\mathcal{H}$-compactification. □

Definition 3.20. The *family of invariant sets* $\mathcal{I}$ is the set of subsets $\mathcal{H} \subset \mathcal{F}$ which satisfy $G(\le) = \bigcap_{h\in\mathcal{H}} G_h$ and are left invariant by i. The set $\mathcal{I}$ is ordered by inclusion.

The next theorem serves to define the family of continuous isotone functions $\mathcal{S}$ which characterizes the smallest compactification.

Theorem 3.21. *If the smallest Hausdorff T_2-preorder compactification exists then it is a $\mathcal{S}$-compactification where $G(\le) = \bigcap_{h\in\mathcal{S}} G_h$, $i(\mathcal{S}) = \mathcal{S}$ and $\mathcal{S} = \bigcap \mathcal{I}$.*

Proof. Suppose that there is a Hausdorff T_2-preorder compactification which is dominated by all the other Hausdorff T_2-preorder compactifications, then by Prop. 3.15 it is equivalent to a $\mathcal{S}$-compactification where $\mathcal{S} \subset \mathcal{F}$ is such that $G(\le) = \bigcap_{h\in\mathcal{S}} G_h$.

By Prop. 3.17 $\mathcal{S}$ can be chosen such that $\mathcal{S} = i(\mathcal{S})$, thus belonging to $\mathcal{I}$. Clearly, $\bigcap \mathcal{I} \subset \mathcal{S}$ because $\mathcal{S} \in \mathcal{I}$. Suppose that $\mathcal{H} \in \mathcal{I}$ and that $f \in \mathcal{F}$, $f \notin \mathcal{H} = i(\mathcal{H})$. This means that f is not extendable as a continuous isotone function to the $\mathcal{H}$-compactified space. If C is the continuous isotone map from the $\mathcal{H}$-compactified space to the $\mathcal{S}$-compactified space (as the $\mathcal{S}$-compactification is dominated by all the other compactifications) one has that if f were extendable to the $\mathcal{S}$-compactified space then by pullbacking the extension to the $\mathcal{H}$-compactified space through C one would get an extension in the $\mathcal{H}$-compactified space. The contradiction proves that $f \notin i(\mathcal{S}) = \mathcal{S}$ thus $\mathcal{S} \subset \mathcal{H}$, and finally $\mathcal{S} \subset \bigcap \mathcal{I}$. □

Remark 3.22. The smallest compactification does not necessarily exist. For instance, if E is non-compact and endowed with the discrete preorder, the $\mathcal{C}$-compactification dominates over the $\mathcal{C}^-$-compactification and the $\mathcal{C}^+$-compactification (see Remark 3.13), indeed $\mathcal{C}^\mp \subset \mathcal{C}$ see Prop. 3.16. Stated in another way, the one-point compactification endowed with the discrete preorder dominates over that in which the added point is less (resp. greater) than any other point (indeed, the former has a smaller preorder). However, $\mathcal{C}^+$ is not contained in $i(\mathcal{C}^-)$ and conversely, thus the $\mathcal{C}^-$– and $\mathcal{C}^+$– compactifications differ. Actually, it is easy to realize that they are minimal, thus there is no smallest compactification.

4. Conclusions

We have investigated the compactification of topological preordered spaces, showing the existence of a largest Hausdorff T_2-preorder compactification for every T_2-preordered Tychonoff space for which the preorder is represented by the continuous isotone functions. An interesting subclass of this family is that of locally compact σ-compact Hausdorff T_2-preordered spaces [19]. It turns out that this largest compactification is essentially the Stone-Čech compactification endowed with a suitable preorder. It can be characterized as the Hausdorff T_2-preorder compactification for which all the continuous function can be continuously extended and the continuous isotone function do so preserving the isotone property. If the preorder is an order or the quotient space is a completely regularly ordered space it is also possible to show a clean relation with Nachbin's T_2-order compactification.

We have considered the problem of identifying the smallest Hausdorff T_2-preorder compactification whenever it exists. We have shown that it corresponds necessarily to the compactification obtained demanding the extendibility of a suitable set of continuous isotone functions. Generically, this set $\mathcal{S}$ is expected to be strictly included in the full set $\mathcal{F}$ of continuous isotone functions with value in [0,1].

The approach followed in this work relies on the study of continuous isotone functions and their extension properties. We close noting that filter approaches are also possible. For instance Choe and Park [2] have constructed a Wallman type preorder compactification which has been subsequently extensively investigated in [9–12, 14] together with some variations. For instance, in [10] the authors show that it is possible to obtain the Nachbin compactification from the Wallman compactification by identifying the points that take the same value on continuous isotone functions. We have followed a similar procedure to show that the Nachbin compactification nE can be obtained from the same functional quotient starting from βE.

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E. Minguzzi (ettore.minguzzi@unifi.it)
Dipartimento di Matematica Applicata "G. Sansone", Università degli Studi di Firenze, Via S. Marta 3, I-50139 Firenze, Italy.