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Epimorphisms and maximal covers in categories of compact spaces

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Abstract

The category $\mathcal{C}$ is "projective complete" if each object has a projective cover (which is then a maximal cover). This property inherits from $\mathcal{C}$ to an epireflective full subcategory $\mathcal{R}$ provided the epimorphisms in $\mathcal{R}$ are also epi in $\mathcal{C}$. When this condition fails, there still may be some maximal covers in $\mathcal{R}$. The main point of this paper is illustration of this in compact Hausdorff spaces with a class of examples, each providing quite strange epimorphisms and maximal covers. These examples are then dualized to a category of algebras providing likewise strange monics and maximal essential extensions.

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1. Introduction

In a category, an essential extension of an object A is a monomorphism $A \xrightarrow{m} B$ for which km monic implies k monic. In recent work [3], the authors have considered the inheritance from a category $\mathcal{C}$ to a monocoreflecive subcategory $\mathcal{V}$ of the property that each object has a unique maximal essential extension. The hypothesis "each monic in $\mathcal{V}$ is also monic in $\mathcal{C}$" was crucial. (The property was deployed to similar ends in [9].) This paper is largely directed at exhibiting in a concrete setting some pathology which can occur in the absence of these hypotheses.

But we shall operate "in dual", as we now describe briefly, and sketch a return to essential extensions in the final §5.

In a category, a cover of the object X is an epimorphism $Y \xleftarrow{g} X$ for which gf epi implies that f is epi. (This definition is dual to "essential extension"). Any projective cover is also a unique maximal cover (2.3). But there are categories with no projectives, and still every object has a unique maximal cover ([3], in dual.)

In compact Hausdorff spaces, Comp, epis are onto and every object has a projective cover (the Gleason cover). For an epireflective subcategory $\mathcal{R}$ of Comp, $\mathcal{R}$ has a non-void projective if and only if epis in $\mathcal{R}$ are onto (3.5) and then the projective covers from Comp are projective covers in $\mathcal{R}$ (3.2).

We begin with a necessary discussion of simple categorical preliminaries, proceed to Comp and two specific epireflective subcategories, then extract what little can be said for an epireflective $\mathcal{R}$ in general. Penultimately, we consider a strongly rigid $E \in$ Comp and the epireflective subcategory $\mathcal{R}(E)$ which E generates. There are epis not onto, and any nonconstant $E \longleftarrow \{0,1\}$ is a maximal cover. Finally, we sketch the dualization of this to a category of algebras, in which any proper $C(E) \longrightarrow \mathbb{R}^2$ is a maximal essential extension.

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2. Preliminaries

The context for 2.1 - 2.7 is a fixed category with no hypotheses at all before 2.4. In the following, $g, h, k, \ldots$ are assumed to be morphisms. The terms "morphism" and "map" will be interchangeable.

Definition 2.1.

(a) A morphism g is an epimorphism (epi) if $hg = kg$ implies $h = k$.

(b) The map g is "covering" if epi, and gf epi implies f epi. (Such g could also be called essential epi (or perhaps co-essential epi).) A cover of object X is a pair (X, g) with $Y \xleftarrow{g} X$ covering. Covers of X, (Y, g) and (Y', g') are equivalent if there is an isomorphism h with $g'h = g$.

(c) Object Y is cover-complete if, (Z, k) a cover of Y implies k is an isomorphism. A maximal cover of X is a cover (Y, g) with Y cover complete. A unique maximal cover of X is a maximal cover which is equivalent to any other maximal cover of X.

(d) Object P is projective if whenever $X \xleftarrow{h} P$ and $X \xleftarrow{g} Y$ is an epi, then there is $Y \xleftarrow{f} P$ with $gf = h$. A projective cover is a cover (P, p) with P projective.

(e) The category is called projectively complete if every object has a projective cover, and (weaker) is said to have enough projectives if for each object X there is $X \xleftarrow{f} P$, f epi and P projective.

The following two elementary propositions are, except for 2.2 (d) and perhaps 2.3 (b), proved (in dual) in [1], 9.14, 9.19, 9.20.

Proposition 2.2.

(a) *An isomorphism is covering.*
(b) *The composition of two covering maps is covering.*
(c) *If g and gf are covering, then f is covering.*
(d) *If gf is covering and f is epi, then f is covering.*

Proof.

(d) Given such gf and f, suppose fh is epi. Note that g is epi (because gf is). So, $g(fh)$ is epi, and $g(fh) = (gf)h$ shows h is epi, since gf is covering.

□

Proposition 2.3.

(a) *A projective object is cover-complete.*
(b) *A projective cover is a unique maximal cover.*

Proof.

(b) Suppose (P, p) is a projective cover of X. It is a maximal cover by (a). If (Y, g) is another cover of X, there is k with $gk = p$ (since P is projective and g is epi). By 2.2 (c), k is covering, thus an isomorphism if Y is cover-complete.

□

To proceed further, we require assumptions.

Two Hypotheses 2.4. *(to be invoked selectively). Let $\mathcal{C}$ be a category, and $\mathcal{R}$ a subcategory (always assumed full and isomorphism-closed).*

The first condition is on $\mathcal{C}$ alone, and is "the other face" of 2.2 (d):

(F°) *If gf is covering and f is epi, then g is covering*

The second condition is on $\mathcal{R} \subset \mathcal{C}$, and is the (frequently invalid) converse to the obvious truth "Any $\mathcal{C}$-epi between $\mathcal{R}$-objects is $\mathcal{R}$-epi":

(S°) *Any $\mathcal{R}$-epi is $\mathcal{C}$-epi.*

The point of this paper is, in the presence of (F°), what happens when (S°) holds (2.7 and §3), and especially what can happen when it fails (§5 , §6).

Proposition 2.5. *If $\mathcal{C}$ has enough projectives (in particular if $\mathcal{C}$ is projectively complete), then $\mathcal{C}$ satisfies (F°).*

Proof. Consider $X \xleftarrow{g} Y \xleftarrow{f} Z$ with gf covering and f epi. Since gf is epi, so is g. Suppose $Y \xleftarrow{t} T$ has gt epi; we want t epi. Take $T \xleftarrow{e} P$ epi with P projective. There is $Z \xleftarrow{h} P$ with $fh = te$ (since f is epi).Then, $(gf)h = g(fh) = g(te) = (gt)e$. The last term is epi, and so also the first term. Thus h is epi (since gf is covering), and so also fh. Since $fh = te$, t is epi. □

Proposition 2.6. *Suppose* (S°)*. If* X*,* $Y \in \mathcal{R}$*, and* $X \xleftarrow{g} Y$ *is* $\mathcal{C}$*-covering, then* g *is* $\mathcal{R}$*-covering.*

Proof. Suppose given $X \xleftarrow{g} Y$ as stated, and $Y \xleftarrow{f} Z$ with $Z \in \mathcal{R}$ and gf R-epi. Then gf is $\mathcal{C}$-epi (by (S°)), so f is $\mathcal{C}$-epi (since g is $\mathcal{C}$-covering), thus also $\mathcal{R}$-epi (as desired). □

We say $(\mathcal{R}, r)$ is epireflective in $\mathcal{C}$ if $\mathcal{R}$ is a subcategory of $\mathcal{C}$, and for each $Y \in \mathcal{C}$ there is $rY \in \mathcal{R}$ (the reflection) and epi $rY \xleftarrow{r_Y} Y$ (the reflection map) for which, whenever $X \xleftarrow{f} Y$ with $X \in \mathcal{R}$, there is $\bar{f}$ with $\bar{f} r_Y = f$. (See [11] for a full account of the theory of epireflective subcategories).

Proposition 2.7. *Suppose that* $(\mathcal{R}, r)$ *is epireflective in* $\mathcal{C}$*, and satisfies* (S°)*.*

(a) *If* P *is projective in* $\mathcal{C}$*, then* rP *is projective in* $\mathcal{R}$*.*
(b) *Suppose further that* $\mathcal{C}$ *satisfies* (F°)*. If* $X \in \mathcal{R}$*, and* (P, p) *is a projective cover in* $\mathcal{C}$ *of* X*, then* $(rP, \bar{p})$ *is a projective cover in* $\mathcal{R}$ *of* X*.*
(c) *If* $\mathcal{C}$ *is projectively complete, then so is* $\mathcal{R}$ *(with projective covers as in (b)).*

Proof.

(c) (from (b)). 2.5 says $\mathcal{C}$ satisfies (F°), so (b) applies.
(a) Suppose given $\mathcal{R}$-epi $X \xleftarrow{g} Y$ and any $X \xleftarrow{f} rP$. By (S°), g is $\mathcal{C}$-epi, so there is f_1 with $gf_1 = fr_P$ (since P is $\mathcal{C}$-projective). Next, there is f_2 with $f_2 r_P = f_1$, and we have $fr_P = gf_1 = g(f_2 r_P) = (gf_2) r_P$. Since r_P is $\mathcal{C}$-epi, $f = gf_2$.
(b) By (a), rP is $\mathcal{R}$-projective. We need that the $\bar{p}$ in $\bar{p} r_P = p$ is $\mathcal{R}$-covering. Since r_P is epi, (F°) says that $\bar{p}$ is $\mathcal{C}$-covering, and thus $\mathcal{R}$-covering by 2.5.

□

Remark 2.8.

(a) [3], 1.2 shows (in dual) that if $\mathcal{C}$ has unique maximal covers, so does epireflective $\mathcal{R}$, assuming the conditions (S°) and (F°). The proofs above of 2.7 (a) and (b) are simplified versions of those in [3]. For 2.7 (c), the present 2.7 (new here) allows suppression of the hypothesis (F°).
(b) If in 2.7, $\mathcal{R}$ already contains every $\mathcal{C}$-projective, then 2.7 (a) and (b) simplify in the obvious way. This is the case for $\mathcal{C} =$ Comp, with $\mathcal{R}$ having (S°); see 3.2 below.

3. Compact Hausdorff Spaces

Comp is the category of compact Hausdorff spaces with continuous functions as maps. A map $X \xleftarrow{f} Y$ in Comp is called irreducible if $f(Y) = X$, but when $F \subsetneq Y$ (F closed), $f(F) \neq X$. The following is mostly due to Gleason

[6]. ((a) is a folk item. (e) follows from (d) and 2.4; it has a short direct proof, and is noted in [8], 2.5.)

Proposition 3.1. *In Comp:*

(a) *Epis are onto. (See comment after 3.3 below.)*
(b) *A map is covering iff it is irreducible.*
(c) *A space is projective iff it is extremally disconnected (every open set has open closure).*
(d) *Any object X has a projective cover (PX, p_X); Comp is projectively complete.*
(e) *(F°) holds.*

The notation (PX, p_X) is reserved for the rest of the paper; this will always denote the projective cover in Comp of $X \in$ Comp. Also, for brevity, we shall let ED stand for the class of extremally disconnected spaces in Comp.

(Considerable literature developed from Gleason's [6], with various new proofs, generalizations, and variants of the theory. See [2], [8], [14] and their bibliographies.)

Now consider a subcategory $\mathcal{A}$ of Comp (which can be identified with its object class). The family of all subobjects (resp., products) of spaces in $\mathcal{A}$ is denoted $S\mathcal{A}$ (resp., $P\mathcal{A}$). (Note that subobjects are closed subspaces.) Kennison [13] has shown that $\mathcal{R}$ is epireflective in Comp iff $\mathcal{R}$ is neither $\varnothing$ nor $\{\varnothing\}$ and $\mathcal{R} = SP\mathcal{R}$. For $\varnothing \neq X \in$ Comp, let $\mathcal{R}(X) = SP\{X\}$; this is the smallest epireflective subcategory containing X.

Let $\{0\}$ (resp., $\{0,1\}$) denote the space with one (resp. two) points. The smallest epireflective is $\mathcal{R}(\{0\}) = \{\varnothing, \{0\}\}$; here, $\{0\} \longleftarrow \varnothing$ is epi, so epis are not onto. We comment further on this shortly. The next largest is $\mathcal{R}(\{0,1\})$: if $\mathcal{R}$ is epireflective and not $\mathcal{R}(\{0\})$, there is $X \in \mathcal{R}$ with $|X| \geq 2$, thus $\{0,1\} \in \mathcal{R}$, so $\mathcal{R}(\{0,1\}) \subset \mathcal{R}$. Note that $\mathcal{R}(\{0,1\}) = \text{Comp}_\circ$, the class of compact zero-dimensional spaces [5], and ED $\subset \text{Comp}_\circ$ [7]. Thus, if $\mathcal{R}$ is epireflective and not $\mathcal{R}(\{0\})$, ED$\subset \mathcal{R}$.

Corollary 3.2. *Suppose $\mathcal{R}$ is epireflective and $\mathcal{R}$-epis are onto (i.e., $\mathcal{R} \subset$ Comp satisfies (S°)). Then $\mathcal{R}$ is projectively complete. In fact, for any $X \in \mathcal{R}$, the $\mathcal{R}$-(projective cover) is (PX, p_X).*

Proof. Apply 3.1, 2.4, and the discussion above. □

Proposition 3.3. *$\text{Comp}_\circ$-epis are onto. 3.2 applies to $\text{Comp}_\circ$.*

Proof. The following takes place in $\text{Comp}_\circ$

The only $\varnothing \longleftarrow Y$ has $Y = \varnothing$ and the map is the identity, which is epi, and technically onto. If $X \neq \varnothing$ then $X \longleftarrow \varnothing$ is not epi (since there are different

$$\{0,1\} \underset{k}{\overset{h}{\leftleftarrows}} X \text{).}$$

Suppose $X \neq \varnothing$, and $X \xleftarrow{g} Y$ is epi. Were g not onto, there would be $p \in X - g(Y)$, and clopen U with $p \notin U \supseteq g(Y)$. Then h constantly 1 and k the characteristic function of U has $h \neq k$ but $hg = kg$. □

(To show Comp-epis are onto, argue similarly using $[0,1]$ instead of $\{0,1\}$, and using complete regularity of X (i.e. the Tietze-Urysohn Theorem).)

Remark 3.4. We do not know if there is epireflective $\mathcal{R}$ different from $\mathrm{Comp}_\circ$ and Comp, for which epis are onto.

The following (closely related to [3], 4.1) shows that, failing "epis are onto", there are no $\neq \varnothing$ projectives. But there still may be some maximal covers, of at least two sorts, as the examples in §5 show.

Proposition 3.5. *Suppose (only) $\{0\} \in \mathcal{R}$. The following statements in $\mathcal{R}$ are equivalent.*

(a) *Epis are onto.*
(b) *$\{0\}$ is projective.*
(c) *There is a non-void projective.*

Proof. (b) $\Rightarrow$ (c) obviously, and (c) $\Rightarrow$ (b) because $\{0\}$ is a retract of any $X \neq \varnothing$, and a retract of a projective is projective.

(a) $\Rightarrow$ (b) because $\{0\}$ is projective in Comp, and if (a) holds, projective in $\mathcal{R}$.

(b) $\Rightarrow$ (a). If $X \xleftarrow{g} Y$ is an epi which is not onto, then there is $p \in X - g(Y)$, and for $X \xleftarrow{h} \{0\}$ defined as $h(0) = p$, there can be no $Y \xleftarrow{f} \{0\}$ with $gf = h$. □

Finally, we clarify the situation for $\varnothing$ and for $\mathcal{R}(\{0\})$.

Note the following for any $\mathcal{R} \subset \mathrm{Comp}$ with $\varnothing \in \mathcal{R}$.

(i) $\varnothing$ is the initial object of $\mathcal{R}$, i.e., for any $X \in \mathcal{R}$, there is unique $X \longleftarrow \varnothing$ (namely, the empty map).
(ii) $\varnothing$ is projective in $\mathcal{R}$.
(iii) If $X \longleftarrow \varnothing$ is epi in $\mathcal{R}$, then this is a projective cover.

Proposition 3.6. *Suppose (only) $\{0\} \in \mathcal{R} = S\mathcal{R}$. The following statements in $\mathcal{R}$ are equivalent.*

(a) $\mathcal{R} = \mathcal{R}(\{0\})$
(b) *$\{0\} \longleftarrow \varnothing$ is epi (and thus a projective cover).*
(c) *For any $X \in \mathcal{R}$, $X \longleftarrow \varnothing$ is epi (and thus a projective cover).*

Proof. The parenthetical remarks follow from the comments above.

(a) $\Rightarrow$ (c): $\varnothing \longleftarrow \varnothing$ is epi, and $\{0\} \longleftarrow \varnothing$ also, since the only map out of $\{0\}$ is the identity.

(c) $\Rightarrow$ (b): since $\{0\} \in \mathcal{R}$.

(b) $\Rightarrow$ (a): If $\mathcal{R} \neq \mathcal{R}(\{0\})$, then there is $X \in \mathcal{R}$ with $|X| \geq 2$, so $\{0,1\} \in \mathcal{R}$ (since $S\mathcal{R} = \mathcal{R}$). Then there are different $\{0,1\} \overset{h}{\underset{k}{\leftleftarrows}} \{0\}$ which compose equally with $\{0\} \longleftarrow \varnothing$, so the latter is not epi. □

Corollary 3.7. *$\mathcal{R}(\{0\})$ is projectively complete, with epis not onto, and is the only epireflective subcategory with these two properties.*

Proof. 3.6, (a) $\Rightarrow$ (c) yields the first statement. If epireflective $\mathcal{R}$ has epis not onto, then by 3.5, the only projective is $\varnothing$. If $\mathcal{R}$ is projectively complete, then the projective cover must be $X \longleftarrow \varnothing$. So these are epi, and 3.6 (c) $\Rightarrow$ (a) says $\mathcal{R} = \mathcal{R}(\{0\})$. □

4. When epis may not be onto

Consider $\mathcal{R} \subset$ Comp. We localize the condition "$\mathcal{R}$-epis are onto". Keep in mind that $\mathcal{R}$ might have no projectives (but any $Y \in \mathcal{R}\cap$ED is still Comp-projective).

Definition 4.1. "X has $e(\mathcal{R})$" means $X \in \mathcal{R}$, and whenever $X \xleftarrow{g} \cdot$ is epi in $\mathcal{R}$, then g is onto.

Proposition 4.2. *Suppose $\mathcal{R} = S\mathcal{R}$.*

(a) *If $X \in \mathcal{R}\cap Comp_\circ$, then X has $e(\mathcal{R})$.*
(b) *If $Y \in \mathcal{R}\cap ED$, then Y is cover-complete in $\mathcal{R}$.*

Proof.

(a) Identical to the proof of 3.3.
(b) If (Z, g) is an $\mathcal{R}$-cover of Y, then g is onto by (a), so there is f with $gf = id_Y$, since Y is Comp-projective, and $f \in \mathcal{R}$ (since Y, $Z \in \mathcal{R}$). So f is an $\mathcal{R}$-section. Also, by 2.2, f is an $\mathcal{R}$-covering map, thus $\mathcal{R}$-epi. So f is an $\mathcal{R}$-isomorphism, so is g, and therefore g is a Comp-isomorphism, thus a homeomorphism.

□

(The converse to 4.2 (a) fails, with $\mathcal{R} = Comp$. But see 4.5 below.)

Proposition 4.3. *Suppose X has $e(\mathcal{R})$.*

(a) *If $Y \in \mathcal{R}$ and $X \xleftarrow{g} Y$ is irreducible, then (Y, g) is an $\mathcal{R}$-cover of X.*
(b) *If also $PX \in \mathcal{R}$, and supposing $\mathcal{R} = S\mathcal{R}$, then (PX, p_X) is the unique maximal $\mathcal{R}$-cover of X.*

Proof. (a) As in the proof of 2.6, *mutatis mutandis*
(b) By (a), (PX, p_X) is an $\mathcal{R}$-cover, and PX is cover-complete. If (Y, g) is another $\mathcal{R}$-cover of X, then g is onto (by $e(\mathcal{R})$), and there is $Y \xleftarrow{f} PX$ with $gf = p_X$ (since PX is Comp-projective). If Y is cover-complete, f is a homeomorphism.

□

Corollary 4.4. *Suppose ED$\subset \mathcal{R} = S\mathcal{R}$. If $X \in \mathcal{R}\cap Comp_\circ$, then (PX, p_X) is the unique maximal $\mathcal{R}$-cover of X.*

Proof. 4.2 (a) and 4.3 (b). □

The following is a qualified converse to 4.2 (a).

Corollary 4.5. *Suppose that ED$\subset \mathcal{R} = S\mathcal{R}$. For $Y \in \mathcal{R}$, the following are equivalent.*

(a) *Y is ED.*
(b) *Y is cover-complete and $Y \in$Comp$_\circ$.*
(c) *Y is cover-complete and Y has $e(\mathcal{R})$.*

Proof. (a) $\Rightarrow$ (b): 4.2 (b) and ED$\subset$Comp$_\circ$.

(b) $\Rightarrow$ (c): 4.2 (a).

(c) $\Rightarrow$ (a): By 4.3 (b) (using ED$\subset \mathcal{R}$ now), (PY, p_Y) is the unique maximal $\mathcal{R}$-cover of Y, so if Y is cover-complete, p_Y is a homeomorphism. □

Here is one (more) triviality valid in (almost) any $\mathcal{R}$.

Proposition 4.6. *Suppose $\{0\} \in \mathcal{R}$. For any $X \in \mathcal{R}$, with $|X| \geq 1$, there are maps $X \xleftarrow{e} \{0\}$ (in $\mathcal{R}$). Such an e is $\mathcal{R}$-epi iff $|X| = 1$.*

Proof. Given such e, there is (the retraction) $X \xrightarrow[r]{} \{0\}$ with $re = id_{\{0\}}$, so e is a section. If $|X| = 1$, then e is onto, thus $\mathcal{R} - epi$. If e is $\mathcal{R}$-epi, it becomes an $\mathcal{R}$-isomorphism, thus a homeomorphism, so $|X| = 1$. □

5. Epireflectives with Epis not onto, and some maximal covers

First, in summary so far of the situation for $\mathcal{R}$ epireflective in Comp: If in $\mathcal{R}$, there are epis not onto, then there are no non-void projectives (3.5). That is the case for $\mathcal{R} = \{\varnothing, \{0\}\}$, but here we have the projective (thus unique maximal) covers $\varnothing \longleftarrow \varnothing$ and $\{0\} \longleftarrow \varnothing$ (3.6). If $\mathcal{R}$ contains the two-point space $\{0,1\}$ then $\mathcal{R} \supseteq$ Comp$_\circ$ and at least has unique maximal covers for $X \in$ Comp$_\circ$, namely the (PX, p_X) (4.4).

We now display a large class of such $\mathcal{R}$ with some very strange epis, and non-unique maximal covers. This will be the $\mathcal{R}(E) = SP\{E\}$, for E as follows.

A space E in Comp will be called strongly rigid if $|E| \geq 2$ and the only continuous $E \longrightarrow E$ are id_E and constants. Cook [4] has several of these, including a metric one M_1.

Note that if E is strongly rigid, then $\{0,1\} \subseteq E$ (since $|E| \geq 2$), E is connected (since a clopen $U \neq \varnothing$, E would yield $E \longrightarrow \{0,1\} \hookrightarrow E$), $|E| \geq c$ (since there are non-constant $E \twoheadrightarrow [0,1]$, using the Tietze-Urysohn Theorem), and $[0,1] \nsubseteq E$ (since $[0,1] \subseteq E$ would yield non-constant $E \longrightarrow [0,1] \hookrightarrow E$, and $[0,1]$ is not strongly rigid).

From Cook's examples, Trnková [15] and Isbell [12] have shown first, that if n is any cardinal, there is strongly rigid E with $|E| \geq n$, and second, that if there is no measurable cardinal, there is a proper class $\mathcal{E}$ of strongly rigid spaces for which, whenever $E_1 \neq E_2$ in $\mathcal{E}$, the only continuous $E_1 \longrightarrow E_2$ are constants (and thus, for $E_1 \neq E_2$ in $\mathcal{E}$, neither of $\mathcal{R}(E_1)$ and $\mathcal{R}(E_2)$ contains the other).

Now let E be any strongly rigid space. In the following, terms epi, cover, ... refer to $\mathcal{R}(E)$.

Of course 4.4 and 4.5 apply here. On the other hand,

Proposition 5.1. *Let F be a closed subspace of E. Label the inclusion $E \xleftarrow{i_F} F$.*

(a) *i_F is epi iff $|F| > 1$.*

(b) *If $|F| = 2$, then (F, i_F) is a cover of E. If $F \in$Comp$_\circ$ and (F, i_F) is a cover of E, then $|F| = 2$.*

(In the second part of (b), the supposition "$F \in$Comp$_\circ$" cannot be dropped, because $E \xleftarrow{id} E$ is a cover.)

Corollary 5.2. *Any nonconstant $E \xleftarrow{g} \{0,1\}$ is a maximal cover of E. Any maximal cover of E is equivalent to one of these. Two of these, with g and g', are equivalent covers of E iff $g(\{0,1\}) = g'(\{0,1\})$.*

In particular, (PE, p_E) is not a cover of E, and there are at least $|E| \geq c$ non-equivalent maximal covers of E;

Proof. (of 5.1)

(a) By 4.6, if $|F| = 1$, then i_F is not epi. Now suppose $|F| > 1$. Suppose $f, g \in \mathcal{R}(E)$ have common codomain - which might as well be supposed of the form E^I - and $fi_F = gi_F$, i.e., $f|_F = g|_F$. Then, for any projection $E^I \xrightarrow{\pi_i} E$, we have $\pi_i f|_F = \pi_i g|F$. We want $f = g$, which is equivalent to $\pi_i f = \pi_i g \ \forall \ i \in I$.

Let $i \in I$. Then each of $\pi_i f$ and $\pi_i g$ is id_E or constant. If $\pi_i f = id_E$, then $\pi_i f|_F$ is not constant (since $|F| \geq 2$), so $\pi_i g|_F$ is not constant, so $\pi_i g = id_E$ also. If $\pi_i f$ is constant, say c, then $\pi_i f|_F = c$ also. So $\pi_i g|_F = c$, and since $|F| \geq 2$, $\pi_i g = c$.

(b) Suppose $|F| = 2$. By (a), i_F is epi. Suppose $i_F f$ is epi. Then f is onto F (since if not, $|range(f)| = 1$, since $|F| = 2$, but then $|range(i_F f)| = 1$ and $i_F f$ is not epi, by 4.6. So f is epi.

Suppose $F \in$Comp$_\circ$.If there are different p_0, p_1, $p_2 \in F$, let $F \xleftarrow{f} \{0,1\}$ be $f(i) = p_i$. Then f is not epi (by 4.2 (a)), but $i_F f$ is epi by (a) above.

□

Proof. (of 5.2)

If $E \xleftarrow{g} \{0,1\}$ is nonconstant, it is a cover because $F \equiv \{g(0), g(1)\} \xleftarrow{g} \{0,1\}$ is a homeomorphism, and thus a maximal cover because F is cover-complete (being ED 4.2).

Suppose $E \xleftarrow{h} Y$ is a maximal cover. Then h is epi, thus nonconstant (6.1). So there are p_0, $p_1 \in Y$ with $h(p_0) \neq h(p_1)$. Define $Y \xleftarrow{f} \{0,1\}$ as $f(i) = p_i$. So hf is a covering-map (by the preceding paragraph), thus f is a covering map (2.2 (d)). Since Y is cover-complete, f is a homeomorphism, so (Y, h) and $(\{0,1\}, hf)$ are equivalent.

Now suppose $E \xleftarrow{g,g'} \{0,1\}$ are non-constant. There are two homeomorphisms h of $\{0,1\}$, the identity and "interchange 0 and 1". And, $range(g) = range(g')$ iff $g' = gh$ for one of these h. □

Remark 5.3. Cook's specific strongly rigid M_1 has these further features:

M_1 has a countable infinity of disjoint subcontinua; if K is any proper subcontinuum of M_1, the only maps $M_1 \longleftarrow K$ are inclusion and constants. (See [4]). Then in the category $\mathcal{R}(M_1)$, in 5.1 and 5.2, $E = M_1$ may be replaced by any proper subcontinuum K of M_1 (as the proofs there show).

6. An application to lattice-ordered groups

We now convert the situations of maximal covers in $\mathcal{R} \subset$Comp to situations of maximal essential extensions in subcategories of a category of algebras. We use terminology categorically dual to the items in 2.1 (a) - (e), respectively, namely (a) monic, (b) essential extension, (c) essentially complete, maximal essential extension (or, essential completion), (d) injective, injective hull, (e) injectively complete.

The category of algebras is W^*, the category of archimedean lattice-ordered groups with distinguished strong order unit, and ℓ-group homomorphisms carrying unit to unit. W^* has monics one-to-one, and is injectively complete; see [3]. Consequently, the dual of 2.7 applies to W^*.

For $X \in$Comp, the continuous real-valued functions $C(X)$, with unit the constant function 1, is a W^*-object, and we have the functor $W^* \xleftarrow{C}$ Comp: for $X \xleftarrow{\tau} Z$ in Comp, $C(X) \xrightarrow{C\tau} C(Z)$ is $C\tau(f) = f \circ \tau$. This has a left adjoint, the Yosida functor: For each $G \in W^*$, there is $YG \in$ Comp and $G \longrightarrow C(YG)$ monic in W^*; for each $G \xrightarrow{\varphi} H$ in W^*, there is unique $YG \xleftarrow{Y\varphi} YH$ in Comp "realizing φ" as $\varphi(g) = g \circ Y\varphi$. Note that $YC(X) \simeq X$, and that φ is one-to-one iff $Y\varphi$ is onto. (See [10]).

Basic features of (Y, C), and some diagram-chasing, convert the situations in Comp discussed in previous sections to "dual" situations in W^*, as follows. (We omit the calculations).

Suppose $\mathcal{R}$ is epireflective in Comp, and $\{0,1\} \in \mathcal{R}$ (so $\text{Comp}_\circ \in \mathcal{R}$). For brevity, set $^*\mathcal{R} = \{G \in W^* | YG \in \mathcal{R}\}$.

Proposition 6.1. (a) *$^*\mathcal{R}$ is monocoreflective in W^*.*

(b) *$C(X) \xrightarrow{\varphi} H$ is monic in $^*\mathcal{R}$ iff $X \xleftarrow{Y\varphi} YH$ is epi in $\mathcal{R}$.*
(c) *$^*\mathcal{R}$ has an injective other than $\{0\}$ iff monics in $^*\mathcal{R}$ are one-to-one iff $\mathcal{R}$-epis are onto. When this occurs, $^*\mathcal{R}$ is injective-complete, with injective hulls $G \longrightarrow C(YG) \longrightarrow C(P(YG))$.*
(d) *If X is ED, then $C(X)$ is essentially complete in $^*\mathcal{R}$.*

(e) *If* $X \in Comp_\circ$*, then* $C(D) \xrightarrow{Cp_X} C(PX)$ *is the unique maximal essential extension of* $C(X)$ *in* ${}^*\mathcal{R}$*.*

Now consider, as in §5, strongly rigid $E \in Comp$ and its generated epireflective $\mathcal{R}(E)$. By 5.1 and 6.1 (b), ${}^*\mathcal{R}(E)$ has monics which are not one-to-one, and thus no $\neq \{0\}$ injectives. 6.1 (d) and (e) hold in ${}^*\mathcal{R}(E)$.

Note that $\{0,1\} \in$ Comp has $C(\{0,1\}) = \mathbb{R}^2 \in W^*$, the self-homeomorphisms of $\{0,1\}$ are the identity and "interchange points", and these correspond to the only self-isomorphisms of $\mathbb{R}^2$, which are the identity, and $H(x,y) = (y,x)$. From 5.2 we obtain

Corollary 6.2. *In* ${}^*\mathcal{R}(E)$*, the maximal essential extensions of* $C(E)$ *are exactly the* W^**-surjections* $C(E) \xrightarrow{\varphi} \mathbb{R}^2$*. Two of these,* φ *and* φ'*, are equivalent iff either* $\varphi = \varphi'$*, or* $\varphi' = \varphi H$*.*

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