A RAFU linear space uniformly dense in $C \left[ a, b \right]$  

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Abstract

In this paper we prove that a RAFU (radical functions) linear space, $\mathcal{C}$, is uniformly dense in $C \left[ a, b \right]$ by means of a $S$-separation condition of certain subsets of $[a,b]$ due to Blasco-Moltó. This linear space is not a lattice or an algebra.

Given an arbitrary function $f \in C \left[ a, b \right]$ we will obtain easily the sequence $(C_n)_n$ of $\mathcal{C}$ that converges uniformly to $f$ and we will show the degree of uniform approximation to $f$ with $(C_n)_n$.

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1. Introduction

Let $K$ be a compact Hausdorff space. The Kakutani-Stone Theorem [10] gives a necessary and sufficient condition for the density of a lattice of $C(K)$ in the topology of the uniform convergence on $K$. The Stone-Weierstrass Theorem [7] provides a necessary and sufficient condition under which an algebra of $C(K)$ is uniformly dense. Nevertheless, the above conditions are not sufficient to ensure the uniform density of a linear space of $C(K)$. Tietze [5], Jameson [4], Mrowka [11], Blasco-Moltó [6], Garrido-Montalvo [8] and recently Gassó-Hernández-Rojas [9] have studied the uniform approximation for linear spaces.

In the Section 2 we will construct a RAFU (Radical functions) linear space, $\mathcal{C}$, in $C \left[ a, b \right]$ and we will prove that $\mathcal{C}$ is uniformly dense in $C \left[ a, b \right]$ by using a $S$-separation condition according to Blasco-Moltó [6]. We will also see that the uniform density of $\mathcal{C}$ in $C \left[ a, b \right]$ is not a consequence of the results given by Kakutani-Stone, Stone-Weierstrass, Tietze, Jameson, or Mrowka.
It is true that Blasco-Moltó showed an example of a linear space, $\mathcal{F}$, uniformly dense in $C[0, 1]$ by using the S-separation condition but some questions were not studied: the linear combinations of elements belonging to $\mathcal{F}$ which approximate uniformly every $f \in C[0, 1]$ and the degree of uniform approximation that $\mathcal{F}$ provides were unknown. In the Section 3 we will solve these problems by using the RAFU linear space $\mathcal{F}$. Moreover, this linear space $\mathcal{F}$ can be used as an example of approximation by series in the work of Gassó-Hernández-Rojas.

2. A RAFU linear space uniformly dense in $C[a, b]$

For each $n \in \mathbb{N}$ we consider the partition $P = \{x_0, x_1, ..., x_n\}$ of $[a, b]$ with $x_j = a + j \cdot \frac{b-a}{n}$, $j = 0, ..., n$ and we define in $[a, b]$ the functions

\[(2.1) \quad C_n(x) = k_1 + \sum_{i=2}^{n} (k_i - k_{i-1}) \cdot F_n(x_{i-1}, x)\]

where $\{k_i\}_{i=1}^{n}$ are a family of real arbitrary numbers and

\[(2.2) \quad F_n(x_{i-1}, x) = \frac{2n^{-1} \sqrt{x_{i-1} - x_0} + 2n^{-1} \sqrt{a - x_{i-1}}}{2n^{-1} \sqrt{x_n - x_{i-1}} + 2n^{-1} \sqrt{x_{i-1} - x_0}}, \quad i = 2, ..., n\]

We designate by $\mathcal{F}_n$ the subset of $C[a, b]$ formed by the functions $C_n$ and we also denote by $\mathcal{F}$ the set $\mathcal{F} = \cup_{n \in \mathbb{N}} \mathcal{F}_n$.

**Proposition 2.1.** The set $\mathcal{F}$ is a linear space included in $C[a, b]$.

**Proof.** It is clear that $\mathcal{F} \subset C[a, b]$. In the first place it is easy to check that $\mathcal{F}_n$ is a linear space $n$-dimensional because $n$ is fixed and hence the values $\{x_i\}_{i=0}^{n}$ are the same points. Moreover, a basis of $\mathcal{F}_n$ is $\{1, F_n(x_1, x), ..., F_n(x_{n-1}, x)\}$.

$\mathcal{F}$ is a linear space. Let $C_p$ and $C_q$ be two elements belonging to $\mathcal{F}$. Then, $C_p \in \mathcal{F}_{p, r}$, $r \in \mathbb{N}$ and $C_q \in \mathcal{F}_{q, s}$, $s \in \mathbb{N}$ by considering zero the coefficients $(k_i - k_{i-1})$ of the functions $F_n(x_{i-1}, x)$ that do not appear on the expressions of $C_p$ or $C_q$. In particular, $C_p$ and $C_q$ belong to the linear space $\mathcal{F}_{p, q}$ and, of course, $C_p + C_q \in \mathcal{F}$. Finally, it is immediate to check that if $C_p \in \mathcal{F}$ and $\lambda \in \mathbb{R}$ then $\lambda \cdot C_p \in \mathcal{F}$. \[\Box\]

**Definition 2.2.** A RAFU linear space is a linear space whose basis is formed by radical functions of the type (2.2). We will say that $\mathcal{F}$ is a RAFU linear space.

The theorems of uniform approximation in $C(K)$ for lattices are known as Kakutani-Stone theorems (the interested reader can see [10], [7], [12]).

The family $\mathcal{F}$ is not a lattice. In fact, in the interval $[-1, 1]$ the function $C(x) = \sqrt{x} \in \mathcal{F}$ but $|C(x)| \notin \mathcal{F}$ because at $x = 0$ its side derivatives do not have the same sign. Therefore, the family $\mathcal{F}$ does not satisfy the Kakutani-Stone theorems.

The theorems of uniform approximation in $C(K)$ for algebras are known as Stone-Weierstrass theorems (the interested reader can see [7], [12]).
A simple count proves that $\mathcal{C}$ is not an algebra, therefore the set $\mathcal{C}$ does not verify the Stone-Weierstrass theorems.

Let $X$ be a topological space and let $C^*(X)$ be the set consisting of all bounded continuous functions and let $C(X)$ be the set consisting of all continuous functions.

**Definition 2.3.** Let $\mathcal{F}$ be a family of $C^*(X)$. We say that

1. A zero-set in $X$ is a set of the form $Z(f) = \{x \in X : f(x) = 0\}$ with $f \in C^*(X)$.
2. The Lebesgue-sets of $f \in C(X)$ are the sets $L_\alpha(f) = \{x \in X : f(x) \leq \alpha\}$ and $L_\beta(f) = \{x \in X : f(x) \geq \beta\}$ where $\alpha$ and $\beta$ are real numbers.
3. $\mathcal{F}$ $S_1$-separates the subsets $A$ and $B$ of $X$ when there is $f \in \mathcal{F}$, $0 \leq f \leq 1$ such that $f(x) = 0$ if $x \in A$ and $f(x) = 1$ if $x \in B$.
4. (Blasco-Moltó [6]). $\mathcal{F}$ $S'_{\varepsilon}$-separates the subsets $A$ and $B$ of $X$ if for each $\delta > 0$, there is $f \in \mathcal{F}$ such that $0 \leq f \leq 1$ for every $x \in X$, $f(A) \subset [0, \delta]$ and $f(B) \subset [1 - \delta, 1]$.
5. (Garrido-Montalvo [7]). $\mathcal{F}$ $S'_{\varepsilon}$-separates the subsets $A$ and $B$ of $X$ if for each $\delta > 0$, there is $f \in \mathcal{F}$ such that $-\delta \leq f \leq 1 + \delta$ for every $x \in X$, $f(A) \subset [-\delta, \delta]$ and $f(B) \subset [1 - \delta, 1 + \delta]$.
6. Given a series of continuous functions $\sum_{i \in I} f_i$ on $X$, the series is locally convergent, for every $x \in X$, if there is a neighborhood $U$ of $x$ such that the series converges uniformly on $U$. For $E \subset C(X)$, $\sum(E)$ is the set of all $f \in C(X)$ such that $f = \sum_{i \in I} f_i$ with $f_i \in E$ for every $i \in I$ and $\sum_{i \in I} f_i$ is a locally convergent series. $\sum(E)$ denotes the uniform closure of $\sum(E)$.

**Theorem 2.4** (Tietze [5], Mrowka [11]). Let $\mathcal{F}$ be a linear space of $C^*(X)$. $\mathcal{F}$ is uniformly dense in $C^*(X)$ if and only if $\mathcal{F}$ $S_1$-separates every pair of disjoint zero-sets in $X$.

**Theorem 2.5** (Jameson [4]). Let $\mathcal{F}$ be a linear space of $C^*(X)$. $\mathcal{F}$ is uniformly dense in $C^*(X)$ if and only if $\mathcal{F}$ $S_1$-separates every pair of disjoint closed subsets in $X$.

By the properties of the functions of the linear space $\mathcal{C}$ it is possible to deduce that we cannot apply to $\mathcal{C}$ the results of Tietze, Mrowka or Jameson.

**Theorem 2.6** (Blasco-Moltó [6]). Let $X$ be a topological space. A linear space $\mathcal{F}$ of $C^*(X)$ is uniformly dense in $C^*(X)$ if and only if $\mathcal{F}$ $S'_\varepsilon$-separates every pair of disjoint zero-sets in $X$.

We go to see that we can apply this theorem to prove the uniform density of $\mathcal{C}$ in $C[a, b]$. Let us consider in $[a, b]$ the step function defined by $f(x) = \begin{cases} k_1 & a \leq x \leq x_1 \\ k_2 & x_1 < x \leq b \end{cases}$, $k_1, k_2 \in \mathbb{R}$. If we calculate, for each $n \in \mathbb{N}$, the expressions of the radical functions $c_n(x) = M_n + N_n \cdot \sqrt[n]{b-x_1}$ that are obtained by the conditions $c_n(a) = k_1$ and $c_n(b) = k_2$, we obtain $N_n = \sqrt[n]{\frac{k_2-k_1}{k_2-x_1} + 2n \cdot \sqrt[n]{b-x_1} - a}$. 

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and \( M_n = k_1 + \frac{(k_2 - k_1) - 2^{n+1} \sqrt{x_1} - x_1}{2^{n+1} \sqrt{b - x_1} + 2^{n+1} \sqrt{b - x_1} - 0} \). In this case, an elementary count shows that the sequence \( (c_n) \) satisfies \( \lim_{n \to +\infty} c_n(x) = \begin{cases} k_1 & a \leq x < x_1 \\ \frac{k_1 + k_2}{2} & x = x_1 \\ k_2 & x < x \leq b \end{cases} \).

Now, we will consider an arbitrary step function in \([a, b] \)

\[
(2.3) \quad f(x) = k_1 \cdot \chi_{[x_0, x_1]} + \sum_{i=2}^{m} k_i \cdot \chi_{[x_{i-1}, x_i]}
\]

where \( k_i \in \mathbb{R}, i = 1, ..., m \) and \( x \in [x_0 = a, x_m = b] \). By an elementary count we can write (2.3) in the form \( f(x) = \sum_{i=1}^{m} f_i(x) \) where \( f_1(x) = k_1 \cdot \chi_{[x_0, x_m]} \) and \( f_p(x) = (k_p - k_{p-1}) \cdot \chi_{[x_{p-1}, x_m]}, p = 2, ..., m \).

For each \( f_p \) we construct its sequence of radical functions \( (c_{p,n}) \). For every \( n \in \mathbb{N} \), the corresponding sequence for \( f_1 \) is \( c_{1,n}(x) = k_1 \) and for \( f_p, p > 1 \), we obtain \( c_{p,n}(x) = M_{p,n} + N_{p,n} \cdot 2^{n+1} \sqrt{x - x_{p-1}} \) where \( N_{p,n} \) and \( M_{p,n} \) are given by \( N_{p,n} = \frac{2^{n+1} \sqrt{x_m - x_{p-1}} + 2^{n+1} \sqrt{x_{p-1} - x_0}}{k_p - k_{p-1}} \) and \( M_{p,n} = \frac{2^{n+1} \sqrt{x_m - x_{p-1}} + 2^{n+1} \sqrt{x_{p-1} - x_0}}{k_p - k_{p-1}} \).

Finally, if we denote by \( (C_{m,n})_n \) to the sequence \( C_{m,n}(x) = \sum_{i=1}^{n} c_{i,n}(x) \) then \( \lim_{n \to +\infty} C_{m,n}(x) = \begin{cases} f(x) & x \in [x_0, x_m] \setminus \{x_1, x_2, ..., x_{m-1}\} \\ f_p(x) & x = x_p, p = 1, ..., m - 1 \end{cases} \) by elementary properties of the limits.

**Proposition 2.7.** Let \( f \) be the function defined by (2.3). For any \( \beta > 0 \) such that \( (x_i - \beta, x_i + \beta) \cap (x_j - \beta, x_j + \beta) = \emptyset \) where \( i \neq j \) and \( i, j \in \{1, ..., m - 1\} \) the limit \( \lim_{n \to +\infty} C_{m,n} = f \) is uniform on \([x_0, x_1 - \beta] \cup [x_1 + \beta, x_2 - \beta] \cup ... \cup [x_{m-1} + \beta, x_m]\).

**Proof.**

1st part. It verifies that \( \lim_{n \to +\infty} 2^{n+1} \sqrt{x} = \begin{cases} -1 & x \in [-M, -\frac{1}{K}] \\ 1 & x \in [\frac{1}{K}, M] \end{cases} \)

where \( M \) and \( K \) are large positive real numbers. Moreover, the limit becomes uniform.

The function \( h_n(x) = 2^{n+1} \sqrt{x} \) is strictly increasing on \( \mathbb{R} \), therefore \( h_n(-M) \leq h_n(x) \leq h_n(-\frac{1}{K}) \) for \( x \in [-M, -\frac{1}{K}] \) and fixed \( \epsilon > 0 \) it is possible to find \( n_{M,K} \in \mathbb{N} \) such that if \( n \geq n_{M,K} \) then \( -1 - \epsilon < h_n(-M) \leq h_n(x) \leq h_n(-\frac{1}{K}) < -1 + \epsilon \). In other words, \(|h_n(x) + 1| < \epsilon \). Analogous, we obtain \(|h_n(x) - 1| < \epsilon \) on \([\frac{1}{K}, M] \).

2nd part. Given a partition \( P = \{a = x_0, x_1, ..., x_m = b\} \) of \([a, b] \) with \( a < x_1 < ... < b \). For each \( n \in \mathbb{N} \) and \( p = 2, ..., m \) we define in \([a, b] \) the function

\[
F_n(x_{p-1}, m, x) = \frac{2^{n+1} \sqrt{x_{p-1} - x_0} + 2^{n+1} \sqrt{x - x_{p-1}}}{2^{n+1} \sqrt{x_m - x_{p-1}} + 2^{n+1} \sqrt{x_{p-1} - x_0}}
\]
Then, it follows that \( \lim_{n \to \infty} F_n(x_{p-1}, 1, x) = \begin{cases} 0 & x < x_{p-1} \\ \frac{1}{2} & x = x_{p-1}, \ p = 2, ..., m \\ 1 & x > x_{p-1} \end{cases} \)
and these limits are uniform on \([x_0, x_1 - \beta] \cup [x_1 + \beta, x_2 - \beta] \cup \ldots \cup [x_{m-1} + \beta, x_m]\).

The first assertion is consequence of the elementary properties of the limits and the second is obtained by applying the first part and take into account that for each \( p = 2, ..., m \) only a root of \( F_n(x_{p-1}, m, x) \) depends upon \( x \).

**3rd part.** By the second part and the definitions of \( C_{m,n} \) and \( f \) we obtain the result which we want to prove. \( \square \)

**Proposition 2.8.** Let \( \beta > 0 \) be such that \( (x_i - \beta, x_i + \beta) \cap (x_j - \beta, x_j + \beta) = \emptyset \) where \( i \neq j \) and \( i, j \in \{1, ..., m-1\} \). Then, for all \( \varepsilon > 0 \), there exists \( n_0 \in \mathbb{N} \) such that for \( n > n_0 \) it follows that:

1. \( |C_{m,n}(x) - f(x)| < \left| k_{j+1} - k_j \right| + \varepsilon \)
2. \( |C_{m,n}(x) - (k_j \cdot (1 - \alpha) + k_{j+1} \cdot \alpha)| < \varepsilon \)

where \( x \in (x_j - \beta, x_j + \beta) \), \( j = 1, ..., m - 1 \) and \( \alpha, \alpha' \in (0, 1) \).

**Proof. 1st Part.** Let \( x \in (x_j - \beta, x_j + \beta) \) be, \( j = 1, ..., m - 1 \). By the Proposition 2.7 the sequence \( (F_n)_n \) converges uniformly to 1 as \( p - 1 < j \) and to 0 as \( p - 1 > j \).

Moreover there exists \( n_0 \in \mathbb{N} \) such that \( \forall n > n_0 \) the function \( (k_{j+1} - k_j) F_n(x_{p-1}, m, x) \) transforms the interval \( (x_j - \beta, x_j + \beta) \) into the interval \( (0, (k_{j+1} - k_j)) \). The rest is obtained by the elementary properties of the limits and the definition of \( C_{m,n}(x) \).

**2nd Part.** It is analogous to the 1st part by considering \( \forall n > n_0 \) the function \( (k_{j+1} - k_j) F_n(x_{p-1}, m, x) \) attains on \( (x_j - \beta, x_j + \beta) \) the values \( (k_{j+1} - k_j) \cdot \alpha \), \( \alpha, \alpha' \in (0, 1) \). \( \square \)

**Theorem 2.9.** The RA FU linear space \( \mathfrak{L} \) is uniformly dense in \( C[a, b] \).

**Proof.** Consider the family \( \mathcal{L} \) of all sets which are finite unions of disjoint compact intervals. First, we will prove that \( \mathfrak{L} \) \( S \)-separates every pair of disjoint sets of \( \mathcal{L} \). Clearly, it suffices to prove the following fact: Given \( \delta > 0 \) and the intervals \([\alpha_i, \beta_i]\), \( 1 \leq i \leq m \), \( m \geq 2 \), \( 0 \leq \alpha_j \leq \beta_j \leq \alpha_{j+1} < 1 \), there is a function \( f \) in \( \mathfrak{L} \) such that \( 0 \leq f(x) \leq 1 \) for every \( x \in [a, b] \), \( f([\alpha_i, \beta_i]) \subset [0, \delta] \) for \( i \) odd and \( f([\alpha_i, \beta_i]) \subset [1 - \delta, 1] \) for \( i \) even, \( 1 \leq i \leq m \).

Consider a partition \( P = (x_i)_{0}^{n} \) of \([a, b]\) with \( x_j = a + j \cdot \frac{b-a}{n} \), \( j = 0, ..., n \) such that \( \beta_j < x_p < \alpha_{j+1} \) for every \( j \) and some \( x_p \). We also consider a step function \( h \) defined in \([a, b]\) from the values \( x_j \) such that \( h(x) = 0 \) or \( h(x) = 1 \) for every \( x \in [a, b] \) but verifying that \( h([x_i, x_i]) = 0 \) when \( [\alpha_i, \beta_i] \subset [x_i, x_i] \) and \( i \) is odd, \( h([x_k, x_i]) = 1 \) when \([\alpha_i, \beta_i] \subset [x_k, x_i] \) and \( i \) is even, \( 1 \leq i \leq m \).

Fixed an appropriate value \( \beta \leq \min \left\{ \frac{|x_n - \beta|}{2}, \frac{|x_{n+1} - x_n|}{2} \right\} \) and given \( \delta > 0 \) we can choose suitable partitions of \([a, b]\) into \( 2kn \) intervals, if it was necessary for some \( k \in \mathbb{N} \), supporting the previous conditions and, by the propositions 2.7 and 2.8, we can obtain a function \( C_{2kn} \in \mathfrak{L} \) such that \( 0 \leq C_{2kn}(x) \leq \delta \).
1 and \( |C_{2kn} - h| < \delta \), that is to say, \( C_{2kn}([\alpha_i, \beta_i]) \subseteq [0, \delta] \) for \( i \) odd and \( C_{2kn}([\alpha_i, \beta_i]) \subseteq [1 - \delta, 1] \) for \( i \) even.

Next, we will prove that \( \mathcal{C} \) \( S \)-separates every pair of disjoint zero-sets \( Z_1 \) and \( Z_2 \) of \( [a, b] \). Since \( \mathcal{L} \) is a basis for the closed sets of \( [a, b] \) we have \( Z_1 = \cap \{ B \in \mathcal{L} : Z_1 \subseteq B \} \). As \( Z_2 \) is compact the family \( \{ Z_2 \} \cup \{ B \in \mathcal{L} : Z_1 \subseteq B \} \) does not have the finite intersection property. Therefore \( Z_2 \cap B_1 \cap ... \cap B_p = \emptyset \), for some \( B_i \in \mathcal{L}, Z_i \subseteq B_i, 1 \leq i \leq p \). Since \( \mathcal{L} \) is closed under finite intersections it follows that \( B' = B_1 \cap ... \cap B_p \in \mathcal{L}, Z_1 \subseteq B' \) and \( B' \cap Z_2 = \emptyset \). In the same way we find \( B'' \in \mathcal{L} \), such that \( Z_2 \subseteq B'' \) and \( B' \cap B'' = \emptyset \). Since \( \mathcal{C} \) \( S \)-separates \( B' \) and \( B'' \), by Blasco-Moltó’s Theorem, \( \mathcal{C} \) is uniformly dense in \( C[a, b] \). \( \square \)

The \( S \)-separation of subsets is equivalent to the \( S' \)-separation of subsets in linear spaces containing constant functions (Garrido-Montalvo [8]). Clearly \( \mathcal{C} \) contains the constant functions, therefore we can also deduce the uniform density of \( \mathcal{C} \) in \( C[a, b] \) by using the \( S' \)-separation condition of every pair of disjoint zero-sets in \( X \).

3. The degree of uniform approximation with the RAFU linear space

Blasco-Moltó [6] proved that the linear subspace \( \mathcal{F} \) of \( C[0,1] \) generated by the functions
\[
\{ \exp((x + \mu)^n) : \mu \in \mathbb{R}, x \in [0,1], n = 0, 1, 3, ..., 2k + 1, ... \}
\]
is uniformly dense in \( C[0,1] \), but the linear combinations which approximate uniformly a function \( f \in C[0,1] \) and the degree of uniform approximation that \( \mathcal{F} \) provides were not studied.

The following result has been proved recently in the XXII CEDYA-XII CMA [2] and solves these two problems by considering the linear space \( \mathcal{C} \).

**Theorem 3.1.** Let \( f \) be a continuous function defined on \( [a, b] \) and let \( P = \{ x_0 = a, x_1, ..., x_n = b \} \) be a partition of \( [a, b] \) with \( x_j = a + j \cdot \frac{b-a}{n}, j = 0, ..., n \). Then,
\[
\| C_n - f \| \leq \frac{M - m}{\sqrt{n}} + \omega \left( \frac{b - a}{n} \right), \quad n \geq 2
\]
where \( \| \cdot \| \) denotes the uniform norm, \( M \) and \( m \) are the maximum and the minimum of \( f \) on \( [a, b] \) respectively, \( \omega(\delta) \) its modulus of continuity and \( C_n(x) \) is defined for all \( x \in [a, b] \) and \( n \in \mathbb{N} \) by \( C_n(x) = f(x) + \sum_{j=2}^{n}[f(x_j) - f(x_{j-1})] \cdot F_n(x_{j-1}, x) \).

Let us observe that the values \( \{ k_i \}_{i=1}^{n} \) of (2.1) becomes \( \{ f(x_i) \}_{i=1}^{n} \) in this case.

**Theorem 3.2** (Gassó-Hernández-Rojas [9]). Let \( A \) be a subset of \( C(X) \) and \( E \) a linear space of \( C(X) \) which \( S \)-separates Lebesgue-sets of \( A \). Then the sublattice generated by \( A \) is contained in \( \sum(E) \).
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The RAFU linear space satisfies the Theorem 3.1 when $X = [a, b]$ because every Lebesgue-set is also a zero-set since $L_{\alpha}(f) = Z((f - \alpha) \lor 0)$ and $L^{\beta}(f) = Z((f - \beta) \land 0)$ and we have proved that $C$ $S$-separates every pair of disjoint zero-sets $Z_1$ and $Z_2$ of $[a, b]$. In this case, if $A = C(X)$ we can say that $C(X)$ is contained in $\sum C$. In fact, given $f \in C[a, b]$, we already knew that $f(x) = \sum_{n=1}^{\infty} c_n(x)$ where $c_n \in C$, $n \in \mathbb{N}$, and the series converges uniformly.

**Example 3.3.** Given the functions $f(x) = e^{-x^2}$ on $[-3, 3]$, $g(x) = \frac{3x}{x^2+1}$ on $[-5, 6]$, $h(x) = 5(x+8)(x+6)(x+2)(x-3)(x-5)$ on $[-10, 6]$ and $l(x) = |x|$ on $[-10, 6]$, the Figure 1 shows the graphics of these functions together with their approximations by means of its respective radical function $C^{75}$ belonging to the RAFU linear space $C$.

4. Conclusions

The RAFU method is an original and unknown procedure of uniform approximation on $C[a, b]$. This method improves the instability of the polynomial interpolation and it is based in the use of radical functions to approximate any continuous function defined in $[a, b]$. We have constructed a linear space $C$ uniformly dense on $C[a, b]$ and this linear space is not a lattice or an algebra. At the moment, the proof of this result was direct but in this work we have proved that $C$ is uniformly dense on $C[a, b]$ by using a $S$-separation condition due to Blasco-Moltó [6] or an equivalent $S'$-separation condition due to Garrido-Montalvo [8]. We already knew another example of a linear space uniformly dense by using these separation conditions [6] but by considering the set $C$, we can know easily the linear combinations of elements belonging to
which approximate uniformly every \( f \in C[a, b] \) and the degree of uniform approximation that \( \mathcal{C} \) provides.

**References**


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