# ANALYSIS OF FALSE WAVES IN NUMERICAL SEA SIMULATIONS

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## ABSTRACT

It is common practice to consider the random sea waves as a succesion of discrete waves characterized by individual amplitude and period. The zero-up-crossing criterion for discretizing waves, as well as other criteria proposed by different authors, has been found to isolate some discrete waves that do not correspond to physical waves. These *false waves* alter the wave statistics of random sea waves. A new orbital criterion is proposed to avoid this problem. The orbital criterion has shown to be consistent and robust with respect to the zero-up-crossing criterion. Furthermore, the new criterion produces a distribution of wave heights in better agreement to the Rayleigh distribution. The mean period of the discrete waves corresponding to the orbital criterion is proved to be $T_{01}$, while the mean period of the zero-up-crossing waves is $T_{02}$. A formula relating the Longuet-Higgins spectral bandwidth $\nu$ whith the relative number of false waves is given.

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## 1. INTRODUCTION

Regular waves can be characterized by amplitude and period, and random waves may be described by the energy spectrum. However, it is common practice to consider the random waves as a succession of "discrete waves" characterized by individual amplitude and period.

Unfortunately, a variety of reasonable criteria for discretizing waves have been proposed by different authors. In fact, any method used to define a discrete wave in regular waves could be extended to the case of random waves.

A number of papers are related to wave statistics and may be affected by the wave discretization procedure. Moreover, a variety of subjective criteria are used for neglecting small waves in the analysis.

This paper describes first the most common wave discretization methods, and the orbital criterion in the complex plane. Secondly, the new concept of "false wave" is introduced and an identification method is justified. Thirdly, the orbital criterion is found the most consistent and robust. Finally, the influence of false waves on the calculation of $H_{1/3}$ is analyzed using numerical simulations. The results using numerical simulations are in good agreement with the observations given by Pires-Silva and Medina (1993) analyzing wave records off coast of Portugal.

## 2. WAVE DISCRETIZATION CRITERIA

### 2.1. The ZUC and the ZDC criteria

In the ZUC (zero-up-crossing) criterion, a discrete wave is limited by two consecutive up-crossings. In the ZDC (zero-down-crossing) criterion, a discrete wave is limited by two consecutive down-crossings. For linear random waves, the ZUC and the ZDC criteria are statistically equivalent. In the following, only the ZUC criterion is analyzed.

Following Rice (1954), Longuet-Higgins (1958) showed that mean period of random waves using the ZUC criterion is $T_{02}$, where $T_{ij}$ is the inverse of the frequency $f_{ij}$ given by:

$$f_{ij} = \sqrt[j-i]{\frac{m_j}{m_i}} \qquad (1)$$

where $m_n$ is the $n^{th}$ moment of the energy spectrum S(f),

$$m_n = \int_0^\infty f^n S(f) df \tag{2}$$

Longuet-Higgins (1975) obtained a symmetric joint distribution of wave heights and periods with a mean period of $T_{01}$. Nolte (1979) improved an approach used by Longuet-Higgins providing an expression for the variance of the periods. Longuet-Higgins (1983) improved his previous approach obtaining an asymmetric joint distribution of wave heights and periods.

A number of small waves are present which may be convenient to neglect for the statistical analysis. Many authors use the ZUC criterion (Rye, 1974; van Vledder, 1983; Thompson and Seeling, 1984; Mansard and Funke, 1984; Mase and Iwagaki, 1986). However, they present different criteria, based on subjective thresholds, for neglecting small invalid waves. The proposed thresholds are defined in absolute or relative terms. Unfortunately, the wave statistics are sensitive to the threshold used to eliminate small waves.

## 2.2. The crest-to-crest criterion

In the crest-to-crest criterion, a discrete wave is limited by two consecutive maxima of the surface displacement function. From Rice (1954), it can be proven that the mean period of the random waves discretized using the crest-to-crest criterion is $T_{24}$. Therefore, the influence of the high frequencies tail is higher than for other discretizing criteria.

A variety of high frequency tails of wave energy spectra are proportional to $f^{-5}$. For these spectra, the fourth moment of S(f) becomes infinite and the crests are infinitely close. These infinitesimal crest-to-crest waves can be removed by cutting the high frequencies tail. However, both $m_4$ and the crest-to-crest criterion are very sensitive to the cutoff frequency. Therefore, the results obtained with this criterion are very unreliable and sensitive to noise.

## 2.3. The orbital criterion

Let $\eta(t)$ be the free sea surface elevation in a fixed point. Let us assume that $\eta(t)$ can be modeled by:

$$\eta(t)=\sum_{i=1}^{N} c_i \cos(2\pi f_i t+\varphi_i) \tag{3}$$

where the frequencies $f_i$ are $i\Delta f$, the phases $\varphi_i$ are random variables distributed uniformly over the interval $[0,2\pi[$, and the amplitudes $c_i$ are such that over any frequency interval $[f_i,f_i+\Delta f[$ is:

$$\frac{1}{2}c_i^2=\int_{f_i}^{f_i+\Delta f} S(f)\,df \tag{4}$$

Following Longuet-Higgins (1975), $\eta(t)$ can be expressed as the real part of a complex function:

$$\eta(t)=\mathbb{R}[AF(t)] \tag{5a}$$

$$AF(t)=\sum_{i=1}^{N} c_i \exp[j(2\pi f_i t+\varphi_i)] \tag{5b}$$

where $j=\sqrt{-1}$ is the imaginary unit. The analytical function $AF(t)$ can be expressed:

$$AF(t) = \eta(t) + j\hat{\eta}(t) \tag{6}$$

where $\hat{\eta}(t)$ is the Hilbert transform of the time series $\eta(t)$, defined as:

$$\hat{\eta}(t) = \sum_{i=1}^{N} c_i \sin(2\pi f_i t + \varphi_i) \tag{7}$$

AF(t) can also be expressed as:

$$AF(t) = A(t)\exp[j\theta(t)] \tag{8a}$$

$$A(t) = \sqrt{\eta^2(t) + \hat{\eta}^2(t)} \tag{8b}$$

$$\theta(t) = \arctan\frac{\hat{\eta}(t)}{\eta(t)} \tag{8c}$$

where A(t) is the wave envelope and $\theta(t)$ is the phase angle. According to Medina and Hudspeth (1987) the analytical function represents the orbital movement of a point floating on the sea surface. The orbital criterion defines a discrete wave as corresponding to a $2\pi$ advance of the phase angle in the complex plane. In the following, the discrete waves defined using this criterion will be referred to as orbital waves.

Fig.1a shows individual waves obtained using the different discretization criteria. Fig.1b represents the orbital movement of a point floating on the sea surface. Only one orbital wave is present between the points A and G, corresponding to a $2\pi$ advance of the phase angle.

[FIGURE 1]

## 3. IDENTIFICATION OF FALSE WAVES

Let us define false waves as any discrete wave that does not correspond to a $2\pi$ advance in the complex plane. It can be proven that false waves lack some properties usually attributed to physical waves.

Selecting the convenient phase origin, orbital waves can be forced to begin and end with zero-up-crossings. Fig.2a shows a piece of a numerical simulation from a JONSWAP spectrum. Fig.2b represents the analytical function AF(t) of the same piece of simulation. Two ZUC waves are shown in the figure that are not orbital waves; they are false waves.

[FIGURE 2]

Longuet-Higgins (1975) presented a random waves model for narrow spectra in which $T_{01}$ was the mean period. Based on the theory of Rice (1954), Hudspeth and Medina (1988) found that the instantaneous wave frequency has a mean value of $f_{01}$, and Longuet-Higgins (1958) found that $T_{02}$ is the mean period using the ZUC criterion. It can be proven that $T_{02}<T_{01}$ for all wave energy spectra, and the difference is due to false waves (see Appendix A).

Table 1 shows the results of the analysis of numerical simulations using both the ZUC and the orbital criteria. The simulations were obtained using a DSA-FFT algorithm (see Tuah and Hudspeth, 1982) and a JONSWAP type spectrum (see Goda, 1985) with N=8192 points and a sample interval $\Delta t$=0.5 seconds. Different values of the peakenhancement parameter $\gamma$ were tested with 300 simulations. Table 1 shows the mean and standard deviation of the ZUC mean period $T_z$ and the orbital mean period $T_r$ normalized by their respective theoretical values $T_{02}$ and $T_{01}$. Table 1 also includes the rate $T_{02}/T_{01}$ and the relative number of false waves, $P_f$.

[TABLE 1]

ZUC and orbital mean periods are found in agreement with the theoretical values $T_{02}$ and $T_{01}$. Moreover, it can be noted that the differences between $T_{01}$ and $T_{02}$ decreases for narrower spectra as well as the number of false waves. The proportion of false waves may be related with the parameter $\nu$ presented by Longuet-Higgins (1975):

$$\nu = \sqrt{\frac{m_2 m_0 - m_1^2}{m_1^2}} = \sqrt{\left(\frac{f_{02}}{f_{01}}\right)^2 - 1} \tag{9}$$

The relative number of false waves $P_f$ is (see Appendix A)

$$P_f = \frac{f_{02} - f_{01}}{f_{02}} = 1 - \frac{1}{\sqrt{1+\nu^2}} \tag{10}$$

The relative number of false waves decreases when the energy spectrum is narrower ($\nu$ decreases) and $P_f$=0 when $\nu$=0. For small values of $\nu$, the expression of $P_f$ may be approximated by

$$P_f \propto \nu^2/2 \tag{11}$$

The error of this formula is only about 6% for a relative broad JONSWAP spectrum with $\gamma$=1.

## 4. ADVANTAGES OF THE ORBITAL CRITERION

### 4.1. Consistency

Fig.3 shows a numerical simulation of two time series corresponding to two points in the sea surface separated 10% of the mean wavelength. Although both time series show a similar profile with a time lag of about 0.8 seconds, there is a ZUC wave in the first time series that does not appear in the second time series. This is a physical inconsistency that can be explained by the presence of a false wave shown in Figs.4a and 4b.

[FIGURE 3]

[FIGURE 4]

Even more, the ZUC criterion requires the use of additional subjective thresholds to remove the smaller waves. The orbital criterion does not require any additional threshold; the waves that do not correspond to a $2\pi$ advance are not considered as actual waves.

### 4.2. Robustness

The ZUC criterion is very sensitive to noise. Fig.5a shows a piece of numerical simulation from a JONSWAP spectrum with $\gamma=1$, and the same record when a 5% of white noise is added. Fig.5b shows the AF(t) analytical function corresponding to the same two simulations. Additional ZUC waves appear in Fig.5a due to the presence of noise. However, these are false waves (see Fig.5b), and they do not appear when using the orbital criterion. Most of the additional ZUC waves due to noise are false waves. Therefore, the influence of the noise on the mean period is expected to be very small when using the orbital criterion.

[FIGURE 5]

For analyzing the sensitivity to noise, the numerical simulations used for table 1 were contaminated with 2% and 5% of white noise. Table 2 shows the rate between the mean period of the simulations with noise ($T_{z,n}$ in the ZUC criterion and $T_{r,n}$ in the orbital criterion) and the mean period of the same simulations without noise ($T_z$ and $T_r$).

[TABLE 2]

The error induced by white noise in the estimation of the mean period is about five times higher when using the ZUC criterion than using the orbital criterion. The error in the estimation of the actual mean period of records with 5% of noise is about 5% when using the orbital criterion; that error is about 20-25% when using the ZUC criterion.

## 5. THE DISTRIBUTION OF WAVE HEIGHTS

Analyzing ergodic Gaussian stochastic processes, Longuet-Higgins (1952) showed that the distribution of the amplitudes of the wave envelope is Rayleigh, as well as the wave height distribution for narrow energy spectra. For broad spectra, the wave height distribution departs from the distribution of the amplitudes of the wave envelope. Therefore, the Rayleigh distribution is less adequate for representing the actual wave height distribution. On the other hand, a broader energy spectrum results in a greater number of false waves, which also contributes to the deviation from the Rayleigh distribution.

The orbital criterion eliminates false waves. Therefore, a better agreement between the actual wave height distribution and the Rayleigh distribution is expected. This idea is tested with the same numerical simulations used in table 1. Table 3 shows the mean rate between the significant wave height of the simulations ($H_{1/3,z}$ in the ZUC criterion and $H_{1/3,r}$ in the orbital criterion) and the theoretical value $H_{m0}$ that corresponds to the Rayleigh distribution.

[TABLE 3]

The orbital criterion gives a better approximation than the ZUC criterion to the Rayleigh distribution. Moreover, it can be concluded from table 3 that the effect of the false waves represents about 25% of the total deviation. Considering both the influences of noise and false waves, the results are in good agreement with the results given by Pires-Silva and Medina (1993) analyzing wave records off coast of Portugal.

## 6. CONCLUSIONS

The ZUC criterion commonly used to discretize wave records has been found to generate discrete waves that do not correspond to physical waves. These "false waves" alter the wave statistics of random waves, showing inconsistencies and high sensitivity to noise. On the other hand, the proposed orbital criterion to discretize random waves has shown to be consistent and robust with respect to the ZUC criterion.

The use of the orbital criterion produces a distribution of wave heights in better agreement to the Rayleigh distribution. The relative number of false waves $P_f$ may be related to the Longuet-Higgins spectral bandwidth parameter $\nu$ by

$$P_f = 1 - \frac{1}{\sqrt{1+\nu^2}} \tag{12}$$

## 7. ACKNOWLEDGEMENTS

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## APPENDIX A: MEAN FREQUENCY OF ORBITAL WAVES AND RELATIVE NUMBER OF FALSE WAVES

[FIGURE A1]

Let us call $\uparrow f$ and $\downarrow f$ to the mean frequencies of zero-up-crossings (such as A, C, G in Fig.A1) and zero-down-crossings (such as B) respectively of a time series $\eta(t)$ corresponding to negative values of its Hilbert transform $\hat{\eta}(t)$. Let us call $f\uparrow$ and $f\downarrow$ to the mean frequencies of zero-up-crossings (such as E) and zero-down-crossings (such as D, F) respectively of the same time series corresponding to positive values of its Hilbert transform. From Rice (1954), it can be concluded that the mean frequency of zeros of the time series is:

$$\uparrow f + \downarrow f + f\uparrow + f\downarrow = 2f_{02} \quad ; \quad f_{02} = \sqrt{m_2/m_0} \tag{A.1}$$

where $m_n$ is the $n^{th}$ moment of the energy spectrum S(f) of the time series $\eta(t)$:

$$m_n = \int_0^{\infty} f^{\,n} S(f)\, df \tag{A.2}$$

As the number of zero-up-crossings and zero-down-crossings has to be the same, the mean frequency of zero-up-crossing waves $f_z$ is:

$$f_z = \uparrow f + f\uparrow = f_{02} \tag{A.3}$$

According to the orbital criterion, a wave begins and ends with zero-up-crossings corresponding to negative values of the Hilbert transform. Moreover, each zero-down-crossing corresponding to a negative value of the Hilbert transform makes a zero-up-crossing with negative Hilbert transform invalid. In Fig.A1, the zero-up-crossing in E is not valid because it corresponds to a positive value of $\hat{\eta}(t)$; on the other hand, C is not a valid zero-up-crossing because it is preceded by a zero-down-crossing B with a negative value of $\hat{\eta}(t)$. Therefore, only one orbital wave exists between points A and G. According to this, the mean frequency of orbital waves $f_r$ is:

$$f_r = \uparrow f - \downarrow f \tag{A.4}$$

Firstly, we are going to calculate the mean frequency of zero-up-crossings of the time series corresponding to negative values of its Hilbert transform, $\uparrow f$. The time series $\eta(t)$ can be modeled by:

$$\eta(t) = \sum_{i=1}^{N} c_i \cos(2\pi f_i t + \varphi_i) \tag{A.5}$$

where the frequencies $f_i$ are $i\Delta f$, the phases $\varphi_i$ are random variables distributed uniformly over the interval $[0,2\pi[$, and the amplitudes $c_i$ are such that over any frequency interval $[f_i,f_i+\Delta f[$ is:

$$\frac{1}{2}c_i^2 = \int_{f_i}^{f_i+\Delta f} S(f)df \tag{A.6}$$

According to (A.5), the Hilbert transform $\hat{\eta}(t)$ and the derivative $\xi(t)$ of the time series $\eta(t)$ can be expressed as:

$$\hat{\eta}(t) = \sum_{i=1}^{N} c_i \sin(2\pi f_i t + \varphi_i) \tag{A.7a}$$

$$\xi(t) = \dot{\eta}(t) = -2\pi \sum_{i=1}^{N} c_i f_i \sin(2\pi f_i t + \varphi_i) \tag{A.7b}$$

When the number of frequencies N is big enough, $\eta(t)$, $\hat{\eta}(t)$ and $\xi(t)$ can be considered as zero mean gaussian processes. The autocovariances and cross-covariances of these processes are:

$$C_{\eta\eta}=\sum_{i=1}^{N}\frac{c_i^2}{2}\rightarrow m_0 \qquad \text{(A.8a)}$$

$$C_{\hat{\eta}\hat{\eta}}=\sum_{i=1}^{N}\frac{c_i^2}{2}\rightarrow m_0 \qquad \text{(A.8b)}$$

$$C_{\xi\xi}=4\pi^2\sum_{i=1}^{N}\frac{c_i^2 f_i^2}{2}\rightarrow 4\pi^2 m_2 \qquad \text{(A.8c)}$$

$$C_{\eta\hat{\eta}}=C_{\eta\xi}=0 \qquad \text{(A.8d)}$$

$$C_{\hat{\eta}\xi}=-2\pi\sum_{i=1}^{N}\frac{c_i^2 f_i}{2}\rightarrow -2\pi m_1 \qquad \text{(A.8e)}$$

Therefore, the matrix of covariances is:

$$[C]=\begin{bmatrix} C_{\eta\eta} & C_{\eta\hat{\eta}} & C_{\eta\xi} \\ C_{\eta\hat{\eta}} & C_{\hat{\eta}\hat{\eta}} & C_{\hat{\eta}\xi} \\ C_{\eta\xi} & C_{\hat{\eta}\xi} & C_{\xi\xi} \end{bmatrix}=\begin{bmatrix} m_0 & 0 & 0 \\ 0 & m_0 & -2\pi m_1 \\ 0 & -2\pi m_1 & 4\pi^2 m_2 \end{bmatrix} \qquad \text{(A.9)}$$

whose determinant and inverse are:

$$|C| = 4\pi^2 m_0 (m_0 m_2 - m_1^2) \qquad \textbf{(A.10a)}$$

$$[C]^{-1} = \frac{1}{|C|}\begin{bmatrix} 4\pi^2(m_0 m_2 - m_1^2) & 0 & 0 \\ 0 & 4\pi^2 m_0 m_2 & 2\pi m_0 m_1 \\ 0 & 2\pi m_0 m_1 & m_0^2 \end{bmatrix} \qquad \textbf{(A.10b)}$$

The joint probability function of $\eta(t)$, $\hat{\eta}(t)$ and $\xi(t)$ is:

$$p(\eta,\hat{\eta},\xi) = \frac{1}{(2\pi)^{3/2}\sqrt{|C|}} \exp\left[-\frac{1}{2}(\eta\ \hat{\eta}\ \xi)[C]^{-1}\begin{pmatrix}\eta \\ \hat{\eta} \\ \xi\end{pmatrix}\right] = K\exp[-A\eta^2 - B\hat{\eta}^2 - D\xi^2 - E\hat{\eta}\xi] \qquad \textbf{(A.11)}$$

where:

$$K = \frac{1}{(2\pi)^{5/2}\sqrt{m_0(m_0 m_2 - m_1^2)}} \tag{A.12a}$$

$$A = \frac{1}{2m_0} \tag{A.12b}$$

$$B = 16\pi^5 m_0 m_2 K^2 \tag{A.12c}$$

$$D = 4\pi^3 m_0^2 K^2 \tag{A.12d}$$

$$E = 16\pi^4 m_0 m_1 K^2 \tag{A.12e}$$

If a zero-up-crossing corresponding to a negative value of the Hilbert transform has to be present in the time interval $[t_0, t_0+dt[$ ,the following conditions have to be verified:

$$\begin{cases} \xi(t_0) \in ]0, \infty[ \\ \eta(t_0) \in ]-\xi(t_0)\,dt, 0] \\ \hat{\eta}(t_0) \in ]-\infty, 0[ \end{cases} \qquad \textbf{(A.13)}$$

Therefore, the probability of such a type of zero-up-crossing to be present in the time interval $[t_0, t_0+dt[$ is:

$$I = \int_{\xi=0}^{\infty} \int_{\hat{\eta}=-\infty}^{0} \int_{\eta=-\xi dt}^{0} p(\eta, \hat{\eta}, \xi)\, d\eta\, d\hat{\eta}\, d\xi \qquad \textbf{(A.14)}$$

This integral can be compared with the one calculated by Rice (1954), which represents the probability of any type of zero-up-crossing, and can be expressed as:

$$I' = \int_{\xi=0}^{\infty} \int_{\hat{\eta}=-\infty}^{\infty} \int_{\eta=-\xi dt}^{0} p(\eta, \hat{\eta}, \xi)\, d\eta\, d\hat{\eta}\, d\xi = \int_{\xi=0}^{\infty} \int_{\eta=-\xi dt}^{0} p(\eta, \xi)\, d\eta\, d\xi \qquad \textbf{(A.15)}$$

Following Rice, as $\eta$ varies in the interval $]-\xi dt, 0]$, it can be considered that:

$$p(\eta, \hat{\eta}, \xi) = p(0, \hat{\eta}, \xi) \qquad \textbf{(A.16)}$$

and then:

$$I = \int_{\xi=0}^{\infty} \int_{\hat{\eta}=-\infty}^{0} \int_{\eta=-\xi dt}^{0} p(0, \hat{\eta}, \xi)\, d\eta\, d\hat{\eta}\, d\xi \qquad \textbf{(A.17)}$$

When (A.11) is used, the integral can be expressed as:

$$I=\int_{\xi=0}^{\infty}\int_{\hat{\eta}=-\infty}^{0}\int_{\eta=-\xi dt}^{0} K\exp[-B\hat{\eta}^2-D\xi^2-E\hat{\eta}\xi]d\eta\, d\hat{\eta}\, d\xi =$$
$$=\int_{\xi=0}^{\infty}\int_{\hat{\eta}=-\infty}^{0} K\exp[-B\hat{\eta}^2-D\xi^2-E\hat{\eta}\xi]\left[\int_{\eta=-\xi dt}^{0} d\eta\right]d\hat{\eta}\, d\xi =$$
$$=\int_{\xi=0}^{\infty}\int_{\hat{\eta}=-\infty}^{0} K\exp[-B\hat{\eta}^2-D\xi^2-E\hat{\eta}\xi]\xi dt\, d\hat{\eta}\, d\xi =$$
$$=\int_{\xi=0}^{\infty} Kdt\,\xi\exp[-D\xi^2]\left[\int_{\hat{\eta}=-\infty}^{0}\exp[-B\hat{\eta}^2-E\xi\hat{\eta}]d\hat{\eta}\right]d\xi \quad \textbf{(A.18)}$$

The inner integral can be solved using the following change of variable:

$$\hat{\eta}=\frac{1}{\sqrt{B}}u-\frac{E\xi}{2B} \quad \textbf{(A.19)}$$

and the result is:

$$\int_{-\infty}^{0}\exp[-B\hat{\eta}^2-E\xi\hat{\eta}]d\hat{\eta}=\int_{-\infty}^{\frac{E\xi}{2\sqrt{B}}}\frac{1}{\sqrt{B}}\exp\left[-u^2+\frac{E^2\xi^2}{4B}\right]du=$$
$$=\frac{\exp\left[\frac{E^2\xi^2}{4B}\right]}{\sqrt{B}}\int_{-\infty}^{\frac{E\xi}{2\sqrt{B}}}\exp[-u^2]du \quad \textbf{(A.20)}$$

A new change of variable $u^2=z^2/2$ can be used, and the result is:

$$\int_{-\infty}^{0}\exp[-B\hat{\eta}^2-E\xi\hat{\eta}]d\hat{\eta}=\frac{\exp\left[\frac{E^2\xi^2}{4B}\right]}{\sqrt{B}}\int_{-\infty}^{\frac{E\xi}{\sqrt{2B}}}\frac{1}{\sqrt{2}}\exp\left[-\frac{z^2}{2}\right]dz=$$
$$=\frac{\sqrt{\pi}\exp\left[\frac{E^2\xi^2}{4B}\right]}{\sqrt{B}}\int_{-\infty}^{\frac{E\xi}{\sqrt{2B}}}\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{z^2}{2}\right]dz= \qquad \text{(A.21)}$$
$$=\frac{\sqrt{\pi}\exp\left[\frac{E^2\xi^2}{4B}\right]}{\sqrt{B}}\phi\left(\frac{E\xi}{\sqrt{2B}}\right)$$

where $\phi(x)$ is the area between $-\infty$ and x under the standard gaussian distribution. The result of using (A.21) in (A.18) is:

$$I=\int_0^{\infty}Kdt\,\xi\exp[-D\xi^2]\frac{\sqrt{\pi}\exp\left[\frac{E^2\xi^2}{4B}\right]}{\sqrt{B}}\phi\left(\frac{E\xi}{\sqrt{2B}}\right)d\xi=$$
$$=\int_0^{\infty}\left\{-\frac{\sqrt{\pi}\,Kdt}{2\sqrt{B}\,[D-E^2/(4B)]}\phi\left(\frac{E\xi}{\sqrt{2B}}\right)\right\}\left\{-2\left(D-\frac{E^2}{4B}\right)\xi\exp\left[-\left(D-\frac{E^2}{4B}\right)\xi^2\right]d\xi\right\}=$$
$$=\left[-\frac{\sqrt{\pi}\,Kdt}{2\sqrt{B}\,[D-E^2/(4B)]}\phi\left(\frac{E\xi}{\sqrt{2B}}\right)\exp\left[-\left(D-\frac{E^2}{4B}\right)\xi^2\right]\right]_0^{\infty}+ \qquad \text{(A.22)}$$
$$+\frac{\sqrt{\pi}\,Kdt}{2\sqrt{B}\,[D-E^2/(4B)]}\frac{E}{\sqrt{2B}}\frac{1}{\sqrt{2\pi}}\int_0^{\infty}\exp\left[-\left(D-\frac{E^2}{4B}\right)\xi^2-\frac{E^2\xi^2}{4B}\right]d\xi=$$
$$=\left[0+\frac{\sqrt{\pi}\,Kdt}{2\sqrt{B}\,[D-E^2/(4B)]}\frac{1}{2}\right]+\frac{KEdt}{4B\,[D-E^2/(4B)]}\int_0^{\infty}\exp[-D\xi^2]d\xi$$

A change of variable $D\xi^2=z^2/2$ can be used, and the result is:

$$\int_0^\infty \exp[-D\xi^2]d\xi = \int_0^\infty \frac{1}{\sqrt{2D}}\exp\left[-\frac{z^2}{2}\right]dz = $$
$$= \frac{\sqrt{\pi}}{\sqrt{D}}\int_0^\infty \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{z^2}{2}\right]dz = \tag{A.23}$$
$$= \frac{\sqrt{\pi}}{\sqrt{D}}[\Phi(\infty)-\Phi(0)] = \frac{\sqrt{\pi}}{2\sqrt{D}}$$

The result of using (A.23) in (A.22) is:

$$I = \frac{\sqrt{\pi}\,Kdt}{4\sqrt{B}[D-E^2/(4B)]} + \frac{KEdt}{4B[D-E^2/(4B)]}\frac{\sqrt{\pi}}{2\sqrt{D}} = $$
$$= \frac{\sqrt{\pi}\sqrt{B}\,Kdt}{4BD-E^2} + \frac{\sqrt{\pi}\,KEdt}{2\sqrt{D}[4BD-E^2]} = \tag{A.24}$$
$$= \frac{\sqrt{\pi}\,Kdt}{2\sqrt{D}[4BD-E^2]}[2\sqrt{BD}+E]$$

Using the expressions (A.12) it can be obtained that:

$$4BD-E^2 = 4(16\pi^5 m_0 m_2 K^2)(4\pi^3 m_0^2 K^2) - (16\pi^4 m_0 m_1 K^2)^2 = $$
$$= 256\pi^8 m_0^3 m_2 K^4 - 256\pi^8 m_0^2 m_1^2 K^4 = 256\pi^8 m_0^2 K^4 (m_0 m_2 - m_1^2)$$
$$\frac{\sqrt{\pi}\,Kdt}{2\sqrt{D}[4BD-E^2]} = \frac{\sqrt{\pi}\,Kdt}{2\sqrt{4\pi^3 m_0^2 K^2}\,256\pi^8 m_0^2 K^4 (m_0 m_2 - m_1^2)} = \tag{A.25}$$
$$= \frac{\sqrt{\pi}\,Kdt}{1024\pi^9\sqrt{\pi}\,m_0^3 K^5 (m_0 m_2 - m_1^2)} = \frac{dt}{1024\pi^9 m_0^3 K^4 (m_0 m_2 - m_1^2)}$$
$$2\sqrt{BD} + E = 2\sqrt{(16\pi^5 m_0 m_2 K^2)(4\pi^3 m_0^2 K^2)} + 16\pi^4 m_0 m_1 K^2 = $$
$$= 16\pi^4 m_0 K^2 \sqrt{m_0 m_2} + 16\pi^4 m_0 m_1 K^2 = 16\pi^4 m_0 K^2 \left(\sqrt{m_0 m_2} + m_1\right)$$

Therefore:

$$I=\frac{dt}{1024\pi^9 m_0^3 K^4(m_0m_2-m_1^2)}16\pi^4 m_0 K^2(\sqrt{m_0m_2}+m_1)=$$
$$=\frac{dt(\sqrt{m_0m_2}+m_1)}{64\pi^5 m_0^2(m_0m_2-m_1^2)}\frac{1}{K^2}=\frac{dt(\sqrt{m_0m_2}+m_1)}{64\pi^5 m_0^2(m_0m_2-m_1^2)}(2\pi)^5 m_0(m_0m_2-m_1^2)= \quad \textbf{(A.26)}$$
$$=\frac{dt(\sqrt{m_0m_2}+m_1)}{2m_0}=dt\left(\frac{\sqrt{m_2/m_0}+m_1/m_0}{2}\right)=dt\left(\frac{f_{02}+f_{01}}{2}\right)$$

where $f_{01}=m_1/m_0$ and $f_{02}$ was already defined in (A.1). As I is the probability that a zero-up-crossing corresponding to a negative value of the Hilbert transform has to be present in the time interval $[t_0,t_0+dt[$, and it is independent of the time position $t_0$, the mean frequency $\uparrow f$ of such a type of zero-up-crossing is:

$$\uparrow f=\frac{I}{dt}=\frac{f_{02}+f_{01}}{2} \quad \textbf{(A.27)}$$

We are going now to relate the frequencies of a zero-down-crossing corresponding to a negative value of the Hilbert transform, $\downarrow f$, and a zero-up-crossing corresponding to a positive value of the Hilbert transform, $f\uparrow$. A zero-crossing of the first type is present in the time interval $[t_0,t_0+dt[$ when:

$$\begin{cases} \xi(t_0)\in]-\infty,0[ \\ \eta(t_0)\in[0,-\xi(t_0)\,dt[ \\ \hat{\eta}(t_0)\in]-\infty,0[ \end{cases} \quad \textbf{(A.28)}$$

and a zero-crossing of the second type is present in the same time interval when:

$$\begin{cases} \xi(t_0)\in ]0,\infty[ \\ \eta(t_0)\in ]-\xi(t_0)\,dt,0] \\ \hat{\eta}(t_0)\in ]0,\infty[ \end{cases} \tag{A.29}$$

The frequency ↓f is:

$$\downarrow f=\frac{1}{dt}\int_{\xi=-\infty}^{0}\int_{\hat{\eta}=-\infty}^{0}\int_{\eta=0}^{-\xi dt}p(\eta,\hat{\eta},\xi)\,d\eta\,d\hat{\eta}\,d\xi \tag{A.30}$$

The frequency f↑ is:

$$f\uparrow=\frac{1}{dt}\int_{\xi=0}^{\infty}\int_{\hat{\eta}=0}^{\infty}\int_{\eta=-\xi dt}^{0}p(\eta,\hat{\eta},\xi)\,d\eta\,d\hat{\eta}\,d\xi \tag{A.31}$$

Using the changes of variables $\eta'=-\eta$, $\xi'=-\xi$ and $\hat{\eta}'=-\hat{\eta}$ in (A.30), it can be obtained that:

$$\begin{aligned}\downarrow f&=\frac{1}{dt}\int_{\xi'=\infty}^{0}\int_{\hat{\eta}'=\infty}^{0}\int_{\eta'=0}^{-\xi' dt}p(-\eta',-\hat{\eta}',-\xi')(-d\eta')(-d\hat{\eta}')(-d\xi')=\\&=\frac{1}{dt}\int_{\xi'=0}^{\infty}\int_{\hat{\eta}'=0}^{\infty}\int_{\eta'=-\xi' dt}^{0}p(-\eta',-\hat{\eta}',-\xi')\,d\eta'\,d\hat{\eta}'\,d\xi'=\\&=\frac{1}{dt}\int_{\xi'=0}^{\infty}\int_{\hat{\eta}'=0}^{\infty}\int_{\eta'=-\xi' dt}^{0}p(\eta',\hat{\eta}',\xi')\,d\eta'\,d\hat{\eta}'\,d\xi'=f\uparrow\end{aligned} \tag{A.32}$$

because $p(-\eta,-\hat{\eta},-\xi)=p(\eta,\hat{\eta},\xi)$, as can be easily proven from (A.11). Finally, from (A.3), (A.4), (A.27) and (A.32) it can be concluded that the mean frequency of the orbital waves is:

$$f_r = \uparrow f - \downarrow f = \uparrow f - f\uparrow = 2\uparrow f - (\uparrow f + f\uparrow) = 2\uparrow f - f_z = 2\left(\frac{f_{02} + f_{01}}{2}\right) - f_{02} = f_{01} \tag{A.33}$$

Therefore, it has been proven that the mean frequency $f_r$ of the orbital waves is $f_{01}$.

On the other hand, a false wave is present when there is either a zero-up-crossing with a positive value of the Hilbert transform (such as E in Fig.A1), or a zero-down-crossing with a negative value of the Hilbert transform (such as B). Therefore, the mean number of false waves in a time unit is:

$$f\uparrow + \downarrow f = (f\uparrow + \uparrow f) - (\uparrow f - \downarrow f) = f_z - f_r = f_{02} - f_{01} \tag{A.34}$$

Because the mean number of ZUC waves in a time unit is $f_{02}$, the relative number of false waves is:

$$P_f = \frac{f_{02} - f_{01}}{f_{02}} \tag{A.35}$$

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| $\gamma$ | $\mu(T_z)/T_{02}$ | $\sigma(T_z)/T_{02}$ | $\mu(T_r)/T_{01}$ | $\sigma(T_r)/T_{01}$ | $T_{02} / T_{01}$ | $P_f$ (%) |
|---|---|---|---|---|---|---|
| 1 | 1.002 | 0.012 | 1.005 | 0.012 | 0.961 | 3.9 |
| 2 | 1.002 | 0.013 | 1.005 | 0.013 | 0.964 | 3.6 |
| 3.3 | 1.002 | 0.012 | 1.004 | 0.012 | 0.967 | 3.3 |
| 5 | 1.001 | 0.012 | 1.003 | 0.011 | 0.970 | 3.0 |
| 7 | 1.002 | 0.013 | 1.004 | 0.011 | 0.973 | 2.7 |
| 10 | 1.002 | 0.012 | 1.003 | 0.011 | 0.976 | 2.4 |

**Table 1.** Statistics of the mean periods of ZUC and orbital waves.

| | 2% of white noise | | 5% of white noise | |
|---|---|---|---|---|
| $\gamma$ | $\mu\ (T_{z,n} / T_z)$ | $\mu\ (T_{r,n} / T_r)$ | $\mu\ (T_{z,n} / T_z)$ | $\mu\ (T_{r,n} / T_r)$ |
| 1 | 0.901 | 0.984 | 0.800 | 0.956 |
| 2 | 0.897 | 0.983 | 0.789 | 0.955 |
| 3.3 | 0.890 | 0.982 | 0.779 | 0.952 |
| 5 | 0.886 | 0.981 | 0.771 | 0.950 |
| 7 | 0.881 | 0.980 | 0.764 | 0.949 |
| 10 | 0.877 | 0.980 | 0.757 | 0.948 |

**Table 2.** Influence of white noise on the mean periods of ZUC and orbital waves.

| $\gamma$ | $\mu(H_{1/3,z}) / H_{m0}$ | $\mu(H_{1/3,r}) / H_{m0}$ |
|---|---|---|
| 1 | 0.952 | 0.963 |
| 2 | 0.957 | 0.967 |
| 3.3 | 0.961 | 0.970 |
| 5 | 0.965 | 0.973 |
| 7 | 0.969 | 0.976 |
| 10 | 0.972 | 0.979 |

**Table 3.** Comparison of $H_{1/3}$ for ZUC and orbital waves.

## CAPTION OF FIGURES

**Figure 1.** Criteria for discretizing waves: a) time series; b) orbital analysis.

**Figure 2.** False waves: a) time series; b) orbital analysis.

**Figure 3.** Time series of two close points.

**Figure 4.** Orbital analysis of the time series of two close points.

**Figure 5.** Influence of white noise: a) time series; b) orbital analysis.

**Figure A1.** Example of different types of zero-crossing.

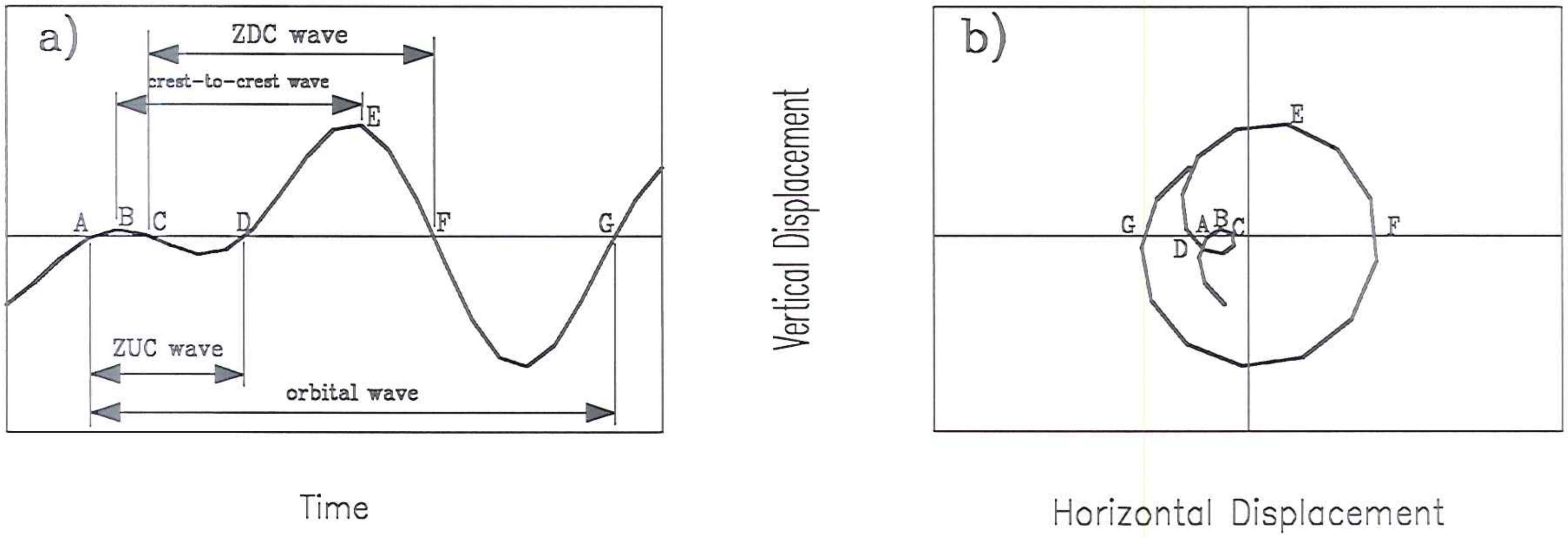


Figure 1. Criteria for discretizing waves: a) time series; b) orbital analysis.

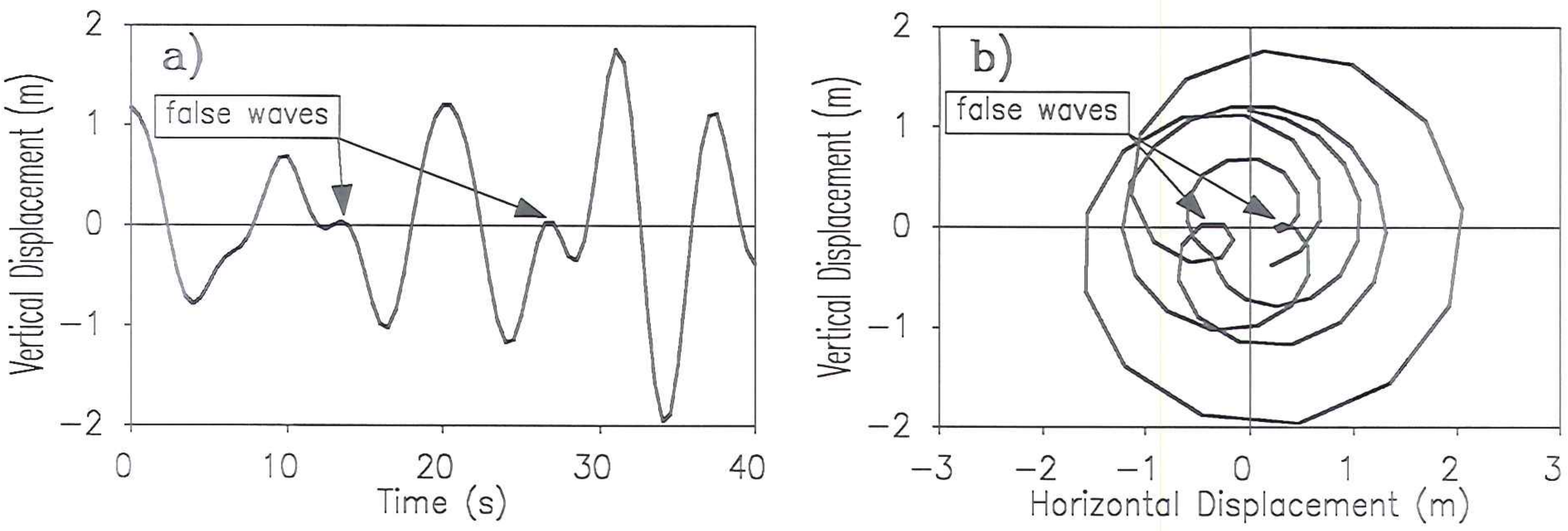


**Figure 2.** False waves: a) time series; b) orbital analysis.

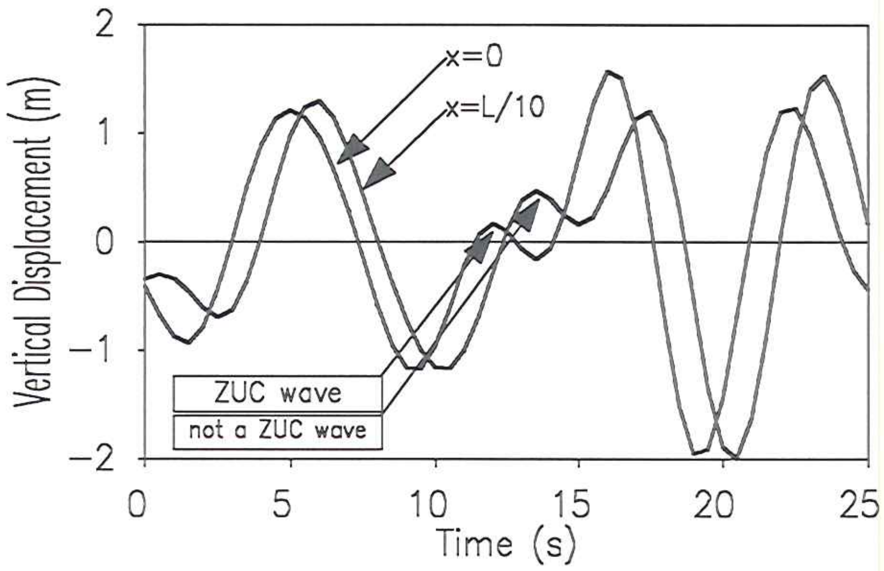


**Figure 3.** Time series of two close points.

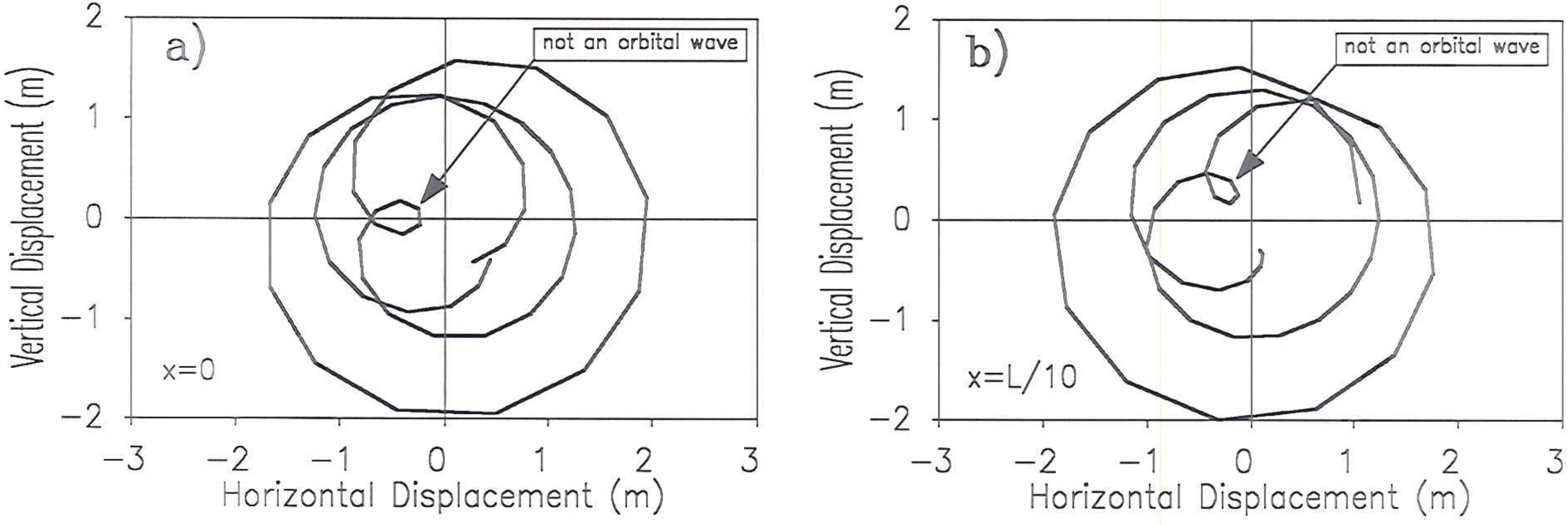


**Figure 4.** Orbital analysis of the time series of two close points.

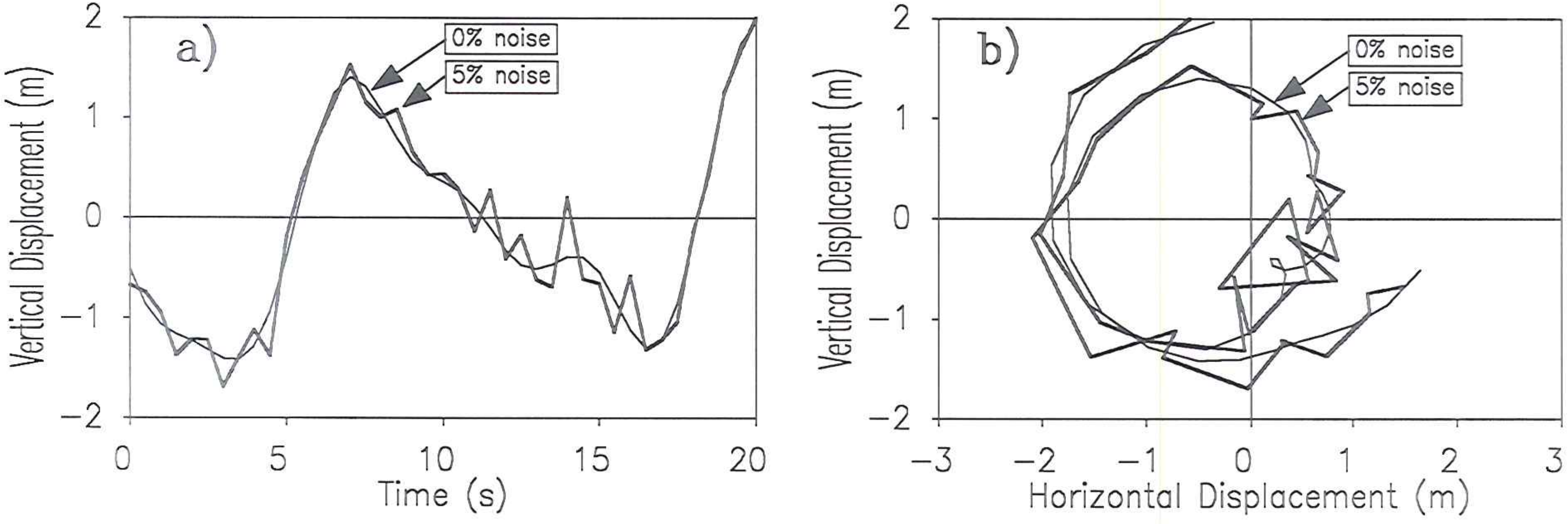


**Figure 5.** Influence of white noise: a) time series; b) orbital analysis.

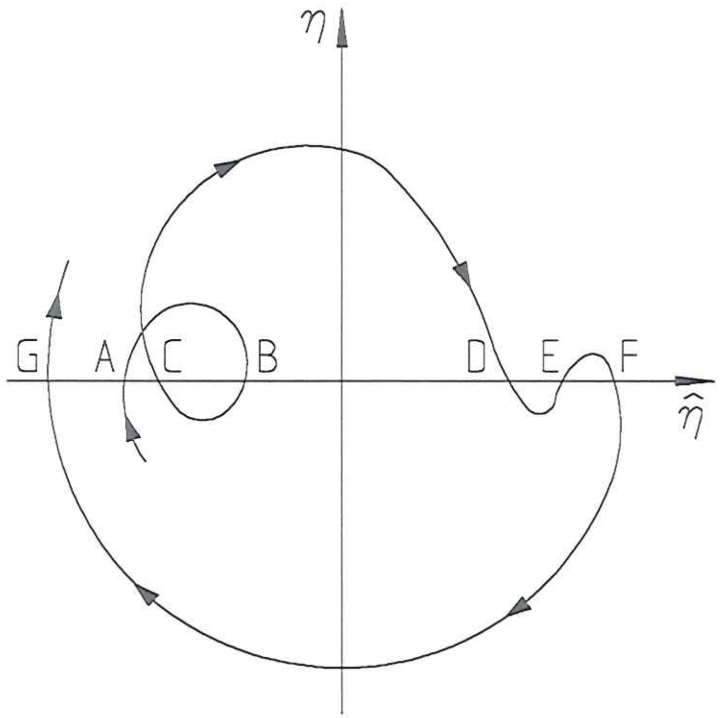


**Figure A1.** Example of different types of zero-crossing.