Letter to the Editor

Comments on “Vibration of simply supported beams under a single moving load: A detailed study of cancellation phenomenon, *International Journal of Mechanical Sciences* 99 (2015) 40–47, doi: 10.1016/j.ijmecsci.2015.05.001, by C.P. Sudheesh Kumar, C. Sujatha, K. Shankar”

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1. Introduction

In a recent paper, Sudheesh Kumar *et al.* analysed, from different points of view, the phenomenon of *cancellation* in simply supported beams under constant moving loads [1]. This article presented some novel facts about this phenomenon that are of interest to scientists and engineers studying bridge dynamics as well as to researchers in disciplines related to the moving load problem. This paper provides corrected versions of certain results in reference [1]. For the sake of clarity, it is also organized in the same sections and subsections.

2. Uniform beam with a single moving point load

2.1. *Forced vibration*

In the previously mentioned paper, Sudheesh Kumar *et al.* refer to an article by Museros *et al.* [2]. To facilitate understanding, some of the results presented in [2] are recalled. Regarding the mathematical expressions, response plots, etc., the notation in [1] is followed. Equations in reference [1] are mentioned as Eq. (N-1), whereas tables are referred to as Table N-1. Similarly, equations from [2] are labelled as Eq. (N-2), etc.

Eq. (5-1) provides the following solution to the problem of the forced motion of the mid-span section during the passage of the load, $0 \leq t \leq L/v$:

$$\frac{w_{\text{forced}}(t)}{w_{\text{static}}} = \sum_{n=1}^{\infty} \frac{1}{n^4\sqrt{(1-K_n^2)^2 + (2\zeta_n K_n)^2}} \{\sin(K_n\omega_n t)$$

$$-\frac{K_n}{\sqrt{1-\zeta_n^2}} \mathrm{e}^{-\zeta_n\omega_n t} \sin\left(\omega_n\sqrt{(1-\zeta_n^2)}t\right)\}, \tag{1}$$

where $w_{\text{static}} = 2P/(\mu L\omega_1{}^2)$ is "the static deflection of the mid-span of the beam". More specifically, w_{static} represents the static deflection of the mid-span section due to the contribution of the fundamental mode. This magnitude is related to the static deflection of the *n*th mode (see Eq. (5-2)) as per

$$q_{n,st} = \frac{2P}{\mu L\omega_n^2} = \frac{w_{\text{static}}}{n^4}. \tag{2}$$

The relation given by Eq. (2) has some practical implications, as shown further on.

The time-dependent modal amplitudes in Eq. (1) should be weighted by the mode shapes $\sin(n\pi x/L)$ evaluated at mid-span (*i.e.* $\sin(n\pi/2)$) in order to rule out the even modes as well as to give the correct sign to the odd modes. Otherwise, the summation in Eq. (1) will yield incorrect results. At this point, it is convenient to remember that the mode shapes $\sin(n\pi x/L)$ are considered to be nondimensional, whereas the modal amplitudes are measured in length units (meters).

The modal amplitudes are analysed in what follows. If extracted from the summation in Eq. (1), and in accordance with Eqs. (6-1) and (7-1), such modal amplitudes are

$$q_n(t) = \frac{q_{n,st}}{\sqrt{(1-K_n^2)^2 + (2\zeta_n K_n)^2}} \{\sin(K_n\omega_n t)$$

$$-\frac{K_n}{\sqrt{1-\zeta_n^2}} \mathrm{e}^{-\zeta_n\omega_n t} \sin\left(\omega_n\sqrt{(1-\zeta_n^2)}t\right)\}. \tag{3}$$

Differentiation of Eq. (3) yields the modal velocity, and subsequent differentiation yields the modal acceleration:

$$\dot{q}_n(t) = \frac{q_{n,st}}{\sqrt{(1-K_n^2)^2 + (2\zeta_n K_n)^2}} K_n\omega_n \{\cos(K_n\omega_n t)$$

$$+\mathrm{e}^{-\zeta_n\omega_n t}[\frac{\zeta_n}{\sqrt{1-\zeta_n^2}} \sin\left(\omega_n\sqrt{(1-\zeta_n^2)}t\right) - \cos\left(\omega_n\sqrt{(1-\zeta_n^2)}t\right)]\}, \tag{4}$$

$$\ddot{q}_n(t) = \frac{q_{n,st}}{\sqrt{(1-K_n^2)^2 + (2\zeta_n K_n)^2}} K_n \omega_n^2 \{-K_n \sin(K_n \omega_n t)$$

$$+\mathrm{e}^{-\zeta_n \omega_n t}[\frac{1-2\zeta_n^2}{\sqrt{1-\zeta_n^2}} \sin\left(\omega_n \sqrt{(1-\zeta_n^2)} t\right) + 2\zeta_n \cos\left(\omega_n \sqrt{(1-\zeta_n^2)} t\right)]\}. \qquad (5)$$

As can be observed, the evaluation of Eqs. (3) and (4) at $t = 0$ gives zero initial response and velocity for each modal amplitude. Conversely, Eq. (5) is not zero at $t = 0$ unless $\zeta_n = 0$, *i.e.*, when damping is present the modal acceleration does not satisfy the initial condition derived from the governing equation of motion (see Eq. (3-1)). Thus, Eq. (5-1) cannot be used for damped beams.

The correct solution to the modal equation of motion, which is valid both for damped and undamped beams, is [3, 4, 5]

$$q_n(t) = \frac{q_{n,st}}{(1-K_n^2)^2 + (2\zeta_n K_n)^2} \{(1-K_n^2)\sin(K_n \omega_n t) - 2\zeta_n K_n \cos(K_n \omega_n t)$$

$$+K_n \mathrm{e}^{-\zeta_n \omega_n t}[\frac{2\zeta_n^2 + K_n^2 - 1}{\sqrt{1-\zeta_n^2}} \sin\left(\omega_n \sqrt{(1-\zeta_n^2)} t\right) + 2\zeta_n \cos\left(\omega_n \sqrt{(1-\zeta_n^2)} t\right)]\}, \qquad (6)$$

where the critical undamped case ($K_n = 1, \zeta_n = 0$) must be excluded. The solution to the critical case can be found, for instance, in references [2] and [6]. Accordingly, the correct modal velocity is

$$\dot{q}_n(t) = \frac{q_{n,st}}{(1-K_n^2)^2 + (2\zeta_n K_n)^2} K_n \omega_n \{(1-K_n^2)\cos(K_n \omega_n t) + 2\zeta_n K_n \sin(K_n \omega_n t)$$

$$-\mathrm{e}^{-\zeta_n \omega_n t}[\frac{\zeta_n(1+K_n^2)}{\sqrt{1-\zeta_n^2}} \sin\left(\omega_n \sqrt{(1-\zeta_n^2)} t\right) + (1-K_n^2)\cos\left(\omega_n \sqrt{(1-\zeta_n^2)} t\right)]\}. \qquad (7)$$

Differentiation of Eq. (7) readily shows that $\ddot{q}_n(0) = 0$.

Both Eqs. (3) and (6) reduce to the same correct result when $\zeta_n = 0$. Therefore, many conclusions regarding undamped beams in reference [1] are correct. Conversely, the formulas related to damped beams are not valid. The corrected versions of these formulas are given below.

2.2. *Free vibration*

The modal amplitude and modal velocity at $t = L/v$ are required to evaluate the free vibration. The exact values must be obtained from Eqs. (6) and (7) for a general damped beam:

$$q_{0n} = \frac{q_{n,st}}{(1-K_n^2)^2 + (2\zeta_n K_n)^2} K_n \{-2\zeta_n \cos(n\pi)$$

$$+\mathrm{e}^{-\zeta_n n\pi/K_n}[\frac{2\zeta_n^2 + K_n^2 - 1}{\sqrt{1-\zeta_n^2}} \sin\left(\frac{n\pi}{K_n}\sqrt{1-\zeta_n^2}\right) + 2\zeta_n \cos\left(\frac{n\pi}{K_n}\sqrt{1-\zeta_n^2}\right)]\}, \qquad (8a)$$

$$\dot{q}_{0n} = \frac{q_{n,st}}{(1-K_n^2)^2 + (2\zeta_n K_n)^2} K_n \omega_n \{(1-K_n^2)\cos(n\pi)$$

$$-\mathrm{e}^{-\zeta_n n\pi/K_n}[\frac{\zeta_n(1+K_n^2)}{\sqrt{1-\zeta_n^2}}\sin\left(\frac{n\pi}{K_n}\sqrt{1-\zeta_n^2}\right) + (1-K_n^2)\cos\left(\frac{n\pi}{K_n}\sqrt{1-\zeta_n^2}\right)]\}. \qquad (8b)$$

Figs. (1) and (2) show the evolution of the (normalised) initial conditions of the free vibration. For the sake of conciseness, only the fundamental mode is shown.

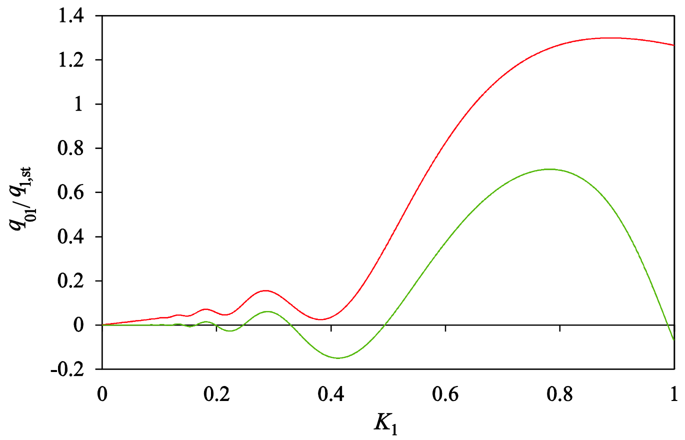

Figure 1. Normalised initial modal amplitude $q_{0n}/q_{n,st}$ of the free vibration $(n = 1, \zeta_n = 0.15)$. ——— Correct solution from Eq. (8a); ——— solution from Eq. (7a-1).

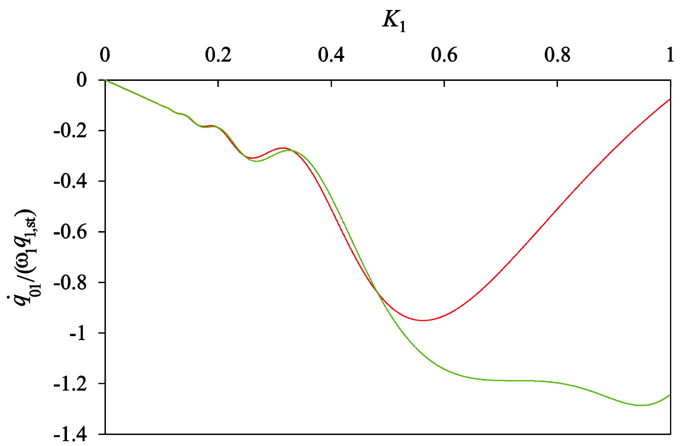

Figure 2. Normalised initial modal velocity $\dot{q}_{0n}/(\omega_n q_{n,st})$ of the free vibration $(n = 1, \zeta_n = 0.15)$. ——— Correct solution from Eq. (8b); ——— solution from Eq. (7b-1).

Eq. (6-1) is the generic expression of the free vibration during interval $t > L/v$. This expression is valid, providing that the time is set to zero when the load departs from the beam:

$$q_n(t) = e^{-\zeta_n \omega_n t}\left[q_{0n}\cos(\omega_{dn}t) + \frac{\zeta_n \omega_n q_{0n} + \dot{q}_{0n}}{\omega_{dn}}\sin(\omega_{dn}t)\right], \tag{9}$$

where $\omega_{dn} = \omega_n\sqrt{1-\zeta_n^2}$ is the damped frequency. For damped beams, Eq. (7c-1) yields an inexact free vibration time-history since it is derived from Eqs. (7a-1) and (7b-1). The correct expression is obtained by substitution of Eqs. (8) in Eq. (9) as follows:

$$q_n(t) = \frac{q_{n,st}}{(1-K_n^2)^2 + (2\zeta_n K_n)^2} K_n e^{-\zeta_n \omega_n t}[C_n\cos(\omega_{dn}t) + D_n\sin(\omega_{dn}t)], \tag{10a}$$

$$C_n = \left(\frac{q_{n,st}}{(1-K_n^2)^2 + (2\zeta_n K_n)^2} K_n\right)^{-1} q_{0n} =$$

$$-2\zeta_n\cos(n\pi) + \mathrm{e}^{-\zeta_n n\pi/K_n}\left[\frac{2\zeta_n^2 + K_n^2 - 1}{\sqrt{1-\zeta_n^2}}\sin\left(\frac{n\pi}{K_n}\sqrt{1-\zeta_n^2}\right) + 2\zeta_n\cos\left(\frac{n\pi}{K_n}\sqrt{1-\zeta_n^2}\right)\right], \tag{10b}$$

$$D_n = \left(\frac{q_{n,st}}{(1-K_n^2)^2 + (2\zeta_n K_n)^2} K_n\right)^{-1}\left(\frac{\zeta_n \omega_n q_{0n} + \dot{q}_{0n}}{\omega_{dn}}\right) =$$

$$-\frac{2\zeta_n^2 + K_n^2 - 1}{\sqrt{1-\zeta_n^2}}\cos(n\pi) + \mathrm{e}^{-\zeta_n n\pi/K_n}\left[\frac{2\zeta_n^2 + K_n^2 - 1}{\sqrt{1-\zeta_n^2}}\cos\left(\frac{n\pi}{K_n}\sqrt{1-\zeta_n^2}\right) - 2\zeta_n\sin\left(\frac{n\pi}{K_n}\sqrt{1-\zeta_n^2}\right)\right]. \tag{10c}$$

In reference [1], the free vibration is subsequently transformed into Eq. (8-1), *i.e.*:

$$q_n(t) = X_n e^{-\zeta_n \omega_n t}\sin(\omega_{dn}t - \phi_n), \tag{11}$$

where the following relations hold:

$$q_{0n} = -X_n\sin(\phi_n), \tag{12a}$$

$$\frac{\zeta_n \omega_n q_{0n} + \dot{q}_{0n}}{\omega_{dn}} = X_n\cos(\phi_n). \tag{12b}$$

The initial amplitude of the free vibration is represented by X_n. Following the transformation given by Eqs. (12), the initial conditions in Eqs. (8) can be combined to give the correct phase angle of the free vibration:

$$\tan(\phi_n) = -\frac{q_{0n}}{\left(\frac{\zeta_n \omega_n q_{0n} + \dot{q}_{0n}}{\omega_{dn}}\right)} = -\frac{C_n}{D_n}. \tag{13}$$

Since the amplitude X_n is a positive number by definition, Eqs. (12) provide the signs of the sine and cosine of ϕ_n. Therefore, the true solution between the two angles in the interval $[0,2\pi)$ that satisfy Eq. (13) can be unequivocally selected: the quadrant of the true solution is always conditioned by the signs of both q_{0n} and $(\zeta_n \omega_n q_{0n} + \dot{q}_{0n})/\omega_{dn}$. This selection of the "arctan" also defines the solution to be taken in [1], where Eq. (10-1) should read as follows:

$$\tan(\phi_n) = \frac{\mathrm{e}^{-\zeta_n n\pi/K_n}\sin\left(\frac{n\pi}{K_n}\sqrt{1-\zeta_n^2}\right)}{\cos(n\pi) - \mathrm{e}^{-\zeta_n n\pi/K_n}\cos\left(\frac{n\pi}{K_n}\sqrt{1-\zeta_n^2}\right)}. \tag{14}$$

However, as previously mentioned, Eq. (14) is only valid for undamped beams. Although Eq. (14) is a version of (10-1) with a corrected sign, it still does not yield the true phase angle in a damped beam for the fundamental mode (see Fig. (3)). The phase angles are given here as the solution of the inverse tangent functions located in the interval $[0,2\pi)$.

The amplitude of the free vibration is obtained from Eqs. (12) as follows:

$$X_n = \sqrt{(q_{0n})^2 + \left(\frac{\zeta_n \omega_n q_{0n} + \dot{q}_{0n}}{\omega_{dn}}\right)^2}\,. \qquad (15)$$

Since the initial conditions given in [1] are not valid for damped beams, one could expect Eq. (9-1) to be incorrect except for $\zeta_n = 0$. However, after some mathematical simplifications, the amplitude given by Eq. (15) turns out to have the same closed-form expression, regardless of whether the initial conditions are Eqs. (7a-1, 7b-1) or Eqs. (8). Therefore Eq. (9-1) is correct, as well as its nondimensional version, Eq. (14-1). For the sake of completeness, the amplitude is repeated below:

$$X_n = \frac{q_{n,st}}{\sqrt{\left(1-K_n^2\right)^2+(2\zeta_n K_n)^2}} \frac{K_n}{\sqrt{1-\zeta_n^2}} \sqrt{1 + e^{-2\zeta_n n\pi/K_n} - 2e^{-\zeta_n n\pi/K_n}\cos(n\pi)\cos\left(\frac{n\pi}{K_n}\sqrt{1-\zeta_n^2}\right)}\,. \qquad (16)$$

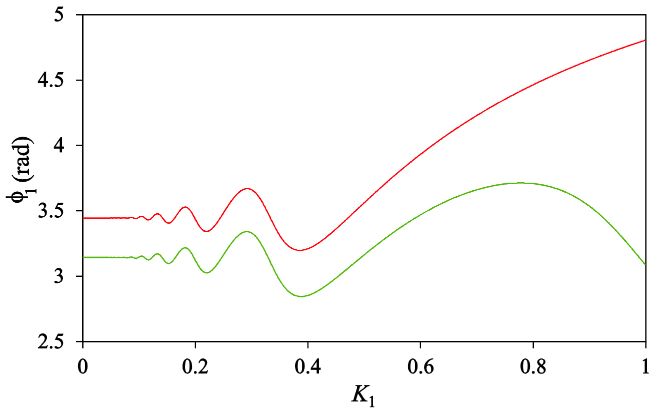

Figure 3. Phase angle ϕ_n of the free vibration ($n = 1, \zeta_n = 0.15$).
—— Correct solution from Eq. (13); —— solution from Eq. (14).

In what follows, a set of numerical values is adopted for purposes of illustration: P=220 kN, L=20 m, m=15000 kg/m, f_1=$\omega_1/2\pi$=7 Hz, ζ_n=0.15, v=120 m/s. Fig. (4) shows the end of the corresponding forced vibration time history and the beginning of the free vibration. For greater clarity, only the first modal amplitude $q_1(t)$ is plotted. The correct solution obtained from Eq. (6) gives rise to initial conditions of the free vibration as per Eqs. (8), with values $q_{01} = 6.82710 \cdot 10^{-5}$ m and $\dot{q}_{01} = -0.0213545$ m/s. Therefore, $(\zeta_1 \omega_1 q_{01} + \dot{q}_{01})/\omega_{d1} = -4.80723 \cdot 10^{-4}$ m. The phase angle is then obtained from Eq. (13), where the signs of the sine/cosine are taken into account, according to Eqs. (12): $\phi_1 = 3.28267$ rad. Finally, the amplitude is computed, based on Eq. (9-1) or Eq. (16): $X_1 = 4.85547 \cdot 10^{-4}$ m.

The solutions given in reference [1] are also depicted in Fig. (4). In this case the initial conditions are $q_{01} = -1.08818 \cdot 10^{-4}$ m and $\dot{q}_{01} = -0.0198588$ m/s. These values lead to $(\zeta_1 \omega_1 q_{01} + \dot{q}_{01})/\omega_{d1} = -4.73194 \cdot 10^{-4}$ m. Four free vibrations are shown in the figure, corresponding to four different phase angles derived from [1]. One of these vibrations features continuous displacement and velocity at $t = L/v = 1/6$ s, This curve corresponds to one of the solutions of the inverse tangent obtained from Eq. (14), particularly the one that satisfies the signs of the sine/cosine in Eqs. (12): $\phi_1 = 2.91556$ rad. The remaining three free vibration curves correspond to $\phi_1 = 2.91556 + \pi$ rad and to the two solutions obtained from Eq. (10-1).

Fig. (4) shows that these four free vibration curves have the same modulus of their initial value. The sign of the initial value is positive for one pair of curves and negative for the other pair. This fact is a direct consequence of the relations between the solutions to Eq. (14) and Eq. (10-1).

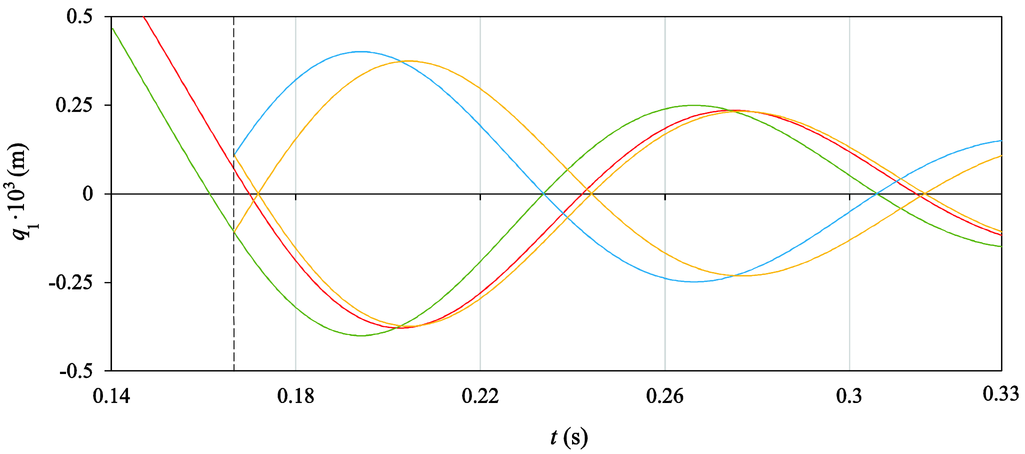

Figure 4. Forced and free vibration $q_1(t)$. ($K_1 = 0.4286, \zeta_n = 0.15$). Correct solution from Eqs: —— (6), (9-1), (13); Other solutions: —— (3), (9-1), (14); —— (3), (9-1), ((14)+π); —— (3), (9-1), ((10-1) and (10-1)+π). End of forced vibration: -----

Regarding the amplitudes given by Eqs. (14-1) and (15-1), it should be highlighted that the normalisation is carried out in Eq. (13-1) with respect to the static response of the first mode $2P/(\mu L \omega_1^2)$, whereas in reference [2], the amplitude is divided by $2P/(\mu L \omega_n^2)$ (see the paragraph before Eq. (7-2)). The relation between them is given in Eq. (2), where the term n^4 arises. This term was not taken into account by the authors of reference [1] in their criticism following Eq. (15-1).

Moreover, the last sentence in section 2.2 from reference [1] states: "This distinction, regarding the modes, has not been made in reference [15] and hence, the reported cancellation speed ratios *for the second mode have no meaning as there are no responses at all to cancel*". In reference [1], the authors purposely focused on the response at mid-span. In contrast, the approach in reference [2] targets both the contribution of even and odd modes. Both types of mode must be dealt with when it is necessary to predict the cancellation speeds of the lowest modes in an experimental test. As is known, these speeds must be avoided in order to produce significant free vibrations and thus be able to more accurately measure the damping ratio of the first mode, second mode, etc. Thus, it is important to emphasise that the cancellation speeds of even modes definitely have a practical application when it comes to testing simply supported bridges (mainly for lowest frequency modes such as the second mode).

3. Maxima and cancellations of free responses

3.1. *Conditions for maxima of free responses*

Reference [1] states that "The maximum dynamic response of a simply supported beam always occurs at its mid-span (i.e, at $x = 0.5\,L$)". In light of the results presented in section 4.4 of reference [6], that statement seems to be quite adequate from a practical viewpoint though it cannot be regarded as a general conclusion. The true maximum response could take place at sections different from $x = 0.5\,L$.

3.2. *Conditions for cancellations of free responses*

In this section of reference [1], a formula is derived for the cancellation speeds of the *n*th (odd) mode that envisages the determination of values $K_n < 1$. Eq. (21-1) and the sentence immediately below read:

$$K_n^i = \frac{n}{2j-1}, \tag{17}$$

"where $j = m + i$; $m = 2n - 1$ and i, j are positive integers". The range of values of positive integers is $i, j \geq 1$. However, as specified in the previous definitions, *j* depends on both *n* and *i*. It is indeed a positive integer, but is not independent, and its lowest value is 2. By substituting the definitions of *j* and *m* in Eq. (21-1), the result obtained is

$$K_n^i = \frac{n}{2(m+i)-1} = \frac{n}{2(2n-1+i)-1} = \frac{n}{4n+2i-3}, \tag{18}$$

where $n = 2k - 1$ (odd modes only), and i, k are positive integers. If the first four cancellations are computed for the first three odd modes from Eq. (18), the following values are obtained (see Table 1).

	***i* = 1**	***i* = 2**	***i* = 3**	***i* = 4**
***n* = 1**	0.3333	0.2000	0.1429	0.1111
***n* = 3**	0.2727	0.2308	0.2000	0.1765
***n* = 5**	0.2632	0.2381	0.2174	0.2000

Table 1. Nondimensional cancellation speeds derived from Eq. (21-1)

As can be observed, the values in Table 1 do not exactly correspond to the values in Table 1-1. Indeed, the former are a subset of the latter. The values in Table 1-1 are correct cancellation values for odd modes such that $K_n < 1$, but they cannot be obtained from Eq. (21-1). Therefore, this equation cannot be used for computing the cancellation speed ratios. Instead, for odd and even modes, Eq. (11-2) in reference [2] gives the correct results and is repeated below for completeness:

$$K_{ni}^{cancel} = \frac{n}{n \pm 2i} > 0\,, \quad i \geq 1. \tag{19}$$

3.3. *Conditions for cancellation of all modes* (*i.e., zero beam response*)

The new result presented in section 3.3 from reference [1] is of interest. Furthermore, it amends a statement in reference [2] that is not always true. More specifically, although there are cases when not all of the modes cancel simultaneously, it is indeed true that for certain velocities, the free response of all modes cancels, thus leading to a zero beam response (in undamped beams).

However, the mathematical proof given in Eqs. (28) should be completed. Eq. (28c-1) is equivalent to Eq. (20), which holds for *any real speed,* regardless of whether total cancellation takes place. Thus, Eq. (28c-1) is a necessary condition for cancellation of all modes, but it does not prove the occurrence of this type of phenomenon. This necessary condition appears in Eq. (4-2):

$$K_n = \frac{n\pi v}{\omega_n L} = \frac{K_1}{n}. \tag{20}$$

Since reference [1] states that the condition for cancellation of odd modes is given by Eq. (21-1), what needs to be demonstrated is that for every mode *n*, particular values of *j* exist such that Eq. (20) is satisfied, and that real cancellation speeds thus exist for all odd modes.

This result can be easily proven both for odd and even modes as follows. If the free vibration of all modes in an undamped beam is cancelled, then the free vibration of the fundamental mode must also vanish. Therefore, if one proves that any cancellation speed of the fundamental mode is also a cancellation speed for the rest of modes, the total zero beam response is demonstrated.

According to Eq. (19), the *i*th cancellation speed of the first mode is always less than unity and is given by

$$K_1 = \frac{1}{1+2i}, i \geq 1. \tag{21}$$

For the *n*th mode $(n > 1)$, if cancellation takes place at the same real speed then $K_n < K_1 < 1$ by virtue of Eq. (20). Thus in Eq. (19), the minus sign must be excluded and the *j*th cancellation speed is

$$K_n = \frac{n}{n+2j}, j \geq 1. \tag{22}$$

According to Eq. (20), the values of *i* and *j* are related by the mode number *n* as per

$$K_n = \frac{K_1}{n} \Rightarrow \frac{n}{n+2j} = \frac{1}{n(1+2i)}. \tag{23}$$

Therefore the *j*th cancellation order of the *n*th mode, corresponding to the same real speed as the *i*th cancellation order of the first mode, is

$$j = \frac{n^2(1+2i)-n}{2}, n > 1, i \geq 1. \tag{24}$$

It is fairly straightforward to prove that the numerator in Eq. (24) is always an even number greater than two. Thus, a positive integer value of *j* greater than one exists for each n and i value. This signifies that all free vibrations will vanish simultaneously when the first mode is cancelled. The corresponding nondimensional speeds for this phenomenon to occur are derived simply from the equations above:

$$K_n = \frac{1}{n(1+2i)}\ , n \geq 1, i \geq 1. \quad (25)$$

Conclusions

The free vibration response due to a point load moving along a simply supported Bernoulli–Euler beam was analysed in this paper. More specifically, this research presented the corrected versions of the initial conditions of the free vibration (displacement and velocity) for damped beams, and also provided the corresponding phase angle. These results make it possible to reproduce the correct values of the response after the load has passed the beam. Furthermore, this paper also provided a complete proof of the existence of total cancellation speeds for undamped beams, and highlighted the mathematical formula for computing all the cancellations for any vibration mode.

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