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Additional Information

Local convergence of a family of iterative methods for Hammerstein equations, CMMSE-15 *

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Abstract

In this paper we give a local convergence result for a uniparametric family of iterative methods for nonlinear equations in Banach spaces. We assume boundedness conditions involving only the first Fréchet derivative, instead of using boundedness conditions for high order derivatives as it is usual in studies of semilocal convergence, which is a drawback for solving some practical problems. The existence and uniqueness theorem that establishes the convergence balls of these methods is obtained. We apply this theory to different examples, including a nonlinear Hammerstein equation that have many applications in chemistry and appears in problems of electro-magnetic fluid dynamics or in the kinetic theory of gases. With these examples we illustrate the advantages of these results. The global convergence of the method is addressed by analysing the behaviour of the methods on complex polynomials of second degree.

Keywords: Nonlinear systems Iterative method Banach space Local convergence Complex dynamics Hammerstein equation

2000 Mathematics Subject Classification: 47H99, 65H10.

1 Introduction

Solving nonlinear equations is an important branch of Numerical Analysis. A great variety of problems in sciences and engineering can be modeled by ordinary differential equations, partial derivative equations, integral equations, etc. [1]. After applying the corresponding

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numerical method many of them become a nonlinear system, $F(x) = 0$, where F is a Fréchet-differentiable operator defined between Banach's spaces.

We deal with the problem of approximating a solution of nonlinear systems. Iterative methods are the common technique used to approximate the solutions.

Many iterative methods can be found in the literature, from the classical Newton or Chebyshev's methods to different variants of these, [2, 3]. In these studies, authors are interested in giving conditions to obtain a starting point that ensures convergence to a root. Precisely, one can find semilocal and local convergence results in order to estimate the radii of the convergence balls.

The main difference between these types of convergence results is that while in semilocal convergence, [7, 9, 4, 6, 10, 8, 5] one imposes conditions on the starting point x_0, in the local convergence, [12, 11], you impose conditions on the solution. Important results about the convergence domains can be obtained with either local or semilocal techniques.

However, most of the local convergence results are obtained under general conditions that, by using using Taylor's expansions, allow us to find the convergence order but not the radii of the convergence balls; see [13, 14, 15], among others.

In [5], a semilocal convergence study for a family of third and fourth order methods for nonlinear systems has been performed. A shortcoming of these proofs is that they assume boundedness conditions for high order derivatives, which can be a drawback for solving some practical problems. As can be easily observed in Example 4.1, that is a logarithmic equation, these type of equations appears for obtaining the equilibrium constants in chemical reactions.

In this paper our aim is to obtain a local convergence result for this family of iterative methods for nonlinear equations in Banach spaces, assuming boundedness conditions involving only the first Fréchet derivative, in order to obtain convergence domains. We apply this theory to different examples in order to illustrate the advantages of our results. We include a nonlinear Hammerstein equation that appears in problems of electro-magnetic fluid dynamics, in the kinetic theory of gases and in the reformulation of boundary value problems with a nonlinear boundary condition, [1, 16].

Finally, we study the dynamics of the method on complex polynomials of second degree, in order to assess the global convergence properties of the family.

2 Preliminary results

Let X, Y be Banach spaces and $F : \Omega \subseteq X \to Y$ a nonlinear operator in an open convex domain $\Omega_0 \subseteq \Omega$. We consider the family iterative methods for solving the nonlinear system $F(x) = 0$ defined by:

$$\begin{aligned} y_k &= x_k - \theta F'(x_k)^{-1} F(x_k) \\ z_k &= y_k - F'(x_k)^{-1} F(y_k) \\ x_{k+1} &= z_k - F'(x_k)^{-1} F(z_k). \end{aligned} \tag{1}$$

with $\theta \in \mathbb{R} \setminus \{0\}$. In [5], the semilocal convergence analysis of the above mentioned method was made under the following assumptions:

F is twice Fréchet differentiable, $x_0 \in \Omega_0$ is such that $\Gamma_0 = F'(x_0)^{-1}$ exists and the following conditions are verified:

$$\begin{aligned} &\|\Gamma_0\| \leq \beta \\ &\|\Gamma_0 F(x_0)\| \leq \eta \\ &\|F''(x)\| \leq M, \quad x \in \Omega_0 \\ &\|F''(x) - F''(y)\| \leq K\|x - y\|, \quad x, y \in \Omega_0. \end{aligned} \tag{2}$$

These conditions involve the existence and boundedness of the Fréchet derivatives $F^{(i)}$ of order $i = 1, 2, 3$, which makes the result very restrictive and difficult to apply to some problems. Our aim is to give a local convergence result for this iterative method relaxing the assumptions of the semilocal convergence case.

From now on, we denote by $B(v, \rho)$, $\overline{B}(v, \rho)$, the open, respectively closed, ball in X with center v and radius $\rho > 0$.

3 Local convergence analysis

In the local convergence analysis of an iterative method, you impose conditions on the values of F and its derivatives at the solution and obtain a ball centered in the solution such that each point in it can be taken as an starting point for the iterative method. Our aim is to increase the radius of this ball.

In a local convergence study, the conditions are usually expressed in the following way:

$F : \Omega_0 \subseteq X \to Y$ is a Fréchet differentiable operator. argyros2015study argyros2015localhl $L_0 > 0$, $L > 0$ and $\theta \in (-\infty, \infty) - \{0\}$ are real numbers verifying that there exists $x^* \in \Omega_0$ such that, $F(x^*) = 0$, and $F'(x^*)^{-1} \in L(Y, X)$ and for all $x, y \in \Omega_0$ the following holds:

$$\begin{aligned} \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| &\leq L_0\|x - x^*\| \\ \|F'(x^*)^{-1}(F'(x) - F'(y))\| &\leq L\|x - y\|. \end{aligned} \tag{3}$$

Always, a third assumption is made, as can be observed in different papers, see [21, 22], that can be written as follows:

$$\|F'(x^*)^{-1}F'(x)\| \leq M, \tag{4}$$

for some $M > 0$.

Considering the remark made in [21], we can drop this condition. Moreover, a difference from this work and as far as we know, this is the first time that the whole process for establishing local convergence is develop without using constant M imposed in (4). This fact allows us to improve the value of the radius of the convergence ball. For this purpose, we prove the following result:

Lemma 3.1 *If operator F verifies all conditions assumed in* (3) *then, the following bounds also hold for all $x \in \Omega_0$ and $t \in [0,1]$:*

$$
\begin{aligned}
\|F'(x^*)^{-1}F'(x)\| &\leq 1 + L_0\|x - x^*\| \\
\|F'(x^*)^{-1}(F'(x^* + t(x - x^*)\| &\leq 1 + L_0\|x - x^*\| \\
\|F'(x^*)^{-1}F(x)\| &\leq (1 + L_0\|x - x^*\|)\|x - x^*\|
\end{aligned}
\tag{5}
$$

Proof: By using the first condition of (3) we have:

$$
\begin{aligned}
\|F'(x^*)^{-1}F'(x)\| = \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I)\| &\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \\
&\leq 1 + L_0\|x - x^*\|,
\end{aligned}
$$

and then it follows that:

$$
\|F'(x^*)^{-1}(F'(x^* + t(x - x^*)))\| \leq 1 + L_0 t\|x - x^*\| \leq 1 + L_0\|x - x^*\|,
$$

being $0 < t < 1$.

Finally, we use the mean value theorem, so

$$
\begin{aligned}
\|F'(x^*)^{-1}F(x)\| = \|F'(x^*)^{-1}(F(x) - F(x^*))\| &\leq \|F'(x^*)^{-1}F'(x^* + t(x - x^*))(x - x^*)\| \\
&\leq (\|1 + L_0\|x - x^*\|)\|x - x^*\|.
\end{aligned}
$$

Now we analyse the family of iterative methods, (1), we look for a ball centred at the solution so that any point of this ball can be taking as starting point for the iterative method and the sequence generates remains in this ball and converges to the solution, specifically we want determine de radius of this ball.

So if we denote $x_0 \in \Omega_0$ the starting point and use the assumption (3) we establish by Banach's lemma the first restriction for the domain of convergence:

$$
\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \leq L_0\|x_0 - x^*\|,
$$

so, we impose that $\|x_0 - x^*\| < \frac{1}{L_0}$, as a result by Banach's Lemma on invertible operators, $F'(x_0)^{-1}$ exists in $L(Y, X)$ and verifies:

$$
\|F'(x_0)^{-1}F'(x^*)\| \leq \frac{1}{1 - L_0\|x_0 - x^*\|}. \tag{6}
$$

Therefore, the method is well defined and now we look for bounding the iterates:

$$
\begin{aligned}
y_0 - x^* &= x_0 - x^* - \theta F'(x_0)^{-1}F(x_0) \\
&= x_0 - x^* - F'(x_0)^{-1}F(x_0) + (1 - \theta)F'(x_0)^{-1}F(x_0) \\
&= -F'(x_0)^{-1}\left(F(x_0) - F'(x_0)(x_0 - x^*)\right) + (1 - \theta)F'(x_0)^{-1}F(x_0) \\
&= -F'(x_0)^{-1}F'(x^*)\int_0^1 F'(x^*)^{-1}[F'(x^* + t(x_0 - x^*)) - F'(x_0)](x_0 - x^*)dt \\
&+ (1 - \theta)F'(x_0)^{-1}F'(x^*)\int_0^1 F'(x^*)^{-1}F'(x^* + t(x_0 - x^*))(x_0 - x^*)dt
\end{aligned}
$$

By taking norms and using (3) and (3.1), we get:

$$\begin{aligned}
\|y_0 - x^*\| &\leq \|F'(x_0)^{-1}F'(x^*)\| \left\| \int_0^1 F'(x^*)^{-1}[F'(x^* + t(x_0 - x^*)) - F'(x_0)](x_0 - x^*)dt \right\| \\
&+ |1-\theta| \|F'(x_0)^{-1}F'(x^*)\| \left\| \int_0^1 F'(x^*)^{-1}F'(x^* + t(x_0 - x^*))(x_0 - x^*)dt \right\| \\
&\leq \frac{1}{1 - L_0\|x_0 - x^*\|} \left[\frac{L}{2}\|x_0 - x^*\| + |1-\theta|(1 + L_0\|x_0 - x^*\|) \right] \|x_0 - x^*\| \\
&\leq g_1(\|x_0 - x^*\|)\|x_0 - x^*\|, \qquad (7)
\end{aligned}$$

where we have used function $g_1(t)$ defined in $[0, 1/L_0]$ as follows:

$$g_1(t) = \frac{1}{1 - L_0 t} \left(\frac{L}{2}t + |1-\theta|(1 + L_0 t) \right).$$

Now, in order to analyze function g1 we define a new function $h_1(t) = g_1(t) - 1$ resulting that $h_1(0) = |1-\theta| - 1 < 0$ if $\theta \in]0, 2[$ and $h_1(1/L_0) \to +\infty$, so by Bolzano's theorem, we take r_1 the smallest root of $h_1(t)$ in $]0, 1/L_0[$ and then we have that:

$$0 \leq g_1(t) \leq 1, \;\; \forall t \in [0, r_1].$$

So, if we come back to (7), we have that:

$$\|y_0 - x^*\| \;\; \leq g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\|.$$

Then, we study bounds for the second step in the first iteration of (1). By using (3), (3.1) and (7) we have:

$$\begin{aligned}
\|z_0 - x^*\| &\leq \|y_0 - x^*\| + \|F'(x_0)^{-1}F(y_0)\| \\
&\leq \|y_0 - x^*\| + \|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(y_0)\|, \\
&\leq \left(1 + \frac{1 + L_0\|y_0 - x^*\|}{1 - L_0\|x_0 - x^*\|} \right) \|y_0 - x^*\| \qquad (8) \\
&\leq \left(1 + \frac{1 + L_0 g_1(\|x_0 - x^*\|)\|x_0 - x^*\|)}{1 - L_0\|x_0 - x^*\|} \right) g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \\
&= g_2\left(\|x_0 - x^*\|\right)\|x_0 - x^*\|,
\end{aligned}$$

where we have used the function $g_2(t)$ defined in $[0, r_1]$ as follows:

$$g_2(t) = (1 + \frac{1 + L_0 t g_1(t)}{1 - L_0 t}) g_1(t).$$

Now, we consider function $h_2(t) = g_2(t) - 1$ resulting that $h_2(0) = 2|1-\theta| - 1 < 0$ if $\theta \in]\frac{1}{2}, \frac{3}{2}[$ and $h_2(r_1) = \frac{2}{1 - L_0 r_1} - 1 > 0$, so by Bolzano's theorem we take r_2 the smallest root of $h_2(t)$ in $]0, r_1[$ and thus we have:

$$0 \leq g_2(t) \leq 1, \;\; \forall t \in [0, r_2].$$

So, coming back to (8), we have:

$$\|z_0 - x^*\| \quad \leq g_2(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\|.$$

Finally, we analyze the last step:

$$\begin{aligned}
\|x_1 - x^*\| &\leq \|z_0 - x^*\| + \|F'(x_0)^{-1}F(z_0)\| \\
&\leq \|z_0 - x^*\| + \|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(z_0)\|, \\
&\leq \left(1 + \frac{1 + L_0\|z_0 - x^*\|}{1 - L_0\|x_0 - x^*\|}\right)\|z_0 - x^*\| \\
&\leq \left(1 + \frac{1 + L_0 g_2(\|x_0 - x^*\|)\|x_0 - x^*\|}{1 - L_0\|x_0 - x^*\|}\right) g_2(\|x_0 - x^*\|)\|x_0 - x^*\| \\
&= g_3(\|x_0 - x^*\|)\|x_0 - x^*\|,
\end{aligned} \tag{9}$$

where we have used the function $g_3(t)$ defined in $[0, r_2]$ as follows:

$$g_3(t) = \left(1 + \frac{1 + L_0 t g_2(t)}{1 - L_0 t}\right) g_2(t).$$

Now, we consider function $h_3(t) = g_3(t) - 1$ resulting that $h_3(0) = 2g_2(0) - 1 < 4|1-\theta| - 1 < 0$ if $\theta \in]\frac{3}{4}, \frac{5}{4}[$ and $h_3(r_2) = \frac{2}{1 - L_0 r_2} - 1 > 0$ so by Bolzano's theorem we take r_3 the smallest root of $h_3(t)$ in $]0, r_2[$ and so we have:

$$0 \leq g_3(t) \leq 1, \quad \forall t \in [0, r_3].$$

So, if we come back to (9), we have:

$$\|x_1 - x^*\| \quad \leq g_3(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\|.$$

We can conclude that for values of $\theta \in]\frac{3}{4}, \frac{5}{4}[$ we have $0 < r_3 < r_2 < r_1 < 1/L_0$ so that, by taking

$$0 < r \leq r_3 : B(x^*, r) \subseteq \Omega_0 \tag{10}$$

and starting from any $x_0 \in B(x^*, r)$ the following relations hold:

$$\begin{aligned}
\|y_0 - x^*\| &\leq g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \\
\|z_0 - x^*\| &\leq g_2(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \\
\|x_1 - x^*\| &\leq g_3(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r
\end{aligned}$$

and thus, the iterates $y_0,\ z_0$ and x_1 remain in $B(x^*, r)$.

The following theorem describes the local convergence analysis of the family of iterative methods (1) using the definitions of the above functions and parameters.

Theorem 3.1 *Let $F : \Omega_0 \subseteq X \to Y$ be a Fréchet differentiable operator. Suppose that $L_0 > 0$, $L > 0$ and $\theta \in]\frac{3}{4}, \frac{5}{4}[$ are real numbers such that there exist $x^* \in \Omega_0$ verifying (3), and let r defined by (10). Then, the sequence $\{x_k\}$ obtained by (1) is well defined for $x_0 \in B(x^*, r)$, remains in $B(x^*, r)$ for each $k = 0, 1, 2, \ldots$ and converges to x^*.*

Furthermore, if there exists $R \in [r, \frac{2}{L_0})$ such that $\overline{B}(x^*, R) \subseteq \Omega_0$, then the limit point x^* is the only solution of the equation $F(x) = 0$ in $\overline{B}(x^*, R)$.

Proof: Obviously the whole process we have presented starting by x_0 obtainig x_1 can be exactly deduced starting from x_k obtaining x_{k+1}, just by substituting x_0, y_0, z_0 and x_1 by x_k, y_k, z_k, and x_{k+1} in the preceding study, we obtain that all the iterates remain in $B(x^*, r)$, verifying for each $k = 0, 1, 2, \ldots,$ the following inequalities:

$$\begin{aligned} \|y_k - x^*\| &\leq g_1(\|x_k - x^*\|)\|x_k - x^*\| \leq \|x_k - x^*\| < r, \\ \|z_k - x^*\| &\leq g_2(\|x_k - x^*\|)\|x_k - x^*\| \leq \|x_k - x^*\| < r, \\ \|x_{k+1} - x^*\| &\leq g_3(\|x_k - x^*\|)\|x_k - x^*\| \leq \|x_k - x^*\| < r \end{aligned}$$

Is easy to obtain that function g_3 is increasing in its domain, so we have:

$$\begin{aligned} \|x_{k+1} - x^*\| \leq g_3(r)\|x_k - x^*\| &\leq g_3(r) g_3(\|x_{k-1} - x^*\|)\|x_{k-1} - x^*\| \\ &\leq g_3(r)^2 g_3(\|x_{k-2} - x^*\|)\|x_{k-2} - x^*\| \leq \ldots \leq g_3(r)^{k+1}\|x_0 - x^*\|. \end{aligned}$$

Then, by taking limits in the last expression and using that $\lim_{k\to\infty} g_3(r)^{k+1} = 0$, we have $\lim\limits_{k\to\infty} x_k = x^*$, and so, the method converges to the solution.

In order to prove the uniqueness part, let $y^* \in B(x^*, R)$, $y^* \neq x^*$ with $F(y^*) = 0$. Let us consider the integral operator $T = \displaystyle\int_0^1 F'(y^* + t(x^* - y^*))dt$. Then by using (3), we have

$$\|F'(x^*)^{-1}(T - F'(x^*))\| \leq \int_0^1 L_0\|y^* + t(x^* - y^*) - x^*\|dt \leq \frac{L_0}{2}\|x^* - y^*\| = \frac{L_0}{2}R < 1,$$

therefore by Banach's Lemma, T^{-1} exists. Then, from the identity

$$0 = F(x^*) - F(y^*) = T(x^* - y^*),$$

we obtain $x^* = y^*$.

4 Numerical examples

In this section the convergence ball for approximating solutions of some nonlinear equations by using methods of the family given by (1) is obtained under weaker hypotheses than before. While the semilocal study presented in [5] involves boundedness conditions for high order derivatives, here we only impose the boundedness condition on the first Fréchet derivative (compare conditions (2) with (3)). Most of the examples have been taken from the literature in order to compare the obtained results.

Example 4.1 *Let function f defined on $D = [-\frac{1}{2}, \frac{5}{2}]$ by*

$$f(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

First we consider a logarithmic equation that is the typical example in studies of local convergence. The successive derivatives are:

$$\begin{aligned} f'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \\ f''(x) &= 6x^2 \ln x^2 + 20x^3 - 12x^3 + 10x, \\ f'''(x) &= 6 \ln x^2 + 60x^2 - 24x + 22. \end{aligned}$$

It can be easily observed that f''' is unbounded on D. So, the results of semilocal convergence involving (2) cannot be applied. However, by applying theorem 3.1 with $x^* = 1$, we have $L_0 = L = 96.6628$. Taking $\theta = 1$, we get:

$$r_3 = 0.002611 < r_2 = 0.004206 < r_1 = 0.006897.$$

Example 4.2 *We study now a nonlinear integral equation of Hammerstein type. These equations have many applications in chemistry and appear in problems of electro-magnetic fluid dynamics, in kinetic theory of gases, and in the reformulation of boundary value problems, [1],[16]. This equation is of the form:*

$$x(s) = u(s) + \int_a^b G(s,t)H(x(t))\,dt, \quad a \leq s \leq b,$$

for $x(s), u(s) \in C[a,b]$ with $-\infty < a < b < \infty$. G is the Green function and H is a polynomial function.

The usual technique to solve these kind of equations consists in expressing it as a nonlinear operator in a Banach space, that is:

$$F(x) = 0,$$

where $F : \Omega \subseteq C[a,b] \to C[a,b]$ with Ω a non-empty open convex subset,

$$[F(x)](s) = x(s) - u(s) - \int_a^b G(s,t)H(x(t))\,dt,$$

considering the uniform norm $\|\nu\| = \max_{s\in[a,b]} |\nu(s)|$.

Observe that in most cases boundedness conditions like (2) cannot be satisfied since $\|F''(x)\|$ or $\|F'''(x)\|$ can be unbounded in a general domain. Thus, an alternative is looking a domain that contains the solution. But it is more convenient using the local convergence results obtained in our study in order to give the radius of a convergence ball.

We apply our theoretical study presented in theorem 3.1 to the particular Hammerstein equation given by:

$$F(x(s)) = x(s) - 5\int_0^1 s\,t\,x(t)^3 dt, \tag{11}$$

with $x(s)$ in $\mathcal{C}[0,1]$. The derivative can be written by:

$$F'(x(s))v(s) = v(s) - 15\int_0^1 s\,t\,x(t)^2 v(s)dt, \tag{12}$$

One solution of this problem is the null function so it is easy to find different values for Lipschitz constans $L_0 = 7.5$ and $L = 15$. By choosing the iterative method from (1) corresponding to $\theta = 1$, the following results are obtained:

$$r_3 = 0.021860 < r_2 = 0.037592 < r_1 = 0.066667.$$

Example 4.3 *Let $X = Y = \mathbb{R}$. Define F on $D = [1,3]$ by*

$$F(x) = \frac{2}{3}x^{\frac{3}{2}} - x$$

Then, $x^* = \frac{9}{4}$, $F'(x^*)^{-1} = 2$, $L_0 = L = 1$.
Choosing

$$\theta = 0.9870,$$

we have

$$r_3 = 0.234583 < r_2 = 0.390015 < r_1 = 0.652346.$$

Example 4.4 *Let $X = Y = \mathbb{R}^3$, $D = \overline{U}(0,1)$. Define F on D for $v = (x,y,z)$ by*

$$F(v) = \left(e^x - 1, \frac{e-1}{2}y^2 + y, z\right).$$

Then the Fréchet derivative is given by

$$F'(v) = \begin{pmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Then, $x^* = (0,0,0)$, $F'(x^*) = F'(x^*)^{-1} = diag\{1,1,1\}$, $L_0 = e-1$, $L = e$, $M = e$. Choosing

$$\theta = 1.0125,$$

we have

$$r_3 = 0.104544 < r_2 = 0.181094 < r_1 = 0.318661.$$

4.1 More results

In Table 4.1 we show for the previous examples the radius of the convergence ball centered at the solution for different values of theta in our family of iterative methods. As you can see the largest radius always corresponds to theta equal one.

Examples	$\theta = 0.85$	$\theta = 0.95$	$\theta = 1$	$\theta = 1.24$
4.1	8.56×10^{-4}	1.942×10^{-3}	2.611×10^{-3}	7.7×10^{-5}
4.2	7.759×10^{-3}	1.6769×10^{-2}	2.1860×10^{-2}	7.28×10^{-4}
4.3	8.2782×10^{-2}	0.187715	0.252388	7.516×10^{-3}
4.4	3.8647×10^{-2}	0.084765	0.111497	3.588×10^{-3}

Table 1: Radii of convergence balls

4.2 Comparing results

Now we compare our results with the ones of recent published papers. First, we consider the method defined in [21] that is a family of Chebyshev-Halley-type methods free from second derivative given by:

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k) \\ z_k &= x_k - \Big(1 + (F(x_k) - 2\alpha F(y_k))^{-1}F(y_k)\Big)F'(x_k)^{-1}F(x_k) \\ x_{k+1} &= z_k - \Big(F'(x_k) + \bar{F}''(x_k)(z_k - x_k)\Big)^{-1}F(z_k). \end{aligned} \tag{13}$$

where $\bar{F}''(x_k) = 2F(y_k)F'(x_k)^2F(x_k)^{-2}$. The order of convergence of these methods was shown to be at least five and for any value of parameter α and if $\alpha = 1$, the order is six.

For this method to work the following condition should be satisfied

$$0 < |\alpha| ML < 1$$

which holds for small values of α. For $\alpha = 1$, this condition is not satisfied for the considered examples. We have to consider smaller values of α that the authors have considered, we have taken $\alpha = 0.00006, 0.0075, 0.009, 0.125$; respectively for examples from 1 to 4.

The second method with which we will compare our results defined in [22] has the following iterative function:

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k) \\ u_k &= y_k + (1-\theta)F'(x_k)^{-1}F(x_k) \\ z_k &= y_k - \gamma A_{\theta,n}F'(x_k)^{-1}F(x_k) \\ x_{k+1} &= z_k - \alpha B_{\theta,n}F'(x_k)^{-1}F(z_k). \end{aligned} \tag{14}$$

where $\alpha, \gamma, \theta \in (\infty, \infty) - \{0\}$, $H_{\theta,n} = \frac{1}{\theta}F'(x_k)^{-1}\Big(F'(u_k) - F'(x_k)\Big)$, $A_{\theta,n} = I - \frac{1}{2}H_{\theta,n}\Big(I - \frac{1}{2}H_{\theta,n}\Big)$ and $B_{\theta,n} = I - H_{1,n} + H_{\theta,n}^2$.

For this method to work, the following conditions should be satisfied.

$$\begin{aligned} M|1-\theta| &< 1 \\ M|\gamma| &< 1 \\ \Big(1 + |\alpha|M\Big)|\gamma|M &< 1 \end{aligned}$$

Examples	θ	γ	α
4.1	1	0.005	0.008
4.2	1	0.575	0.03
4.3	0.987	3/5	0.001
4.4	1.0125	0.3	0.03

Table 2: Value of the parameters

We have chosen the values of these parameters, see Table 4.2, in such a way that above conditions are satisfied.

From Table 4.2, it can be observed that the present method gives larger radii of convergence than the existing methods. So we conclude that the fact of dropping constant M in the local convergence study has the advantage of obtaining greater convergence balls.

Examples	Present method	Method (13)	Method (14)
4.1	0.002611	4.4×10^{-9}	2.501628×10^{-4}
4.2	0.021860	0.007218	0.004999
4.3	0.234583	0.105844	0.059554
4.4	0.104544	0.001728	0.027226

Table 3: Comparison of radii of the convergence balls

5 Dynamics

The dynamics of the family of iterative methods 1 has been studied in [5] for systems of nonlinear equations in the real plane. Here we study its dynamical behaviour for complex polynomials of second degree proving scaling and conjugacy results. Similar studies have been performed in [18, 19, 20] for other families of iterative methods. The dynamics of the relaxed Newton's method has been studied in [17].

The iterates obtained starting from $z_0 \in \mathbb{C}$ can be denoted by $\{z_0, R(z_0), R^2(z_0), \ldots, R^n(z_0), \ldots\}$, where R is a rational function defined on the Riemann sphere $\hat{\mathbb{C}}$. This set is called the *orbit* of z_0.

Let $z \in \hat{\mathbb{C}}$ be a *fixed point* of the rational function R, that is to say $R(z) = z$. The *basin of attraction* of z consists of the points whose orbit tends to z. The behaviour of the orbits near a fixed point z depends on the derivative $R'(z)$. If $|R'(z)| < 1$, the fixed point z is *attracting* and if $|R'(z)| > 1$, it is *repelling*. If $R'(z) = 0$, the fixed point is *superattracting*.

The set of points $z_0 \in \hat{\mathbb{C}}$ such that their families $R_n(z_0)$, $n \in \mathbb{N}$ are normal in some neighbourhood $U(z_0)$ is the *Fatou set*, $\mathcal{F}(R)$ and its complement in $\hat{\mathbb{C}}$ is the *Julia set* $\mathcal{J}(R)$. Roughly speaking, the orbits of the points in $\mathcal{F}(R)$ present a stable behaviour whereas the orbits of the points in $\mathcal{J}(R)$ have chaotic behaviour. In particular, the Fatou set contains the attraction basins of the attracting fixed points whereas the Julia set contains the boundaries of the attraction basins.

Given an analytic function $f(z)$, consider the function associated to a step of the iterative method (1) $M_f : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$, such that $M_f(x_k) = x_{k+1}$. The following scaling result holds for M_f:

Theorem 5.1 *Let f be an analytic function on $\hat{\mathbb{C}}$, and $A(z) = \alpha z + \beta$, with $\alpha \neq 0$, an affine map. If $g(z) = \lambda(f \circ A)(z)$, $\lambda \neq 0$, then M_f is analytically conjugated to M_g by A, that is, $A \circ M_g \circ A^{-1} = M_f$.*

Any polynomial of second degree is conjugated by an affine transformation to a polynomial of the form $f(z) = z^2 + c$, $c \in \mathbb{C}$, so that in order to study the dynamics of M_f on quadratic polynomials, it suffices to consider only polynomials of this form.

Then, if $f(z) = z^2 + c$, with $c \neq 0$, M_f has the form

$$M_f(z) = -\frac{16z^4\left(c+z^2\right)\left(c+5z^2\right) + 8z^2\left(c+z^2\right)^3\theta^2 + \left(c+z^2\right)^4\theta^4}{128z^7}. \tag{15}$$

The equation $M_f(z) = z$ can be written as

$$\frac{16z^4\left(c+z^2\right)\left(c+5z^2\right) + 8z^2\left(c+z^2\right)^3\theta^2 + \left(c+z^2\right)^4\theta^4}{128z^7} = 0, \tag{16}$$

so that, M_f has eight fixed points. Six of them depend on θ and the two remaining are the roots of $f(z)$, $\pm\sqrt{-c}$, which do not depend on θ. These two fixed points are

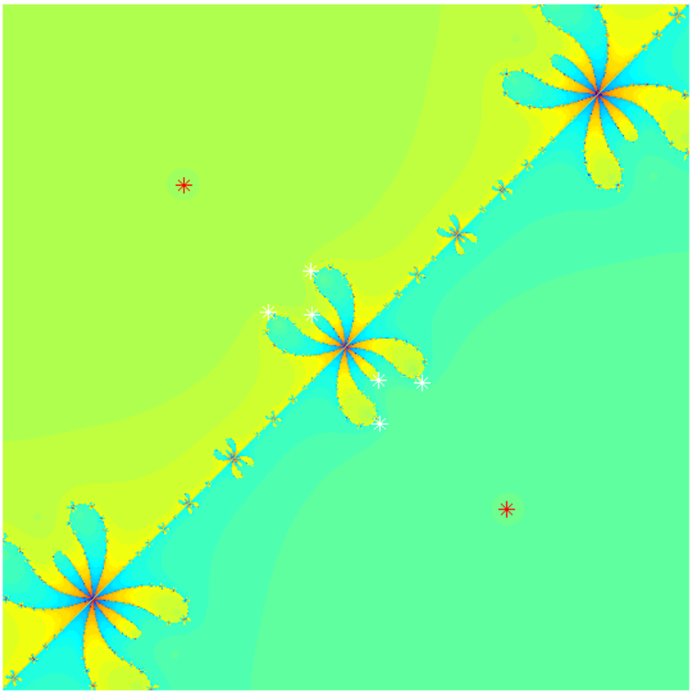

Figure 1: Attraction basins for $c = i$ and $\theta = 0.5$.

superattracting, because

$$M_f'(z) = \frac{\left(c+z^2\right)^2\left(48z^4 - 8z^2\left(-5c+z^2\right)\theta^2 + \left(7c-z^2\right)\left(c+z^2\right)\theta^4\right)}{128z^8} \tag{17}$$

and then, $M_f'(\pm\sqrt{-c}) = 0$. The character of the remaining fixed points depends on θ. Figures 1 and 2 show the attraction basins of M_f for $f(z) = z^2 + i$ and values of θ for which the roots of the polynomial are the only attracting fixed points. The roots, marked in red, are in its attraction basin and the other repelling fixed pints, marked in white, are in the boundary of the basins, in the Julia set.

The dynamical study can be simplified further by using the idea of analytical conjugation. If B(z) is a Möbius map

$$B(z) = \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \alpha\delta - \beta\gamma \neq 0, \tag{18}$$

the rational maps M and N are analytically conjugated via B if $N = BMB^{-1}$. Then, $\mathcal{F}(N) = B(\mathcal{F}(M)$ and $\mathcal{J}(N) = B(\mathcal{J}(M)$.

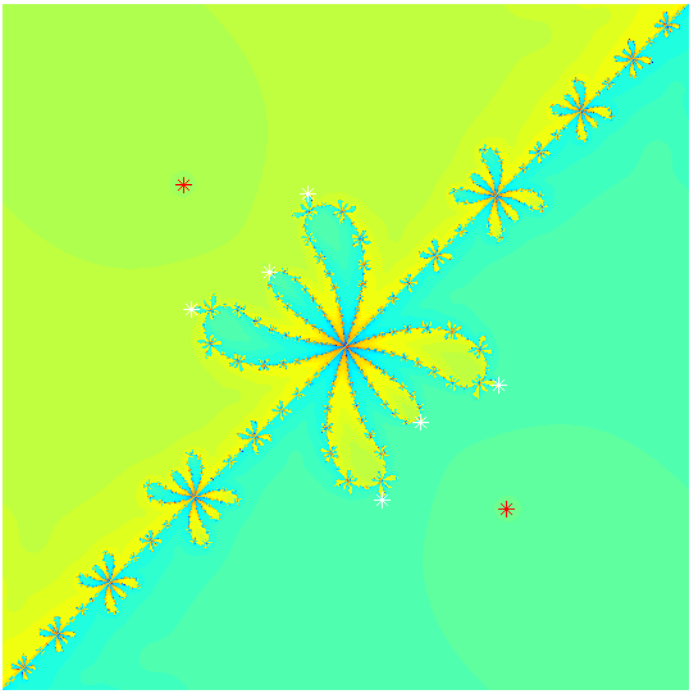

Figure 2: Attraction basins for $c = i$ and $\theta = 0.5$.

Theorem 5.2 *Let $f(z)$ be a quadratic polynomial with simple roots. The fixed point operator $M_f(z)$ associated to the family of iterative methods* (1) *verifies:*

1. *$M_f(z)$ is analytically conjugated with*

$$N_f(z) = \frac{z^3\left(-(1+z)^4(2+z) + 2(1+z)^2\theta^2 + z\theta^4\right)}{-(1+z)^4(1+2z) + 2z^3(1+z)^2\theta^2 + z^4\theta^4}. \tag{19}$$

2. *The Julia set of this operator contains the unit circle.*

3. *The Fatou set consists of the attraction basins of 0 and infinity. Both are superattracting fixed points.*

Proof:

1. Due to the scaling theorem 5.1, we suppose $f(z) = z^2 + c$. Then, the Möbius transform
$$B(z) = \frac{z - \sqrt{-c}}{z + \sqrt{-c}}$$
has the following properties:
$$B(\infty) = 1, \quad B(\sqrt{-c}) = 0, \text{ and } B(-\sqrt{-c}) = \infty.$$
By conjugating M_f with B one gets (19), which does not depend on c.

2. It is easy to check that the unit circle $z : |z| = 1$ is invariant under N_f.

3. Expression (19) shows that 0 and ∞ are superattracting fixed points of third order.

Figures 3 and 4 depict the Julia sets of N_f for $\theta = 0.5$ and $\theta = 1.5$, respectively. The Julia set of Newton's method for polynomials of second degree reduces to the unit circle. In our case, the set is more involved, but still includes it.

The figures show that most of the points in the complex plane belong to the attraction basin of a root, so that, they are suitable as starting points for the iterations of the considered family of methods, when dealing with quadratic polynomials. This implies a good global behaviour of the family, in comparison with other iterative methods.

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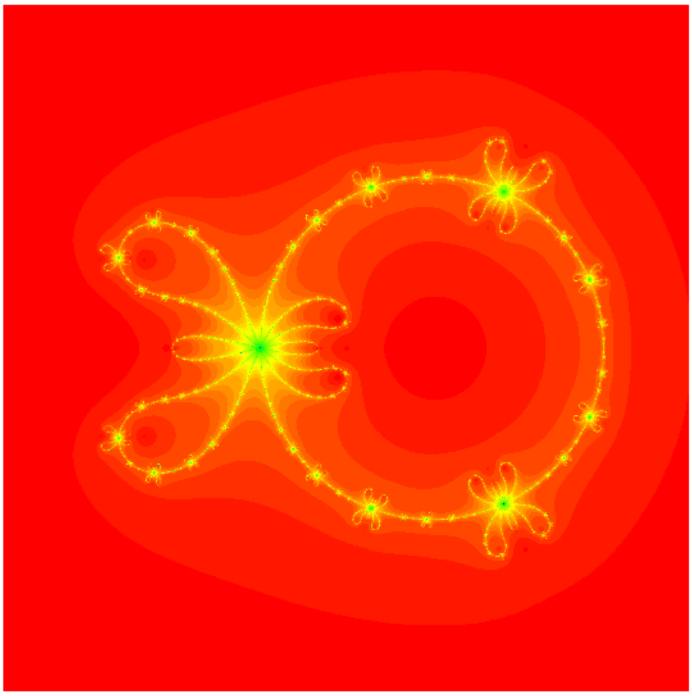

Figure 3: Julia set of N_f for $\theta = 0.5$

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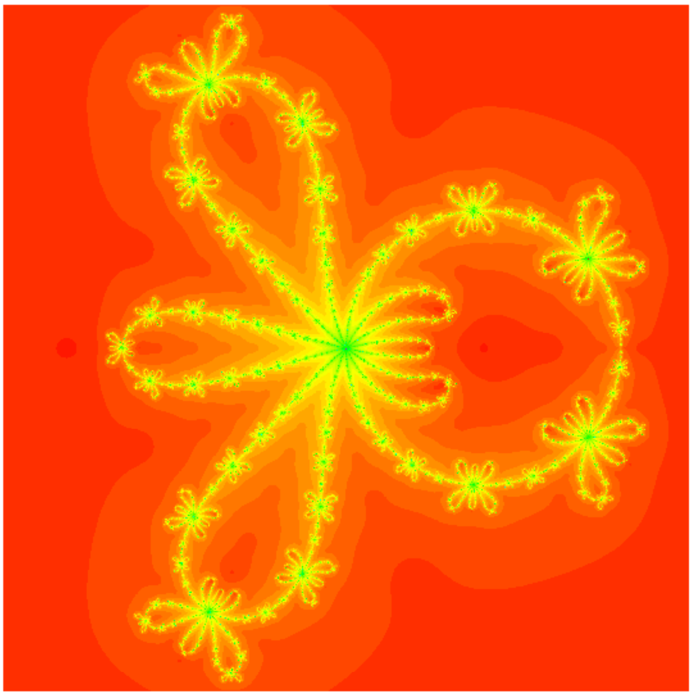

Figure 4: Julia set of N_f for $\theta = 1.5$

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