Nikodym boundedness property for webs in σ-algebras*

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Abstract

A subset $\mathscr{B}$ of an algebra $\mathscr{A}$ of subsets of Ω is said to have the *property N* if a $\mathscr{B}$-pointwise bounded subset M of $\mathrm{ba}(\mathscr{A})$ is uniformly bounded on $\mathscr{A}$, where $\mathrm{ba}(\mathscr{A})$ is the Banach space of the real (or complex) finitely additive measures of bounded variation defined on $\mathscr{A}$ with the norm variation. Moreover $\mathscr{B}$ is said to have the *property sN* if for each increasing countable covering $(\mathscr{B}_m)_m$ of $\mathscr{B}$ there exists $\mathscr{B}_n$ which has the property N and $\mathscr{B}$ is said to have *property wN* if given the increasing countable coverings $(\mathscr{B}_{m_1})_{m_1}$ of $\mathscr{B}$ and $(\mathscr{B}_{m_1m_2\ldots m_pm_{p+1}})_{m_{p+1}}$ of $\mathscr{B}_{m_1m_2\ldots m_p}$, for each $p, m_i \in \mathbb{N}$, $1 \leqslant i \leqslant p+1$, there exists a sequence $(n_i)_i$ such that each $\mathscr{B}_{n_1n_2\ldots n_r}$, $r \in \mathbb{N}$, has property N. For a σ-algebra $\mathscr{S}$ of subsets of Ω it has been proved that $\mathscr{S}$ has property N (Nikodym-Grothendieck), property sN (Valdivia) and property $w(sN)$ (Kakol-López-Pellicer). We give a proof of property wN for a σ-algebra $\mathscr{S}$ which is independent of properties N and sN. This result and the equivalence of properties wN and w^2N enable us to give some applications to localization of bounded additive vector measures.

Keywords: Bounded set; finitely additive scalar (vector) measure; inductive limit; NV-tree; σ-algebra; web Nikodym property

MSC: 28A60, 46G10

*This work was supported for the second named author by the Spanish Ministerio de Economía y Competitividad under grant MTM2014-58159-P.

1 Introduction

Let Ω be a set and $\mathscr{A}$ a set-algebra of subsets of Ω. If $\mathscr{B}$ is a subset of $\mathscr{A}$ then $L(\mathscr{B})$ is the normed space of the real or complex linear hull of the set of characteristics functions $\{e_C : C \in \mathscr{B}\}$ endowed with the supremum norm $\|\cdot\|$. The dual of $L(\mathscr{A})$ with the dual norm is named $L(\mathscr{A})'$ and it is isometric to the Banach space $\mathrm{ba}(\mathscr{A})$ of finitely additive measures on $\mathscr{A}$ with bounded variation provided with the variation norm, i.e., $|\cdot| := |\cdot|(\Omega)$, being the isometry the map Θ: $\mathrm{ba}(\mathscr{A}) \to L(\mathscr{A})'$ such that, for each $\mu \in \mathrm{ba}(\mathscr{A})$, $\Theta(\mu)$ is the linear form named also by μ and defined by $\mu(e_C) := \mu(C)$, for each $C \in \mathscr{A}$, [2, Chpater 1]. A norm in $L(\mathscr{A})$ equivalent to the supremum norm is defined by the Minkowski functional of $\mathrm{absco}(\{e_C : C \in \mathscr{A}\})$ ([12, Propositions 1 and 2]), which dual norm is the $\mathscr{A}$-supremum norm, i.e., $\|\mu\| := \sup\{|\mu(C)| : C \in \mathscr{A}\}$, $\mu \in \mathrm{ba}(\mathscr{A})$.

In this paper duality is referred to the dual pair $\langle L(\mathscr{A}), \mathrm{ba}(\mathscr{A})\rangle$ and we follow notations of [7]. Then the weak * dual of a locally convex space E is $(E', \tau_s(E))$, whence the topology $\tau_s(L(\mathscr{A}))$ is the topology $\tau_s(\mathscr{A})$ of pointwise convergence in the elements of $\mathscr{A}$, the cardinal of a set C is denoted by $|C|$, $\mathbb{N}$ is the set $\{1, 2, \ldots\}$ of positive integers, the closure of a set is marked by an overline, the convex (absolutely convex) hull of a subset M of a topological vector space is represented by $\mathrm{co}(M)$ ($\mathrm{absco}(M)$) and $\mathrm{absco}(M) = \mathrm{co}(\cup\{rM : |r| = 1\})$.

A subset $\mathscr{B}$ of a set-algebra $\mathscr{A}$ has the Nikodym property*, property N in brief, if each $\mathscr{B}$-pointwise bounded subset M of* $\mathrm{ba}(\mathscr{A})$ *is bounded in* $\mathrm{ba}(\mathscr{A})$ (see [10, Definition 2.4] or [13, Definition 1]). If $\mathscr{B}$ has property N the polar set $\{e_C : C \in \mathscr{B}\}^\circ$ is bounded in $\mathrm{ba}(\mathscr{A})$, hence the bipolar set $\{e_C : C \in \mathscr{B}\}^{\circ\circ} = \overline{\mathrm{absco}\{e_C : C \in \mathscr{B}\}}$ is a neighborhood of zero in $L(\mathscr{A})$ and then $L(\mathscr{B})$ is dense in $L(\mathscr{A})$. Notice also that a subset $\mathscr{B}$ of an algebra $\mathscr{A}$ has property N if each $\mathscr{B}$-pointwise bounded, $\tau_s(A)$-closed and absolutely convex subset M of $\mathrm{ba}(A)$ is uniformly bounded in $\mathscr{A}$. The algebra of finite and co-finite subsets of $\mathbb{N}$ fails to have property N and Schachermayer proved that the algebra $\mathscr{J}(I)$ of Jordan measurable subsets of $I := [0, 1]$ has property N [10, Corollary 3.5] (see a generalization of this property in [4, Corollary]).

A subset $\mathscr{B}$ of a set-algebra $\mathscr{A}$ has the strong Nikodym property, property sN in brief, *if for each increasing covering $\cup_m \mathscr{B}_m$ of $\mathscr{B}$ there exists $\mathscr{B}_n$ which has property N*. Valdivia proved that the algebra $\mathscr{J}(K)$ of Jordan measurable subsets of a compact k-dimensional interval $K := \Pi\{[a_i, b_i] : 1 \leqslant i \leqslant k\}$ in $\mathbb{R}^k$ has property sN [13, Theorem 2].

An *increasing web* in a set A is a family $\mathscr{W} := \{A_{m_1 m_2 \ldots m_p} : (m_1, m_2, \ldots, m_p) \in \cup_s N^s\}$ of subsets of A such that $(A_{m_1})_{m_1}$ and $(A_{m_1 m_2 \ldots m_p m_{p+1}})_{m_{p+1}}$ are, respectively, increasing coverings of A and $A_{m_1 m_2 \ldots m_p}$, for each $p, m_i \in N$, $1 \leqslant i \leqslant p+1$ [7, Chapter 7, 35.1], and each sequence $(A_{m_1 m_2 \ldots m_p})_p$ is a *strand* in $\mathscr{W}$. *A subset $\mathscr{B}$ of a set-algebra $\mathscr{A}$ has the* web Nikodym property, property wN in brief, *if for each increasing web $\{\mathscr{B}_t : t \in \cup_s N^s\}$ in $\mathscr{B}$ there exists a strand composed of sets which have property N*. In general, if B is a set and $\mathfrak{P}$ is a property verified in the elements of a family of subsets of B then *B has property $w\mathfrak{P}$ if each increasing web $\{B_t : t \in \cup_s N^s\}$ in $\mathscr{B}$ has a strand composed of sets which have property $\mathfrak{P}$*.

Property $w(w\mathfrak{P})$ is named as property $w^2\mathfrak{P}$. The next straightforward proposition states that properties $w\mathfrak{P}$ and $w^2\mathfrak{P}$ are equivalent.

Proposition 1. *Let $(B_m)_m$ be an increasing covering of a set B which verifies property $w\mathfrak{P}$. There exists B_n which has property $w\mathfrak{P}$, whence B has property $w^2\mathfrak{P}$.*

Proof. Let us suppose that $(B_m)_m$ is an increasing covering of a set B such that each B_m does not have property $w\mathfrak{P}$. Then, for each natural number m there exists an increasing web $\mathscr{W}_m := \{B^m_{m_1 m_2 \ldots m_p} : p, m_1, m_2, \ldots, m_p \in N\}$ in B_m such that every strand in $\mathscr{W}_m$ contains a set $B^m_{m_1 m_2 \ldots m_p}$ without property $\mathfrak{P}$. If $B_{m_1 m_2 \ldots m_p} := B^{m_1}_{m_2 m_3 \ldots m_p}$ we get that $\mathscr{W} := \{B_{m_1 m_2 \ldots m_p} : p, m_1, m_2, \ldots, m_p \in N\}$ is an increasing web in B without strands consisting of sets with property $\mathfrak{P}$, whence B does not have property $w\mathfrak{P}$. This proves the first affirmation which readily implies that if B verifies property $w\mathfrak{P}$ then every increasing web in B contains a strand consisting of sets with property $w\mathfrak{P}$, whence properties $w\mathfrak{P}$ and $w^2\mathfrak{P}$ are equivalent in B. □

Let $\mathscr{S}$ be a σ-algebra of subsets of a set Ω. It has been sequentially shown that (i) $\mathscr{S}$ has property N (Nikodym-Dieudonné-Grothendieck theorem [9], [3] and [1, page 80, named as Nikodym-Grothendieck boundedness theorem]), (ii) $\mathscr{S}$ has property sN ([12, Theorem 2]) and (iii) $\mathscr{S}$ has property $w(sN)$ (very recently in [6, Theorem 2]). The aim of this paper is to present in the next section a proof of the property that each σ-algebra $\mathscr{S}$ has property wN independent of any property related to Nikodym boundedness property, as properties N or sN, and using very elementary locally convex space theory.

Last section deals with some applications to bounded vector measures deduced from the property wN of each σ-algebra $\mathscr{S}$ and from the equivalence stated in Proposition 1.

Following the characterization of sN-property of a set-algebra A by the locally convex property of $L(A)$ given in [13, Theorem 3] it is possible to get a characterization of wN property of a set-algebra A by the locally convex properties considered in [5] and [8]. In fact Theorem 1 is equivalent to Theorem 2.7 of [8], totally stated in the locally convex theory frame.

2 NV-trees and property *wN*

Given two elements, $t = (t_1, t_2, \ldots, t_p)$ and $s = (s_1, s_2, \ldots, s_q)$, and two subsets, T and U, of $\cup_s \mathbb{N}^s$ then p is the *length of* t, for each $1 \leqslant i \leqslant p$ the *section of length* i of t is $t(i) := (t_1, t_2, \ldots, t_i)$; if $i > p$, $t(i) := \emptyset$; $T(m) := \{t(m) : t \in T\}$, for each $m \in \mathbb{N}$; $t \times s := (t_1, t_2, \ldots, t_p, t_{p+1}, t_{p+2}, \ldots, t_{p+q})$, with $t_{p+j} := s_j$, for $1 \leqslant j \leqslant q$, and $T \times U := \{t \times u : t \in T, u \in U\}$.

Each $t \times s \in U$ *is an extension of* t *in* U and a sequence $(t^n)_n$ of elements $t^n = (t^n_1, t^n_2, \ldots, t^n_n, \ldots) \in T$ is an *infinite chain in* T if for each $n \in \mathbb{N}$ the element t^{n+1} is an extension of the section $t^n(n)$ in T, i.e., $\emptyset \neq t^n(n) = t^{n+1}(n)$, and length of t^n is at least n, for each $n \in \mathbb{N}$. If $t = (t_1)$ then t and the products $T \times t$ and $t \times T$ are represented by t_1, $T \times t_1$ and $t_1 \times T$.

Let $\emptyset \neq U \subset \cup_n \mathbb{N}^n$. U is *increasing* at $t = (t_1, t_2, \ldots, t_p) \in \cup_s \mathbb{N}^s$ if U contains elements $t^1 = (t^1_1, t^1_2, \ldots)$ and $t^i = (t_1, t_2, \ldots, t_{i-1}, t^i_i, t^i_{i+1}, \ldots)$, $1 < i \leqslant p$, such that $t_i < t^i_i$, for each $1 \leqslant i \leqslant p$. *$U$ is increasing* (*increasing respect to a subset V of* $\cup_s \mathbb{N}^s$) *if U is* *increasing at each* $t \in U$ (*at each* $t \in V$). Clearly U is increasing if $|U(1)| = \infty$ and $|\{n \in \mathbb{N} : t(i) \times n \in U(i+1)\}| = \infty$, for each $t = (t_1, t_2, \ldots, t_p) \in U$ and $1 \leqslant i < p$.

Next definition deals with a particular type of increasing trees (see [6, Definition 2]).

Definition 1. An NV-tree T is an increasing subset of $\cup_{s\in\mathbb{N}}\mathbb{N}^s$ without infinite chains such that for each $t = (t_1, t_2, \ldots, t_p) \in T$ the length of each extension of $t(p-1)$ in T is p and $\{t(i) : 1 \leqslant i \leqslant p\} \cap T = \{t\}$.

An NV-tree T is *trivial* if $T = T(1)$ and then T is an infinite subset of $\mathbb{N}$.

The sets $\mathbb{N}^i$, $i \in \mathbb{N}\backslash\{1\}$, and the set $\cup\{(i) \times \mathbb{N}^i : i \in \mathbb{N}\}$ are non trivial NV-trees. The finite product of NV-trees is an NV-tree.

If T is an increasing subset of $\cup_{s\in\mathbb{N}}\mathbb{N}^s$ and $\{B_u : u \in \cup_s\mathbb{N}^s\}$ is an increasing web in B then $(B_{u(1)})_{u\in T}$ is an increasing covering of B, because for each $u = (u_1, u_2, \ldots, u_p) \in T$ and each $i < p$ the sequence $(B_{u(i)\times n})_{u(i)\times n\in T(i+1)}$ is an increasing covering of $B_{u(i)}$, hence if T does not contain infinite chains and $b \in B$ there exists $t \in T$ such that $b \in B_t$. Therefore $B = \cup\{B_t : t \in T\}$.

Each increasing subset S of an NV-tree T is an NV-tree, whence if $(S_n)_n$ is a sequence of subsets of an NV-tree T such that each S_{n+1} is increasing respect to S_n then $\cup_n S_n$ is an NV-tree. This hereditary property and Proposition 7 in [6] imply next Proposition 2 and we give a proof as a help for the reader.

Proposition 2. *Let* U *be a subset of an* NV*-tree* T*. If* U *does not contain an* NV*-tree then* $T\backslash U$ *contains an* NV*-tree.*

Proof. This proposition is obvious if T is a trivial NV-tree. Whence we suppose that T is a non-trivial NV-tree and then there exists $m'_1 \in T(1)$ such that for each $n \geqslant m'_1$ the set $\{v \in \cup_s\mathbb{N}^s : n \times v \in U\}$ does not contain an NV-tree. We define $Q_1 := \emptyset$ and $Q'_1 := \{n \in T(1)\backslash T : m'_1 \leqslant n\}$.

Let us suppose that we have obtained for each j, with $2 \leqslant j \leqslant i$, two disjoint subsets Q_j and Q'_j of $T(j)$, with $Q_j \subset T\backslash U$ and $Q'_j \cap T = \emptyset$, such that for each $t \in Q_j \cup Q'_j$ the section $t(j-1) \in Q'_{j-1}$ and $A_{t(j-1)} := \{n \in \mathbb{N} : t(j-1) \times n \in Q_j \cup Q'_j\}$ is an infinite set such that $t \in Q_j$ implies that $t(j-1) \times A_{t(j-1)} \subset Q_j$ and from $t \in Q'_j$ it follows that $t(j-1) \times A_{t(j-1)} \subset Q'_j$ and that the set $\{v \in \cup_s\mathbb{N}^s : t \times v \in U\}$ does not contain an NV-tree. Then we define $S_{t(j-1)} := A_{t(j-1)}$ and $S'_{t(j-1)} := \emptyset$ in the first case and $S_{t(j-1)} := \emptyset$, $S'_{t(j-1)} := A_{t(j-1)}$ in the second case.

As for each $t \in Q'_i (\subset T(i)\backslash T)$ the set $\{v \in \cup_s\mathbb{N}^s : t \times v \in U\}$ does not contain an NV-tree and it is a subset of the NV-tree $T_t := \{v \in \cup_s\mathbb{N}^s : t \times v \in T\}$, the following two cases may happen:

i. Either the NV-tree T_t is trivial and then there exists $m_{i+1} \in \mathbb{N}$ such that the infinite set $S_t := \{n \in \mathbb{N} : m_{i+1} \leqslant n,\ t \times n \in T(i+1)\}$ verifies that $t \times S_t \subset T\backslash U$. In this case we define $S'_t := \emptyset$.

ii. Or the NV-tree T_t is non-trivial and then there exists $m'_{i+1} \in \mathbb{N}$ such that the infinite set $S'_t := \{n \in \mathbb{N} : m'_{i+1} < n,\ t \times n \in T(i+1)\}$ verifies that $t \times S'_t \subset T(i+1)\backslash T$ and for each $t \times n \in t \times S'_t$ the set $\{v \in \cup_s\mathbb{N}^s : t \times n \times v \in U\}$ does not contain an NV-tree. Now we define $S_t := \emptyset$.

The induction finish by setting $Q_{i+1} := \cup\{t \times S_t : t \in Q'_i\}$ and $Q'_{i+1} := \cup\{t \times S'_t : t \in Q'_i\}$. Then $Q_{i+1} \subset T(i+1) \cap (T \backslash U)$, $Q'_{i+1} \subset T(i+1) \backslash T$, and each $t \in Q_{i+1} \cup Q'_{i+1}$ verifies the above indicated properties when $t \in Q_j \cup Q'_j$, changing j by $i+1$.

As T does not contain infinite chains for each $(t_1, t_2, \ldots, t_i) \in Q'_i$ there exists $q \in \mathbb{N}$ and $(t_{i+1}, \ldots, t_{i+q}) \in \mathbb{N}^q$ such that $(t_1, t_2, \ldots, t_i, t_{i+1}, \ldots, t_{i+q}) \in Q_{i+q}$, whence $(\cup_{j>i} Q_j)(i) = Q'_i$. This implies that the subset $W := \cup\{Q_j : j \in \mathbb{N}\}$ of $T \backslash U$ has the increasing property, because from $W(k) = Q_k \cup Q'_k$, for each $k \in \mathbb{N}$, we get that $|W(1)| = |Q'_1| = \infty$ and if $t = (t_1, t_2, \ldots, t_p) \in W$ then $(t_1, t_2, \ldots, t_i) \in Q'_i$, if $1 < i < p$, and $(t_1, t_2, \ldots, t_p) \in Q_p$, whence the infinite subsets $S'_{t(i-1)}$ and $S_{t(p-1)}$ of $\mathbb{N}$ verify that $t(i-1) \times S'_{t(i-1)} \subset Q'_i \subset W(i)$ and $t(p-1) \times S_{t(p-1)} \subset Q_p \subset W$. Therefore W is an NV-tree contained in $T \backslash U$. □

Definition 2. A property $\mathfrak{P}$ is hereditary increasing in a set A if for each pair of subsets B and C of A such that B verifies property $\mathfrak{P}$ and $B \subset C \subset A$ then C also has property $\mathfrak{P}$.

Example 1. The properties wN, sN and N are hereditary increasing properties in a set-algebra $\mathscr{A}$.

Proof. Let $\mathscr{B} \subset \mathscr{C} \subset \mathscr{A}$. It is obvious that if $\mathscr{B}$ has property N then $\mathscr{C}$ has also property N. Whence if $\mathscr{B}$ has property sN and if $\cup_m \mathscr{C}_m$ is an increasing covering of $\mathscr{C}$ then there exists $\mathscr{C}_n$ such that $\mathscr{C}_n \cap \mathscr{B}$ has property N, therefore $\mathscr{C}_n$ has property N and we get that $\mathscr{C}$ has also property sN.

If $\mathscr{B}$ has property wN and $\{\mathscr{C}_{m_1 m_2 \ldots m_p} : p, m_1, m_2, \ldots, m_p \in \mathbb{N}\}$ is an increasing web in $\mathscr{C}$, then there exists a sequence $(n_i)_i$ such that each $\mathscr{C}_{n_1 n_2 \ldots n_i} \cap \mathscr{B}$ has property N, $i \in \mathbb{N}$, whence $(\mathscr{C}_{n_1 n_2 \ldots n_i})_i$ is a strand in $\mathscr{C}$ consisting of sets which have property N. □

Proposition 3. *Let $\mathfrak{P}$ be an hereditary increasing property in A and let $\mathscr{B} := \{B_{m_1 m_2 \ldots m_p} : p, m_1, m_2, \ldots, m_p \in \mathbb{N}\}$ be an increasing web in A without strands consisting of sets with property $\mathfrak{P}$. Then there exists an NV-tree T such that for each $t = (t_1, t_2, \ldots, t_q) \in T$ the set B_t does not have property $\mathfrak{P}$ and if $p > 1$ then $B_{t(i)}$ has property $\mathfrak{P}$, for each $i = 1, 2, \ldots, p-1$.*

Proof. If each B_{m_1}, $m_1 \in \mathbb{N}$, does not have property $\mathfrak{P}$ the proposition is obvious with $T := \mathbb{N}$. Hence we may suppose that there exists $m'_1 \in \mathbb{N}$ such that B_{t_1} has property $\mathfrak{P}$ for each $t_1 \geqslant m'_1$ and then we write $Q_1 := \emptyset$ and $Q'_1 := \{t_1 \in \mathbb{N} : t_1 \geqslant m'_1\}$.

Let us assume that for each j, with $2 \leqslant j \leqslant i$, we have obtained by induction two disjoint subsets Q_j and Q'_j of $\mathbb{N}^j$ such that for each $t = (t_1, t_2, \ldots, t_j) \in Q_j \cup Q'_j$ the section $t(j-1) = (t_1, t_2, \ldots, t_{j-1}) \in Q'_{j-1}$, if $t \in Q_j$ then the set B_t does not have property $\mathfrak{P}$ and $t(j-1) \times \mathbb{N} \subset Q_j$ and then we define $S_{t(j-1)} := \mathbb{N}$ and $S'_{t(j-1)} = \emptyset$; otherwise, if $t \in Q'_j$ then the set B_t has property $\mathfrak{P}$ and $S'_{t(j-1)} := \{n \in \mathbb{N} : t(j-1) \times n \in Q_j \cup Q'_j\}$ is a co-finite subset of $\mathbb{N}$ such that $t(j-1) \times S'_{t(j-1)} \subset Q'_j$. In this case we define $S_{t(j-1)} := \emptyset$.

If $t := (t_1, t_2, \ldots, t_i) \in Q'_i$ then, by induction, $B_{t_1 t_2 \ldots t_i}$ has property $\mathfrak{P}$ and as $(B_{t_1 t_2 \ldots t_i n})_n$ is an increasing covering of $B_{t_1 t_2 \ldots t_i}$ it may happen that either $B_{t_1 t_2 \ldots t_i n}$ does not have property $\mathfrak{P}$ for each $n \in \mathbb{N}$ and then we define $S_{t_1 t_2 \ldots t_i} := \mathbb{N}$ and $S'_{t_1 t_2 \ldots t_i} := \emptyset$, or there exists $m'_{i+1} \in \mathbb{N}$ such that $B_{t_1 t_2 \dots t_i n}$ has property $\mathfrak{P}$ for each $n \geqslant m'_{i+1}$ and in this second case we define $S_{t_1 t_2 \dots t_i} := \emptyset$ and $S'_{t_1 t_2 \dots t_i} := \{n \in \mathbb{N} : m'_{i+1} \leqslant n\}$.

We finish this induction procedure by setting $Q_{i+1} := \cup\{t \times S_t : t \in Q'_i\}$ and $Q'_{i+1} := \cup\{t \times S'_t : t \in Q'_i\}$. By construction Q_{i+1} and Q'_{i+1} verify the above indicated properties of Q_j and Q'_j replacing j by $i+1$.

The hypothesis that for each sequence $(m_i)_i \in \mathbb{N}^{\mathbb{N}}$ there exists $j \in \mathbb{N}$ such that $B_{m_1 m_2 \dots m_j}$ does not have property $\mathfrak{P}$ implies that $T := \cup\{Q_i : i \in \mathbb{N}\}$ does not contain infinite chains, because if $(m_1, m_2, \dots, m_p) \in Q_p$ then $(m_1, m_2, \dots, m_{p-1}) \in Q'_p$, hence $B_{m_1 m_2 \dots m_{p-1}}$ has property $\mathfrak{P}$. Therefore for each $(t_1, t_2, \dots, t_k) \in Q'_k$ there exists an extension $(t_1, t_2, \dots, t_k, t_{k+1}, \dots, t_{k+q}) \in Q_{k+q}$, whence $T(k) = Q_k \cup Q'_k$, for each $k \in \mathbb{N}$. Then the set T has the increasing property, because $|T(1)| = |Q'_1| = \infty$ and if $t = (t_1, t_2, \dots, t_p) \in T$ the sets $S'_{t(i-1)}$, $1 < i < p$, are co-finite subsets of $\mathbb{N}$, $S_{t(p-1)} := \mathbb{N}$, $t(i-1) \times S'_{t(i-1)} \subset Q'_i \subset T(i)$ and $t(p-1) \times S'_{t(p-1)} \subset Q_p \subset T$. By construction, if $t = (t_1, t_2, \dots, t_p) \in T$ then $t(i) \in Q'_i$, if $1 \leqslant i < p$, and $t \in Q_p$, whence $B_{t(i)}$ has property $\mathfrak{P}$, for each $i = 1, 2, \dots, p-1$, B_t does not have property $\mathfrak{P}$, $\{t(i) : 1 \leqslant i \leqslant p\} \cap T = \{t\}$ and the extensions of $t(p-1)$ in T are the elements of $t(p-1) \times \mathbb{N}$, whose lengths are p.

□

Definition 3 ([6, Definition 1])**.** Let B be an element of the algebra $\mathscr{A}$ of subsets of Ω. A subset M of $\mathrm{ba}(\mathscr{A})$ is deep B-unbounded if each finite subset $\mathscr{Q}$ of $\{e_A : A \in \mathscr{A}\}$ verifies that

$$\sup\{|\mu(C)| : \mu \in M \cap \mathscr{Q}^\circ,\, C \in \mathscr{A},\, C \subset B\} = \infty.$$

The proof of the next proposition is straightforward.

Proposition 4 ([6, Proposition 5])**.** *If a subset M of $\mathrm{ba}(\mathscr{A})$ is deep B-unbounded and $\{B_i \in \mathscr{A} : 1 \leqslant i \leqslant q\}$ is a partition of B then there exists j, $1 \leqslant j \leqslant q$, such that M is deep B_j-unbounded.*

Proposition 5 ([6, Proposition 4])**.** *Let $\mathscr{A}$ be an algebra of subsets of Ω and let $(\mathscr{B}_m)_m$ be an increasing sequence of subsets of $\mathscr{A}$ such that each $\mathscr{B}_m$ does not have N-property and $\overline{\mathrm{span}\{e_C : C \in \cup_m \mathscr{B}_m\}} = L(\mathscr{A})$. There exists $n_0 \in \mathbb{N}$ such that for each $m \geqslant n_0$ there exists a deep Ω-unbounded $\tau_s(\mathscr{A})$-closed absolutely convex subsets M_m of $\mathrm{ba}(\mathscr{A})$ which is pointwise bounded in $\mathscr{B}_m$, i.e., $\sup\{|\mu(C)| : \mu \in M_m\} < \infty$ for each $C \in \mathscr{B}_m$. In particular this proposition holds if $\cup_m \mathscr{B}_m = \mathscr{A}$ or if $\cup_m \mathscr{B}_m$ has N-property.*

Proposition 6. *Let $\mathscr{B} := \{\mathscr{B}_{m_1 m_2 \dots m_p} : p, m_1, m_2, \dots, m_p \in \mathbb{N}\}$ be an increasing web in a set- algebra $\mathscr{A}$. If $\mathscr{B}$ does not contain strands consisting of sets with property N then there exists an NV-tree T such that for each $t \in T$ there exists a deep Ω-unbounded $\tau_s(\mathscr{A})$-closed absolutely convex subset M_t of $\mathrm{ba}(\mathscr{S})$ which is B_t-pointwise bounded.*

Proof. By Proposition 3 with $\mathfrak{P} = N$ there exists an NV-tree T_1 such that for each $t = (t_1, t_2, \dots, t_p) \in T_1$ the set $\mathscr{B}_t$ does not have property N and if $p > 1$ then $\mathscr{B}_{t(i)}$ has property N, for each $i = 1, 2, \dots, p-1$. If $p = 1$ the conclusion follows from Proposition 5 in the case $\cup_{m_1} \mathscr{B}_{m_1} = \mathscr{A}$, being $T := T_1 \setminus \{1, 2, \dots, n_0 - 1\}$, where n_0 is the natural number in Proposition 5. If $p > 1$ then $\mathscr{B}_{t(p-1)} = \cup_m \mathscr{B}_{t(p-1) \times m}$ has property

N and the conclusion follows again from Proposition 5 in the case that $\cup_m \mathscr{B}_m$ has N-property, being T the NV-tree obtained after deleting in T_1 the elements $t(p-1) \times \{1,2,\dots,n_0(t)-1\}$, for each $t = (t_1,t_2,\dots,t_p) \in T_1$ where $n_0(t)$ is the natural number of Proposition 5 for the increasing sequence $(\mathscr{B}_{t(p-1)\times m})_m$. $\square$

Next Proposition 7 is given in [6, Proposition 8] as a currently version of Propositions 2 and 3 in [13]. Also Proposition 8 is contained in [6, Propositions 9 and 10]. In both propositions we present a sketch of the proofs for the sake of completeness and as a new help to the reader.

Proposition 7 ([6, Proposition 8]). *Let $\{B,Q_1,\dots,Q_r\}$ be a subset of the algebra $\mathscr{A}$ of subsets of Ω and let M be a deep B-unbounded absolutely convex subset of* $\mathrm{ba}(\mathscr{A})$. *Then given a positive real number α and a natural number $q > 1$ there exists a finite partition $\{C_1,C_2,\dots,C_q\}$ of B by elements of $\mathscr{A}$ and a subset $\{\mu_1,\mu_2,\dots,\mu_q\}$ of M such that $|\mu_i(C_i)| > \alpha$ and $\Sigma_{1\leqslant j\leqslant r}\mu_i(Q_j) \leqslant 1$, for $i = 1,2,\dots,q$.*

Proof. It is enough to proof the case $q = 2$, because then there exists C_i, $i \in \{1,2\}$, such that M is deep C_i-unbounded by Proposition 4. Let $\mathscr{Q} = \{\chi_B, \chi_{Q_1}, \chi_{Q_2},\dots,\chi_{Q_r}\}$. As rM is deep B-unbounded, i.e., $\sup\{|\mu(D)| : \mu \in rM \cap \mathscr{Q}^\circ,\ D \subset B,\ D \in \mathscr{A}\} = \infty$, there exists $C_1 \subset B$, with $C_1 \in \mathscr{A}$, and $\mu \in rM \cap \mathscr{Q}^\circ$ such that $|\mu(C_1)| > r(1+\alpha)$. Then $\mu_1 = r^{-1}\mu \in M$, $|\mu_1(B)| \leqslant r^{-1} \leqslant 1$ and $\Sigma_{1\leqslant j\leqslant r}\left|\mu_1(Q_j)\right| \leqslant r^{-1}r = 1$. Clearly $C_2 := B\backslash C_1$ and $\mu_2 := \mu_1$ verify that $|\mu_1(C_2)| \geqslant |\mu_1(C_1)| - |\mu_1(B)| > 1+\alpha-1 = \alpha$. $\square$

Proposition 8 ([6, Propositions 9 and 10]). *Let $\{B,Q_1,\dots,Q_r\}$ be a subset of an algebra $\mathscr{A}$ of subsets of Ω and let $\{M_t : t \in T\}$ be a family of deep B-unbounded absolutely convex subsets of* $\mathrm{ba}(\mathscr{A})$*, indexed by an NV-tree T. Then for each positive real number α and each finite subset $\{t^j : 1 \leqslant j \leqslant k\}$ of T there exist a set $B_1 \in \mathscr{A}$, a measure $\mu_1 \in M_{t^1}$ and an increasing tree T_1, such that*

1. *$B_1 \subset B$, $\{t^j : 1 \leqslant j \leqslant k\} \subset T_1 \subset T$ and M_t is deep $(B\backslash B_1)$-unbounded for each $t \in T_1$.*
2. *$|\mu_1(B_1)| > \alpha$ and $\Sigma\{|\mu_1(Q_i)| : 1 \leqslant i \leqslant r\} \leqslant 1$.*

Proof. Let $t^j := (t_1^j, t_2^j, \dots, t_{p_j}^j)$, for $1 \leqslant j \leqslant k$. By Proposition 7 applied to B, α, $q := 2 + \Sigma_{1\leqslant j\leqslant k}\, p_j$ and M_{t^1} there exist a partition $\{C_1,C_2,\dots,C_q\}$ of B by elements of $\mathscr{A}$ and $\{\lambda_1,\lambda_2,\dots,\lambda_q\} \subset M_{t^1}$ such that:

$$\left|\lambda_k(C_k^1)\right| > \alpha \quad \text{and} \quad \Sigma_{1\leqslant i\leqslant r}\,|\lambda_k(Q_i)| \leqslant 1 \text{ for } k = 1,2,\dots,q. \tag{1}$$

From Proposition 4 it follows that if M is deep B-unbounded there exists an $i_M \in \{1,2,\dots,q\}$ such that M is deep C_{i_M}-unbounded, hence if M_u is deep B-unbounded for each $u \in U$ and $V_i := \{u \in U : M_u \text{ is deep } C_i\text{-unbounded}\}$, $1 \leqslant i \leqslant q$, then $U = \cup_{1\leqslant i\leqslant q} V_i$. Whence if U is an NV-tree there exists i_0, with $1 \leqslant i_0 \leqslant q$, such that V_{i_0} contains an NV-tree U_{i_0} by Proposition 2.

Therefore there exists C_{i^j} and C_{i_0}, with $\{i^j, i_0\} \subset \{1,2,\dots,q\}$, and an NV-tree $T_{i_0} \subset T$ such that M_{t^j} is deep C_{i^j}-unbounded, for each $j \in \{1,2,\dots,k\}$, and M_t is deep C_{i_0}-unbounded for each $t \in T_{i_0}$.

For each $t^j = (t_1^j, t_2^j, \ldots, t_{p_j}^j) \notin T_{i_0}$, $1 \leqslant j \leqslant k$, and each section $t^j(m-1)$ of t^j, with $2 \leqslant m \leqslant p_j$, the set $W_m^j := \{v \in \cup_s \mathbb{N}^s : t^j(m-1) \times v \in T\}$ is an NV-tree such that $M_{(t_1^j, t_2^j, \ldots, t_{m-1}^j) \times w}$ is deep B-unbounded for each $w \in W_m^j$, whence there exists $i_m^j \in \{1, 2, \ldots, q\}$ and an NV-tree V_m^j contained in W_m^j such that $M_{(t_1^j, t_2^j, \ldots, t_{m-1}^j) \times w}$ is deep $C_{i_m^j}$-unbounded for each $w \in V_m^j$.

Let D be the union $D := C_{i_0} \cup (\cup \{C_{i^j} \cup C_{i_m^j} : j \in S, 2 \leqslant m \leqslant p_j\})$ and let T_1 be the union of T_{i_0} and the sets $\{t^j\} \cup \{(t_1^j, t_2^j, \ldots, t_{m-1}^j) \times V_m^j : 2 \leqslant m \leqslant p_j\}$, such that $t^j \notin T_{i_0}$ and $1 \leqslant j \leqslant k$. By construction T_1 has the increasing property and if $t \in T_1$ the set M_t is deep D-unbounded.

The number of sets defining D is less or equal than $q-1$, hence there exists C_h such that $D \subset B \backslash C_h$ and we get that T_1 is an NV-tree such that M_t is deep $B \backslash C_h$-unbounded for each $t \in T_1$ and, by (1), this proof is done with $B_1 := C_h^1$ and $\mu_1 := \lambda_h$. □

Corollary 1 ([6, Proposition 10]). *Let $\{B, Q_1, \ldots, Q_r\}$ be a subset of an algebra $\mathscr{A}$ of subsets of Ω and $\{M_t : t \in T\}$ a family of deep B-unbounded absolutely convex subsets of* ba$(\mathscr{A})$*, indexed by an increasing tree T. Then for each positive real number α and each finite subset $\{t^j : 1 \leqslant j \leqslant k\}$ of T there exist k pairwise disjoint sets $B_j \in \mathscr{A}$, k measures $\mu_j \in M_{t^j}$, $1 \leqslant j \leqslant k$, and an increasing tree T^* such that:*

1. *$\cup\{B_j : 1 \leqslant j \leqslant k\} \subset B$, $\{t^j : 1 \leqslant j \leqslant k\} \subset T^* \subset T$ and M_t is deep $(B \backslash \cup_{1 \leqslant j \leqslant k} B_j)$-unbounded for each $t \in T^*$.*
2. *$|\mu_j(B_j)| > \alpha$ and $\Sigma\{|\mu_j(Q_i)| : 1 \leqslant i \leqslant r\} \leqslant 1$, for $j = 1, 2, \ldots, k$.*

Proof. Apply k times Proposition 8. □

In Theorem 1 we need the sequence $(i_n)_n := (1, 1, 2, 1, 2, 3, \ldots)$ obtained with the first components of the sequence $\{(1,1), (1,2), (2,1), (1,3), (2,2), (3,1), \ldots\}$ generated writing the elements of $\mathbb{N}^2$ following the diagonal order.

Theorem 1. *A σ-algebra $\mathscr{S}$ of subsets of a set Ω has property wN.*

Proof. Let us suppose that $\mathscr{S}$ is a σ-algebra of subsets of a set Ω which does not have property *wN*. Then there would exists in $\mathscr{S}$ an increasing web $\{\mathscr{B}_{m_1 m_2 \ldots m_p} : p, m_1, m_2, \ldots, m_p \in \mathbb{N}\}$ without strands consisting of sets with Property N. By Proposition 6 there exists an NV-tree T such that for each $t \in T$ there exists a deep Ω-unbounded $\tau_s(\mathscr{A})$-closed absolutely convex subset M_t of ba$(\mathscr{S})$ which is B_t-pointwise bounded.

By induction it is easy to determine an NV-tree $\{t^i : i \in \mathbb{N}\}$ contained in T and a strictly increasing sequence of natural numbers $(k_j)_j$ such that for each $(i, j) \in \mathbb{N}^2$ with $i \leqslant k_j$ there exists a set $B_{ij} \in \mathscr{A}$ and $\mu_{ij} \in M_{t^i}$ that verify

$$\Sigma_{s,v}\{|\mu_{ij}(B_{sv})| : s \leqslant k_v, \ 1 \leqslant v < j\}) < 1, \tag{2}$$

$$|\mu_{ij}(B_{ij})| > j, \tag{3}$$

and $B_{ij} \cap B_{i'j'} = \emptyset$ if $(i, j) \neq (i', j')$.

In fact, select $t^1 \in T$. Corollary 1 with $B := \Omega$ and $\alpha = 1$ provides $B_{11} \in \mathscr{S}$, $\mu_{11} \in M_{t^1}$ and an NV-tree T_1 such that $|\mu_{11}(B_{11})| > 1$, $t^1 \in T_1 \subset T$ and M_t is deep $\Omega \backslash B_{11}$-unbounded for each $t \in T_1$. Define $k_1 := 1$, $S^1 := \{t^1\}$ and $B^1 := B_{11}$.

Let us suppose that we have obtained the natural numbers $k_1 < k_2 < k_3 < \cdots < k_n$, the NV-trees $T_1 \supset T_2 \supset T_3 \supset \cdots \supset T_n$, the elements $\{t^1, t^2, \ldots, t^{k_n}\}$ such that $S^j := \{t^i : i \leqslant k_j\} \subset T_j$ and $S_j := \{t^{k_{j-1}+1}, \ldots, t^{k_j}\}$ has the increasing property respect to S^{j-1}, for each $1 < j \leqslant n$, together with the measures $\mu_{ij} \in M_{t^i}$ and the pairwise disjoint elements $B_{ij} \in \mathscr{S}$, $i \leqslant k_j$ and $j \leqslant n$, such that $\left|\mu_{ij}(B_{ij})\right| > j$ and $\Sigma_{s,v}\{\left|\mu_{ij}(B_{sv})\right| : s \leqslant k_v,$ $1 \leqslant v < j\} < 1$, if $i \leqslant k_j$ and $j \leqslant n$, in such a way that the union $B^j := \cup\{B_{sv} : s \leqslant k_v,$ $1 \leqslant v \leqslant j\}$ verifies that M_t is deep $\Omega \backslash B^j$-unbounded for each t belonging to the NV-tree T_j, for each $j < n$.

To finish the induction procedure select a subset $S_{n+1} := \{t^{k_n+1}, \ldots, t^{k_{n+1}}\}$ of $T_n \backslash \{t^i : i \leqslant k_n\}$ which has the increasing property respect to S^n and apply again Corollary 1 to $\Omega \backslash B^n$, $\{B_{sv} : s \leqslant k_v, 1 \leqslant v \leqslant n\}$, T_n, the finite subset $S^{n+1} := \{t^i : i \leqslant k_{n+1}\}$ of T_n and $n+1$. Then, for each $i \leqslant k_{n+1}$, we obtain $B_{in+1} \in \mathscr{A}$, $B_{in+1} \subset \Omega \backslash B^n$, and $\mu_{in+1} \in M_{t^i}$ such that $|\mu_{in+1}(B_{in+1})| > n+1$, $\Sigma_{s,v}\{|\mu_{in+1}(B_{sv})| : s \leqslant k_v,\ 1 \leqslant v \leqslant n\} < 1$, $B_{in+1} \cap B_{i'n+1} = \emptyset$, if $i \neq i'$, and the union $B^{n+1} := \cup\{B_{sv} : s \leqslant k_s,\ 1 \leqslant v \leqslant n+1\}$ has the property that T_n contains an increasing tree T_{n+1} such that $S^{n+1} \subset T_{n+1}$ and M_t is deep $\Omega \backslash B^{n+1}$-unbounded for each $t \in T_{n+1}$.

With a new easy induction we obtain a subset $J := \{j_1, j_2, \ldots, j_n, \ldots\}$ of $\mathbb{N}$ such that $j_n < j_{n+1}$, for $n \in \mathbb{N}$, and for each $(i, j) \in \mathbb{N} \times J$ with $i \leqslant k_j$ we have

$$\Sigma_{s,v}\{\left|\mu_{ij}(B_{sv})\right| : s \leqslant k_v,\ j < v \in J\} < 1$$

because if the variation $\left|\mu_{ij}\right|(\Omega) < s \in \mathbb{N}$, $\{N_u, 1 \leqslant u \leqslant s\}$ is a partition of $\mathbb{N} \backslash \{1, 2, \ldots, j\}$ in s infinite subsets and $B_u := \cup\{B_{sv} : s \leqslant k_v,\ v \in N_u\}$, $1 \leqslant u \leqslant s_1$, then the inequality $\Sigma\{\left|\mu_{ij}\right|(B_u) : 1 \leqslant u \leqslant s_1\} < s_1$ implies that there exists u', with $1 \leqslant u' \leqslant s_1$, such that $\left|\mu_{ij}\right|(B_{u'}) < 1$, whence

$$\Sigma_{s,v}\{\left|\mu_{ij}(B_{sv})\right| : s \leqslant k_v, v \in N_{u'}\} < 1,$$

and then the sequence $(B_{i_n j_n}, \mu_{i_n j_n})_n$ verifies for each $n \in \mathbb{N}$ that:

$$\Sigma_s\{\left|\mu_{i_n j_n}(B_{i_s j_s})\right| :\ s < n\}) < 1, \tag{4}$$

$$\left|\mu_{i_n j_n}(B_{i_n j_n})\right| > j_n, \tag{5}$$

and

$$\left|\mu_{i_n j_n}(\cup_s\{B_{i_s j_s} :\ n < s\})\right| < 1. \tag{6}$$

As S^{n+1} has the increasing property respect to S^n we have that $\{t^i : i \in \mathbb{N}\}$ is an NV-tree contained in T, hence $\cup\{\mathscr{B}_{t^i} : i \in \mathbb{N}\} = \mathscr{S}$. The relation $H := \cup\{B_{i_s j_s} :\ s = 1, 2, \ldots\} \in \mathscr{S}$ implies that there exists $r \in \mathbb{N}$ such that $H \in \mathscr{B}_{t^r}$. Then for each strictly increasing sequence $(n_p)_p$ such that $i_{n_p} = r$ we have that $\{\mu_{i_{n_p} j_{n_p}} : p \subset \mathbb{N}\}$ is a subset of M_{t^r}. As M_{t^r} is B_{t^r}-pointwise bounded we get that

$$\sup\left\{\left|\mu_{i_{n_p} j_{n_p}}(H)\right| : p \in \mathbb{N}\right\} < \infty. \tag{7}$$

The sets $C_p := \cup_s\{B_{i_s j_s} : s < n_p\}$, $B_{i_{n_p} j_{n_p}}$ and $D_p := \cup_s\{B_{i_s j_s} : n_p < s\}$ are a partition of the set H. By (4), (5) and (6), $\left|\mu_{i_{n_p} j_{n_p}}(C)\right| < 1$, $\mu_{i_{n_p} j_{n_p}}(B_{i_{n_p} j_{n_p}}) > j_{n_p} > n_p$ and $\left|\mu_{i_{n_p} j_{n_p}}(D)\right| < 1$, for each $p \in \mathbb{N}\backslash\{1\}$. Therefore the inequality

$$\left|\mu_{i_{n_p} j_{n_p}}(H)\right| > -\left|\mu_{i_{n_p} j_{n_p}}(C)\right| + \mu_{i_{n_p} j_{n_p}}(B_{i_{n_p} j_{n_p}}) - \left|\mu_{i_{n_p} j_{n_p}}\right|(D) > j_{n_p} - 2$$

implies that

$$\lim_p \left|\mu_{i_{n_p} j_{n_p}}(H_0)\right| = \infty,$$

contradicting (7). □

The following corollary extends Corollary 13 in [6]. A family $\{B_{m_1 m_2 \ldots m_i} : i, m_j \in \mathbb{N}, 1 \leqslant j \leqslant i \leqslant p\}$ of subsets of A is an *increasing p-web in A* if $(B_{m_1})_{m_1}$ is an increasing covering of A and $(B_{m_1 m_2 \ldots m_{i+1}})_{m_{i+1}}$ is an increasing covering of $B_{m_1 m_2 \ldots m_i}$, for each $m_j \in \mathbb{N}$, $1 \leqslant j \leqslant i < p$ (this definition comes from [7, Chapter 7, 35.1]).

Corollary 2. *Let $\mathscr{S}$ be a σ-algebra of subsets of Ω and let $\{\mathscr{B}_{m_1 m_2 \ldots m_i} : i, m_j \in \mathbb{N}, 1 \leqslant j \leqslant i \leqslant p\}$ be an increasing p-web in $\mathscr{S}$. Then there exists $\mathscr{B}_{n_1 n_2 \ldots n_p}$ such that if*

$$\{\mathscr{B}_{n_1 n_2 \ldots n_p m_{p+1} m_{p+1} \ldots m_{p+k}} : k, m_{p+l} \in \mathbb{N}, 1 \leqslant l \leqslant k \leqslant q)$$

is an increasing q-web of $\mathscr{B}_{n_1 n_2 \ldots n_p}$ there exists $(n_{p+1}, n_{p+2}, \ldots, n_{p+q}) \in \mathbb{N}^q$ such that each $\tau_s(\mathscr{B}_{n_1 n_2 \ldots n_p n_{p+1} \ldots n_{p+q}})$-Cauchy sequence $(\mu_n \in \mathrm{ba}(\mathscr{S}))_n$ is $\tau_s(\mathscr{S})$-convergent.

Proof. By Proposition 1 with $\mathfrak{P} = N$ and Theorem 1 there exists $\mathscr{B}_{n_1 n_2 \ldots n_p}$ which has property wN. Hence there exists $\mathscr{B}_{n_1 n_2 \ldots n_p n_{p+1} \ldots n_{p+q}}$ which has property N. Then if $(\mu_n)_n \subset \mathrm{ba}(\mathscr{S})$ is a $\tau_s(\mathscr{B}_{n_1 n_2 \ldots n_p n_{p+1} \ldots n_{p+q}})$-Cauchy sequence we have that $(\mu_n)_n$ has no more than one $\tau_s(\mathscr{S})$-adherent point, whence $(\mu_n)_n$ is $\tau_s(\mathscr{S})$-convergent. As $\overline{L(\mathscr{B}_{n_1 n_2 \ldots n_p n_{p+1}})} = L(\mathscr{S})$ the sequence $(\mu_n)_n$ has no more that one $\tau_s(\mathscr{S})$-adherent point, whence $(\mu_n)_n$ is $\tau_s(\mathscr{S})$-convergent. □

3 Applications

In this section we obtain some applications of Theorem 1 to bounded finitely additive vector measures.

A *bounded finitely additive vector measure*, or simple *bounded vector measure*, μ defined in an algebra $\mathscr{A}$ of subsets of Ω with values in a topological vector space E is a map $\mu : \mathscr{A} \to E$ such that $\mu(\mathscr{A})$ is a bounded subset of E and $\mu(B \cup C) = \mu(B) + \mu(C)$, for each pairwise disjoint subsets $B, C \in \mathscr{A}$. Then the E-valued linear map $\mu : L(\mathscr{A}) \to E$ defined by $\mu(e_B) := \mu(B)$, for each $B \in \mathscr{A}$, is continuous.

A locally convex space $E(\tau)$ is the *p-inductive limit* of the family of locally convex spaces $\mathscr{E} := \{E_{m_1 m_2 \ldots m_i}(\tau_{m_1 m_2 \ldots m_i}) : i, m_j \in \mathbb{N}, 1 \leqslant j \leqslant i \leqslant p\}$ if $E(\tau)$ is the inductive limit of $(E_{m_1}(\tau_{m_1}))_{m_1}$ and moreover, each $E_{m_1 m_2 \ldots m_i}(\tau_{m_1 m_2 \ldots m_i})$ is the inductive limit of the sequence $(E_{m_1 m_2 \ldots m_i m_{i+1}}(\tau_{m_1 m_2 \ldots m_{i+1}}))_{m_{i+1}}$, for each $m_j \in \mathbb{N}$, $1 \leqslant j \leqslant i < p$. Then $\mathscr{E}$ is a *defining p-increasing web* for $E(\tau)$ *with steps* $E_{m_1 m_2 \ldots m_i}(\tau_{m_1 m_2 \ldots m_i})$. *$E(\tau)$ is a*

p-(LF) (or p-(LB)) space if $E(\tau)$ admits a defining p-increasing web $\mathscr{E}$ such that each $E_{m_1m_2\ldots m_p}(\tau_{m_1m_2\ldots m_p})$ is a Fréchet (or Banach) space and we say that $\mathscr{E}$ is a *defining p-(LF) (or p-(LB)) increasing web for* $E(\tau)$.

Next proposition extends [12, Theorem 4] and [6, Proposition 10].

Proposition 9. *Let μ be a bounded vector measure defined in a σ-algebra $\mathscr{S}$ of subsets of Ω with values in a topological vector space $E(\tau)$. Suppose that $\{E_{m_1m_2\cdots m_i} : m_j \in \mathbb{N}, 1 \leqslant j \leqslant i \leqslant p\}$ is an increasing p-web in E. Then there exists $E_{n_1n_2\cdots n_p}$ such that if $E_{n_1n_2\cdots n_p}(\tau_{n_1n_2\cdots n_p})$ is an q-(LF)-space, the topology $\tau_{n_1n_2\cdots n_p}$ is finer than the relative topology $\tau|_{E_{n_1n_2\cdots n_p}}$ and $\{E_{n_1n_2\ldots n_pm_{p+1}\ldots m_{p+i}}(\tau_{m_1m_2\ldots m_im_{p+1}\ldots m_{p+i}}) : i, m_{p+j} \in \mathbb{N}, 1 \leqslant j \leqslant i \leqslant q\}$ a defining q-(LF) increasing web for $E_{n_1n_2\cdots n_p}(\tau_{n_1n_2\cdots n_p})$ there exists $(n_{p+1}, n_{p+2}, \ldots, n_{p+q}) \in \mathbb{N}^q$ such that $\mu(\mathscr{S})$ is a bounded subset of $E_{n_1n_2\ldots n_pn_{p+1}\ldots n_{p+q}}(\tau_{n_1n_2\ldots n_pn_{p+1}\ldots n_{p+q}})$.*

Proof. Let $\mathscr{B}_{m_1m_2\ldots m_i} := \mu^{-1}(E_{m_1m_2\ldots m_i})$ for each $m_j \in \mathbb{N}$, $1 \leqslant j \leqslant i \leqslant p$. By Proposition 1 and Theorem 1 there exists $(n_1, n_2, \ldots, n_p) \in \mathbb{N}^p$ such that $\mathscr{B}_{n_1n_2\ldots n_p}$ has *wN*-property. Let $\{E_{n_1n_2\ldots n_pm_{p+1}\ldots m_{p+i}}(\tau_{m_1m_2\ldots m_im_{p+1}\ldots m_{p+i}}) : i, m_j \in \mathbb{N}, 1 \leqslant j \leqslant i \leqslant q\}$ be a defining p-(LF) increasing web for $E_{n_1n_2\ldots n_p}(\tau_{n_1n_2\ldots n_p})$ and let $\mathscr{B}_{n_1n_2\ldots n_pm_{p+1}\ldots m_{p+i}} := \mu^{-1}(E_{n_1n_2\ldots n_pm_{p+1}\ldots m_{p+i}})$, for each $i, m_{p+j} \in \mathbb{N}, 1 \leqslant j \leqslant i \leqslant q$. As

$$\{\mathscr{B}_{n_1n_2\ldots n_pm_{p+1}\ldots m_{p+i}} : i, m_{p+j} \in \mathbb{N}, 1 \leqslant j \leqslant i \leqslant q\}$$

is an increasing q-web of $\mathscr{B}_{n_1n_2\ldots n_p}$ and this set has *wN*-property then there exists a subset $\mathscr{B}_{n_1n_2\ldots n_pn_{p+1}\ldots n_{p+q}}$ which has property *N*, whence $L(\mathscr{B}_{n_1n_2\ldots n_pn_{p+1}\ldots n_{p+q}})$ is a dense subspace of $L(\mathscr{S})$ and then the map with closed graph

$$\mu|_{L(\mathscr{B}_{n_1n_2\ldots n_pn_{p+1}\ldots n_{p+q}})} : L(\mathscr{B}_{n_1n_2\ldots n_pn_{p+1}\ldots n_{p+q}}) \to E_{n_1n_2\ldots n_pn_{p+1}\ldots n_{p+q}}(\tau_{n_1n_2\ldots n_pn_{p+1}\ldots n_{p+q}})$$

has a continuous extension υ to $L(\mathscr{S})$ with values in $E_{n_1n_2\ldots n_pn_{p+1}\ldots n_{p+q}}(\tau_{n_1n_2\ldots n_pn_{p+1}\ldots n_{p+q}})$ (by [10, 2.4 Definition and (N_2)] and [11, Theorems 1 and 14]). Since $\mu : L(\mathscr{S}) \to E(\tau)$ is continuous, $\upsilon(A) = \mu(A)$, for each $A \in \mathscr{S}$.
Whence $\mu(\mathscr{S})$ is a bounded subset of $E_{n_1n_2\ldots n_pn_{p+1}\ldots n_{p+q}}(\tau_{n_1n_2\ldots n_pn_{p+1}\ldots n_{p+q}})$. □

Corollary 3. *Let μ be a bounded vector measure defined in a σ-algebra $\mathscr{S}$ of subsets of Ω with values in an inductive limit $E(\tau) = \Sigma_m E_m(\tau_m)$ of an increasing sequence $(E_m(\tau_m))_m$ of q-(LF) spaces. There exists $n_1 \in \mathbb{N}$ such that for each defining q-(LF) increasing web for $E_{n_1}(\tau_{n_1})$, $\{E_{n_1m_{1+1}\ldots m_{1+i}}(\tau_{n_1m_{1+1}\ldots m_{1+i}}) : i, m_{1+j} \in \mathbb{N}, 1 \leqslant j \leqslant i \leqslant q\}$ there exists $(n_{1+i})_{1 \leq i \leq q}$ in $\mathbb{N}^q$ such that $\mu(\mathscr{S})$ is a bounded subset of $E_{n_1n_{1+1}\ldots n_{1+q}}, (\tau_{n_1n_{1+1}\ldots n_{1+q}})$.*

A sequence $(x_k)_k$ in a locally convex space E is *subseries convergent* if for every subset J of $\mathbb{N}$ the series $\Sigma\{x_k : k \in J\}$ converges. The following corollary is a generalization of the localization property given in [12, Corollary 1.4] and it follows from Corollary 3.

Corollary 4. *Let $(x_k)_k$ be a subseries convergent sequence in an inductive limit $E(\tau) = \Sigma_m E_m(\tau_m)$ of an increasing sequence $(E_m(\tau_m))_m$ of q-(LF) spaces. There exists $n_1 \in \mathbb{N}$ such that for each defining q-(LF) increasing web $\{E_{n_1m_{1+1}\ldots m_{1+i}}(\tau_{n_1m_{1+1}\ldots m_{1+i}}) : i, m_{1+j} \in \mathbb{N}, 1 \leqslant j \leqslant i \leqslant q\}$ for $E_{n_1}(\tau_{n_1})$ there exists $(n_{1+1}, n_{1+2}, \ldots, n_{1+q}) \in \mathbb{N}^q$ such that $\{x_k : k \in \mathbb{N}\}$ is a bounded subset of $E_{n_1n_{1+1}\ldots n_{1+q}}(\tau_{n_1n_{1+1}\ldots n_{1+q}})$.*

Proof. As $(x_k)_k$ is subseries convergent, then the additive vector measure $\mu: 2^{\mathbb{N}} \to E(\tau)$ defined by $\mu(J) := \Sigma_{k\in J} x_k$, for each $J \in 2^{\mathbb{N}}$, is bounded, because $(f(x_k))_k$ is subseries convergent for each $f \in E'$, whence $\Sigma_{n=1}^{\infty} |f(x_n)| < \infty$. Therefore we may apply Corollary 3. □

Proposition 10. *Let μ be a bounded vector measure defined in a σ-algebra $\mathscr{S}$ of subsets of Ω with values in a topological vector space $E(\tau)$. Suppose that $\{E_{m_1m_2\ldots m_i} : m_j \in \mathbb{N}, 1 \leqslant j \leqslant i \leqslant p\}$ is an increasing p-web in E. There exists $E_{n_1n_2\ldots n_p}$ such that if $\{E_{n_1n_2\ldots n_p m_{p+1}\ldots m_{p+i}} : i, m_{p+j} \in \mathbb{N},\ 1 \leqslant j \leqslant i \leqslant q\}$ is a q-increasing web in $E_{n_1n_2\ldots n_p}$ with the property that each relative topology $\tau|_{E_{n_1n_2\ldots n_p m_{p+1}\ldots m_{p+q}}}, (m_{p+1}, \ldots, m_{p+q}) \in \mathbb{N}^q$ is sequentially complete, then there exists $(n_{p+1}, \ldots, n_{p+q}) \in \mathbb{N}^q$ such that $\mu(\mathscr{S}) \subset E_{n_1n_2\ldots n_p n_{p+1}\ldots nq}$.*

Proof. Let $\mathscr{B}_{m_1m_2\ldots m_i} := \mu^{-1}(E_{m_1m_2\ldots m_i})$ for each $m_j \in \mathbb{N}$, $1 \leqslant j \leqslant i \leqslant p$. By Proposition 1 and Theorem 1 there exists $(n_1, n_2, \ldots, n_p) \in \mathbb{N}^p$ such that $\mathscr{B}_{n_1n_2\ldots n_p}$ has *wN*-property. Let $\{E_{n_1n_2\ldots n_p m_{p+1}\ldots m_{p+i}} : i, m_j \in \mathbb{N}, 1 \leqslant j \leqslant i \leqslant q\}$ be a increasing q-web in $E_{n_1n_2\ldots n_p}$ and let $\mathscr{B}_{n_1n_2\ldots n_p m_{p+1}\ldots m_{p+i}} := \mu^{-1}(E_{n_1n_2\ldots n_p m_{p+1}\ldots m_{p+i}})$, for each $i, m_{p+j} \in \mathbb{N}, 1 \leqslant j \leqslant i \leqslant q$. As

$$\{\mathscr{B}_{n_1n_2\ldots n_p m_{p+1}\ldots m_{p+i}} : i, m_{p+j} \in \mathbb{N},\ 1 \leqslant j \leqslant i \leqslant q\}$$

is an increasing q-web of $\mathscr{B}_{n_1n_2\ldots n_p}$ there exists $\mathscr{B}_{n_1n_2\ldots n_p n_{p+1}\ldots n_{p+q}}$ which has property N, whence $L(\mathscr{B}_{n_1n_2\ldots n_p n_{p+1}\ldots n_{p+q}})$ is a dense subspace of $L(\mathscr{S})$ and then the continuous map

$$\mu|_{L(\mathscr{B}_{n_1n_2\cdots n_p n_{p+1}})} : L(\mathscr{B}_{n_1n_2\ldots n_p n_{p+1}\ldots n_{p+q}}) \to E_{n_1n_2\ldots n_p n_{p+1}\ldots n_{p+q}}(\tau|_{E_{n_1n_2\cdots n_p n_{p+1}\cdots n_{p+q}}})$$

has a continuous extension υ to $L(\mathscr{S})$ with values in $E_{n_1n_2\ldots n_p n_{p+1}\ldots n_{p+q}}(\tau|_{E_{n_1n_2\cdots n_p n_{p+1}\cdots n_{p+q}}})$. The continuity of $\mu : L(\mathscr{S}) \to E(\tau)$ implies that $\upsilon(A) = \mu(A)$, for each $A \in \mathscr{S}$. Whence $\mu(\mathscr{S})$ is a subset of $E_{n_1n_2\ldots n_p n_{p+1}\ldots n_{p+q}}$. □

Corollary 5. *Let μ be a bounded additive vector measure defined in a σ-algebra $\mathscr{S}$ of subsets of Ω with values in an inductive limit $E(\tau) = \Sigma_{m_1} E_{m_1}(\tau_{m_1})$ of an increasing sequence $(E_m(\tau_m))_m$ of countable dimensional topological vector spaces. Then there exists n_1 such that for each q-increasing web $\{E_{n_1m_{1+1}\ldots m_{1+i}} : i, m_{1+j} \in \mathbb{N}, 1 \leqslant j \leqslant i \leqslant q\}$ in E_{n_1} such that the dimension of each $E_{n_1m_{1+1}\ldots m_{1+q}}$ is finite there exists $E_{n_1n_{1+1}\ldots n_{1+q}}$ which contains the set.*

Proof. As the relative topology $\tau|_{E_{n_1m_{1+1}\cdots m_{1+q}}}$ is complete we may apply Proposition 10. □

Acknowledgement

To Professor Manuel Valdivia (1928-2014), whose paper *On Nikodym boundedness property*, RACSAM 2013, give us many suggestions for this work.

We thank to an anonymous referee his proof that properties *wN* and w^2N are equivalent that we generalize in Proposition 1. We are also grateful to two both referees by their suggestions and indications.

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