Nikodym boundedness property for webs in $\sigma$-algebras

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Abstract

A subset $\mathcal{B}$ of an algebra $\mathcal{A}$ of subsets of $\Omega$ is said to have the property $N$ if a $\mathcal{B}$-pointwise bounded subset $M$ of $ba(\mathcal{A})$ is uniformly bounded on $\mathcal{A}$, where $ba(\mathcal{A})$ is the Banach space of the real (or complex) finitely additive measures of bounded variation defined on $\mathcal{A}$ with the norm variation. Moreover $\mathcal{B}$ is said to have the property $sN$ if for each increasing countable covering $\{\mathcal{B}_m\}_{m \in \mathbb{N}}$ of $\mathcal{B}$ there exists $\mathcal{B}_n$ which has the property $N$ and $\mathcal{B}$ is said to have property $wN$ if given the increasing countable coverings $\{\mathcal{B}_m\}_{m \in \mathbb{N}}$ of $\mathcal{B}$ and $\{\mathcal{B}_{m_1, m_2, \ldots, m_{p+1}}\}_{m_{p+1} \in \mathbb{N}}$ of $\mathcal{B}_{m_1, m_2, \ldots, m_r}$, for each $p, m_i \in \mathbb{N}$, $1 \leq i \leq p + 1$, there exists a sequence $(n_i)$ such that each $\mathcal{B}_{n_1, n_2, \ldots, n_r}$, $r \in \mathbb{N}$, has property $N$. For a $\sigma$-algebra $\mathcal{S}$ of subsets of $\Omega$ it has been proved that $\mathcal{S}$ has property $N$ (Nikodym-Grothendieck), property $sN$ (Valdivia) and property $w(sN)$ (Kakol-López-Pellicer). We give a proof of property $wN$ for a $\sigma$-algebra $\mathcal{S}$ which is independent of properties $N$ and $sN$. This result and the equivalence of properties $wN$ and $w^2N$ enable us to give some applications to localization of bounded additive vector measures.

Keywords: Bounded set; finitely additive scalar (vector) measure; inductive limit; NV-tree; $\sigma$-algebra; web Nikodym property

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1 Introduction

Let $\Omega$ be a set and $\mathcal{A}$ a set-algebra of subsets of $\Omega$. If $\mathcal{B}$ is a subset of $\mathcal{A}$ then $L(\mathcal{B})$ is the normed space of the real or complex linear hull of the set of characteristics functions $\{e_C : C \in \mathcal{B}\}$ endowed with the supremum norm $\|\cdot\|$. The dual of $L(\mathcal{A})$ with the dual norm is named $L(\mathcal{A})'$ and it is isometric to the Banach space $ba(\mathcal{A})$ of finitely additive measures on $\mathcal{A}$ with bounded variation provided with the variation norm, i.e., $\|\cdot\| := \|\cdot|\Omega\|$, being the isometry the map $\Theta : ba(\mathcal{A}) \to L(\mathcal{A})'$ such that, for each $\mu \in ba(\mathcal{A})$, $\Theta(\mu)$ is the linear form named also by $\mu(e_C) := \mu(C)$, for each $C \in \mathcal{A}$ [2, Chapter 1]. A norm in $L(\mathcal{A})'$ equivalent to the supremum norm is defined by the Minkowski functional of $\text{absco}(\{e_C : C \in \mathcal{A}\})$ ([12, Proposition 1 and 2]), which dual norm is the $\mathcal{A}$-supremum norm, i.e., $\|\mu\| := \sup\{|\mu(C)| : C \in \mathcal{A}\}$, $\mu \in ba(\mathcal{A})$.

In this paper duality is referred to the dual pair $(L(\mathcal{A}), \text{ba}(\mathcal{A}))$ and we follow notations of [7]. Then the weak * dual of a locally convex space $E$ is $(E', \tau_\pi(E))$, whence the topology $\tau_\pi(\mathcal{A})$ is the topology $\pi(\mathcal{A})$ of pointwise convergence in the elements of $\mathcal{A}$, the cardinal of a set $C$ is denoted by $|C|$, $\mathbb{N}$ is the set $\{1, 2, \ldots\}$ of positive integers, the closure of a set is marked by an overline, the convex (absolutely convex) hull of a subset $M$ of a topological vector space is represented by $\text{co}(M)$ ($\text{absco}(M)$) and $\text{absco}(M) = \text{co}(\cup\{rM : |r| = 1\})$.

A subset $\mathcal{B}$ of a set-algebra $\mathcal{A}$ has the Nikodym property, property $\mathcal{N}$ in brief, if each $\mathcal{B}$-pointwise bounded subset $M$ of $\text{ba}(\mathcal{A})$ is bounded in $\text{ba}(\mathcal{A})$ (see [10, Definition 2.4] or [13, Definition 1]). If $\mathcal{B}$ has property $\mathcal{N}$ the polar set $\{e_C : C \in \mathcal{B}\}^\circ$ is bounded in $\text{ba}(\mathcal{A})$, hence the bipolar set $\{e_C : C \in \mathcal{B}\}^{\circ\circ} = \text{absco}\{e_C : C \in \mathcal{B}\}$ is a neighborhood of zero in $L(\mathcal{A})$ and then $L(\mathcal{B})$ is dense in $L(\mathcal{A})$. Notice also that a subset $\mathcal{B}$ of an algebra $\mathcal{A}$ has property $\mathcal{N}$ if each $\mathcal{B}$-pointwise bounded, $\tau_\pi(\mathcal{A})$-closed and absolutely convex subset $M$ of $\text{ba}(\mathcal{A})$ is uniformly bounded in $\mathcal{A}$. The algebra of finite and co-finite subsets of $\mathbb{N}$ fails to have property $\mathcal{N}$ and Schachermayer proved that the algebra $\mathcal{J}(I)$ of Jordan measurable subsets of $I := [0, 1]$ has property $\mathcal{N}$ [10, Corollary 3.5] (see a generalization of this property in [4, Corollary 1]).

A subset $\mathcal{B}$ of a set-algebra $\mathcal{A}$ has the strong Nikodym property, property $s\mathcal{N}$ in brief, if for each increasing covering $\cup_m \mathcal{B}_m$ of $\mathcal{B}$ there exists $\mathcal{B}_n$ which has property $\mathcal{N}$. Valdivia proved that the algebra $\mathcal{J}(K)$ of Jordan measurable subsets of a compact $k$-dimensional interval $K := \Pi\{[a_i, b_i] : 1 \leq i \leq k\}$ in $\mathbb{R}^k$ has property $s\mathcal{N}$ [13, Theorem 2].

An increasing web in a set $A$ is a family $\mathcal{W} := \{A_{m_1m_2 \ldots m_p} : (m_1, m_2, \ldots, m_p) \in \cup_i \mathcal{N}^p\}$ of subsets of $A$ such that $(A_{m_1})_{m_1}$ and $(A_{m_1m_2 \ldots m_{p+1},})_{m_{p+1}}$ are, respectively, increasing coverings of $A$ and $A_{m_1m_2 \ldots m_p}$, for each $p, m_i \in \mathcal{N}, 1 \leq i \leq p + 1$ [7, Chapter 7, 35.1], and each sequence $(A_{m_1m_2 \ldots m_p})_p$ is a strand in $\mathcal{W}$. A subset $\mathcal{B}$ of a set-algebra $\mathcal{A}$ has the web Nikodym property, property $w\mathcal{N}$ in brief, if for each increasing web $\{\mathcal{B}_t : t \in \cup_i \mathcal{N}^i\}$ in $\mathcal{B}$ there exists a strand composed of sets which have property $\mathcal{N}$. In general, if $B$ is a set and $\mathfrak{B}$ is a property verified in the elements of a family of subsets of $B$ then $B$ has property $w\mathcal{B}$ if each increasing web $\{B_t : t \in \cup_i \mathcal{N}^i\}$ in $\mathcal{B}$ has a strand composed of sets which have property $\mathfrak{B}$.

Property $w(\mathfrak{w}\mathcal{B})$ is named as property $w^2\mathcal{B}$. The next straightforward proposition states that properties $w\mathcal{B}$ and $w^2\mathcal{B}$ are equivalent.
Proposition 1. Let \((B_m)_m\) be an increasing covering of a set \(B\) which verifies property \(w\mathcal{P}\). There exists \(B_n\) which has property \(w\mathcal{P}\), whence \(B\) has property \(w^2\mathcal{P}\).

Proof. Let us suppose that \((B_m)_m\) is an increasing covering of a set \(B\) such that each \(B_m\) does not have property \(w\mathcal{P}\). Then, for each natural number \(m\) there exists an increasing web \(\mathcal{W}_m:=\{B_{m_1,m_2,...,m_p}: p,m_1,m_2,...,m_p \in \mathbb{N}\}\) in \(B_m\) such that every strand in \(\mathcal{W}_m\) contains a set \(B_{m_1,m_2,...,m_p}\) of \(w\mathcal{P}\). If \(B_{m_1,m_2,...,m_p}\) and \(B_{m_1,m_2,...,m_p}\) we get that \(\mathcal{W}:=\{B_{m_1,m_2,...,m_p}: p,m_1,m_2,...,m_p \in \mathbb{N}\}\) is an increasing web in \(B\) without strands consisting of sets with property \(\mathcal{P}\), whence \(B\) does not have property \(w\mathcal{P}\). This proves the first affirmation which readily implies that if \(B\) verifies property \(w\mathcal{P}\) then every increasing web in \(B\) contains a strand consisting of sets with property \(w\mathcal{P}\), whence properties \(w\mathcal{P}\) and \(w^2\mathcal{P}\) are equivalent in \(B\).

Let \(\mathcal{P}\) be a \(\sigma\)-algebra of subsets of a set \(\Omega\). It has been sequentially shown that (i) \(\mathcal{P}\) has property \(N\) (Nikodym-Dieudonné-Grothendieck theorem [9], [3] and [1, page 80, named as Nikodym-Grothendieck boundedness theorem]), (ii) \(\mathcal{P}\) has property \(sN\) ([12, Theorem 2]) and (iii) \(\mathcal{P}\) has property \(w(sN)\) (very recently in [6, Theorem 2]).

The aim of this paper is to present in the next section a proof of the property that each \(\sigma\)-algebra \(\mathcal{P}\) has property \(wN\) independent of any property related to Nikodym boundedness property, as properties \(N\) or \(sN\), and using very elementary locally convex space theory.

Last section deals with some applications to bounded vector measures deduced from the property \(wN\) of each \(\sigma\)-algebra \(\mathcal{P}\) and from the equivalence stated in Proposition 1.

Following the characterization of \(sN\)-property of a set-algebra \(A\) by the locally convex property of \(L(A)\) given in [13, Theorem 3] it is possible to get a characterization of \(wN\) property of a set-algebra \(A\) by the locally convex properties considered in [5] and [8]. In fact Theorem 1 is equivalent to Theorem 2.7 of [8], totally stated in the locally convex theory frame.

2 NV-trees and property \(wN\)

Given two elements, \(t = (t_1, t_2, \ldots, t_p)\) and \(s = (s_1, s_2, \ldots, s_q)\), and two subsets, \(T\) and \(U\), of \(\bigcup \mathbb{N}^n\), then \(p \in \mathbb{N}\) is the length of \(t\), for each \(1 \leq i \leq p\) the section of length \(i\) of \(p\) is \(t(i):= (t_1, t_2, \ldots, t_i)\); if \(p > 0\), \(t(t):= \emptyset\); \(T(m):= \{t(m) : t \in T\}\), for each \(m \in \mathbb{N}\); \(t \times s := \{(t_1, t_2, \ldots, t_p, s_1, s_2, \ldots, s_q)\}\), with \(t_{p+j} := s_j\), for \(1 \leq j \leq q\), and \(T \times U := \{t \times u : t \in T, u \in U\}\).

Each \(t \times u \in U\) is an extension of \(t\) in \(U\) and a sequence \((t^n)\) of elements \(t^n = (t^n_1, t^n_2, \ldots, t^n_n) \in T\) is an infinite chain in \(T\) if for each \(n \in \mathbb{N}\) the element \(t^{n+1}\) is an extension of the section \(t^n\) in \(T\), i.e., \(\emptyset \neq t^n(n) = t^{n+1}(n)\), and length of \(t^n\) is at least \(n\), for each \(n \in \mathbb{N}\). If \(t = (t_1)\) then \(t\) and the products \(T \times t\) and \(t \times T\) are represented by \(t_1, T \times t_1\) and \(t_1 \times T\).

Let \(\emptyset \neq U \subset \bigcup \mathbb{N}^n\). \(U\) is increasing at \(t = (t_1, t_2, \ldots, t_p) \in \bigcup \mathbb{N}^n\) if \(U\) contains elements \(t^i = (t_1^i, t_2^i, \ldots, t_p^i)\) and \(t' = (t_1, t_2, \ldots, t_{i-1}, t'_i, t_{i+1}, \ldots)\), \(1 \leq i \leq p\), such that \(t_i < t'_i\), for each \(1 \leq i \leq p\). \(U\) is increasing (increasing respect to a subset \(V\) of \(\bigcup \mathbb{N}^n\)) if \(U\) is
increasing at each \( t \in U \) (at each \( t \in V \)). Clearly \( U \) is increasing if \( |U(1)| = \infty \) and 
\[
|\{n \in \mathbb{N} : t(i) \times n \in U(i+1)\}| = \infty, \text{for each } t = (t_1, t_2, \ldots, t_p) \in U \text{ and } 1 \leq i < p.
\]

Next definition deals with a particular type of increasing trees (see [6, Definition 2]).

**Definition 1.** An NV-tree \( T \) is an increasing subset of \( \cup_{i \in \mathbb{N}} N_i \) without infinite chains such that for each \( t = (t_1, t_2, \ldots, t_p) \in T \) the length of each extension of \( t(p-1) \) in \( T \) is \( p \) and \( \{t(i) : 1 \leq i \leq p\} \cap T = \{t\} \).

An NV-tree \( T \) is trivial if \( T = T(1) \) and then \( T \) is an infinite subset of \( \mathbb{N} \).

The sets \( N_i, i \in \mathbb{N} \setminus \{1\} \), and the set \( \cup \{(i) \times N_i : i \in \mathbb{N}\} \) are non-trivial NV-trees. The finite product of NV-trees is an NV-tree.

If \( T \) is an increasing subset of \( \cup_{i \in \mathbb{N}} N_i \) and \( \{B_u : u \in \cup_{i \in \mathbb{N}} N_i\} \) is an increasing web in \( B \) then \( (\cup_{i \in T(j)} B_u) \) is an increasing covering of \( B \), because for each \( n = (u_1, u_2, \ldots, u_p) \in T \) and each \( i < p \) the sequence \( (\cup_{i \in T(j)} B_u) \times n \) is an increasing covering of \( B(1) \), hence if \( T \) does not contain infinite chains and \( b \in B \) there exists \( t \in T \) such that \( b \in B_t \). Therefore \( B = \cup \{B_t : t \in T\} \).

Each increasing subset \( S \) of an NV-tree \( T \) is an NV-tree, whence if \( (S_n)_n \) is a sequence of subsets of an NV-tree \( T \) such that each \( S_{n+1} \) is increasing relative to \( S_n \) then \( \cup_n S_n \) is an NV-tree. This hereditary property and Proposition 7 in [6] imply next Proposition 2 and we give a proof as a help for the reader.

**Proposition 2.** Let \( U \) be a subset of an NV-tree \( T \). If \( U \) does not contain an NV-tree then \( T \setminus U \) contains an NV-tree.

**Proof.** This proposition is obvious if \( T \) is a trivial NV-tree. Whence we suppose that \( T \) is a non-trivial NV-tree and then there exists \( m'_i \in T(1) \) such that for each \( n \geq m'_i \) the set \( \{v \in \cup_{i \in \mathbb{N}} : n \times v \in U\} \) does not contain an NV-tree. We define \( Q_0 := \emptyset \) and \( Q_i := \{n \in T(1) \setminus T : m'_i \leq n\} \).

Let us suppose that we have obtained for each \( j \), with \( 2 \leq j \leq i \), two disjoint subsets \( Q_j \) and \( Q'_j \) of \( T(j) \), with \( Q_j \subset T \setminus U \) and \( Q'_j \cap T = \emptyset \), such that for each \( t \in Q_j \cup Q'_j \) the section \( t(j-1) \in Q'_{j-1} \) and \( A_{t(j-1)} := \{n \in \mathbb{N} : t(j-1) \times n \in Q_j \cup Q'_j\} \) is an infinite set such that \( t \in Q_j \) implies that \( t(j-1) \times A_{t(j-1)} \subset Q_j \) and from \( t \in Q'_j \) it follows that \( t(j-1) \times A_{t(j-1)} \subset Q'_j \), and that the set \( \{v \in \cup_{i \in \mathbb{N}} : t \times v \in U\} \) does not contain an NV-tree. Then we define \( S_{j-1} := A_{t(j-1)} \) and \( S'_{j-1} := \emptyset \) in the first case and \( S_{j-1} := \emptyset \) in the second case.

As for each \( t \in Q'_j \subset T(j) \setminus T \) the set \( \{v \in \cup_{i \in \mathbb{N}} : t \times v \in U\} \) does not contain an NV-tree and it is a subset of the NV-tree \( T_i := \{v \in \cup_{i \in \mathbb{N}} : t \times v \in T\} \), the following two cases may happen:

- **i.** Either the NV-tree \( T_i \) is trivial and then there exists \( m_{i+1} \in \mathbb{N} \) such that the infinite set \( S_{i+1} := \{n \in \mathbb{N} : m_{i+1} \leq n, t \times n \in T(i+1)\} \) verifies that \( t \times S_{i+1} \subset T \setminus U \). In this case we define \( S_{i+1} := \emptyset \).

- **ii.** Or the NV-tree \( T_i \) is non-trivial and then there exists \( m_{i+1} \in \mathbb{N} \) such that the infinite set \( S_{i+1} := \{n \in \mathbb{N} : m_{i+1} \leq n, t \times n \in T(i+1)\} \) verifies that \( t \times S_{i+1} \subset T(i+1) \setminus T \) and for each \( t \times n \in t \times S_{i+1} \) the set \( \{v \in \cup_{i \in \mathbb{N}} : t \times n \times v \in U\} \) does not contain an NV-tree. Now we define \( S_{i+1} := \emptyset \).
The induction finish by setting $Q_{i+1} := \bigcup \{ t \times S_i : t \in Q'_i \}$ and $Q'_i := \bigcup \{ t \times S'_i : t \in Q'_i \}$. Then $Q_{i+1} \subset T(i+1) \cap (T \setminus U)$, $Q'_{i+1} \subset T(i+1) \setminus T$, and each $t \in Q_{i+1} \cup Q'_{i+1}$ verifies the above indicated properties when $i \in \omega \setminus Q'_i$, changing $j$ by $i+1$.

As $T$ does not contain infinite chains for each $t_1, t_2, \ldots, t_i \in Q'_i$, there exists $q \in \mathbb{N}$ and $(t_{i+1}, \ldots, t_{iq}) \in \mathbb{N}^q$ such that $(t_1, t_2, \ldots, t_i, t_{i+1}, \ldots, t_{iq}) \in Q_{iq}$, whence $(\bigcup_{j \geq 1} Q'_j)(i) = Q'_i$. This implies that the subset $W := \bigcup \{ Q'_j : j \in \mathbb{N} \}$ of $T \setminus U$ has the increasing property, because from $W(k) = Q'_k \cup Q'_s$, for each $k \in \mathbb{N}$, we get that $|W(1)| = |Q'_1| = \infty$ and if $t = (t_1, t_2, \ldots, t_p) \in W$ then $(t_{i+1}, \ldots, t_i) \in Q'_p$, if $1 \leq i < p$, and $(t_1, t_2, \ldots, t_p) \in Q'_p$, whence the infinite subsets $S'_i$ and $S'_j$ of $\mathbb{N}$ verify that $t(i-1) \times S'_i \subset Q'_i \subset W(i)$ and $t(p-1) \times S'_j \subset Q'_j \subset W$. Therefore $W$ is an NV-tree contained in $T \setminus U$. □

**Definition 2.** A property $\mathfrak{P}$ is hereditary increasing in a set $A$ if for each pair of subsets $B$ and $C$ of $A$ such that $B$ verifies property $\mathfrak{P}$ and $B \subset C \subset A$ then $C$ also has property $\mathfrak{P}$.

**Example 1.** The properties $wN$, $sN$ and $N$ are hereditary increasing properties in a set-algebra $\mathfrak{A}$.

**Proof.** Let $B \subset C \subset \mathfrak{A}$. It is obvious that if $B$ has property $N$ then $C$ has also property $N$. Whence if $B$ has property $sN$ and if $\cup_{p\in \omega} C_p$ is an increasing covering of $C$ then there exists $C_p$ such that $C_p \cap B$ has property $N$, therefore $C_p$ has property $N$ and we get that $C$ has also property $sN$.

If $B$ has property $wN$ and $(C_{p,m_1,\ldots,m_p} : p, m_1, m_2, \ldots, m_p \in \mathbb{N})$ is an increasing web in $C$, then there exists a sequence $(n_i)$ such that each $C_{n, m_2, \ldots, m_p} \cap B$ has property $N$, $i \in \mathbb{N}$, whence $(C_{n, m_2, \ldots, m_p})$ is a strand in $C$ consisting of sets which have property $N$. □

**Proposition 3.** Let $\mathfrak{P}$ be an hereditary increasing property in $A$ and let $B := \{ B_{m_1,\ldots,m_p} : p, m_1, m_2, \ldots, m_p \in \mathbb{N} \}$ be an increasing web in $A$ without strands consisting of sets with property $\mathfrak{P}$. Then there exists an NV-tree $T$ such that for each $t = (t_1, t_2, \ldots, t_p) \in T$ the set $B_t$ does not have property $\mathfrak{P}$ and if $p > 1$ then $B_{t(i)}$ has property $\mathfrak{P}$, for each $i = 1, 2, \ldots, p - 1$.

**Proof.** If each $B_{m_1}$, $m_1 \in \mathbb{N}$, does not have property $\mathfrak{P}$ the proposition is obvious with $T := \mathbb{N}$. Hence we may suppose that there exists $m'_1 \in \mathbb{N}$ such that $B_{m'_1}$ has property $\mathfrak{P}$ for each $t_1 \geq m'_1$ and then we write $Q_1 := \emptyset$ and $Q'_1 := \{ t_1 \in \mathbb{N} : t_1 \geq m'_1 \}$.

Let us assume that for each $j$, with $2 \leq j \leq i$, we have obtained by induction two disjoint subsets $Q_j$ and $Q'_j$ of $\mathbb{N}^j$ such that for each $t = (t_1, t_2, \ldots, t_j) \in Q_j \cup Q'_j$ the assertion $t(j - 1) = (t_1, t_2, \ldots, t_{j-1}) \in Q'_{j-1}$, if $t \in Q_j$ then the set $B_t$ does not have property $\mathfrak{P}$ and $t(j - 1) \times Q'_{j-1} \subset Q_j$ and then we define $S_{t(j-1)} := \emptyset$ and $S'_{t(j-1)} := \emptyset$; otherwise, if $t \in Q'_j$ then the set $B_t$ has property $\mathfrak{P}$ and $S'_{t(j-1)} := \{ n \in \mathbb{N} : t(j - 1) \times n \in Q_j \cup Q'_j \}$ is a co-finite subset of $\mathbb{N}$ such that $t(j - 1) \times S'_{t(j-1)} \subset Q'_j$. In this case we define $S_{t(j-1)} := \emptyset$.

If $t := (t_1, t_2, \ldots, t_i) \in Q'_i$ then, by induction, $B_{t_{i-2},\ldots,i}$ has property $\mathfrak{P}$ and as $(B_{t_{i-2},\ldots,i})_n$ is an increasing covering of $B_{t_{i-2},\ldots,i}$ it may happen that either $B_{t_{i-2},\ldots,i}$ does not have property $\mathfrak{P}$ for each $n \in \mathbb{N}$ and then we define $S_{t_{i-2},\ldots,i} := \emptyset$ and $S'_{t_{i-2},\ldots,i} := \emptyset$, or there
exists $m'_1, m'_2, \ldots, m'_n \in \mathbb{N}$ such that $B_{1, i_2, \ldots, i_n}$ has property $\mathcal{P}$ for each $n \geq m'_i$, and in this second case we define $S_{t_1, \ldots, t_n} := \emptyset$ and $S'_{t_1, \ldots, t_n} := \{n \in \mathbb{N} : m'_n \leq n\}$.

We finish this induction procedure by setting $Q_{i+1} := \cup \{t \times S_t : t \in Q_i\}$ and $Q'_{i+1} := \cup \{t \times S'_t : t \in Q'_i\}$. By construction $Q_{i+1}$ and $Q'_{i+1}$ verify the above indicated properties of $Q_i$ and $Q'_i$, replacing $j$ by $i + 1$.

The hypothesis that for each sequence $(m_i)_i \in \mathbb{N}^\mathbb{N}$ there exists $j \in \mathbb{N}$ such that $B_{m_1, m_2, \ldots, m_p} \in \mathcal{Q}_p$ implies that $T := \cup \{Q_i : i \in \mathbb{N}\}$ does not contain infinite chains, because if $(m_1, m_2, \ldots, m_p) \in \mathcal{Q}_p$ then $(m_1, m_2, \ldots, m_{p-1}) \in \mathcal{Q}'_{p-1}$, hence $B_{m_1, m_2, \ldots, m_{p-1}}$ has property $\mathcal{P}$. Therefore for each $(t_1, t_2, \ldots, t_k) \in \mathcal{Q}'_k$ there exists an extension $(t_1, t_2, \ldots, t_k, k+1, \ldots, k+q) \in \mathcal{Q}_{k+q}$, whence $T(k) = Q_k \cup Q'_k$, for each $k \in \mathbb{N}$. Then the set $T$ has the increasing property, because $|T(1)| = |T'_1| = \infty$ and if $t = (t_1, t_2, \ldots, t_p) \in T$ the sets $S'_t(i-1), 1 < i < p$, are co-finite subsets of $\mathbb{N}$, $S'_t(p-1) := \mathbb{N}$, $t(i-1) \times S'_t(i-1) \subseteq Q'_t \subseteq T(t)$ and $t(p) \times S'_t(p-1) \subseteq Q_p \subseteq T$. By construction, if $t = (t_1, t_2, \ldots, t_p) \in T$ then $i(i) \in \mathcal{Q}'_p$ if $1 \leq i < p$, and $t \in \mathcal{Q}_p$, whence $\mathcal{B}_i$ has property $\mathcal{P}$, for each $i = 1, 2, \ldots, p-1$. $B_i$ does not have property $\mathcal{P}$, $(i(i) : 1 \leq i < p) \cap T = \emptyset$ and the extensions of $t(p-1)$ in $T$ are the elements of $t(p-1) \times \mathbb{N}$, whose lengths are $p$.

\[ \square \]

**Definition 3** (6, Definition 1). Let $B$ be an element of the algebra $\mathcal{A}$ of subsets of $\Omega$. A subset $M$ of $\text{ba}(\mathcal{A})$ is deep $B$-unbounded if each finite subset $\mathcal{D}$ of $\{e_A : A \in \mathcal{A}\}$ verifies that

$$\sup \{|\mu(C)| : \mu \in M \cap \mathcal{D}, C \in \mathcal{A}, C \subseteq B\} = \infty.$$ 

The proof of the next proposition is straightforward.

**Proposition 4** (6, Proposition 5). If a subset $M$ of $\text{ba}(\mathcal{A})$ is deep $B$-unbounded and $\{B_i \in \mathcal{A} : 1 \leq i \leq q\}$ is a partition of $B$ then there exists $j, 1 \leq j \leq q$, such that $M$ is deep $B_j$-unbounded.

**Proposition 5** (6, Proposition 4). Let $\mathcal{A}$ be an algebra of subsets of $\Omega$ and let $(\mathcal{P}_m)_m$ be an increasing sequence of subsets of $\mathcal{A}$ such that each $\mathcal{P}_m$ does not have $N$-property and span$\{e_C : C \in \cup_m \mathcal{P}_m\} = L(\mathcal{A})$. There exists $n_0 \in \mathbb{N}$ such that for each $m \geq n_0$ there exists a deep $\Omega$-unbounded $\tau(\mathcal{A})$-closed absolutely convex subsets $M_m$ of $\text{ba}(\mathcal{A})$ which is pointwise bounded in $\mathcal{P}_m$, i.e., $\sup \{|\mu(C)| : \mu \in M_m\} < \infty$ for each $C \in \mathcal{P}_m$. In particular this proposition holds if $\cup_m \mathcal{P}_m = \mathcal{A}$ or if $\cup_m \mathcal{P}_m$ has $N$-property.

**Proposition 6.** Let $\mathcal{B} := \{\mathcal{P}_m, m_1, m_2, \ldots, m_p \in \mathbb{N}\}$ be an increasing web in a set-algebra $\mathcal{A}$. If $\mathcal{B}$ does not contain strands consisting of sets with property $N$ then there exists an $N$-tree $T$ such that for each $t \in T$ there exists a deep $\Omega$-unbounded $\tau(\mathcal{A})$-closed absolutely convex subset $M_t$ of $\text{ba}(\mathcal{A})$ which is $B_t$-pointwise bounded.

**Proof.** By Proposition 3 with $\mathcal{B} = N$ there exists an $N$-tree $T_1$ such that for each $t = (t_1, t_2, \ldots, t_p) \in T_1$ the set $\mathcal{B}_t$ does not have property $N$ and if $p > 1$ then $\mathcal{B}_t(i)$ has property $N$, for each $i = 1, 2, \ldots, p-1$. If $p = 1$ the conclusion follows from Proposition 5 in the case $\cup_{m_1} \mathcal{P}_{m_1} = \mathcal{A}$, being $T := T_1 \setminus \{1, 2, \ldots, n_0 = 1\}$, where $n_0$ is the natural number in Proposition 5. If $p > 1$ then $\mathcal{B}_{(p-1)} = \cup_m \mathcal{B}_{(p-1) \times m}$ has property
N and the conclusion follows again from Proposition 5 in the case that \( \cup_m \mathcal{B}_m \) has \( N \)-property, being \( T \) the NV-tree obtained after deleting in \( T_1 \) the elements \( t(p-1) \times \{1, 2, \ldots, n_0(t)-1\} \), for each \( t = (t_1, t_2, \ldots, t_p) \in T_1 \) where \( n_0(t) \) is the natural number of Proposition 5 for the increasing sequence \((\mathcal{B}_{l(p-1)}m)\).

Next Proposition 7 is given in [6, Proposition 8] as a currently version of Propositions 2 and 3 in [13]. Also Proposition 8 is contained in [6, Propositions 9 and 10]. In both propositions we present a sketch of the proofs for the sake of completeness and as a new help to the reader.

**Proposition 7** ([6, Proposition 8]). Let \( \{B, Q_1, \ldots, Q_r\} \) be a subset of the algebra \( \mathcal{A} \) of subsets of \( \Omega \) and let \( M \) be a deep \( B \)-unbounded absolutely convex subset of \( \text{ba}(\mathcal{A}) \).

Then given a positive real number \( \alpha \) and a natural number \( q > 1 \) there exists a finite partition \( \{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_q\} \) of \( B \) by elements of \( \mathcal{A} \) and a subset \( \{\mu_1, \mu_2, \ldots, \mu_q\} \) of \( M \) such that \( |\mu_i(C_1)| > \alpha \) and \( \sum_{1 \leq j \leq r} |\mu_i(Q_j)| < 1 \), for \( i = 1, 2, \ldots, q \).

**Proof.** It is enough to prove the case \( q = 2 \), because then there exists \( C_i, i \in \{1, 2\} \), such that \( M \) is deep \( C_i \)-unbounded by Proposition 4. Let \( \mathcal{D} = \{X_B, X_{Q_1}, X_{Q_2}, \ldots, X_{Q_r}\} \). As \( rM \) is deep \( B \)-unbounded, i.e., \( \sup \{\mu(D) : \mu \in rM \cap \mathcal{P}^r, D \subset B \} = 1 \) in \( \mathcal{A} \) exists \( C_1 \subset B \), with \( C_1 \in \mathcal{A} \), and \( \mu \in rM \cap \mathcal{P}^r \) such that \( |\mu(C_1)| > r(1 + \alpha) \). Then \( \mu_1 = r^{-1} \mu \in M \), \( |\mu_1(B)| \leq r^{-1} \leq 1 \) and \( \sum_{1 \leq i \leq r} |\mu_1(Q_j)| \leq r^{-1} r = 1 \). Clearly \( C_2 := B \setminus C_1 \) and \( \mu_2 := \mu_1 \) verify that \( |\mu_1(C_2)| \leq |\mu_1(C_1)| + |\mu_1(B)| > 1 + \alpha - 1 = \alpha \).

**Proposition 8** ([6, Propositions 9 and 10]). Let \( \{B, Q_1, \ldots, Q_r\} \) be a subset of an algebra \( \mathcal{A} \) of subsets of \( \Omega \) and let \( \{M_t : t \in T\} \) be a family of deep \( B \)-unbounded absolutely convex subsets of \( \text{ba}(\mathcal{A}) \), indexed by an NV-tree \( T \). Then for each positive real number \( \alpha \) and each finite subset \( \{t^j : 1 \leq j \leq k\} \) of \( T \) there exist a set \( B_1 \in \mathcal{A} \), a measure \( \mu_1 \in M_{t^1} \) and an increasing tree \( T_1 \), such that

1. \( B_1 \subset B \), \( \{t^j : 1 \leq j \leq k\} \subset T_1 \subset T \) and \( M_t \) is deep \((B^B_{B_1})\)-unbounded for each \( t \in T_1 \),

2. \( |\mu_1(B_1)| > \alpha \) and \( \sum \{ |\mu_1(Q_j)| : 1 \leq i \leq r \} \leq 1 \).

**Proof.** Let \( t^j := (t_1^j, t_2^j, \ldots, t_p^j) \), for \( 1 \leq j \leq k \). By Proposition 7 applied to \( B, \alpha, q := 2 + \sum_{1 \leq j \leq k} p_j \) and \( M_{t^j} \) there exist a partition \( \{C_1, C_2, \ldots, C_q\} \) of \( B \) by elements of \( \mathcal{A} \) and \( \{\lambda_1, \lambda_2, \ldots, \lambda_q\} \subset \{M_{t} : t \in T_1\} \) such that:

\[
|\lambda_k(C^k_1)| > \alpha \quad \text{and} \quad \sum_{1 \leq j \leq r} |\lambda_k(Q_j)| \leq 1 \quad \forall k = 1, 2, \ldots, q. \tag{1}
\]

From Proposition 4 it follows that if \( M \) is deep \( B \)-unbounded there exists an \( i_M \in \{1, 2, \ldots, q\} \) such that \( M \) is deep \( C_{i_M} \)-unbounded, hence if \( M_k \) is deep \( B \)-unbounded for each \( u \in U \) and \( V_i := \{u \in U : M_u \text{ is deep } C_j \text{-unbounded}\}, 1 \leq i \leq q \), then \( U' = \bigcup_{1 \leq i \leq q} V_i \).

Whence if \( U \) is an NV-tree there exists \( i_0 \), with \( 1 \leq i_0 \leq q \), such that \( V_{i_0} \) contains an NV-tree \( U_{i_0} \) by Proposition 2.

Therefore there exists \( C_{i_0} \) and \( C_{i_0}' \), with \( \{i', i_0\} \subset \{1, 2, \ldots, q\} \), and an NV-tree \( T_{i_0} \subset T \) such that \( M_{i'_0} \) is deep \( C_{i_0}' \)-unbounded, for each \( j \in \{1, 2, \ldots, k\} \), and \( M_t \) is deep \( C_{i'_0} \)-unbounded for each \( t \in T_{i_0} \).
For each \( t^j = (t^j_1, t^j_2, \ldots, t^j_k) \notin T_0 \), \( 1 \leq j \leq k \), and each section \( t^j(m-1) \) of \( t^j \), with \( 2 \leq m \leq p_j \), the set \( W^j_{m} := \{ v \in \bigcup_i \mathbb{N}^i : t^j(m-1) \times v \in T \} \) is an NV-tree such that \( M(t^j_1, t^j_2, \ldots, t^j_{m-1}) \times w \) is deep B-unbounded for each \( w \in W^j_{m} \), whence there exists \( t^j_m \in \{1, 2, \ldots, q\} \) and an NV-tree \( V^j_{m} \) contained in \( W^j_{m} \) such that \( M(t^j_1, t^j_2, \ldots, t^j_{m-1}) \times w \) is deep \( C^j_{m} \)-unbounded for each \( w \in V^j_{m} \).

Let \( D \) be the union \( D := C_{h} \cup \left( \bigcup \{ C_i \cup C_{i_{m}} : j \in S, 2 \leq m \leq p_j \} \right) \) and let \( T_1 \) be the union of \( T_0 \) and the sets \( \{ t^j : 1 \leq j \leq k \} \) of \( T \) that exist \( k \) pairwise disjoint sets \( B_j \in \mathcal{A} \), \( k \) measures \( \mu_j \in M_j \), \( 1 \leq j \leq k \), and an increasing tree \( T^* \) such that:

1. \( \bigcup \{ B_j : 1 \leq j \leq k \} \subset B \), \( \{ t^j : 1 \leq j \leq k \} \subset T^* \subset T \) and \( M_j \) is deep \( \big( B \setminus \bigcup_{1 \leq i \leq j \leq k} B_j \big) \)-unbounded for each \( t \in T^* \).

2. \( |\mu_j(B_j)| > \alpha \) and \( \Sigma \left( |\mu_j(Q)| : 1 \leq i \leq r \right) \leq 1 \), for \( j = 1, 2, \ldots, k \).

**Proof.** Apply \( k \) times Proposition 8. \( \square \)

In Theorem 1 we need the sequence \( (i_n)_{n} := (1, 1, 2, 1, 2, 3, \ldots) \) obtained with the first components of the sequence \( \{(1,1), (1,2), (2,1), (1,3), (2,2), (3,1), \ldots\} \) generated writing the elements of \( \mathbb{N}^2 \) following the diagonal order.

**Theorem 1.** A \( \sigma \)-algebra \( \mathcal{A} \) of subsets of a set \( \Omega \) has property \( \omega \mathcal{N} \).

**Proof.** Let us suppose that \( \mathcal{A} \) is a \( \sigma \)-algebra of subsets of a set \( \Omega \) which does not have property \( \omega \mathcal{N} \). Then there would exist in \( \mathcal{A} \) an increasing web \( \{ \mathcal{R}_{m_1, m_2, \ldots, m_p} : p, m_1, m_2, \ldots, m_p \in \mathbb{N} \} \) without strands consisting of sets with Property \( N \). By Proposition 6 there exists an NV-tree \( T \) such that for each \( t \in T \) there exists a deep \( \Omega \)-unbounded \( \tau_{x}(\mathcal{A}) \)-closed absolutely convex subset \( M_t \) of \( \mathfrak{b}(\mathcal{A}) \) which is \( B_t \)-pointwise bounded.

By induction it is easy to determine an NV-tree \( \{ t^i : i \in \mathbb{N} \} \) contained in \( T \) and a strictly increasing sequence of natural numbers \( \langle k_i \rangle \), such that for each \( i, j \in \mathbb{N} \) with \( i \leq k_j \) there exists a set \( B_{ij} \in \mathfrak{b}(\mathcal{A}) \) and \( \mu_{ij} \in M_{ij} \) that verify

\[
\Sigma_{x,v} \left( |\mu_{ij}(B_{xv})| : s \leq k_v, 1 \leq v < j \right) < 1, \quad (2)
\]

\[
|\mu_{ij}(B_{ij})| > j, \quad (3)
\]

and \( B_{ij} \cap B_{i'j'} = \emptyset \) if \( (i, j) \neq (i', j') \).
In fact, select $t^1 \in T$. Corollary 1 with $B := \Omega$ and $\alpha = 1$ provides $B_{11} \in \mathcal{F}$, $\mu_{11} \in M_r$ and an NV-tree $T_1$ such that $|\mu_{11}(B_{11})| > 1$, $t^1 \in T_1 \subset T$ and $M_r$ is deep $\Omega \setminus B_{11}$-unbounded for each $t \in T_1$. Define $k_1 := 1$, $S^1 := \{t^1\}$ and $B^1 := B_{11}$.

Let us suppose that we have obtained the natural numbers $k_1 < k_2 < k_3 < \cdots < k_n$, the NV-trees $T_1 \supset T_2 \supset T_3 \supset \cdots \supset T_n$, the elements $\{t^1, t^2, \ldots, t^n\}$ such that $S^t := \{t^i : i \leq k_j\} \subset T_j$ and $S^j := \{t^{k_j+1}, \ldots, t^n\}$ has the increasing property respect to $S^{j-1}$, for each $1 < j < n$, together with the measures $\mu_j \in M_r$ and the pairwise disjoint elements $B_{ij} \in \mathcal{F}$, $i \leq k_j$ and $j \leq n$, such that $|\mu_j(B_{ij})| > j$ and $\Sigma_{s,v}|\mu_j(B_{sv})| : s \leq k_v$, $1 \leq v < j$, $j = 1, \ldots, n$ in such a way that the union $B^j := \cup\{B_{sv} : s \leq k_v, 1 \leq v < j\}$ verifies that $M_r$ is deep $\Omega \setminus B^j$-unbounded for each $t$ belonging to the NV-tree $T_j$, for each $j < n$.

To finish the induction procedure select a subset $S_{n+1} := \{t^{k_{n+1}}, \ldots, t^{k_n}\}$ of $T_n \setminus \{t^i : i \leq k_n\}$ which has the increasing property respect to $S^n$ and apply again Corollary 1 to $\Omega \setminus B^n$, $\{B_{sv} : s \leq k_v, 1 \leq v \leq n\}$, $T_n$, the finite subset $S^{n+1} := \{t^i : i \leq k_{n+1}\}$ of $T_n$ and $n+1$. Then, for each $i \leq k_{n+1}$, we obtain $B_{im+1} \in \mathcal{F}$, $B_{im+1} n \subset \Omega \setminus B^n$, and $\mu_{im+1} \in M_r$ such that $|\mu_{im+1}(B_{im+1})| > n+1$, $\Sigma_{s,v}|\mu_{im+1}(B_{sv})| : s \leq k_v$, $1 \leq v \leq n \leq 1$, $B_{im+1} \cap B_{im+1} = \emptyset$, $i \neq i'$, and the union $B^{n+1} := \cup\{B_{sv} : s \leq k_v, 1 \leq v \leq n+1\}$ has the property that $T_n$ contains an increasing tree $T_{n+1}$ such that $S^{n+1} \subset T_{n+1}$ and $M_r$ is deep $\Omega \setminus B^{n+1}$-unbounded for each $t \in T_{n+1}$.

With a new easy induction we obtain a subset $S := \{j_1, j_2, \ldots, j_m\}$ of $\mathbb{N}$ such that $j_n < j_{n+1}$, for $n \in \mathbb{N}$, and for each $(i, j) \in \mathbb{N} \times J$ with $i \leq k_j$ we have

$$
\Sigma_{s,v}|\mu_j(B_{sv})| : s \leq k_v, 1 \leq v \leq J < 1
$$

because if the variation $|\mu_j|\Omega < s \in \mathbb{N}$, $\{N_u, 1 \leq u \leq s\}$ is a partition of $\mathbb{N} \setminus \{1, 2, \ldots, j\}$ in $s$ infinite subsets and $B_u := \cup\{B_{sv} : s \leq k_v, u \in N_v\}$, $1 \leq u \leq s_1$, then the inequality $\Sigma\{|\mu_j|B_{sv} > j_u, 1 \leq u \leq s_1\}$ implies that there exists $u'$, with $1 \leq u' \leq s_1$, such that $|\mu_j|B_{u'} < 1$, whence

$$
\Sigma_{s,v}|\mu_j(B_{sv})| : s \leq k_v, 1 \leq v \leq J < 1,
$$

and then the sequence $(B_{i_n}, \mu_{i_n})_{n}$ verifies for each $n \in \mathbb{N}$ that:

$$
\Sigma\{|\mu_{i_n}B_{i_n} : s \leq n\} < 1,
$$

and

$$
|\mu_{i_n}B_{i_n} > j_n | < 1.
$$

As $S^{n+1}$ has the increasing property respect to $S^n$ we have that $\{t^i : i \in \mathbb{N}\}$ is an NV-tree contained in $T$, hence $\cup\{B_{i_n} : i \in \mathbb{N}\} \subset \mathcal{F}$. The relation $H := \cup\{B_{i_n} : s = 1, 2, \ldots \} \subset \mathcal{F}$ implies that there exists $r \in \mathbb{N}$ such that $H \in \mathcal{F}$. Then for each strictly increasing sequence $(n_p)$ such that $n_p = r$ we have that $\{\mu_{i_n}B_{i_p} : p \in \mathbb{N}\}$ is a subset of $M_r$. As $M_r$ is $B_r$-pointwise bounded we get that

$$
\sup\{|\mu_{i_n}B_{i_p}(H)) : p \in \mathbb{N}\} < \infty.
$$
The sets \( C_p := \bigcup \{ B_{k,i} : s < n_p \}, B_{k,p} \) and \( D_p := \bigcup \{ B_{k,i} : n_p < s \} \) are a partition of the set \( H \). By (4), (5) and (6), \( |\mu_{l,p,j}(C)| < 1, \mu_{l,p,j}(B_{l,p,j}) > j_p > n_p \) and \( \mu_{l,p,j}(D) < 1 \), for each \( p \in \mathbb{N}\setminus\{1\} \). Therefore the inequality
\[
|\mu_{l,p,j}(H)| > -|\mu_{l,p,j}(C)| + \mu_{l,p,j}(B_{l,p,j}) - |\mu_{l,p,j}(D)| > j_p - 2
\]
implies that
\[
\lim_p |\mu_{l,p,j}(H_0)| = \infty,
\]
contradicting (7).

The following corollary extends Corollary 13 in [6]. A family \( \{ B_{m,j} : i, m_j \in \mathbb{N}, 1 \leq i \leq \frac{m_j}{p} \} \) of subsets of \( A \) is an increasing \( p \)-web in \( A \) if \( \binom{B_{m,1}}{m_1} \) is an increasing covering of \( A \) and \( \binom{B_{m,j+1}}{m_{j+1}} \) is an increasing covering of \( B_{m,j+1} \), for each \( m_j \in \mathbb{N}, 1 \leq j \leq i < p \) (this definition comes from [7, Chapter 7, 35.1]).

**Corollary 2.** Let \( \mathcal{S} \) be a \( \sigma \)-algebra of subsets of \( \Omega \) and let \( \{ B_{m,j} : i, m_j \in \mathbb{N}, 1 \leq j \leq p \} \) be an increasing \( p \)-web in \( \mathcal{S} \). Then there exists \( B_{n,m,j} \) such that if
\[
\{ B_{n,m,i} \}_{i=1}^{m} : k, m_i + l \in \mathbb{N}, 1 \leq l \leq q)
\]
is an increasing \( q \)-web of \( B_{n,m,j} \), there exists \( (n_p+1, n_{p+2}, \ldots, n_{p+q}) \in \mathbb{N}^q \) such that each \( \tau_i(\mathcal{S}_{n,m,i}) \) is Cauchy sequence \( (\mu_n \in \text{ba}(\mathcal{S})) \) is \( \tau_i(\mathcal{S}) \)-convergent.

**Proof.** By Proposition 1 with \( \mathcal{Q} = N \) and Theorem 1 there exists \( B_{n,m,j} \) which has property \( \omega N \). Hence there exists \( B_{n,m,i} \) which has property \( N \). Then if \( (\mu_n)_n \subset \text{ba}(\mathcal{S}) \) is a \( \tau_i(\mathcal{S}_{n,m,i}) \)-Cauchy sequence we have that \( (\mu_n)_n \) has no more than one \( \tau_i(\mathcal{S}) \)-adherent point, whence \( (\mu_n)_n \) is \( \tau_i(\mathcal{S}) \)-convergent. As \( L(\mathcal{S}_{n,m,i}) \) the sequence \( (\mu_n)_n \) has no more than one \( \tau_i(\mathcal{S}) \)-adherent point, whence \( (\mu_n)_n \) is \( \tau_i(\mathcal{S}) \)-convergent.

**3 Applications**

In this section we obtain some applications of Theorem 1 to bounded finitely additive vector measures.

A **bounded finitely additive vector measure**, or simple **bounded vector measure**, \( \mu \) defined in an algebra \( \mathcal{A} \) of subsets of \( \Omega \) with values in a topological vector space \( E \) is a map \( \mu : \mathcal{A} \rightarrow E \) such that \( \mu(\mathcal{A}) \) is a bounded subset of \( E \) and \( \mu(B \cup C) = \mu(B) + \mu(C) \), for each pair disjoint subsets \( B, C \in \mathcal{A} \). Then the \( E \)-valued linear map \( \mu : L(\mathcal{A}) \rightarrow E \) defined by \( \mu(\mathcal{E}) := \mu(B) \), for each \( B \in \mathcal{A} \), is continuous.

A locally convex space \( E(\mathcal{S}) \) is the \( p \)-inductive limit of the family of locally convex spaces \( \mathcal{S} := \{ E_{m,j} : i, m_j \in \mathbb{N}, 1 \leq j \leq i \} \) if \( E(\mathcal{S}) \) is the inductive limit of \( E_{m_j}(\tau_{m_j}) \) and moreover, each \( E_{m,j} \) is the inductive limit of the sequence \( E_{m,j+1}(\tau_{m+1})m_{j+1} \), for each \( m_j \in \mathbb{N}, 1 \leq j < i \). Then \( \mathcal{S} \) is a **defining \( p \)-increasing web** for \( E(\mathcal{S}) \) with steps \( E_{m,j}(\tau_{m,j}) \). \( E(\mathcal{S}) \) is a
Corollary 3. Let \( \mu \) be a bounded vector measure defined in a \( \sigma \)-algebra \( \mathcal{F} \) of subsets of \( \Omega \) with values in a topological vector space \( E(\tau) \). Suppose that \( \{ E_{m1,m2} \} : m_j \in \mathbb{N}, 1 \leq j \leq i \leq p \} \) is an increasing \( p \)-web in \( E \). Then there exists \( E_{m1,m2} \) such that \( E_{m1,m2} \) is a \( \mu \)-relative topology \( \tau_{m1,m2} \) is finer than the relative topology \( \tau_{m1,m2} \) and \( \{ E_{m1,m2} \} : i, m_{p+j} \in \mathbb{N}, 1 \leq j \leq q \} \) defining an \( \mu \)-increasing web for \( E_{m1,m2} \) there exists \( (n_{p+1}, n_{p+2}, \ldots, n_{p+q}) \in \mathbb{N}^q \) such that \( \mu(\mathcal{F}) \) is a bounded subset of \( E_{m1,m2} \).

Next proposition extends [12, Theorem 4] and [6, Proposition 10].

**Proposition 9.** Let \( \mu \) be a bounded vector measure defined in a \( \sigma \)-algebra \( \mathcal{F} \) of subsets of \( \Omega \) with values in a topological vector space \( E(\tau) \). Suppose that \( \{ E_{m1,m2} \} : m_j \in \mathbb{N}, 1 \leq j \leq i \leq p \} \) is an increasing \( p \)-web in \( E \). Then there exists \( E_{m1,m2} \) such that \( E_{m1,m2} \) is an \( \mu \)-(LF)-space, the topology \( \tau_{m1,m2} \) is finer than the relative topology \( \tau_{m1,m2} \) and \( \{ E_{m1,m2} \} : i, m_{p+j} \in \mathbb{N}, 1 \leq j \leq q \} \) defining an \( \mu \)-(LF)-increasing web for \( E_{m1,m2} \) there exists \( (n_{p+1}, n_{p+2}, \ldots, n_{p+q}) \in \mathbb{N}^q \) such that \( \mu(\mathcal{F}) \) is a bounded subset of \( E_{m1,m2} \).

**Proof.** Let \( B_{m1,m2} := \mu^{-1}(E_{m1,m2}) \) for each \( m_j \in \mathbb{N}, 1 \leq j \leq i \leq p \). By Proposition 1 and Theorem 1 there exists \( (n_1, n_2, \ldots, n_p) \in \mathbb{N}^p \) such that \( B_{m1,m2} \) has \( \mu \)-property. Let \( \{ E_{m1,m2} \} : i, m_j \in \mathbb{N}, 1 \leq j \leq i \leq q \} \) be an increasing \( \mu \)-(LF)-increasing web for \( E_{m1,m2} \) and let \( B_{m1,m2} := \mu^{-1}(E_{m1,m2}) \), for each \( i, m_j \in \mathbb{N}, 1 \leq j \leq i \leq q \). As \( \{ B_{m1,m2} \} : i, m_j \in \mathbb{N}, 1 \leq j \leq i \leq q \} \) is an increasing \( q \)-web of \( B_{m1,m2} \) and this set has \( \mu \)-property then there exists a subset \( B_{m1,m2} \) which has property \( \mathcal{N} \), whence \( L(B_{m1,m2}) \) is a dense subspace of \( L(\mathcal{F}) \) and then the map with closed graph

\[
\mu|_{L(B_{m1,m2})} : L(B_{m1,m2}) \to E_{m1,m2}
\]

has a continuous extension \( \vartheta \) to \( L(\mathcal{F}) \) with values in \( E_{m1,m2} \). Since \( \mu \) is \( \mu \)-continuous, \( \vartheta(\mathcal{F}) = \mu(\mathcal{F}) \). Whence \( \mu(\mathcal{F}) \) is a bounded subset of \( E_{m1,m2} \).

**Corollary 3.** Let \( \mu \) be a bounded vector measure defined in a \( \sigma \)-algebra \( \mathcal{F} \) of subsets of \( \Omega \) with values in an inductive limit \( E(\tau) = \sum_{m} E_{m}(\tau_{m}) \) of increasing sequence \( E_{m}(\tau_{m}) \), of \( \mu \)-(LF) spaces. There exists \( n_1 \in \mathbb{N} \) such that for each defining \( \mu \)-(LF) increasing web for \( E_{n_1} \) \( \{ E_{n_1,m1,m2} \} : i, m_{1+j} \in \mathbb{N}, 1 \leq j \leq q \} \) there exists \( (n_{m1,m2}) \in \mathbb{N}^q \) such that \( \mu(\mathcal{F}) \) is a bounded subset of \( E_{n_1,m1,m2} \).

A sequence \( \{ x_k \} \) in a locally convex space \( E \) is a subsequence convergent in \( \mathcal{F} \) if for every subset \( J \) of \( \mathbb{N} \) such that \( \{ x_k \} \) converges. The following corollary is a generalization of the localization property given in [12, Corollary 1.4] and it follows from Corollary 3.

**Corollary 4.** Let \( \{ x_k \} \) be a subsequence convergent sequence in an inductive limit \( E(\tau) = \sum_{m} E_{m}(\tau_{m}) \) of an increasing sequence \( E_{m}(\tau_{m}) \), of \( \mu \)-(LF) spaces. There exists \( n_1 \in \mathbb{N} \) such that for each defining \( \mu \)-(LF) increasing web \( \{ E_{n_1,m1,m2} \} : i, m_{1+j} \in \mathbb{N}, 1 \leq j \leq q \} \) for \( E_{n_1} \) there exists \( (n_{m1,m2}) \in \mathbb{N}^q \) such that \( \{ x_k \} \) is a bounded subset of \( E_{n_1,m1,m2} \).
Proof. As \((x_k)_k\) is subseries convergent, then the additive vector measure \(\mu : 2^\mathbb{N} \rightarrow E(\tau)\) defined by \(\mu (J) := \sum_{k \in J} x_k\), for each \(J \in 2^\mathbb{N}\), is bounded, because \((f(x_k))_k\) is subseries convergent for each \(f \in E'\), whence \(\sum_{n=1}^{\infty} |f(x_n)| < \infty\). Therefore we may apply Corollary 3. \(\square\)

**Proposition 10.** Let \(\mu\) be a bounded vector measure defined in a \(\sigma\)-algebra \(\mathcal{S}\) of subsets of \(\Omega\) with values in a topological vector space \(E(\tau)\). Suppose that \(\{ E_{m_1 m_2 \ldots m_n} : m_j \in \mathbb{N}, 1 \leq j \leq i \leq p \}\) is an increasing \(p\)-web in \(E\). There exists \(E_{n_1 n_2 \ldots n_p}\) such that if \(\{ E_{n_1 n_2 \ldots n_{mp+1} \ldots n_{qj+1}} : i, m_p+j \in \mathbb{N}, 1 \leq j \leq i \leq q \}\) is a \(q\)-increasing web in \(E_{m_1 m_2 \ldots m_p}\) with the property that each relative topology \(\tau |_{E_{n_1 n_2 \ldots n_{mp+1} \ldots n_{qj+1}}} \) is subseries convergent, then there exists \(E_{n_1 n_2 \ldots n_{mp+1} \ldots n_{qj+1}}\) is complete, we may apply Proposition 10.

**Corollary 5.** Let \(\mu\) be a bounded additive vector measure defined in a \(\sigma\)-algebra \(\mathcal{S}\) of subsets of \(\Omega\) with values in an inductive limit \(E(\tau) = \sum_m E_m(\tau_m)\) of an increasing sequence \((E_m(\tau_m))_m\) of countable dimensional topological vector spaces. Then there exists \(n_1\) such that for each \(q\)-increasing web \(\{ E_{n_1 n_1 \ldots n_{mp+1} \ldots m_{qj+1}} : i, m_p+j \in \mathbb{N}, 1 \leq j \leq i \leq q \}\) in \(E_{n_1}\) such that the dimension of each \(E_{n_1 n_1 \ldots n_{mp+1} \ldots m_{qj+1}}\) is finite there exists \(E_{n_1 n_1 \ldots n_{mp+1} \ldots m_{qj+1}}\) which contains the set.

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References


