Schur complement of general H-matrices*

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Abstract

It is well-known that the Schur complement of some H-matrices is an H-matrix. In this paper, the Schur complement of any general H-matrix is studied. In particular it is proved that the Schur complement, if it exists, is an H-matrix and it is studied to which class of H-matrix the Schur complement belongs to. In addition, some results are given for singular irreducible H-matrices and for the Schur complement of nonsingular irreducible H-matrices.

1 Introduction

M-matrices and the more general class of H-matrices have been applied in different problems of mathematics and other sciences. One of the most important applications of these kind of matrices is in Numerical Linear Algebra; more concretely in the solution of linear systems by the LU factorization and by the Schur complement as well as in the construction of preconditioners.

Furthermore, there are studies aimed to determine if the L and U factors or the Schur complement of a matrix preserve some properties of the initial matrix A, such as diagonally dominance, M-matrix or H-matrix properties including irreducibility. Despite the existence of many studies on these topics it would be interesting to conclude the subject studying the heritability of the H-matrix properties of the Schur complements, which is the goal of this paper.

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The concept of nonsingular M-matrix and H-matrix was introduced by Ostrowski [18] in the study of the convergence of iteration processes and spectral theory. Later, these definitions were extended by Fiedler and Ptak to possible singular M-matrices [8] and H-matrices [9]. Moreover, the study of nonsingular or general M-matrices and H-matrices was widely extended (see [23], [3] and the references therein). Different characterizations of M-matrices and H-matrices are referred to the properties of diagonal dominance (strict, general, double, weak, ...) or nonsingularity of a principal submatrix and to the irreducibility of the original matrix. On the other hand, the partition of H-matrices according to the singularity or not of the matrix is established in reference [4], where the general set of H-matrices is split in three different classes with different properties related with the above subjects for each class of H-matrices.

With regards to the Schur complement of H-matrices, the initial reference is [7] where Ky Fan stated that the Schur complement (with respect to an index) of an invertible M-matrix is an invertible M-matrix. This result was extended to general Schur complements by Crabtree [6] (see also [26]) and for nonsingular H-matrices by Polman [19]. Also, from the results of Johnson [14] and Smith [22] the irreducibility of the Schur complement of general M-matrices with respect to an invertible principal submatrix was characterized (see also [25]). Further, in the book [21] there is a detailed study of different classes of matrices which are closed for the Schur complement, in particular the invertible class of H-matrices.

Moreover, with respect to the LU factorization it seems that the first result is in a characterization by Fiedler and Ptak [8] of a symmetric permutation of a nonsingular M-matrix (see also Varga [23]). Other related interesting references are Kuo [15] obtaining the LU factorization of irreducible possible singular M-matrices, Varga and Cai [25] extending the earlier result to particular reducible M-matrices, or Funderlic, Neumann and Plemmons [10, 11] obtaining the LU factorization of generalized diagonally dominant H-matrices. Neumann [17] completes those results among Schur complements and LU factorizations. In this context Ahac, Buoni and Olesky [2, 1, 5] give a stable algorithm for computing the LU factorization of H-matrices.

In this paper we are dealing with general M-matrices (matrices written as $A = rI - B$, $r \geq \rho(B)$ and B nonnegative) and with general H-matrices (matrices whose comparison matrix $\mathcal{M}(A)$ is a general M-matrix. See [3]). In [4] a partition of the set of all H-matrices in three classes is given: The *invertible* class $\mathcal{H}_I$ containing all (nonsingular) H-matrices such that their comparison matrix is a nonsingular M-matrix; the *mixed* class $\mathcal{H}_M$ formed for all H-matrices having singular comparison matrix but with at least one equimodular matrix (that is, a matrix with the same comparison matrix) being nonsingular; and the *singular* class $\mathcal{H}_S$ with all (reducible) H-matrices such that all their equimodular matrices are singular.

The goal of this paper is to establish that the Schur complement of a general H-matrix, if it exists, is also an H-matrix. Moreover we establish that $S_\alpha(A)$ belongs to the same H-matrices class than the original matrix A if A belongs to $\mathcal{H}_I$ or $\mathcal{H}_S$ or if A is a singular matrix in $\mathcal{H}_M$. For nonsingular matrices in $\mathcal{H}_M$ we give some conditions on the graph of A to determine when $S_\alpha(A)$ remains in $\mathcal{H}_M$ or improves to $\mathcal{H}_I$.

The background on Schur complements of general M-matrices and H-matrices in $\mathcal{H}_I$ is contained in section 2, where we include some results that will be used later. The Schur complement of H-matrices in $\mathcal{H}_M$ and $\mathcal{H}_S$ are studied in sections 3 and 4 respectively. The paper ends with a collection of the main results.

2 Background and Schur complements on $\mathcal{H}_I$

Let $A \in \mathbb{C}^{n\times n}$ and let $\alpha, \beta \subseteq \langle n\rangle = \{1, 2, \dots, n\}$. As usual, $A(\alpha, \beta)$ denotes the submatrix of A with row and column indices all those in α and β, respectively, and $A(\alpha)$ denotes the principal submatrix of A with row and column indices in α. Through the paper, we denote the cardinality of the set α by $|\alpha|$ and the complementary subset of α in $\langle n\rangle$ by α'. Finally, we write the strict set inclusion by $X \subset Y$.

Given a nonsingular proper principal submatrix $A(\alpha)$, we will denote by $S_\alpha(A)$ the Schur complement of A with respect to $A(\alpha)$. That is, if $|\alpha| = k$, $S_\alpha(A)$ is the $(n-k)\times(n-k)$ matrix

$$S_\alpha(A) = A(\alpha') - A(\alpha', \alpha)A(\alpha)^{-1}A(\alpha, \alpha'). \tag{1}$$

Recall that properties of H-matrices are strongly related with generalized diagonally dominance. In order to preserve these properties, we should restrict our study to Schur complements with respect to principal submatrices. In this sense we have to consider symmetric permutations of A. Let P be the permutation matrix such that

$$PAP^T = \begin{bmatrix} A(\alpha) & A(\alpha, \alpha') \\ A(\alpha', \alpha) & A(\alpha') \end{bmatrix},$$

then, the Schur complement is part of the block LU factorization

$$PAP^T = LU = \begin{bmatrix} I & 0 \\ A(\alpha', \alpha)A(\alpha)^{-1} & I \end{bmatrix} \begin{bmatrix} A(\alpha) & A(\alpha, \alpha') \\ 0 & S_\alpha(A) \end{bmatrix} \tag{2}$$

and

$$\det(A) = \det(P^T AP) = \det(A(\alpha)) \det(S_\alpha(A)).$$

We can easily state the generalization of the results of [7, Lemma 1] and [6, Lemma 1], given for nonsingular M-matrices, to general M-matrices as follows.

Theorem 1. *Let A be an M-matrix and let $\alpha \subset \langle n \rangle$ such that $A(\alpha)$ is nonsingular, then $S_\alpha(A)$ is an M-matrix.*

To study the Schur complement of general H-matrices we need an interesting inequality obtained in [1, Lemma 1] and [16, Theorem 1] with some different conditions. This result can be applied to singular H-matrices.

Lemma 1. *Let A be an H-matrix and let $\alpha \subset \langle n \rangle$ such that $A(\alpha) \in \mathcal{H}_I$, then*

$$S_\alpha(\mathcal{M}(A)) \leq \mathcal{M}(S_\alpha(A)). \tag{3}$$

and $S_\alpha(A)$ is an H-matrix.

Proof. Since $\mathcal{M}(A)(\alpha) = \mathcal{M}(A(\alpha))$ is a nonsingular M-matrix and $A(\alpha)$ is nonsingular both two Schur complements can be computed.

The proof of the inequality (3) follows the steps of [16, Theorem 1] considering that what is really used is that $[\mathcal{M}(A)(\alpha)]^{-1} \geq 0$. Further, $S_\alpha(\mathcal{M}(A))$ is an M-matrix from Theorem 1, and following the inequality (3), $\mathcal{M}(S_\alpha(A))$ is also an M-matrix and so $S_\alpha(A)$ is an H-matrix. □

With Lemma 1 we can give the result for H-matrices in $\mathcal{H}_I$.

Corollary 1 ([19, Lemma 3])**.** *Let $A \in \mathcal{H}_I$ and let $\alpha \subset \langle n \rangle$, then $S_\alpha(A) \in \mathcal{H}_I$.*

Despite Lemma 1 applies to general H-matrices, the generalization is not complete since there could be H-matrices with nonsingular $A(\alpha)$ but with singular $\mathcal{M}(A(\alpha))$. Then $S_\alpha(A)$ exists but we can not apply Lemma 1 to conclude that $S_\alpha(A)$ is an H-matrix. However if the matrix is an irreducible M-matrix we have the next result.

Theorem 2. *Let A be an irreducible M-matrix and let $\alpha \subset \langle n \rangle$, then the Schur complement $S_\alpha(A)$ exists and it is an M-matrix. Moreover, $S_\alpha(A)$ is irreducible unless A is singular and $|\alpha| = n - 1$.*

Proof. Any principal submatrix of an irreducible M-matrix is a nonsingular M-matrix. Then, $A(\alpha)$ is nonsingular and $S_\alpha(A)$ is an M-matrix. If A is singular and $|\alpha| = n - 1$, then $S_\alpha(A) = [0]$. [1] In other case, $S_\alpha(A)$ is irreducible ([22, Lemma 2.1 (ii)]). □

[1] As usual the 1×1 null matrix is considered reducible.

Theorem 2 can not be stated for H-matrices as it can be seen in the following example.

Example 1. *Consider the irreducible matrix* $A \in \mathcal{H}_I$

$$A = \begin{bmatrix} 3 & -1 & -1 \\ 1 & 3 & 1 \\ -1 & \frac{1}{3} & 3 \end{bmatrix}$$

Taking $\alpha = \{1\}$ *the Schur complement is the reducible matrix*

$$S_\alpha(A) = \begin{bmatrix} 10/3 & 4/3 \\ 0 & 8/3 \end{bmatrix}.$$

3 Schur complements in $\mathcal{H}_M$

We remain that any matrix $A \in \mathcal{H}_M$, singular or not, has nonzero diagonal elements, its comparison matrix $\mathcal{M}(A)$ is singular but at least one equimodular matrix is nonsingular, see [4] for details. The matrix A can be irreducible or not, then we distinguish two cases.

3.1 Irreducible case

We prove that in this case the Schur complement always exists and is an H-matrix (related results has been done in [10, 11, 1]). We recall that if A is a (nonsingular) M-matrix any principal submatrix $A(\alpha)$ is a (nonsingular) M-matrix, see [3]. The same happens with H-matrices, but the point is that some principal submatrix of an H-matrix (outside the invertible class) can belong to a different class than the original H-matrix. We shall deal with this point in the mixed irreducible case in next theorem.

Theorem 3. *Let* $A \in \mathcal{H}_M$ *be an irreducible matrix. Then the Schur complement with respect to any* $\alpha \subset \langle n \rangle$ *can be computed and* $S_\alpha(A)$ *is an* H*-matrix.*

Moreover,

1. *If* $S_\alpha(\mathcal{M}(A)) < \mathcal{M}(S_\alpha(A))$*, then* $S_\alpha(A) \in \mathcal{H}_I$ *(and* A *is nonsingular).*

2. *If* A *is singular and* $|\alpha| = n - 1$*, then* $S_\alpha(A) = [0] \in \mathcal{H}_S$ *and it is reducible.*

3. *Otherwise,* $S_\alpha(\mathcal{M}(A)) = \mathcal{M}(S_\alpha(A))$*, and* $S_\alpha(A) \in \mathcal{H}_M$ *is an irreducible matrix (singular or not).*

Proof. Since $\mathcal{M}(A)$ is an irreducible M-matrix, we can compute the Schur complement of $\mathcal{M}(A)$ with respect to any proper principal submatrix, which is a singular M-matrix by Theorem 2. Moreover, as all proper principal submatrices of $\mathcal{M}(A)$ are nonsingular M-matrices, all proper principal submatrices of A are in $\mathcal{H}_I$, and then by Lemma 1 the inequality (3) holds and $S_\alpha(A)$ is an H-matrix.

To end the proof we distinguish three cases.

Case 1. Consider $S_\alpha(\mathcal{M}(A)) < \mathcal{M}(S_\alpha(A))$. If $S_\alpha(A)$ is irreducible[2], then $\mathcal{M}(S_\alpha(A))$ is an irreducible Z-matrix greater than the M-matrix $S_\alpha(\mathcal{M}(A))$. Then, $\mathcal{M}(S_\alpha(A))$ is a nonsingular M-matrix and consequently $S_\alpha(A) \in \mathcal{H}_I$, so A is nonsingular.

If $S_\alpha(A)$ is reducible and $S_\alpha(\mathcal{M}(A)) < \mathcal{M}(S_\alpha(A))$, since both matrices are M-matrices we can write

$$S_\alpha(\mathcal{M}(A)) = rI - B < rI - C = \mathcal{M}(S_\alpha(A))$$

where $r \geq \max_{i\in\langle n\rangle} c_{ii}$, and $0 \leq C < B$. By Theorem 2 as we said before $S_\alpha(\mathcal{M}(A))$ is singular and irreducible, then by [24, Corollary 2.5] we have $r = \rho(B) > 0$. Since B is irreducible and C is reducible, we can assume, without lost of generality that

$$C = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1k} \\ 0 & C_{22} & \cdots & C_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & C_{kk} \end{bmatrix} < B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1k} \\ B_{21} & B_{22} & \cdots & B_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ B_{k1} & B_{k2} & \dots & B_{kk} \end{bmatrix}$$

where all diagonal blocks C_{ii} are irreducible or the 1×1 null matrix. The case $C_{ii} = [0]$ for all $i = 1, 2, \ldots, k$, with $k = n$, implies $\rho(C) = 0$ and so $r > \rho(C)$. Then $\mathcal{M}(S_\alpha(A))$ is a nonsingular M-matrix and so $S_\alpha(A) \in \mathcal{H}_I$.

In other case let C_{ii} be the irreducible diagonal block with larger spectral radius, so $\rho(C) = \rho(C_{ii})$. The corresponding block B_{ii} is a principal submatrix of the irreducible nonnegative matrix B, then applying [24, Lemma 2.6] $\rho(B_{ii}) < \rho(B)$. Thus

$$\rho(C) = \rho(C_{ii}) \leq \rho(B_{ii}) < \rho(B) = r$$

then $\mathcal{M}(S_\alpha(A))$ is a nonsingular M-matrix, and $S_\alpha(A) \in \mathcal{H}_I$ and, so, A is nonsingular.

Case 2. It is straightforward.

[2]Note that the case $S_\alpha(\mathcal{M}(A)) = [0] < \mathcal{M}(S_\alpha(A))$ is included here.

Case 3. From Theorem 2, $S_\alpha(\mathcal{M}(A))$ is a singular and irreducible M-matrix, and, since $S_\alpha(\mathcal{M}(A)) = \mathcal{M}(S_\alpha(A))$, $S_\alpha(A)$ is an H-matrix not included in $\mathcal{H}_I$. By the irreducibility, $S_\alpha(\mathcal{M}(A))$ is not in $\mathcal{H}_S$ (see [4]). Then $S_\alpha(A)$ remains in $\mathcal{H}_M$. □

Observe from the last theorem that Schur complements of an H-matrix in $\mathcal{H}_M$ does not decrease the quality of the original matrix, that is, the Schur complement remains in $\mathcal{H}_M$ or improve to $\mathcal{H}_I$, unless A is singular and the cardinality of α is maximum. In this special case one obtains the null matrix which belongs to $\mathcal{H}_S$. (Note that in the set of 1×1 matrices there are only two classes of H-matrices, $\mathcal{H}_I$ and $\mathcal{H}_S$.)

Moreover, the case 2 of Theorem 3 is not the unique situation in which the Schur complement of an irreducible matrix in $\mathcal{H}_M$ can be reducible, as the following example shows. Recall that Example 1 illustrates an analogous situation for a matrix in $\mathcal{H}_I$. Then one can conclude that, while the property of being an H-matrix is inherited by Schur complements, irreducibility is only inherited by Schur complements of M-matrices.

Example 2. *Consider the irreducible matrix $B \in \mathcal{H}_M$*

$$B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}.$$

For $\alpha = \{1\}$, we obtain

$$S_\alpha(B) = \begin{bmatrix} 3/2 & 1/2 \\ 0 & 2 \end{bmatrix}$$

which is a reducible matrix in $\mathcal{H}_I$.

The above example illustrates the following conclusion, according with Theorem 2 and Case 1 of Theorem 3.

Corollary 2. *Let $A \in \mathcal{H}_M$ be an irreducible matrix. If $S_\alpha(A) \neq [0]$ is reducible, then $S_\alpha(A) \in \mathcal{H}_I$.*

Theorem 3 gives the following characterization of singular irreducible H-matrices in $\mathcal{H}_M$.

Corollary 3. *Let A be an irreducible H-matrix in $\mathcal{H}_M$. The matrix A is singular if and only if $S_\alpha(\mathcal{M}(A)) = \mathcal{M}(S_\alpha(A))$ holds for all $\alpha \subset \langle n \rangle$.*

Proof. The if part is obtained choosing α such that $|\alpha| = n - 1$. The only if part follows from Theorem 3, cases 2 and 3. □

Singular irreducible H-matrices can be characterized also using their comparison matrix. In particular when the H-matrix has positive diagonal elements we have the following result in collaboration with Schneider [20].

Theorem 4. *Let A be a singular irreducible H-matrix with positive diagonal entries. Then, A is diagonally similar to its comparison matrix.*

Proof. Let $D = \operatorname{diag}(h_{11}, h_{22}, \ldots, h_{nn})$, then the matrix $H_1 = D^{-1}A$ is a singular irreducible H-matrix with all diagonal entries equal to 1.

Let us write $H_1 = I + K$, since H_1 is singular, then $-1 \in \sigma(K)$. On the other hand $\mathcal{M}(H_1) = M_1 = I - |K|$ is a singular M-matrix. Then $\rho(|K|) = 1$. By [12, Theorem 8.1.18] $\rho(K) \leq \rho(|K|) = 1$. Then, -1 is the maximum eigenvalue of K in absolute value, so $\rho(K) = 1$. Now, taking $K = B$ and $|K| = A$ in the Wielandt's Theorem [12, Theorem 8.4.5] there exists a diagonal unitary matrix D_1 such that

$$K = -1D_1|K|D_1^{-1}$$

then, $H_1 = I + K = I - D_1|K|D_1^{-1} = D_1M_1D_1^{-1}$. Finally, we recover the matrix A as

$$A = DH_1 = DD_1M_1D_1^{-1} = D_1DM_1D_1^{-1} = D_1MD_1^{-1}$$

where $M = \mathcal{M}(A)$. □

In addition, we can give the general characterization of singular and irreducible H-matrices in $\mathcal{H}_M$. There is another proof given by Johnson [13].

Corollary 4. *Let A be an irreducible H-matrix in $\mathcal{H}_M$. Then, A is singular if and only if A is diagonally equivalent to its comparison matrix, that is, there exist unitary diagonal matrices D_1 and D_2 such that*

$$A = D_1\mathcal{M}(A)D_2.$$

Proof. Let $H = D_A^{-1}A$, where $D_A = \operatorname{diag}(A)$. Now, applying Theorem 4 to the matrix H we have a diagonal unitary matrix D_1 such that

$$H = D_1\mathcal{M}(H)D_1^{-1}.$$

Then

$$A = D_AH = D_AD_1\mathcal{M}(H)D_1^{-1} = D_AD_1\mathcal{M}(D_A^{-1}A)D_1^{-1} = D_2\mathcal{M}(A)D_3$$

Note that $D_2 = D_AD_1\mathcal{M}(D_A^{-1})$ and $D_3 = D_1^{-1}$ are unitary diagonal matrices. □

Going back to Schur complements, Corollary 3 and Corollary 4 tell us that all Schur complements of a singular irreducible H-matrix are in $\mathcal{H}_M$ (unless one obtains the null complement) and they are equimodular to the corresponding Schur complement of $\mathcal{M}(A)$.

Therefore, if $A \in \mathcal{H}_M$ is nonsingular, then the inequality (3) is strict for at least one α and these Schur complements are in $\mathcal{H}_I$. But, it is not straightforward for all α except for some dense matrices as we will see in Corollary 5. On the contrary, there exist nonsingular matrices such that the strict inequality holds only when $|\alpha| = n-1$. Below we give an example of this limit case.

Example 3. *Consider the irreducible nonsingular matrix* $A \in \mathcal{H}_M$

$$A = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

For all $\alpha \subset \langle n \rangle$ *such that* $|\alpha| < 3$ *the equality* $S_\alpha(\mathcal{M}(A)) = \mathcal{M}(S_\alpha(A))$ *holds, but for any* α *such that* $|\alpha| = 3$ *one has* $[0] = S_\alpha(\mathcal{M}(A)) < \mathcal{M}(S_\alpha(A))$.

We study more in depth the above case. Avoiding trivial cases $n = 1$ or 2, consider the irreducible matrices

$$A = D + M \tag{4}$$

where D is diagonal with nonzero diagonal entries and $|M|$ is a cyclic matrix of index n (see [24]). That is, the graph of M is (only) a cycle of length n.

Theorem 5. *Let* $A \in \mathcal{H}_M \cup \mathcal{H}_I$. *If* A *admits the partition* (4) *with* $n > 2$,

Let $A = D + M$ *be a matrix of size* $n > 2$, *where* D *is diagonal with nonzero diagonal entries and* $|M|$ *is a cyclic matrix of index* n *and let* $\alpha \subset \langle n \rangle$.

1. *If* $A \in \mathcal{H}_M \cup \mathcal{H}_I$ *and* $|\alpha| < n-1$, *then* $S_\alpha(\mathcal{M}(A)) = \mathcal{M}(S_\alpha(A))$.

2. *If* $A \in \mathcal{H}_M$ *is nonsingular and* $|\alpha| = n-1$, *then* $S_\alpha(A) \in \mathcal{H}_I$.

Proof. Without loss of generality suppose that $\gamma = \{1, 2, \ldots, n, 1\}$ is the cycle of the graph of M. Let $\alpha = \{k\}$. Then, all entries of $S_\alpha(A)$, $\widetilde{a}_{ij}$, remains unchanged except exactly one entry depending on k:

- $\widetilde{a}_{n,2} = 0 - \frac{a_{n,1}a_{1,2}}{a_{11}} \neq 0 \quad$ if $\quad k = 1$

- $\widetilde{a}_{n-1,1} = 0 - \frac{a_{n-1,n}a_{n,1}}{a_{nn}} \neq 0 \quad$ if $\quad k = n$

- $\widetilde{a}_{k-1,k+1} = 0 - \frac{a_{k-1,k}a_{k,k+1}}{a_{kk}} \neq 0 \quad$ if $\quad k \neq 1$ and $k \neq n$

and the same happens for the Schur complement of the comparison matrix. Then, if $\widetilde{m}_{ij}$ with $i, j \in \alpha'$ denotes the entries of $S_\alpha(\mathcal{M}(A))$, one has $|\widetilde{m}_{ij}| = |\widetilde{a}_{ij}|$, so $S_\alpha(\mathcal{M}(A)) = \mathcal{M}(S_\alpha(A))$.

Moreover, $S_\alpha(A)$ admits the partition (5) where the graph of the $(n-1) \times (n-1)$ cyclic matrix $\widetilde{M}$ is obtained deleting the k vertex, so $S_\alpha(A)$ satisfies initial hypothesis. Proceeding by induction until $|\alpha| = n-2$ the proof of first part of the theorem is complete.

Finally, when $A \in \mathcal{H}_M$ is nonsingular, since A is irreducible, there exists some α such that $S_\alpha(\mathcal{M}(A)) < \mathcal{M}(S_\alpha(A))$ and, necessarily, $|\alpha| = n-1$. $\square$

If $\alpha = \{i_1, i_2, \ldots, i_p\}$, $S_\alpha(A)$ can be done recursively computing the Schur complements $S_{i_p}(S_{i_{p-1}}(\cdots S_{i_1}(A)))$ where $|\alpha_k| = 1$. Then, a characterization of $S_\alpha(\mathcal{M}(A)) = \mathcal{M}(S_\alpha(A))$ for $|\alpha| = 1$ is given in the following result where the matrix A can be irreducible or not.

Recall that the entries of the matrix may be complex, so the sign of the nonzero complex entry a_{ij} is $\text{sign}(a_{ij}) = a_{ij}/|a_{ij}|$.

Theorem 6. *Let $A \in \mathcal{H}_I$ or $A \in \mathcal{H}_M$, and let $\alpha = \{k\}$. Then $S_\alpha(\mathcal{M}(A)) = \mathcal{M}(S_\alpha(A))$ holds if and only if for all $i, j \neq k$ the two following conditions are satisfied*

1. $i = j$: $\quad \text{sign}(a_{ii}) = \text{sign}(a_{ik}a_{ki}a_{kk}^{-1}) \quad$ *or* $\quad a_{ik}a_{ki} = 0$
2. $i \neq j$: $\quad \text{sign}(a_{ij}) = -\text{sign}(a_{ik}a_{kj}a_{kk}^{-1}) \quad$ *or* $\quad a_{ij}a_{ik}a_{kj} = 0.$

Moreover, in this case,

$$\text{sign}(\widetilde{a}_{ij}) = \text{sign}(a_{ij}) \quad \textit{if} \quad a_{ij} \neq 0$$

where $\widetilde{a}_{ij} = a_{ij} - a_{ik}a_{kj}/a_{kk} \in S_\alpha(A), \quad i, j \in \alpha'$.

Proof. Let denote by m_{ij} and $\widetilde{m}_{ij}$ the entries of $\mathcal{M}(A)$ and $S_\alpha(\mathcal{M}(A))$, respectively. Then $S_\alpha(\mathcal{M}(A)) = \mathcal{M}(S_\alpha(A))$ if and only if $|\widetilde{m}_{ij}| = |\widetilde{a}_{ij}|$, $i, j \neq k$. We study element by element considering three cases.
(i) Clearly if $a_{ik} = 0$ all entries in the i-th row of A and $\mathcal{M}(A)$ remain unchanged, then $|\widetilde{m}_{ij}| = |\widetilde{a}_{ij}|$ for all $j \neq k$ and conditions 1 and 2 are fulfilled. Moreover, $\text{sign}(\widetilde{a}_{ij}) = \text{sign}(a_{ij})$ for $j \neq k$.

The same applies in the case $a_{kj} = 0$ to the elements of the j-th column.
(ii) If $a_{ij} = 0 \quad (i \neq j)$ the corresponding entries in the Schur complements of A and $\mathcal{M}(A)$ are

$$\widetilde{a}_{ij} = -\frac{a_{ik}a_{kj}}{a_{kk}}, \quad \text{and} \quad \widetilde{m}_{ij} = -\frac{m_{ik}m_{kj}}{m_{kk}},$$

and then, $|\widetilde{a}_{ij}| = |\widetilde{m}_{ij}|$.

Condition 2 is fulfilled, and condition 1 does not happen since $i \neq j$ and A is an H-matrix in $\mathcal{H}_I$ or in $\mathcal{H}_M$, that is $a_{ii} \neq 0$.
(iii) Otherwise, $a_{ij}a_{ik}a_{kj} \neq 0$, and so, we have

$$\widetilde{a}_{ij} = a_{ij} - \frac{a_{ik}a_{kj}}{a_{kk}}, \quad \text{and} \quad \widetilde{m}_{ij} = m_{ij} - \frac{m_{ik}m_{kj}}{m_{kk}}.$$

When $i = j$, recalling that $\mathcal{M}(A)$ is an M-matrix we have $m_{ii} \geq \frac{m_{ik}m_{ki}}{m_{kk}} > 0$. Then $|\widetilde{a}_{ii}| = |a_{ii} - \frac{a_{ik}a_{ki}}{a_{kk}}| = \widetilde{m}_{ii}$ if and only if $\operatorname{sign}(a_{ii}) = \operatorname{sign}(a_{kk}^{-1}a_{ik}a_{ki})$. Moreover, $\operatorname{sign}(\widetilde{a}_{ii}) = \operatorname{sign}(a_{ii})$.

When $i \neq j$, since $m_{ij} < 0$ and $\frac{m_{ik}m_{kj}}{m_{kk}} > 0$, $|\widetilde{a}_{ij}| = |\widetilde{m}_{ij}|$ if and only if $\operatorname{sign}(a_{ij}) = -\operatorname{sign}(a_{ik}a_{kj}a_{kk}^{-1})$. In this case also, the equality $\operatorname{sign}(\widetilde{a}_{ij}) = \operatorname{sign}(a_{ij})$ holds. □

Corollary 5. *Let $A \in \mathcal{H}_M$ be nonsingular such that $a_{ij} \neq 0$ for all $i, j \in \langle n \rangle$. Then $S_\alpha(\mathcal{M}(A)) < \mathcal{M}(S_\alpha(A))$ and $S_\alpha(A) \in \mathcal{H}_I$ for all $\alpha \subset \langle n \rangle$.*

Proof. OPCION 1: (Note that A is nonsingular and irreducible and $\mathcal{M}(A)$ is singular, then, by Theorem 3, if $\alpha \subset \langle n \rangle$, $S_\alpha(A) \in \mathcal{H}_I$ if and only if $S_\alpha(\mathcal{M}(A)) < \mathcal{M}(S_\alpha(A))$. Moreover, if $\alpha_1 \subset \alpha_2 \subset \langle n \rangle$ and $S_{\alpha_1}(A) \in \mathcal{H}_I$, then $S_{\alpha_2}(A) \in \mathcal{H}_I$ by Corollary 1. Then, proving that $S_\alpha(A) \in \mathcal{H}_I$ when $\alpha = \{k\}$ is enough.)

OPCION 2: (First we prove that $S_\alpha(A) \in \mathcal{H}_I$ when $\alpha = \{k\}$.)

Without loss of generality we assume $k = 1$ and we proceed by contradiction.

OPCION 2: (Suppose, by Theorem 3, that $S_{\{1\}}(\mathcal{M}(A)) = \mathcal{M}(S_{\{1\}}(A))$.)

OPCION 1: (Suppose that $S_{\{1\}}(\mathcal{M}(A)) = \mathcal{M}(S_{\{1\}}(A))$.)

Then, conditions 1 and 2 of Theorem 6 can be written as

$$\frac{\operatorname{sign}(a_{1i})\operatorname{sign}(a_{i1})}{\operatorname{sign}(a_{11})} = \operatorname{sign}(a_{ii}), \quad \forall i \neq 1,$$

$$\frac{\operatorname{sign}(a_{i1})\operatorname{sign}(a_{1j})}{\operatorname{sign}(a_{11})} = -\operatorname{sign}(a_{ij}), \quad \forall i, j \neq 1, i \neq j.$$

From both conditions we obtain

$$\frac{\operatorname{sign}(a_{1i})\operatorname{sign}(a_{ij})}{\operatorname{sign}(a_{1j})} = -\operatorname{sign}(a_{ii}), \quad \forall i, j \neq 1, i \neq j.$$

Since $A = [a_{ij}] = [|a_{ij}|\operatorname{sign}(a_{ij})]$, let $D = \operatorname{diag}(\operatorname{sign}(a_{1i}))$. Then,

$$DAD^{-1} = \left[\frac{|a_{ij}|\operatorname{sign}(a_{ij})\operatorname{sign}(a_{1i})}{\operatorname{sign}(a_{1j})}\right] = [p_{ij}]$$

We have $p_{ij} = -|a_{ij}|\,\mathrm{sign}(a_{ii})$, for $i, j \neq 1$, $i \neq j$, and $p_{1i} = |a_{1i}|\,\mathrm{sign}(a_{11})$, $p_{i1} = |a_{i1}|\,\mathrm{sign}(a_{ii})$ and $p_{ii} = |a_{ii}|\,\mathrm{sign}(a_{ii})$, $\forall i$. Therefore, constructing the diagonal matrices

$$D_2 = \mathrm{diag}(-1, 1, 1, \ldots, 1) \quad \text{and} \quad D_3 = \mathrm{diag}(\mathrm{sign}(a_{ii})^{-1})$$

we obtain $D_2 D_3 D A D^{-1} D_2 = \mathcal{M}(A)$, which is singular and so A, contradicting the hypothesis.

OPCION 1: (END PROOF)

OPCION 2: (For a general subset $\alpha = \{i_1, i_2, \ldots, i_p\} \in \langle n \rangle$, since $S_\alpha(A)$ is a Schur complement of $S_{\{i_k\}}(A) \in \mathcal{H}_I$, by Corollary 1, $S_\alpha(A) \in \mathcal{H}_I$.) □

Remark. The result of Theorem 3 has a nice application to preconditioning linear systems with H-matrices in $\mathcal{H}_M$. The proofs, in literature of preconditioning, to assure that a preconditioner can be computed breakdown-free are based on the breakdown-free computing of the preconditioner of the comparison matrix, which is assumed to be nonsingular or to admit an LU factorization into M-matrices [5, Theorem 2.2]. Then, this technique and the corresponding results cannot be applied to a nonsingular irreducible H-matrix in $\mathcal{H}_M$. Since Theorem 3 shows that in some moment of the computation the Schur complement will switch to an H-matrix in $\mathcal{H}_I$, then the same technique as before could be applied.

3.2 Reducible case

Without lost of generality we can assume that $A \in \mathcal{H}_M$ is already in its normal form

$$A = (A_{ij}) \qquad i > j \Rightarrow A_{ij} = 0 \tag{5}$$

where A_{ii}, for $i = 1, 2, \ldots, p$, are irreducible square H-matrices (see [4, Theorem 5, Theorem 7]), that is, there are not null diagonal blocks and then $A_{ii} \in \mathcal{H}_M$ or $A_{ii} \in \mathcal{H}_I$. Moreover, at least one diagonal block is in $\mathcal{H}_M$ and its comparison matrix is singular. All properties that we want to study (see [4] again) depend only of these diagonal blocks (the offdiagonal blocks does not influence on the Schur complement). So we reduce our study to the block diagonal submatrices.

We will denote by β_i the subset of $\langle n \rangle$ such that $A(\beta_i)$ is the submatrix A_{ii} in the main diagonal, and given an $\alpha \subset \langle n \rangle$ we shall denote by $\alpha_i = \alpha \cap \beta_i$.

Theorem 7. *Let $A \in \mathcal{H}_M$ reducible, and let $\alpha \subset \langle n \rangle$. Then,*

1. *The Schur complement of A with respect to $A(\alpha)$ exists if and only for all i such that $\alpha_i = \beta_i$ the submatrix A_{ii} is nonsingular.*

2. *If the above condition is satisfied then $S_\alpha(A)$ is an H-matrix. In addition,*

   (a) *If there exists some i such that A_{ii} is singular and $|\alpha_i| = |\beta_i| - 1$, then $S_\alpha(A) \in \mathcal{H}_S$.*

   (b) *If, either $\alpha_i \neq \emptyset, \beta_i$ and $S_{\alpha_i}(A_{ii}) \in \mathcal{H}_I$ either $\alpha_i = \beta_i$, for all i such that $A_{ii} \in \mathcal{H}_M$, then $S_\alpha(A) \in \mathcal{H}_I$.*

   (c) *In other case $S_\alpha(A)$ is in $\mathcal{H}_M$.*

Proof. 1. It is clear that $A(\alpha)$ is nonsingular if and only if all blocks $A(\alpha_i)$ are nonsingular. If the block A_{ii} is nonsingular it is clear that $A(\alpha_i)$ is nonsingular. Moreover, if the block A_{ii} is singular and $\alpha_i \neq \beta_i$ then $A(\alpha_i)$ is nonsingular since it is a proper submatrix of an irreducible H-matrix.

2. This part follows from Theorem 3 applied to the blocks in $\mathcal{H}_M$, and the observations that (i) the Schur complement is in $\mathcal{H}_S$ if at least one of its blocks is in $\mathcal{H}_S$, (ii) the Schur complement is in $\mathcal{H}_I$ if all its blocks are in $\mathcal{H}_I$, and (iii) the Schur complement is in $\mathcal{H}_M$ if none of its blocks is in $\mathcal{H}_S$ and at least one is in $\mathcal{H}_M$. □

4 Schur complements in $\mathcal{H}_S$

If $A \in \mathcal{H}_S$, then it has some null diagonal entries and it is reducible (see [4]). Then its normal form has at least one diagonal block which is a 1×1 null matrix (see [4]). Then the following result is straightforward.

Theorem 8. *Let A be an H-matrix in $\mathcal{H}_S$ and let $\alpha \subset \langle n \rangle$. If the submatrix A_{ii} is nonsingular for all i such that $\alpha_i = \beta_i$, then the Schur complement of A with respect to $A(\alpha)$ exists and is an H-matrix in $\mathcal{H}_S$.*

Note that from conditions of Theorem 8 null diagonal entries are not in $A(\alpha)$. Then, they are unchanged in the Schur complement.

Remark. For matrices in $\mathcal{H}_S$ and for reducible matrices in $\mathcal{H}_M$, the Schur complement can not be computed when α contains all indices of a singular diagonal block, $A(\alpha_i)$, of its normal form. Nevertheless, deleting in A rows and columns corresponding to α_i, one could compute the Schur complement of $A(\alpha_i')$ with respect to $A(\alpha \setminus \alpha_i)$ and will be an H-matrix. Adding the deleted rows and columns one obtain an H-matrix of size n. Note that this remark applies when the set α contains all indices of more than one singular diagonal block.

Then, considering this remark, one can conclude that any Schur complement of any H-matrix is an H-matrix.

5 Conclusions

We have proved that the Schur complement of a general H-matrix is also an H-matrix, if it can be computed, which is the case in $\mathcal{H}_I$ or irreducible H-matrices in $\mathcal{H}_M$. In addition, considering the remark of Section 4, the result extends to general H-matrices set.

Furthermore the irreducibility can be lost for H-matrices even in $\mathcal{H}_I$. When the Schur complement of an irreducible H-matrix becomes reducible, then it is in $\mathcal{H}_S$ or in $\mathcal{H}_I$ if A is singular or not respectively. Moreover it has been proved that a singular irreducible matrix in $\mathcal{H}_M$ is diagonally equivalent to its comparison matrix, and its Schur complements are equimodular to the corresponding $S_\alpha(\mathcal{M}(A))$.

We have studied the class to which the Schur complement belongs and our main results are collected in Table 1. In addition we have found conditions to guarantee that any Schur complement of a nonsingular irreducible H-matrix in $\mathcal{H}_M$ belongs to $\mathcal{H}_I$, $S_\alpha(\mathcal{M}(A)) < \mathcal{M}(S_\alpha(A))$, or, on the contrary, belongs to $\mathcal{H}_M$, $S_\alpha(\mathcal{M}(A)) = \mathcal{M}(S_\alpha(A))$, except for 1×1 Schur complements.

Class of A	Invertibility of A	$A(\alpha)$	Class of $S_\alpha(A)$
$\mathcal{H}_I$	Invertible	Invertible	$\mathcal{H}_I$
$\mathcal{H}_M$	Invertible	Invertible	$\mathcal{H}_M$ or $\mathcal{H}_I$
	Singular	$(\star)$	$\mathcal{H}_M$ or $\mathcal{H}_S$ $(\heartsuit)$
$\mathcal{H}_S$	Singular	$(\star)$	$\mathcal{H}_S$

Table 1: Summary of the results.

$(\star)$ If A is invertible or irreducible, then $A(\alpha)$ is always invertible. In the reducible case, $A(\alpha)$ is nonsingular if condition 1 in Theorem 7 is fulfilled.

$(\heartsuit)$ The Schur complement is in $\mathcal{H}_S$ if A is irreducible and $|\alpha| = n-1$, or A is reducible and there exists some i, with A_{ii} singular and $|\alpha_i| = |\beta_i| - 1$.

In brief the class of the Schur complement, provided it exists, maintains or improves the initial class of A, except in the case of the Schur complement of a singular (block) matrix is computed with respect to an $(n-1) \times (n-1)$ principal submatrix.

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