Weakly Cauchy filters and quasi-uniform completeness

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Abstract

It is well-known that the notion of a Smyth complete quasi-uniform space provides an appropriate notion of completeness to study many interesting quasi-metric spaces which appear in Theoretical Computer Science. We observe that several of these spaces actually possess a stronger form of completeness based on the use of weakly Cauchy filters in the sense of H.H. Corson and we develop a theory of completion and completeness for this kind of filters. In parallel, we also study a more general notion of completeness based on the use of certain stable filters. Thus our results extend and generalize important theorems of A. Császár, J.R. Isbell and N.R. Howes on uniform completeness.

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1. Introduction

The notion of a Smyth complete quasi-uniform space provides an appropriate tool to explain completeness properties of many quasi-metric spaces which appear in Theoretical Computer Science (see [28], [29], [30], [27], [25], etc.) However, several of these spaces possess a stronger kind of completeness based on the use of weakly Cauchy filters in the sense of H.H. Corson [3] and P. Fletcher and W.F. Lindgren [8] (see Section 2). The success of such spaces calls for a study of this type of completeness. The main purpose of this paper is to start such a study. In parallel,

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we also study a more general form of completeness based on the use of certain stable filters.

For the sake of generality we will work in the setting of quasi-uniform spaces. Thus we introduce the notions of a Corson complete, of a Császár complete, of a Corson completable and of a Császár completable quasi-uniform space. Every Corson complete (resp. completable) quasi-uniform space is Császár complete (resp. completable) and every T_o Császár complete (resp. completable) quasi-uniform space is Smyth complete (resp. completable). We characterize the quasi-uniform spaces which are Corson (resp. Császár) completable and deduce that every totally bounded quasi-uniform space is Corson completable. We prove that a quasi-uniform space is Corson (resp. Császár) complete if and only if it is bicomplete and Corson (resp. Császár) completable and discuss these kinds of completeness when one uses nets instead of filters. In this way, our results generalize well-known theorems on uniform completeness of A. Császár [4], J.R. Isbell [12] and N.R. Howes [11].

Terms and undefined concepts are used as in [9] and in [16].

Given a quasi-uniform space $(X,\mathcal{U})$ we shall denote by $\mathcal{U}^s$ the coarsest uniformity finer than $\mathcal{U}$ and its conjugate $\mathcal{U}^{-1}$ (i.e. $\mathcal{U}^s = \mathcal{U} \vee \mathcal{U}^{-1}$). If $U \in \mathcal{U}$ we denote by U^s the entourage of $\mathcal{U}^s$, $U \cap U^{-1}$.

Every quasi-uniformity $\mathcal{U}$ on a set X induces a topology $T(\mathcal{U}) = \{A \subseteq X :$ for each $x \in A$ there is $U \in \mathcal{U}$ such that $U(x) \subseteq A\}$, where $U(x) = \{y \in X : (x,y) \in U\}$.

According to [9], a quasi-uniform space $(X,\mathcal{U})$ is called bicomplete if $(X,\mathcal{U}^s)$ is a complete uniform space. A bicompletion of $(X,\mathcal{U})$ is a bicomplete quasi-uniform space $(Y,\mathcal{V})$ which has a $T(\mathcal{V}^s)$-dense subspace quasi-unimorphic to $(X,\mathcal{U})$. It was shown in [26] and in [9] that every quasi-uniform space $(X,\mathcal{U})$ admits a bicompletion $(\widetilde{X},\widetilde{\mathcal{U}})$ such that if $(X,\mathcal{U})$ is a T_o quasi-uniform space then $(\widetilde{X},\widetilde{\mathcal{U}})$ is T_o and it is the unique (up to quasi-unimorphism) bicompletion of $(X,\mathcal{U})$.

In the context of this paper, a quasi-metric on a set X is a nonnegative real-valued function d on $X \times X$ such that for all $x,y,z \in X$: *(i)* $d(x,y) = d(y,x) = 0 \Leftrightarrow x = y$ and *(ii)* $d(x,y) \leq d(x,z) + d(z,y)$.

If d is a quasi-metric on a set X and $x \in X$, the set $\{y \in X : d(x,y) < r\}$ is called the open r-sphere about x and is denoted by $S_d(x,r)$. The conjugate d^{-1} of the quasi-metric d is given by $d^{-1}(x,y) = d(y,x)$. Then we shall denote by d^s the metric defined on X by $d^s = d \vee d^{-1}$.

Every quasi-metric d on a set X generates a quasi-uniformity $\mathcal{U}_d$ on X which has as a base the family of sets of the form $U_n = \{(x,y) \in X \times X : d(x,y) < 2^{-n}\}$, for $n = 0, 1, 2, \ldots$ (see [9, p. 3]). The uniformity $(\mathcal{U}_d)^s$ will be denoted simply by $\mathcal{U}_d^s$.

2. Corson complete and Császár complete quasi-uniform spaces

Let us recall that a filter $\mathcal{F}$ on a quasi-uniform space $(X, \mathcal{U})$ is left K-Cauchy [24] provided that for each $U \in \mathcal{U}$ there is an $F \in \mathcal{F}$ such that $U(x) \in \mathcal{F}$ for all $x \in F$. $\mathcal{F}$ is said to be a stable filter [5] if for each $U \in \mathcal{U}$, $\cap_{F \in \mathcal{F}} U(F) \in \mathcal{F}$. Finally, $\mathcal{F}$ is called a weakly Cauchy filter ([3], [8]) if for each $U \in \mathcal{U}$ there is $x \in X$ such that $U(x) \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. From a Computer Science point of view the following characterization of weakly Cauchy filters seems to be more visual: A filter $\mathcal{F}$ on a quasi-uniform space $(X, \mathcal{U})$ is weakly Cauchy if and only if for each $U \in \mathcal{U}$, $\cap_{F \in \mathcal{F}} U^{-1}(F) \neq \emptyset$ ([8, Proposition 2.1]).

Let $(X, \mathcal{U})$ be a quasi-uniform space. In the following, a weakly Cauchy filter on $(X, \mathcal{U})$ will be called a *Corson filter* and a stable filter on $(X, \mathcal{U}^{-1})$ will be called a *Császár filter*.

It is known that every left K-Cauchy filter on $(X, \mathcal{U})$ is a Császár filter and that every Császár filter is a Corson filter. It is also known that the converse implications do not hold.

In [17] H.P.A. Künzi proved that a quasi-uniform space $(X, \mathcal{U})$ is Smyth complete if and only if each left K-Cauchy filter is $T(\mathcal{U}^s)$-convergent to a unique point of X and that $(X, \mathcal{U})$ is Smyth completable if and only if every left K-Cauchy filter is a Cauchy filter on the uniform space $(X, \mathcal{U}^s)$.

Definition 1. A quasi-uniform space $(X, \mathcal{U})$ is called *Corson (Császár) complete* if each Corson (Császár) filter has a $T(\mathcal{U}^s)$-cluster point.

Definition 2. **Let** $(X, \mathcal{U})$ be a quasi-uniform space. A *Corson (Császár) completion* of $(X, \mathcal{U})$ is a Corson (Császár) complete quasi-uniform space $(Y, \mathcal{V})$ in which $(X, \mathcal{U})$ can be quasi-uniformly embedded as a $T(\mathcal{V}^s)$-dense subspace. In this case we say that $(X, \mathcal{U})$ is *Corson (Császár) completable.*

It immediately follows from the definitions that every Corson complete quasi-uniform space is Császár complete and that every Corson completable quasi-uniform space is Császár completable. It is also clear that every Császár complete quasi-uniform space is bicomplete. (See [11, p. 33-35] for an example of a Császár complete metric space which is not Corson complete.)

Example 1. Let $\mathbb{Z}$ be the set of integers. The Khalimsky line (used in image processing) consists of $\mathbb{Z}$ with the topology generated by all sets of the form $\{2n-1, 2n, 2n+1\}, n \in \mathbb{Z}$ (see, for instance, [14], [15]). Then the quasi-metric d defined on $\mathbb{Z}$ by $d(2n, 2n-1) = d(2n, 2n+1) = d(n,n) = 0$ for all $n \in \mathbb{Z}$ and $d(x,y) = 1$ otherwise, generates the topology of the Khalimsky line. Let $\mathcal{F}$ be a Corson filter on $(\mathbb{Z}, \mathcal{U}_d)$. Then there is $x \in \mathbb{Z}$ such that $S_d(x,1) \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. Since $S_d(x,1)$ is a finite set there is a $y \in S_d(x,1) \cap (\cap_{F \in \mathcal{F}} F)$, so y is a $T(\mathcal{U}_d^s)$-cluster point of $\mathcal{F}$. We conclude that $(\mathbb{Z}, \mathcal{U}_d)$ is Corson complete.

Example 2. Let $\mathbb{R}^+$ be set of nonnegative real numbers. Let d be the quasi-metric defined on $\mathbb{R}^+$ by $d(x,y) = (y-x) \vee 0$. The functions of the dual complexity space ([25], [27]) are $\mathbb{R}^+$-valued functions and the Smyth completeness of $(\mathbb{R}^+, \mathcal{U}_d)$ plays an important role in showing that the (dual) complexity quasi-metric space is also Smyth complete [25, Theorem 3]. (Recall that the complexity space was introduced in [27] as a part of the development of a topological foundation for the complexity analysis of alghoritms.) We shall show that $(\mathbb{R}^+, \mathcal{U}_d)$ is actually Corson complete. Let $\mathcal{F}$ be a Corson filter on $(\mathbb{R}^+, \mathcal{U}_d)$. Then there is an $x \in \mathbb{R}^+$ such that $[0, x+1) \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. It follows that there is a point in $[0, x+1]$ which is a cluster point of $\mathcal{F}$ with respect to the Euclidean topology on $\mathbb{R}^+$. Hence $(\mathbb{R}^+, \mathcal{U}_d)$ is Corson complete.

Example 3. S.G. Matthews introduced in [22] the notions of a *Scott-like topology* in order to present a topological study of those totally order sequences (i.e. chains) of increasing information, which appear in Computer Science, having the property that its least upper bound is intended to capture the notion of the amount of information defined by the chain but it cannot contain more information than can be derived from the members of the chain (see [22, p. 184-185]). A typical example of this situation is the topology $T[\leq]$ defined on the set X of the positive integers $\mathbb{N}$ with infinity ∞ as follows $T[\leq] = \{\{n, n+1, ..., \infty\} : n \in \mathbb{N}\}$. (Here, $\leq$ denotes the usual partial ordering on X). Thus $T[\leq]$ is a Scott-like topology in the sense of Matthews. This topology can be generated by a quasi-metric d on X such that (X, d^s) is a compact metric space and, thus. $(X, \mathcal{U}_d)$ is Corson complete. In fact, for each $x, y \in X$ put $d(x,y) = 0$ if $x \leq y$, and $d(x,y) = 1/y$ otherwise. Clearly $T(d) = T[\leq]$ and every sequence of distinct points in X converges to ∞ with respect to the topology $T(\mathcal{U}_d^s)$.

Example 4. Let Σ^F be the set of finite words in a finite alphabet Σ. For any $x, y \in \Sigma^F$ set $d(x,y) = 0$ if x is a prefix of y, and $d(x,y) = 2^{-(l(x,y)+1)}$ where $l(x,y) = \sup\{n \in \omega : x(k) = y(k) \text{ whenever } k \leq n\}$ otherwise (see, for instance, [28], [17].) It is well-known that $(\Sigma^F, \mathcal{U}_d)$ is a totally bounded nonbicomplete quasi-uniform space. By Corollary 1 (see below), the space is Corson completable. The bicompletion of $(\Sigma^F, \mathcal{U}_d)$ is the space $(\Sigma^\infty, \mathcal{U}_{\tilde{d}})$, where Σ^∞ consists of all finite and infinite words in Σ and $\tilde{d}$ is the quasi-metric defined from d on Σ^∞ in the obvious manner (see [28, Example 2.4] or [17, Example 8(b)]). Since $(\Sigma^\infty, \mathcal{U}_{\tilde{d}})$ is also totally bounded it follows from Corollary 1 and Theorem 2 below that it is Corson complete.

Lemma 1. *Let $(Y, \mathcal{V})$ be a quasi-uniform space and let X be a $T(\mathcal{V}^s)$-dense subset of Y. If $\mathcal{F}$ is a Corson (Császár) filter on $(Y, \mathcal{V})$, then the filter base*

$$\mathcal{G} = \{U^s(F) \cap X : F \in \mathcal{F}, U \in \mathcal{V}\}$$

generates a Corson (Császár) filter on $(X, \mathcal{V} \mid X)$.

Proof. First let $\mathcal{F}$ be a Corson filter on $(Y, \mathcal{V})$. Let $U \in \mathcal{V}$. We shall show that there is $x \in X$ such that $U(x) \cap G \neq \emptyset$ for all $G \in \mathcal{G}$. Choose a $V \in \mathcal{V}$ such that $V^3 \subseteq U$. Then there exists a $y \in Y$ such that $V(y) \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. Since X is $T(\mathcal{V}^s)$-dense in Y, there exists an $x \in V^s(y) \cap X$. Given $G \in \mathcal{G}$, we have $G = W^s(F) \cap X$ for some $F \in \mathcal{F}$ and some $W \in \mathcal{V}$. Then for any $b \in V(y) \cap F$ there is some $a \in (V \cap W)^s(b) \cap X$. So $a \in G$. Furthermore $(x, a) \in U$ since $(x, y) \in V, (y, b) \in V$ and $(b, a) \in V$. We have shown that $U(x) \cap G \neq \emptyset$ for all $G \in \mathcal{G}$. Hence $\mathcal{G}$ is a Corson filter on $(X, \mathcal{V} \mid X)$.

Now let $\mathcal{F}$ be a Császár filter on $(Y, \mathcal{V})$. Let $U \in \mathcal{V}$. We shall show that $B \subseteq \cap_{G \in \mathcal{G}} U^{-1}(G)$ for some $B \in \mathcal{G}$. Choose a $V \in \mathcal{V}$ such that $V^3 \subseteq U$. Then there exists an $A \in \mathcal{F}$ such that $A \subseteq \cap_{F \in \mathcal{F}} V^{-1}(F)$. Set $B = V^s(A) \cap X$. Then $B \in \mathcal{G}$. For any $b \in B$ and $G \in \mathcal{G}$ we have $G = W^s(F) \cap X$ for some $F \in \mathcal{F}$ and some $W \in \mathcal{V}$, and $b \in V^s(a)$ for some $a \in A$. Let $y \in F$ such that $a \in V^{-1}(y)$. Thus $(b, y) \in V^2$. Choose an $L \in \mathcal{V}$ with $L \subseteq V \cap W$ and let $z \in L^s(y) \cap X$. Then $z \in (V \cap W)^s(y) \cap X \subseteq W^s(F) \cap X = G$ and $(b, z) \in V^3 \subseteq U$. We conclude that $B \subseteq U^{-1}(G)$ for all $G \in \mathcal{G}$. This completes the proof. □

Theorem 1. *A quasi-uniform space $(X, \mathcal{U})$ is Corson (Császár) completable if and only if every Corson (Császár) filter on $(X, \mathcal{U})$ is contained in a Cauchy* *filter on the uniform space* $(X,\mathcal{U}^s)$.

Proof. Suppose that $(X,\mathcal{U})$ is Corson (Császár) completable. Then there is a quasi-unimorphism f from $(X,\mathcal{U})$ to a $T(\mathcal{V}^s)$-dense subspace of a Corson (Császár) complete quasi-uniform space $(Y,\mathcal{V})$. Let $\mathcal{F}$ be Corson (Császár) filter on $(X,\mathcal{U})$. Clearly $f(\mathcal{F})$ is a Corson (Császár) filter base on $(Y,\mathcal{V})$, so it has a $T(\mathcal{V}^s)$-cluster point $y \in Y$. Then $fil\{f^{-1}(V^s(y)) \cap F : F \in \mathcal{F},\ V \in \mathcal{V}\}$ is a Cauchy filter on $(X,\mathcal{U}^s)$ which contains $\mathcal{F}$.

Conversely, let $(\widetilde{X},\widetilde{\mathcal{U}})$ be a bicompletion of $(X,\mathcal{U})$. If $\mathcal{F}$ is a Corson (Császár) filter on $(\widetilde{X},\widetilde{\mathcal{U}})$ it follows from Lemma 1 that the filter base $\mathcal{G} = \{U^s(F) \cap X : F \in \mathcal{F},\ U \in \widetilde{\mathcal{U}}\}$ is Corson (Császár) on $(X,\mathcal{U})$. So $\mathcal{G} \subseteq \mathcal{H}$ for some Cauchy filter $\mathcal{H}$ on $(X,\mathcal{U}^s)$. Denote by $\widetilde{\mathcal{H}}$ the filter generated on $\widetilde{X}$ by $\mathcal{H}$. Then $\widetilde{\mathcal{H}}$ is $T(\widetilde{\mathcal{U}}^s)$-convergent to some point $y \in \widetilde{X}$. Hence y is a $T(\widetilde{\mathcal{U}}^s)$-cluster point of $\mathcal{F}$. We conclude that $(X,\mathcal{U})$ is Corson (Császár) completable. □

Remark . It follows from the preceding result that if $(X,\mathcal{U})$ is a T_o Corson (Császár) completable quasi-uniform space, then its bicompletion is the unique Corson (Császár) completion of $(X,\mathcal{U})$.

Recall that a quasi-uniform space $(X,\mathcal{U})$ is precompact provided that for each $U \in \mathcal{U}$ there is a finite subset A of X such that $U(A) = X$. $(X,\mathcal{U})$ is said to be totally bounded if the uniform space $(X,\mathcal{U}^s)$ is precompact (see [9], [16]). It immediately follows from [9, Proposition 3.14] that every ultrafilter on a precompact quasi-uniform space is a Corson (ultra) filter.

Corollary 1. *A quasi-uniform space is totally bounded if and only if its precompact and Corson completable.*

Proof. Let $(X,\mathcal{U})$ be a precompact Corson completable quasi-uniform space. Let $\mathcal{F}$ be an ultrafilter on X. By the precompactness of $(X,\mathcal{U})$, $\mathcal{F}$ is a Corson (ultra) filter. So, by Theorem 1, $\mathcal{F}$ is a Cauchy filter on the uniform space $(X,\mathcal{U}^s)$. Therefore, $(X,\mathcal{U})$ is totally bounded.

Conversely, it follows from Theorem 1 that every totally bounded quasi-uniform space is Corson completable and it is well-known that every totally bounded quasi-uniform space is (hereditarily) precompact. □

Corollary 2. *A quasi-uniform space is totally bounded if and only if it is* *hereditarily precompact and Császár completable.*

Proof. Let $(X,\mathcal{U})$ be a hereditarily precompact Császár completable quasi-uniform space. Let $\mathcal{F}$ be an ultrafilter on X. By the hereditary precompactness of $(X,\mathcal{U})$, $\mathcal{F}$ is a Császár (ultra) filter ([6], [19]). So, by Theorem 1, $\mathcal{F}$ is a Cauchy filter on the uniform space $(X,\mathcal{U}^s)$. Therefore, $(X,\mathcal{U})$ is totally bounded. The converse follows from Corollary 1 and the well-known fact that every totally bounded quasi-uniform space is hereditarily precompact. □

Corollary 3. *Every Császár completable quasi-uniform space is Smyth completable.*

Proof. Let $\mathcal{F}$ be a left K-Cauchy filter on a Császár completable quasi-uniform space $(X,\mathcal{U})$. We shall prove that $\mathcal{F}$ is a Cauchy filter on the uniform space $(X,\mathcal{U}^s)$. By Theorem 1, there is a Cauchy filter $\mathcal{G}$ on $(X,\mathcal{U}^s)$ such that $\mathcal{F} \subseteq \mathcal{G}$. Let $U \in \mathcal{U}$. Choose $V \in \mathcal{U}$ such that $V^4 \subseteq U$. There is $F_V \in \mathcal{F}$ such that $V(x) \in \mathcal{F}$ for all $x \in F_V$. There also exists $p \in X$ such that $V^s(p) \in \mathcal{G}$. Choose a point $y \in F_V \cap V^s(p)$. Then $F_V \cap V(y) \in \mathcal{F}$. For any pair $a, b \in F_V \cap V(y)$, there is $c \in V(a) \cap V^s(p)$ since $V(a) \in \mathcal{F} \subseteq \mathcal{G}$. Moreover $b \in V^2(p)$. So $(a,b) \in V^4 \subseteq U$. We have shown that $\mathcal{F}$ is a Cauchy filter on $(X,\mathcal{U}^s)$. Consequently, $(X,\mathcal{U})$ is Smyth completable. □

Corollary 4. *Every Corson completable quasi-uniform space is Smyth completable.*

Related to Corollaries 1 and 2 we give an example of a compact Császár complete locally symmetric quasi-metric space which is not totally bounded.

Example 5. Let d be the quasi-metric defined on the set ω on nonegative integer numbers by $d(0,n) = 1/n$ for all $n \in \mathbb{N}$, $d(n,x) = 1$ for all $n \in \mathbb{N}$ and for all $x \neq n$ and $d(x,x) = 0$ for all $x \in \omega$. Then $(\omega,\mathcal{U}_d)$ is a compact quiet (in the sense of [7]) locally symmetric Császár complete quasi-uniform space. However $(\omega,\mathcal{U}_d)$ is not totally bounded.

Theorem 2. *A quasi-uniform space $(X,\mathcal{U})$ is Corson (Császár) complete if and only if it is bicomplete and Corson (Császár) completable.*

Proof. Suppose that $(X,\mathcal{U})$ is bicomplete and Corson (Császár) completable. Let $\mathcal{F}$ be a Corson (Császár) filter on $(X,\mathcal{U})$. By Theorem 1, $\mathcal{F} \subseteq \mathcal{G}$ for some Cauchy filter $\mathcal{G}$ on $(X,\mathcal{U}^s)$. Hence $\mathcal{G}$ is $T(\mathcal{U}^s)$-convergent to a point $x \in X$. So x is a $T(\mathcal{U}^s)$-cluster point of $\mathcal{F}$. We conclude that $(X,\mathcal{U})$ is Corson (Császár) complete. The converse is obvious. □

Corollary 5. *A T_o quasi-uniform space $(X,\mathcal{U})$ is Corson (Császár) complete if and only if it is Smyth complete and Corson (Császár) completable.*

Proof. Suppose that $(X,\mathcal{U})$ is a T_o Corson (Császár) complete quasi-uniform space. Let $\mathcal{F}$ be a left K-Cauchy filter on $(X,\mathcal{U})$. ~~Then $\mathcal{F}$ is a Császár filter, so~~ By corollary 3 and Theorem 2 it is $T(\mathcal{U}^s)$-convergent to a unique point of X. Hence $(X,\mathcal{U})$ is Smyth complete. The converse follows from Theorem 2. □

Proposition 1. *Let $(X,\mathcal{U})$ be a quasi-uniform space. Then the uniform space $(X,\mathcal{U}^s)$ is compact if and only if $(X,\mathcal{U})$ is precompact and Corson complete.*

Proof. Suppose that $(X,\mathcal{U})$ is precompact and Corson complete. Then $(X,\mathcal{U})$ is bicomplete and, by Corollary 1, it is totally bounded. We conclude thart $(X,\mathcal{U}^s)$ is a compact uniform space. The converse is obvious. □

Let $(X,\mathcal{U})$ be a T_1 quasi-uniform space. In our next result we characterize compactness of $(X,\mathcal{U})$ in terms of the Hausdorff-Bourbaki quasi-uniformity of $(X,\mathcal{U})$ and convergent Corson filters.

Let us recall that if $(X,\mathcal{U})$ is a quasi-uniform space and $\mathcal{P}_o(X)$ denotes the collection of all nonempty subsets of X, then the Hausdorff-Bourbaki quasi-uniformity of $(X,\mathcal{U})$ is defined as the quasi-uniformity $\mathcal{U}_*$ on $\mathcal{P}_o(X)$ which has as a base the the family of sets of the form

$$\{(A,B) \in \mathcal{P}_o(X) \times \mathcal{P}_o(X) : B \subseteq U(A) \text{ and } A \subseteq U^{-1}(B)\}$$

whenever $U \in \mathcal{U}$ ([1], [21]).

Similarly to the uniform case, we say that a filter $\mathcal{F}$ on a quasi-uniform space $(X,\mathcal{U})$ is convergent in the quasi-uniform space $(\mathcal{P}_o(X),\mathcal{U}_*)$ if the net $(F)_{(F\in\mathcal{F},\supseteq)}$ is $T(\mathcal{U}_*)$-convergent in $\mathcal{P}_o(X)$.

Theorem 3. *A T_1 quasi-uniform space $(X,\mathcal{U})$ is compact if and only if every Corson filter on $(X,\mathcal{U})$ is convergent in the quasi-uniform space $(\mathcal{P}_o(X),\mathcal{U}_*)$.*

Proof. Suppose that $(X,\mathcal{U})$ is a compact quasi-uniform space. Let $\mathcal{F}$ be a filter on X. Denote by C be the set of clusters points of $\mathcal{F}$. Since $(X,\mathcal{U})$ is compact, $C \neq \emptyset$. We shall show that $\mathcal{F}$ converges to C in $(\mathcal{P}_o(X),\mathcal{U}_*)$. Fix $U \in \mathcal{U}$. We claim that there is an $F_U \in \mathcal{F}$ such that $F_U \subseteq U(C)$. Assume the contrary. Then $\{F \setminus U(C) : F \in \mathcal{F}\}$ is a filter base on X. Thus it has a cluster point $x \in X \setminus C$, which is a contradiction. Therefore, for each $F \in \mathcal{F}$ such that $F \subseteq F_U$, we have $F \subseteq U(C)$. Moreover $C \subseteq U^{-1}(F)$ for all $F \in \mathcal{F}$. We have shown that every (Corson) filter on $(X,\mathcal{U})$ is convergent in $(\mathcal{P}_o(X),\mathcal{U}_*)$.

Conversely, suppose that there is a filter $\mathcal{F}$ on X without $T(\mathcal{U})$-cluster point. Fix a point $y \in X$. Consider the filter $\mathcal{G} = \{F \cup \{y\} : F \in \mathcal{F}\}$. Clearly $\mathcal{G}$ is a Corson filter on $(X,\mathcal{U})$, so the net $(G)_{(G\in\mathcal{G},\supseteq)}$ converges to some $C \in \mathcal{P}_o(X)$ with respect to the topology $T(\mathcal{U}_*)$. Since for each $U \in \mathcal{U}$ there is a $G \in \mathcal{G}$ such that $C \subseteq U^{-1}(H)$ for every $H \in \mathcal{G}$ with $H \subseteq G$, it follows that each point of C is a $T(\mathcal{U})$-cluster point of $\mathcal{G}$. Thus $C \subseteq cl_{T(\mathcal{U})}\{y\}$. So $C = \{y\}$. Therefore given $U \in \mathcal{U}$ there is an $F \in \mathcal{F}$ such that $F \cup \{y\} \subseteq U(y)$. Hence $\mathcal{F}$ is $T(\mathcal{U})$-convergent to y. This contradiction concludes the proof. □

Remark 2. Note that the condition T_1 is only used in the proof of the sufficiency in the above theorem. Note also that this condition can be generalized assuming simply that there is a point $y \in X$ such that $\cap\{U(y) : U \in \mathcal{U}\} = \{y\}$.

In [31] A. Weil introduced the notion of a uniformly locally compact uniform space. The following elegant characterization of uniformly locally compact (quasi-) uniform spaces was obtained by Fletcher and Lindgren [8] (see also [9, Theorem 5.33] and [23, Theorem 4]): A (quasi-) uniform space $(X,\mathcal{U})$ is uniformly locally compact if and only it is locally compact and every weakly Cauchy filter has a $T(\mathcal{U})$-cluster point. In Theorem 4 below we obtain the analogue of this result in the context of Corson complete quasi-uniform spaces.

Let us recall that an open cover $\mathcal{C}$ of a topological space is said to be directed provided that if A and B are members of $\mathcal{C}$ there is a $C \in \mathcal{C}$ such that $A \cup B \subseteq C$.

Lemma 2. *A quasi-uniform space $(X,\mathcal{U})$ is Corson complete if and only if whenever $\mathcal{C}$ is a directed open cover of $(X, T(\mathcal{U}^s))$ there is a $U \in \mathcal{U}$ such that $\{U(x) : x \in X\}$ refines $\mathcal{C}$.*

Proof. Suppose that $(X,\mathcal{U})$ is a Corson complete quasi-uniform space and let

$\mathcal{C}$ be a directed open cover of the topological space $(X, T(\mathcal{U}^s))$. We may assume without loss of generality that $X \notin \mathcal{C}$. Then $\{X \setminus C : C \in \mathcal{C}\}$ is a filter base of $T(\mathcal{U}^s)$-closed sets that obviously has no $T(\mathcal{U}^s)$-cluster point. Hence this filter base cannot be a Corson filter. Thus there is a $U \in \mathcal{U}$ such that for each $x \in X$ there is $C(x) \in \mathcal{C}$ for which $U(x) \cap (X \setminus C(x)) = \emptyset$. We conclude that $\{U(x) : x \in X\}$ refines $\mathcal{C}$.

Conversely, suppose that there is a Corson filter $\mathcal{F}$ on $(X,\mathcal{U})$ without $T(\mathcal{U}^s)$-cluster point. Then $\mathcal{C} = \{X \setminus cl_{T(\mathcal{U}^s)}F : F \in \mathcal{F}\}$ is a directed open cover of $(X, T(\mathcal{U}^s))$. Let $U \in \mathcal{U}$ be such that $\{U(x) : x \in X\}$ refines $\mathcal{C}$. Then there is an $x \in X$ such that $U(x) \cap F \neq \emptyset$ for all $F \in \mathcal{F}$ and there is an $F_o \in \mathcal{F}$ such that $U(x) \subseteq X \setminus cl_{T(\mathcal{U}^s)}F_o$. This contradiction concludes the proof. □

Definition 3. A quasi-uniform space $(X,\mathcal{U})$ is called *Corson locally compact* if there is a $U \in \mathcal{U}$ such that for each $x \in X$, $cl_{T(\mathcal{U}^s)}U(x)$ is a $T(\mathcal{U}^s)$-compact subset of X.

Theorem 4. *A quasi-uniform space $(X,\mathcal{U})$ is Corson locally compact if and only if $(X,\mathcal{U})$ is Corson complete and $(X, T(\mathcal{U}^s))$ is locally compact.*

Proof. Suppose that $(X,\mathcal{U})$ is a Corson complete quasi-uniform space such that $(X, T(\mathcal{U}^s))$ is a locally compact topological space. For each $x \in X$ there is a $T(\mathcal{U}^s)$-open neighborhood G_x of x such that $cl_{T(\mathcal{U}^s)}G_x$ is a $T(\mathcal{U}^s)$-compact subset of X. Let $\mathcal{C}$ be the $T(\mathcal{U}^s)$-open cover of X consisting of all finite unions of members of $\{G_x : x \in X\}$. By Lemma 2 there is a $U \in \mathcal{U}$ such that $\{U(x) : x \in X\}$ refines $\mathcal{C}$. Fix $x \in X$. Then there is a finite subset A of X such that $U(x) \subseteq \cup_{a \in A} G_a$. Since $cl_{T(\mathcal{U}^s)}G_a$ is $T(\mathcal{U}^s)$-compact for each $a \in A$, we deduce that $cl_{T(\mathcal{U}^s)}U(x)$ is a $T(\mathcal{U}^s)$-compact subset of X, so $(X,\mathcal{U})$ is Corson locally compact.

Conversely, if $(X,\mathcal{U})$ is Corson locally compact, then $(X, T(\mathcal{U}^s))$ is, clearly, locally compact. Now let $\mathcal{F}$ be a Corson filter on $(X,\mathcal{U})$. There is a $U \in \mathcal{U}$ such that $cl_{T(\mathcal{U}^s)}U(x)$ is $T(\mathcal{U}^s)$-compact for all $x \in X$. Let $y \in X$ such that $U(y) \cap F \neq \emptyset$ for all $F \in \mathcal{F}$. Then $\mathcal{G} = \{F \cap cl_{T(\mathcal{U}^s)}U(y) : F \in \mathcal{F}\}$ is a filter on the $T(\mathcal{U}^s)$-compact set $cl_{T(\mathcal{U}^s)}U(y)$. So $\mathcal{G}$ has a $T(\mathcal{U}^s)$-cluster point. Since $\mathcal{F} \subseteq \mathcal{G}$, $\mathcal{F}$ has a $T(\mathcal{U}^s)$-cluster point. We conclude that $(X,\mathcal{U})$ is Corson complete. □

Remark 3. It follows from Theorem 4 that the quasi-uniform spaces $(X,\mathcal{U}_d)$ of Examples 1, 2 and 3 are actually Corson locally compact.

According to [13], a family $\mathcal{H}$ of subsets of a topological space (X, T) is called well-monotone provided that the partial order $\subseteq$ of set inclusion is a well-order on $\mathcal{H}$. The compatible quasi-uniformity $\mathcal{WM}$ on (X, T) which has as a subbase the set of all binary relations of the form

$$W = \cup\{\{x\} \times (\cap\{G : x \in G \in \mathcal{H}\}) : x \in X\}$$

where $\mathcal{H}$ is a well-monotone open cover of (X, T) is called the well-monotone (open covering) quasi-uniformity of (X, T).

A quasi-uniformity $\mathcal{U}$ on a set X is left K-complete provided that every left K-Cauchy filter is convergent in $(X, T(\mathcal{U}))$ [24]. It is known [24, Lemma 1] that every cluster point of a left K-Cauchy filter $\mathcal{F}$ is a limit point of $\mathcal{F}$. It is proved in [20, Proposition 1] that the well-monotone quasi-uniformity of each topological space is left K-complete. The technique of the proof of [18, Lemma 1(a)] permits us to state the following more general result.

Proposition 2. *Let (X, T) be a topological space. Then every Császár filter on $(X, \mathcal{WM})$ has a cluster point in (X, T).*

Proof. Let $\mathcal{F}$ be a Császár filter on $(X, \mathcal{WM})$. We shall prove that $adh\mathcal{F} \in \mathcal{F}$, where $adh\mathcal{F}$ denotes the set of all cluster points of $\mathcal{F}$ in (X, T). Assume the contrary. Then there exists a mimimal (infinite) cardinal number m so that there is a subcollection $\mathcal{E}$ of $\mathcal{F}$ consisting of closed subsets of (X, T) such that $card(\mathcal{E}) = m$ and $\cap\mathcal{E} \notin \mathcal{F}$. We can suppose that $\mathcal{E} = \{F_\alpha : \alpha < m\}$. For each $\beta < m$ put $E_\beta = \cap\{F_\alpha : \alpha < \beta\}$. (In particular let $E_o = X$.) Set $\mathcal{C} = \{X \setminus E_\beta : \beta < m\} \cup \{X\}$ and $W(x) = \cap\{D : x \in D \in \mathcal{C}\}$ for all $x \in X$. Thus $W = \cup\{\{x\} \times W(x) : x \in X\} \in \mathcal{WM}$ and $E_\beta \in \mathcal{F}$ for each $\beta < m$. Since $\mathcal{F}$ is a Császár filter there is an $F_o \in \mathcal{F}$ such that $F_o \subseteq W^{-1}(F)$ for all $F \in \mathcal{F}$. Since $\cap\mathcal{E} \notin \mathcal{F}$ there is $x \in F_o \setminus \cap\mathcal{E}$. Hence there exists $\beta < m$ such that $x \in X \setminus E_\beta$. On the other hand, $x \in W^{-1}(y)$ for some $y \in E_\beta$ since $E_\beta \in \mathcal{F}$. Thus $y \in W(x) \subseteq X \setminus E_\beta$, a contradiction. We conclude that $adh\mathcal{F} \neq \emptyset$, so $\mathcal{F}$ has a cluster point in (X, T). □

Given a topological space (X, T) we denote by $\mathcal{FN}$ its fine quasi-uniformity (see, for instance, [9]). Thus we immendiately deduce from Proposition 2 the following

Corollary 6. *Let (X, T) be a topological space. Then every Császár filter on*

$(X, \mathcal{FN})$ *has a cluster point in* (X, T).

Remark 4. Corollary 6 should be compared with [2, Corollary 4] (see also [12, Chapter VII]) which states that the fine uniformity of each paracompact topological space has the property that each stable filter has a cluster point.

3. Corson completeness and Császár completeness via nets

In this section we will restate Theorems 1 and 2 above in terms of Corson (resp. Császár) nets having a cluster point in $(X, \mathcal{U}^s)$.

The net properties which are analogous to Corson filter and Császár filter respectively, are the following (compare [12], [10], [2]):

A net $(x_\alpha)_{\alpha\in\Lambda}$ in a quasi-uniform space $(X, \mathcal{U})$ is called a *Corson net* if for each $U \in \mathcal{U}$ there is $x \in X$ such that $(x_\alpha)_{\alpha\in\Lambda}$ is frequently in $U(x)$. $(x_\alpha)_{\alpha\in\Lambda}$ is called a *Császár net* if for each $U \in \mathcal{U}$ there is $\beta \in \Lambda$ such that for each $\gamma \geq \beta$ some subnet of $(x_\alpha)_{\alpha\in\Lambda}$ is contained in $U^{-1}(x_\gamma)$.

By using standard techniques on the relationship between filters and nets, one can easily deduce from Theorem 1 and Theorem 2 respectively, the following results:

Theorem 5. *A quasi-uniform space* $(X, \mathcal{U})$ *is Corson (Császár) completable if and only if every Corson (Császár) net in* $(X, \mathcal{U})$ *has a Cauchy subnet in the uniform space* $(X, \mathcal{U}^s)$.

Theorem 6. *A quasi-uniform space* $(X, \mathcal{U})$ *is Corson (Császár) complete if and only if every Corson (Császár) net has a cluster point in the uniform space* $(X, \mathcal{U}^s)$.

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