A note on extreme points of $C^{\infty}$-smooth balls in polyhedral spaces

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Abstract

Morris [Mo83] proved that every separable Banach space $X$ that contains an isomorphic copy of $c_0$ has an equivalent strictly convex norm such that all points of its unit sphere $S_X$ are unpreserved extreme, i.e., they are no longer extreme points of $B_X$. We use a result of Hájek [Ha95] to prove that any separable infinite-dimensional polyhedral Banach space has an equivalent $C^{\infty}$-smooth and strictly convex norm with the same property as in Morris’ result. We additionally show that no point on the sphere of a $C^2$-smooth equivalent norm on a polyhedral infinite-dimensional space can be strongly extreme, i.e., there is no point $x$ on the sphere for which a sequence $(h_n)$ in $X$ with $\|h_n\| \not\to 0$ exists such that $\|x \pm h_n\| \to 1$.

1 Introduction

It is known that in non-superreflexive spaces, there exist no equivalent $C^2$-smooth norms that would be at the same time locally uniformly rotund (cf e.g. [FHHMZ, Exercise 9.16]). We show in this note that yet, in separable polyhedral spaces—all of which non-superreflexive—, there exist $C^{\infty}$-smooth norms with various degrees of rotundity weaker than local uniform rotundity.

If $(X, \| \cdot \|)$ is a normed space, its closed unit ball (its unit sphere) will be denoted alternatively by $B_X, B_{\| \cdot \|}$, or even $B_{(X, \| \cdot \|)}$ (respectively $S_X, S_{\| \cdot \|}$, or $S_{(X, \| \cdot \|)}$), according to the circumstances. If $x \in X$ and $\delta > 0$, we put $B_X(x; \delta), B_{\| \cdot \|}(x; \delta)$, or even $B_{(X, \| \cdot \|)}(x; \delta)$, for $x + \delta B_X$. The norm on $X$, its dual norm on $X^*$, and its bidual norm on $X^{**}$, are denoted by the same notation. For standard notation, results, and undefined terms we refer, e.g., to [FHHMZ].

Extreme points of $B_X$ that are not extreme of $B_{X^{**}}$ are called unpreserved. On the other side, points in $S_X$ that are extreme points of $B_{X^{**}}$ are called preserved extreme points (see Figure 1). Obviously, every preserved extreme point of $B_X$ is itself an extreme point of $B_X$.

The preserved extreme points coincide with the $w$-strongly extreme points of $B_X$ (see [GLT92] and references therein). A point $x \in S_X$ is called (w-) strongly extreme of $B_X$ if given two sequences $\{y_n\}$ and $\{z_n\}$ in $B_X$ such that $(y_n + z_n) \to 2x$, then $y_n \to x$ (respectively, $y_n \overset{w}{\to} x$). A norm $\| \cdot \|$ such that all points in $S_{\| \cdot \|}$ are strongly extreme is said to be midpoint locally uniformly rotund (for this notion, see, e.g., [LPT09] and references therein).

Solving a question by Phelps, Katznelson (see the reference in [Mo83]) proved that the closed unit ball of the disk algebra contains unpreserved extreme points.

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Let \( x \in S_X \). The point \( x \) is said to be strongly exposed (by a functional \( f \in S_{X^*} \)) if \( f(x) = 1 \) and \( \text{diam} S(f, \delta) \to 0 \) as \( \delta \downarrow 0 \), where \( S(f, \delta) := \{ x \in B_X : f(x) > 1 - \delta \} \) is a section of \( B_X \) determined by \( f \). The point \( x \) is said to be denting if for every \( \epsilon > 0 \) it is contained in a section of \( B_X \) having diameter less than \( \epsilon \). It is easy to show that strongly exposed \( \Rightarrow \) denting \( \Rightarrow \) strongly extreme \( \Rightarrow \) w'-strongly extreme (= preserved extreme) \( \Rightarrow \) extreme, and that if \( X \) is locally uniformly rotund, then every point in \( S_X \) is strongly exposed. For an example showing how big the gap between being strongly or w'-strongly extreme is, see Theorem 4. It is simple to show that a denting point of \( S_{X^*} \) must belong to \( X \), hence the example in Remark 5.2 hints also at the difference between being strongly extreme and denting.

Morris proved in [Mo83] the following result.

(M1) Any separable Banach space \( X \) containing an isomorphic copy of \( c_0 \) can be renormed in such a way that all points of \( S_X \) are unpreserved extreme points. (Observe that the new norm is then strictly convex.)

The space \( c_0 \) has the property that the set \( \text{Ext}(B_{X^*}) \) of extreme points of the closed dual unit ball is countable. The set \( \text{Ext}(B_{X^*}) \) is an example of a James boundary, i.e., a subset of \( B_{X^*} \) where each element \( x \in X \) attains its supremum on \( B_{X^*} \). A Banach space with a countable James boundary has a separable dual space (this follows, e.g., from the fact that a countable James boundary is strong, i.e., its closed convex hull is the closed dual unit ball ([Ro81], see also [Go87]).

A Banach space \( X \) is called polyhedral if the ball of every finite-dimensional subspace (equivalently every two-dimensional subspace, see [K59]) of \( X \) has only a finite number of extreme points. Every polyhedral separable space has a countable James boundary ([Fo80], see also [Ve00]).

An example of polyhedral space is \( c_0 \) in its canonical norm ([K60], see also [GM72] and [Go01]). The argument in [Go01] is so nice that we cannot help but to reproduce it here. It relies on the fact that the \( \| \cdot \|_\infty \)-norm on \( c_0 \) depends locally on a finite number of coordinates (see the precise definition of this term below). Let \( E \) be a finite-dimensional subspace of \( c_0 \). For each \( x \in S_E \) there exists \( \varepsilon(x) > 0 \) and a finite subset \( F(x) \) of \( X^* \) such that \( \| y \|_\infty = \sup \{ \| y \cdot x^* \| : x^* \in F(x) \} \) for all \( y \in B_E(x; \varepsilon(x)) \). Since \( S_E \) is compact, there are \( x_1, \ldots, x_n \) in \( S_E \) such that

\[
S_E \subseteq \bigcup_{i=1}^n B_E(x_i, \varepsilon(x_i)).
\]

Put \( F := \bigcup_{i=1}^n F(x_i) \). Then \( F \) is a finite subset of \( X^* \) such that

\[
\| x \|_\infty = \sup \{ \| y \cdot x^* \| : x^* \in F \}
\]

for all \( x \in E \), hence \( E \) is isometric to a subspace of \( (\mathbb{R}^|F|, \| \cdot \|_\infty) \), a polyhedral space.

On the other side, the space \( c \) in its canonical norm is not polyhedral. The following argument was kindly provided by L. Veselý (personal communication): Consider the points \( P_n := \exp \{ i(1 - 1/n)\pi/4 \} \) in the plane, for all \( n \in \mathbb{N} \) (see Figure 2). Let \( a_n x + b_n y = 1 \) be the equation of the line through \( P_n \) and \( P_{n+1} \) for all \( n \in \mathbb{N} \), and \( a_0 x + b_0 y = 1 \) the equation of the line through \( P_\infty := \exp(\pi/4) \) and \( P_0 := (-1, 0) \). Then \( a := (a_n)_{n\geq0} \) and \( b := (b_n)_{n\geq0} \) are elements in \( c \), and

Figure 1: In (i), all points in \( S_X \) are preserved extreme, none in (ii)
their linear span $L$ is isometric to a plane equipped with the norm whose closed unit ball is the set \text{conv} \{±P_1, ±P_2, \ldots, ±P_∞\}.
There is no infinite-dimensional reflexive polyhedral space ([L64]). Actually, no infinite-dimensional $C(K)$ space in its canonical norm is polyhedral—although such space has, if $K$ is a countable compact topological space, obviously, a countable James boundary—. As seen below (see (H)), every $C(K)$ space with $K$ a countable and compact topological space is isomorphic to a polyhedral space.

We will need the following result:

(Z) Banach spaces with a countable James boundary are $c_0$-saturated, i.e., each closed subspace contains an isomorphic copy of $c_0$ ([Fo77], [PWZ81], see also [FHHMZ, Theorem 10.9]).

In this note we slightly modify Morris technique by means of a result of P. Hájek ([Ha95]), see also [FHHMZ, Theorem 10.12]) on normed spaces with a countable James boundary—a characterization quoted below as (H)— to add, under these circumstances, smoothness—in fact, $C^∞$-smoothness— to the kind of renorming shown by Morris.

The norm $\| \cdot \|$ of a Banach space is said to depend locally on a finite number of coordinates if given any $x_0 \in S_X$ there exists $\delta > 0$, continuous linear functionals $\{ψ_1, ψ_2, \ldots, ψ_n\} \subset X^*$, and a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ such that, for every $x \in B(x_0; δ)$ we have $\|x\| = f(ψ_1(x), ψ_1(x), \ldots, ψ_1(x))$. The result of Hájek [Ha95] (see also [FHHMZ, Theorem 10.12]) mentioned above, an improvement of results in [Fo77] and [PWZ81], is the equivalence (i) to (iv) in the following. For the property (v) see [FLP01, Proposition 6.19] and, e.g., [Ve00].

(H) For a Banach space $X$, the following are equivalent: (i) $X$ has a countable James boundary. (ii) $X$ has a James boundary that can be covered by a countable number of $\| \cdot \|$-compact subsets of $X^*$. (iii) $X$ is separable and has an equivalent norm that depends locally on a finite number of coordinates. (iv) $X$ is separable and has an equivalent norm that is $C^∞$-smooth away from the origin and depends locally on a finite number of coordinates. (v) $X$ is separable and isomorphic to a polyhedral Banach space.

The following result appears in [Mo83], with a different argument, as an ingredient of the proof of (M1) above; it will also be used in the proof of our main result.

(M2) There exists an infinite-dimensional $w^*$-closed subspace $M_0$ of $ℓ_∞$ such that $M_0 \cap c_0 = \{0\}$.
To see this, first note that every separable Banach space is isometric to a subspace of $ℓ_∞$, thus in particular $ℓ_∞$ contains an isometric copy $Z$ of a given infinite-dimensional separable reflexive space. By a result of Rosenthal (see, e.g., [FHHMZ, Lemma 4.62]), $Z$ is $w^*$-closed. Observe that $Z \cap c_0$ must be finite-dimensional, as any infinite-dimensional subspace of $c_0$ contains a copy of $c_0$. Then, a finite-codimensional subspace $M_0$ of $Z$ is what we need to finish the proof.

2 The results

Theorem 1 Let $(X, \| \cdot \|_0)$ be a Banach space having a countable James boundary. Then there exists an equivalent (strictly convex) norm $\| \cdot \|$ on $X$ that is $C^∞$-smooth away from the origin and
such that every point in \( S_{\| \cdot \|} \) is an unpreserved extreme point of \( B_{\| \cdot \|} \).

**Proof.** By (H) above, the space \( X \) has an equivalent \( C^\infty \)-smooth norm \( \| \cdot \| \) that depends locally on a finite number of coordinates. Moreover, it contains an isomorphic copy \( Z \) of \( c_0 \) (see (Z) above).

The space \( Z^{**} \) can be canonically identified to a closed subspace of \( X^{**} \). Let \( M \) be a \( w^* \)-closed infinite-dimensional subspace of \( Z^{**} \) such that \( M \cap Z = \{0\} \); it exists thanks to (M2) above. It is clear, then, that \( M \cap X = \{0\} \).

Let \( N := M \perp X^* \) (the orthogonal is taken with respect to the duality \( \langle X^{**}, X^* \rangle \)). Find a sequence \( \{\phi_n\} \in N \) such that \( \text{span} \{\phi_n : n \in \mathbb{N}\} = N \) and \( \sum_{n=1}^{\infty} \|\phi_n\|^2 < +\infty \). Define a linear operator \( T : X \to l_2 \) by \( T x := ((x, \phi_n))_{n=1}^{\infty} \) for \( x \in X \); then \( T \) is clearly bounded and one-to-one, and the mapping \( x \to \| T x \|_2 \) from \( X \) into \( \mathbb{R} \) is certainly \( C^\infty \)-smooth away from the origin.

Define a norm \( \| \cdot \| \) on \( X \) by

\[
\|x\| := \|x\| + \|Tx\|_2 \quad \text{for all} \quad x \in X.
\]

Clearly \( \| \cdot \| \) is strictly convex (see e.g. [DGZ, Chapter II]) and \( C^\infty \)-smooth away from the origin.

Let us show that every point \( x_0 \) in \( S_{\| \cdot \|} \) is unpreserved extreme. Find \( \delta > 0 \) such that \( \| \cdot \| \) depends on \( B_{\| \cdot \|}(x_0; \delta) \) on finitely many coordinates \( \{\psi_1, \psi_2, \ldots, \psi_n\} \), i.e., \( \|x\| = f(\psi_1(x), \psi_2(x), \ldots, \psi_n(x)) \) for \( x \in B_{\| \cdot \|}(x_0; \delta) \), where \( f : \mathbb{R}^n \to \mathbb{R} \) is a continuous function. Due to the fact that \( M \) is infinite-dimensional, we can find \( h^{**} \in M \cap \bigcap_{k=1}^{\infty} \ker \psi_k \) with \( 0 < \|h^{**}\| \leq \delta \).

Find a net \( \{h_i : i \in I, \leq \} \) in \( B_{\| \cdot \|}(0; \delta) \) that \( w^* \)-converges to \( h^{**} \). Observe that \( x_0 + h_i \in B_{\| \cdot \|}(x_0; \delta) \), hence

\[
\|x_0 + h_i\| = f(\psi_1(x_0 + h_i), \psi_2(x_0 + h_i), \ldots, \psi_n(x_0 + h_i)), \quad \text{for all} \quad i \in I.
\]

Note that \( \psi_k(x_0 + h_i) \to \psi_k(x_0 + h^{**}) \) for all \( k = 1, 2, \ldots, n \), and so, by (2),

\[
\|x_0 + h_i\| = f(\psi_1(x_0 + h_i), \psi_2(x_0 + h_i), \ldots, \psi_n(x_0 + h_i))
\]

\[
\to f(\psi_1(x_0 + h^{**}), \psi_2(x_0 + h^{**}), \ldots, \psi_n(x_0 + h^{**}))
\]

\[
= f(\psi_1(x_0), \psi_2(x_0), \ldots, \psi_n(x_0)) = \|x_0\|.
\]

Since

\[
x_0 + h_i \stackrel{w^*}{\to} x_0 + h^{**},
\]

we get from (3) and (4) that \( \|x_0 + h^{**}\| \leq \|x_0\| \). In the same way we get \( \|x_0 - h^{**}\| \leq \|x_0\| \), so finally by a standard convexity argument, \( \|x_0\| = \|x_0 + h^{**}\| = \|x_0 - h^{**}\| \). Regarding the norm \( \| \cdot \| \), we have then

\[
\|x_0 + h^{**}\| = \|x_0 + h^{**}\| + \|T(x_0 + h^{**})\|,
\]

as it is easy to show, hence, since \( T(h^{**}) = 0 \),

\[
\|x_0 + h^{**}\| = \|x_0\| + \|T(x_0)\| = \|x_0\| = 1.
\]

Analogously,

\[
\|x_0 - h^{**}\| = \|x_0\| = 1.
\]

Equations (5) and (6) together show that \( x_0 \) is an unpreserved extreme point of \( B_{\| \cdot \|} \).

The following result extends what formerly was known for \( C^2 \)-smooth LUR norms (see, e.g., [FFHMZ, Exercise 9.16]) and later for \( C^2 \)-smooth norms with a strongly exposed point on its unit sphere [FWZ83, Theorem 3.3].

**Theorem 2** Let \( (X, \| \cdot \|) \) be an infinite-dimensional \( C^2 \)-smooth Banach space. If there exists a strongly extreme point of \( B_{\| \cdot \|} \), then \( X \) is superreflexive.

**Proof.** Assume that \( x \) is a strongly extreme point of \( B_X \). The \( C^2 \)-differentiability of \( \| \cdot \| \) implies that there exists \( \delta > 0 \) such that the first derivative of \( \| \cdot \| \) is uniformly continuous on a \( 2\delta \)-ball around \( x \).

Let \( g \) be the supporting functional to the ball at \( x \). For \( h \in g^{-1}(0) \), let \( f(h) = \|x + h\| + \|x - h\| - 2 \).
Then $f(h) \geq 0$, $f(0) = 0$ and $\inf_{\|h\|=\delta} f > 0$. Indeed, otherwise there exists a sequence $\{h_n\}_{n=1}^\infty$ in $g^{-1}(0)$ such that $\|h_n\| = \delta$ for all $n \in \mathbb{N}$, and $f(h_n) \to 0$, meaning that $\|x + h_n\| \to 1$ and $\|x - h_n\| \to 1$, as $\|x \pm h_n\| \geq g(x \pm h_n) = g(x) = 1$. Thus, by the definition of the strong extremality of $x$, $\|h_n\| \to 0$, a contradiction. Hence, by standard methods we can construct a bump function (i.e. a function with bounded nonempty support) on $g^{-1}(0)$ with uniformly continuous derivative, meaning that $X$ is superreflexive (see, e.g., [FHHMZ, Theorem 9.19]).

**Corollary 3** Let $(X, \| \cdot \|)$ be an infinite-dimensional $C^2$-smooth Banach space. Assume that $X$ does contain an isomorphic copy of $c_0$ (in particular, assume that $X$ is isomorphic to a polyhedral space). Then no point of $S_{\| \cdot \|}$ is a strongly extreme point of $B_{\| \cdot \|}$.

**Proof.** Otherwise, according to Theorem 2, the space $X$ would be superreflexive. This is impossible since $X$ contains an isomorphic copy of $c_0$. In case that $X$ is isomorphic to a polyhedral space, so it is every separable subspace of $X$, thus the containment of $c_0$ follows from (Z) and (H) above. □

**Theorem 4** Let $X$ be a separable infinite-dimensional polyhedral Banach space. Then there exists an equivalent norm $\| \cdot \|$ on $X$ such that every point in $S_{\| \cdot \|}$ is preserved extreme non-strongly extreme of $B_{\| \cdot \|}$.

**Proof.** Let $\| \cdot \|$ be an equivalent $C^2$-smooth norm on $X$ (such a norm always exists, see (H) above). Let $\{f_i : i \in \mathbb{N}\}$ be a countable norm-dense subset of $B_{\| \cdot \|}$ (recall that $X$ is Asplund). Then the equivalent norm $\| \cdot \|$ on $X$ defined by $\|x\| := (\|x\|^2 + \sum \frac{1}{2} f_i^2(x))^2$ for all $x \in X$, is weakly uniformly rotund, i.e., whenever $x_n, y_n$ are in $S_{\| \cdot \|}$ and $\|x_n + y_n\| \to 2$, then $x_n - y_n \to 0$ in the weak topology of $X$. This means that, in particular, the bidual norm of $\| \cdot \|$ is rotund (indeed, assume that $2x^{* *} = y^{* *} + z^{* *}$ for some $x^{* *} \in S_{\| \cdot \|}$, where $y^{* *}$ and $z^{* *}$ are both in $B_{\| \cdot \|}$) and $y^{* *} \neq z^{* *}$. Since $X^{*}$ is separable, there exist sequences $\{y_n\}$ and $\{z_n\}$ in $B_{\| \cdot \|}$ such that $y_n \to y^{* *}$ and $z_n \to z^{* *}$ in the $w^{*}$-topology. This leads immediately to a contradiction. Moreover, the norm $\| \cdot \|$ on $X$ is clearly $C^2$-smooth. Thus all points in $S_{\| \cdot \|}$ are preserved extreme points and yet, no point there is strongly extreme point of $B_{\| \cdot \|}$ by Corollary 3 (indeed, $X$ is not superreflexive, as it contains an isomorphic copy of $c_0$). □

**Remark 5**

1. Note that, in the setting of Theorem 4, no point in $S_{\| \cdot \|}$ is a point where the norm and weak topologies coincide, as otherwise, by a result in [LLT88], such a point would be a strongly extreme point of $B_{\| \cdot \|}$.

2. The James space $J$ can be renormed by a norm the second bidual norm of which has the property that all its points on its sphere are strongly extreme points ([MOTV01]), see also [LPT09]). None of the points in $S_{\cdot \cdot \cdot X}$ can be denting. Recall that a space is reflexive if its dual space admits an equivalent Fréchet differentiable dual norm ([FHHMZ, Corollary 7.26]).

3. The space $\ell_\infty$ cannot be renormed so that all points on the sphere would be preserved extreme points ([HMS]).

4. Hájek ([Ha98]) showed that, if $\Gamma$ is uncountable, then there exists no $C^2$-smooth and strictly convex norm on $c_0(\Gamma)$.

5. We refer to, e.g., [HMZ12], for a survey on related topics.

**References**


