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Interplay between various graphs on $C(X)_{\mathcal{P}}$

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Abstract

This article focuses on the study of zero-divisor graph $\Gamma(C(X)_{\mathcal{P}})$, annihilator graph $AG(C(X)_{\mathcal{P}})$ and weakly zero-divisor graph $W\Gamma(C(X)_{\mathcal{P}})$ on the ring $C(X)_{\mathcal{P}}$ of all real-valued functions on a topological space X that are continuous outside a member of an ideal $\mathcal{P}$ of closed subsets of X. We establish that if $C(X)_{\mathcal{P}}$ properly contains the ring $C(X)$ of real-valued continuous functions on X, then the radius of $\Gamma(C(X)_{\mathcal{P}})$ is 2 and it is not triangulated. Moreover, in this situation, both $\Gamma(C(X)_{\mathcal{P}})$ and $AG(C(X)_{\mathcal{P}})$ are not hypertriangulated and the dominating number of $AG(C(X)_{\mathcal{P}})$ is 2. Furthermore, $W\Gamma(C(X)_{\mathcal{P}})$ fails to be a complete graph under the hypothesis $C(X)_{\mathcal{P}} \supsetneq C(X)$. We establish a connection between the complemented-ness of $\Gamma(C(X)_{\mathcal{P}})$ and the Von-Neumann regularity of $C(X)_{\mathcal{P}}$ under the assumption that $C(X)_{\mathcal{P}} \supseteq \{\chi_{\{p\}} \colon p \in X\}$. We realise that any two of these three graphs coincide if and only if $|X| = 2$ and in this case, the graphs are complete bipartite. We also note that the phenomena of $W\Gamma(C(X)_{\mathcal{P}})$ being triangulated, hypertriangulated and complemented depend solely on the cardinality of X.

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1. Introduction

Let X be a Tychonoff space and $C(X)$ denote the ring of real-valued continuous functions on X. Let the collection of all real-valued functions on X be denoted by $\mathbb{R}^X$ and for $f \in \mathbb{R}^X$, the set of points of discontinuities of f by D_f. As usual, the closure (interior) of a subset A of X is denoted by $\overline{A}$ (resp. *int* A) in X. A collection $\mathcal{P}$ of closed subsets of X is said to be an '**ideal of closed sets**' if it satisfies the following conditions:

(i) $\mathcal{P}$ is closed under taking finite union.
(ii) If A, B are closed sets in X such that $A \subseteq B$ and $B \in \mathcal{P}$, then $A \in \mathcal{P}$.

The following collections form ideals of closed sets in X:

- $\mathcal{P}_f$ = the collection of all finite subsets of X,
- $\mathcal{K}$ = the collection of all closed compact subsets of X and
- $\mathcal{P}_{nd}$ = the collection of all closed nowhere dense subsets of X.

The triplet $(X, \tau, \mathcal{P})$ is called a $\tau\mathcal{P}$-space ([7]) and the collection, $C(X)_{\mathcal{P}} = \{f \in \mathbb{R}^X : \overline{D_f} \in \mathcal{P}\}$ [7] forms a commutative ring with unity, under the operations of pointwise additions and multiplications. Note that the ring $C(X)_{\mathcal{P}}$ is an abstraction of various known rings of functions.

- If $\mathcal{P} = \{\emptyset\}$, then $C(X)_{\mathcal{P}} = C(X)$.
- If $\mathcal{P} = \mathcal{P}_{nd}$, then $C(X)_{\mathcal{P}} = T'(X)$, which was introduced by Ahmadi Zand in 2010 [2].
- If $\mathcal{P} = \mathcal{P}_f$, then $C(X)_{\mathcal{P}} = C(X)_F$, which was introduced by Gharabaghi et. al. in 2018 [9].
- The notation $C(X)_K$ is used to denote $C(X)_{\mathcal{K}}$ in [7].

For $f \in C(X)_{\mathcal{P}}$, the '**zero-set**', $Z_{\mathcal{P}}(f)$ is the set $\{x \in X : f(x) = 0\}$. If $f \in C(X)$, we write $Z(f)$ instead of $Z_{\mathcal{P}}(f)$. The collection $\{Z(f) : f \in C(X)\}$ is as usual denoted by $Z[X]$ and $\{Z_{\mathcal{P}}(f) : f \in C(X)_{\mathcal{P}}\}$ is denoted by $Z_{\mathcal{P}}[X]$.

On a commutative ring R, Anderson and Livingston defined the **zero-divisor graph**, $\Gamma(R)$ on the vertex set $V(R)$, consisting of all non-zero zero-divisors of the ring R and two vertices x and y in $V(R)$ to be adjacent in $\Gamma(R)$ if $xy = 0$ [1]. In 2013, Badawi defined the **annihilator graph** $AG(R)$ on the same vertex set $V(R)$ such that two vertices $x, y \in V(R)$ are adjacent if $Ann(x) \cup Ann(y) \subsetneq Ann(xy)$ [5], where $Ann(z)$ denotes the collection $\{z' \in R : zz' = 0\}$ for $z \in R$. Another graph, called the **weakly zero-divisor graph**, denoted by $W\Gamma(R)$, was defined on $V(R)$ by Nikmehr et. al. [16]. Two distinct vertices x and y are adjacent in $W\Gamma(R)$ if there exist non-zero elements $a \in Ann(x)$ and $b \in Ann(y)$ such that $ab = \mathbf{0}$. It follows that $\Gamma(R)$ is a spanning subgraph of $AG(R)$ and $AG(R)$ is a spanning subgraph of $W\Gamma(R)$. Recall that a graph G_2 is said to be the **spanning subgraph** of a graph G_1 if they have the same vertex set and whenever two vertices are adjacent in G_2, they are also adjacent in G_1.

In this article, we wish to study the above mentioned graphs defined on the collection $V(C(X)_{\mathcal{P}})$ (or simply, V) of all non-zero divisors of zero of the ring $C(X)_{\mathcal{P}}$. In order to do that efficiently, we first recollect some definitions. Let G be a graph defined on a vertex set V_0. For $f, g \in V_0$, the **distance** $d(f, g)$ is the length of the shortest path between f and g. The **diameter** $diam(G) = sup\{d(f, g)\colon f, g \in V_0\}$ and the **girth** $gr(G)$ is the length of the shortest cycle in G. If G has no cycles, $gr(G) = \infty$. Furthermore, for $f \in V_0$, the **associated number**, $e(f)$ of f is defined as $max\{d(f, g)\colon g \in V_0 \setminus \{f\}\}$. An $f \in V_0$ is said to be a **center** of G if it has the smallest associated number. The associated number of such a center is called the **radius** of G and is denoted by $\rho(G)$. G is said to be **triangulated** if each vertex of G is a vertex of a triangle and **hypertriangulated** if each edge in G is an edge of a triangle. A subset of V_0 is said to be a **stable set** if no two vertices in the set are adjacent. G is called a **bipartite graph** if V_0 can be decomposed into two stable sets. A bipartite graph is **complete** if each pair of vertices from distinct stable sets are adjacent. For two vertices $f, g \in V_0$, $\boldsymbol{c(f, g)}$ denotes the length of the shortest cycle (if any) containing f and g, and $c(f, g) = \infty$ if there does not exist any cycle in G containing f and g. Two distinct vertices $f, g \in V_0$ are said to be **orthogonal** (denoted as $\boldsymbol{f \perp g}$) if f is adjacent to g and there does not exist any $h \in V_0$ which is adjacent to both f and g. G shall be called a **complemented** graph if for each $f \in V_0$, there exists $g \in V_0$ such that $f \perp g$. A complemented graph G is called **uniquely complemented** if $g, h \in V_0$ are adjacent to the same set of vertices in G, whenever $f \perp g$ and $f \perp h$ in G for some $f \in V_0$. A vertex is said to be a **universal vertex** in G if it is adjacent to all other vertices. A subset V_1 of V_0 is said to be a **dominating set** in G if every vertex outside V_1 is adjacent to some vertex in V_1. The **dominating number**, $\boldsymbol{dt(G)} = \min\{|V_1|\colon V_1$ is a dominating set in $G\}$. The smallest cardinal number $\aleph$ such that every complete subgraph of G is of cardinality $\leq \aleph$ is said to be the **clique number** of G, and is denoted by $\boldsymbol{\omega G}$. For more graph theoretic terms, one can refer [6].

We observe that if R has no non-zero divisors of zero, then $V(R) = \emptyset$ and all the three graphs $\Gamma(R)$, $AG(R)$ and $W\Gamma(R)$ are then null-graphs. If X is a topological space with exactly one point, then $C(X)_{\mathcal{P}}$ is isomorphic to $\mathbb{R}$, for any choice of ideal $\mathcal{P}$ and hence all the above mentioned graphs on $C(X)_{\mathcal{P}}$ are null-graphs. So, to avoid this case, we shall always assume $|X| \geq 2$.

Throughout this article, we have made use of the space $X_{\mathcal{P}}$, which is generated by $Z_{\mathcal{P}}[X]$ as a base for closed sets. First note that if $\mathcal{P} = \{\emptyset\}$, then the space $X_{\mathcal{P}}$ is the original space X. We observe that the space $X_{\mathcal{P}}$ is a T_3 space. We also note that $X_{\mathcal{P}}$ contains isolated points if either X has isolated points or $C(X)_{\mathcal{P}} \supsetneq C(X)$. Next we establish a relation between the Goldie dimension of the ring $C(X)_{\mathcal{P}}$ and the cellularity of the space $X_{\mathcal{P}}$. Furthermore, we describe the space of minimal prime ideals of $C(X)_{\mathcal{P}}$ and establish a condition under which the space is compact. We note that a few results in [3] and [11] can be established as particular cases of some theorems in this section, when we choose $\mathcal{P} = \{\emptyset\}$.

Section 3 of this article is devoted to the study of the zero-divisor graph $\Gamma(C(X)_{\mathcal{P}})$ (or simply, Γ) of the ring $C(X)_{\mathcal{P}}$. In particular, we calculate the diameter, girth and radius of the graph. It has been noted in this section that the radius of the graph Γ is 2 if $C(X)_{\mathcal{P}} \supsetneq C(X)$. Moreover, we have observed that if $C(X)_{\mathcal{P}} \supsetneq C(X)$, then Γ is neither triangulated nor hypertriangulated. This brings out a contrast between the zero-divisor graph of $C(X)$ and Γ (see Example 3.15). We note that Γ is a complete bipartite graph if $|X| = 2$. We further observe that the clique number of Γ and the Goldie dimension of $C(X)_{\mathcal{P}}$ are equal. In particular, we explicitly show that the clique number of $\Gamma(C(X)_F)$, $\Gamma(T'(X))$ and $\Gamma(C(X)_K)$ are equal and equal to the cardinality of X. We next establish that the compactness of the space of minimal prime ideals of $C(X)_{\mathcal{P}}$ is equivalent to the complemented-ness of the graph Γ. We further recall a condition equivalent to the Von-Neumann regularity of the ring $C(X)_{\mathcal{P}}$ (see [7]) to establish that if $C(X)_{\mathcal{P}} \supseteq \{\chi_{\{p\}} : p \in X\}$, then the complemented-ness of the graph Γ is equivalent to the Von-Neumann regularity of the ring $C(X)_{\mathcal{P}}$.

Section 4 of this article deals with the annihilator graph $AG(C(X)_{\mathcal{P}})$ of the ring $C(X)_{\mathcal{P}}$. For simplicity, we write AG instead of $AG(C(X)_{\mathcal{P}})$ when there is no ambiguity. We realise that AG coincides with Γ if and only if $|X| = 2$ and then and only then AG is a complete bipartite graph. Moreover, we note that the distance between any two non-adjacent vertices is 2 and it follows that the diameter of the graph is 2. We further establish that AG is hypertriangulated if and only if $C(X)_{\mathcal{P}} = C(X)$ and X has no isolated points. However, AG is triangulated if and only if $|X| \geq 3$. We also observe a correspondence between the dominating number of AG and the phenomenon of AG being hypertriangulated.

The final section involves the weakly zero-divisor graph of the ring $C(X)_{\mathcal{P}}$. We denote it by $W\Gamma$, instead of $W\Gamma(C(X)_{\mathcal{P}})$. Just as we have seen in case of Γ and AG, $W\Gamma$ is also a complete bipartite graph, which coincides with Γ and AG if and only if $|X| = 2$. We are able to describe the distance between two vertices, diameter of $W\Gamma$ and the dominating number of the graph. We further show that $W\Gamma$ is a complete graph if and only if $C(X)_{\mathcal{P}} = C(X)$ and X has no isolated points. In particular, whenever $C(X)_{\mathcal{P}}$ properly contains $C(X)$, $W\Gamma$ is not a complete graph. We realise that $W\Gamma$ is triangulated if and only if $|X| \geq 3$ and this is equivalent to the hypertriangulation of $W\Gamma$ as well. Moreover, the above statements are equivalent to saying that $W\Gamma$ is not complemented. Finally, we describe some of these results for the particular case of $C(X)$.

2. Prerequisites on the ring $C(X)_{\mathcal{P}}$ and the space $X_{\mathcal{P}}$

We initiate this section with the recollection that the set of all zero-sets of functions in $C(X)_{\mathcal{P}}$, denoted by $Z_{\mathcal{P}}[X]$, forms a base for closed sets for the topology $\tau_{\mathcal{P}}$ on X, which is finer than the original topology on X [8]. In short we denote the pair $(X, \tau_{\mathcal{P}})$ as $X_{\mathcal{P}}$. For a subset A of X, $cl_{X_{\mathcal{P}}}A$ ($int_{X_{\mathcal{P}}}A$) denotes the closure (resp. interior) of A in $X_{\mathcal{P}}$. We first observe a topological property of the space $X_{\mathcal{P}}$ which shall be useful throughout this article.

Theorem 2.1. *$X_{\mathcal{P}}$ is a T_3-topological space and the collection $\{int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \colon f \in C(X)_{\mathcal{P}}\}$ forms a base for open sets for the space $X_{\mathcal{P}}$.*

Proof. Suppose K is a closed set in $X_{\mathcal{P}}$ and $x_0 \notin K$. Then there exists $f \in C(X)_{\mathcal{P}}$ such that $x_0 \in X \setminus Z_{\mathcal{P}}(f) \subseteq X \setminus K$. Without loss of generality assume that $f(x_0) > 0$ and let $r = \frac{f(x_0)}{3}$. Define $U = \{x \in X \colon f(x) < r\}$ and $V = \{x \in X \colon f(x) > r\}$. Then U and V are disjoint basic open sets in $X_{\mathcal{P}}$ such that $x_0 \in V$ and $K \subseteq U$. This ensures that $X_{\mathcal{P}}$ is a T_3-space.

Moreover, there exist $g, h \in C(X)_{\mathcal{P}}$ such that $U = X \setminus Z_{\mathcal{P}}(g)$ and $V = X \setminus Z_{\mathcal{P}}(h)$. Also since $x_0 \in V \subseteq X \setminus U \subseteq X \setminus K$, it follows that $x_0 \in int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g) \subseteq X \setminus K$. $\square$

The following observation helps us make certain compelling deductions regarding the rings $C(X)_F$, $T'(X)$ and $C(X)_K$.

Observation 2.2. The assumption that the characteristic function $\chi_{\{p\}} \in C(X)_{\mathcal{P}}$ for each $p \in X$ (for eg.: $C(X)_F$, $C(X)_K$, $T'(X)$, etc) ensures that $X \setminus \{p\} = Z_{\mathcal{P}}(\chi_{\{p\}}) \in Z_{\mathcal{P}}[X]$, which is a base for closed sets $X_{\mathcal{P}}$ and so $X_{\mathcal{P}}$ is the discrete space. In this context,

Next, we note that often the existence of even a single function of the form $\chi_{\{p\}}$, for some $p \in X$, in $C(X)_{\mathcal{P}}$ describes certain behaviours of graphs on the ring $C(X)_{\mathcal{P}}$. Such an existence is clearly guaranteed if X has an isolated point. Moreover, we have the following useful lemma.

Lemma 2.3. *$C(X)_{\mathcal{P}}$ contains $\chi_{\{p\}}$ for some $p \in X$ if and only if either X has an isolated point or there exists a non-isolated point $p \in X$ with $\{p\} \in \mathcal{P}$. Moreover, the latter holds if and only if $C(X) \subsetneq C(X)_{\mathcal{P}}$. In particular, a point $p \in X$ is isolated in the space $X_{\mathcal{P}}$ if and only if either p is an isolated point in X or $\{p\} \in \mathcal{P}$.*

Proof. If $C(X) \ni \chi_{\{p\}}$, for some $p \in X$, then p is an isolated point in X. If $\chi_{\{p\}} \in C(X)_{\mathcal{P}} \setminus C(X)$, then p is a non-isolated point in X and so $D_{\chi_{\{p\}}} = \{p\}$. Therefore, $\{p\} \in \mathcal{P}$. The converse is straightforward.

Moreover, if $\{p\} \in \mathcal{P}$, where p is non-isolated, then $\chi_{\{p\}} \in C(X)_{\mathcal{P}} \setminus C(X)$. Conversely let $f \in C(X)_{\mathcal{P}} \setminus C(X)$. Then $\emptyset \neq D_f \subseteq \overline{D_f} \in \mathcal{P}$. Let $p \in D_f$, then it follows that p is a non-isolated point in X and $\{p\} \in \mathcal{P}$. The rest is immediate. $\square$

Recall that the cellularity $c(X)$ of a space X is defined as the smallest cardinal number $\aleph (\geq \aleph_0)$ such that every family of pairwise disjoint non-empty open subsets of X has cardinality $\leq \aleph$ and a cardinal number $\aleph$ is called the Goldie dimension, $dim(R)$ of a commutative ring R if a direct sum of non-zero ideals in R consists of m terms where $m \leq \aleph$.

We now realise that the Goldie dimension, $dim(C(X)_{\mathcal{P}})$ of $C(X)_{\mathcal{P}}$ is closely related to the cellularity of the space $X_{\mathcal{P}}$. This relation shall be useful in discussing the clique number of the zero-divisor graph of $C(X)_{\mathcal{P}}$.

Theorem 2.4. *For a topological space X, $dim(C(X)_{\mathcal{P}}) = c(X_{\mathcal{P}})$.*

Proof. Let $\bigoplus_{\alpha\in\Lambda} I_\alpha$ be a direct sum of non-zero ideals in $C(X)_{\mathcal{P}}$. Note that for $\alpha,\beta \in \Lambda$ with $\alpha \neq \beta$, $f_\alpha f_\beta = \mathbf{0}$ as $I_\alpha \cap I_\beta = \{\mathbf{0}\}$. Thus, the collection $\{X \setminus Z_{\mathcal{P}}(f_\alpha) : \alpha \in \Lambda\}$ is a family of pairwise disjoint open sets in $X_{\mathcal{P}}$. Thus, we have $c(X_{\mathcal{P}}) \geq |\Lambda|$ and hence $c(X_{\mathcal{P}}) \geq dim(C(X)_{\mathcal{P}})$.

Next suppose $\{G_i : i \in I\}$ is a family of pairwise disjoint open subsets of $X_{\mathcal{P}}$. Then for each $i \in I$, there exists $f_i \in C(X)$ such that $\emptyset \neq X \setminus Z_{\mathcal{P}}(f_i) \subseteq G_i$. Set $I_i =< f_i >$ for each $i \in I$. We claim that the family $\{I_i : i \in I\}$ of non-zero ideals in $C(X)_{\mathcal{P}}$, constitutes an independent family. In other words, for $i \in I$, $I_i \cap \sum_{j\in I\setminus\{i\}} I_j = \emptyset$. Indeed, if $f \in I_i \cap \sum_{j\in I\setminus\{i\}} I_j$, then $f = f_i h = \sum_{k=1}^{m} f_{j_k} h_k$, where $j_1, j_2, \cdots, j_m \in I \setminus \{i\}$ and $h, h_k \in C(X)_{\mathcal{P}}$ for $k = 1, 2, \cdots, m$. This ensures that $f^2 = (f_i h)(\sum_{k=1}^{m} f_{j_k} h_k) = \sum_{k=1}^{m} f_i f_{j_k} h h_k = \mathbf{0}$, since $f_i f_j = \mathbf{0}$ for $i \neq j$ (as $G_i \cap G_j = \emptyset$ for $i \neq j$). Thus, we have a direct sum of ideals in $C(X)_{\mathcal{P}}$, $\bigoplus_{i\in I} I_i$ and so $|I| \leq dim(C(X)_{\mathcal{P}})$. It follows that $c(X_{\mathcal{P}}) \leq dim(C(X)_{\mathcal{P}})$. □

Next, we wish to discuss the compactness of the space of minimal prime ideals of $C(X)_{\mathcal{P}}$.

We initiate this discussion by revisiting some notations and results regarding minimal prime ideals and the space of minimal prime ideals of a commutative ring R.

Lemma 2.5. *The intersection of all minimal prime ideals in R is the set of all nilpotent elements in R.*

Lemma 2.6 ([12, Lemma 3.1])**.** *A prime ideal P of R is minimal if and only if for each $x \in P$, there exists $a \in R \setminus P$ such that ax is a nilpotent element in R.*

We borrow the notation, $\mathscr{P}(R)$ from [11] to denote the family of all minimal prime ideals of R.

Notations 2.7.

- For any subset S of R, the **hull** $h(S)$ of S is the collection of all minimal prime ideals of R that contain the set S.
- For a subfamily $\mathscr{L}$ of $\mathscr{P}(R)$, the **kernel** $k(\mathscr{L})$ is the intersection of all members of $\mathscr{L}$.

It is easy to verify that $\mathscr{P}(R)$ becomes a topological space with the understanding that a set $\mathscr{L} \subseteq \mathscr{P}(R)$ is closed if $\mathscr{L} = h(k(\mathscr{L}))$.

The annihilator of a subset S of R, $Ann(S)$ is defined as $Ann(S) = \{y \in R \colon xy = 0 \text{ for all } x \in S\}$. A ring R is said to be reduced if it does not contain any non-zero nilpotents. Clearly, any subring of $\mathbb{R}^X$ is a reduced ring. A reduced ring R is said to satisfy the annihilator condition, and is said to be an a.c. ring if for $x, y \in R$, there exists $z \in R$ such that $Ann(x) \cap Ann(y) = Ann(z)$ [4]. We observe that for $f, g \in C(X)_{\mathcal{P}}$, $Ann(f) \cap Ann(g) = Ann(f^2 + g^2)$. Therefore, $C(X)_{\mathcal{P}}$ is an a.c. ring.

Theorem 2.8 ([11, Theorem 2.3])**.** *In a reduced ring R, for $a \in R$, $h(a) = \mathscr{P}(R) \setminus h(Ann(a))$.*

Theorem 2.9 ([11, Theorem 2.7])**.** *If R is a reduced ring and $S \subseteq R$, we have:*

$$Ann(S) = kh(Ann(S)) = \bigcap \{P \in \mathscr{P}(R) \colon Ann(S) \subseteq P\}$$

Lemma 2.10 ([11, Lemma 3.1])**.** *For $x, y \in R$, where R is a reduced ring, $Ann(Ann(x)) = Ann(y)$ if and only if $h(x) = h(Ann(y))$.*

Theorem 2.11 ([11, Theorem 3.4])**.** *The following conditions are equivalent for a reduced ring R.*

(1) $\mathscr{P}(R)$ is compact and R is an a.c. ring.

(2) For $x \in R$, there exists $y \in R$ such that $Ann(Ann(y)) = Ann(x)$. (Equivalently, For $x \in R$, there exists $y \in R$ such that $h(y) = h(Ann(x))$.)

We use the above results to see when is $\mathscr{P}(C(X)_{\mathcal{P}})$ compact.

Theorem 2.12. *The space $\mathscr{P}(C(X)_{\mathcal{P}})$ is compact if and only if for each $f \in C(X)_{\mathcal{P}}$ there exists $g \in C(X)_{\mathcal{P}}$ such that $Z_{\mathcal{P}}(f) \cup Z_{\mathcal{P}}(g) = X$ and $int_{X_{\mathcal{P}}}(Z_{\mathcal{P}}(f) \cap Z_{\mathcal{P}}(g)) = \emptyset$.*

Proof. Let us suppose that $\mathscr{P}(C(X)_{\mathcal{P}})$ is compact. Since $C(X)_{\mathcal{P}}$ is an a.c. ring, it follows from Theorem 2.11 that for $f \in C(X)_{\mathcal{P}}$, there exists $g \in C(X)_{\mathcal{P}}$ such that $h(g) = h(Ann(f))$. Let $P \in \mathscr{P}(C(X)_{\mathcal{P}})$ be such that $Ann(f) \subseteq P$. Since $h(Ann(f)) = h(g)$, $P \in h(g)$ and it follows that $g \in P$. By Theorem 2.9, $g \in Ann(f)$ and hence $Z_{\mathcal{P}}(f) \cup Z_{\mathcal{P}}(g) = X$. Next note that $h(g) = h(Ann(f))$ implies that $h(f) \cap h(g) = \emptyset$ since otherwise there exists a minimal prime ideal P of $C(X)_{\mathcal{P}}$ containing both f and g. As $g \in P$, $Ann(f) \subseteq P$. But it follows from Lemma 2.6 that there exists $t \in Ann(f) \setminus P$ which is a not possible. Again, $h(f) \cap h(g) = \emptyset$ implies that $h(f^2 + g^2) = \emptyset$. Indeed, if $P \in h(f^2 + g^2)$, then $f^2 + g^2 \in P$. By Lemma 2.6, there exists $t \notin P$ such that $(f^2 + g^2)t = \mathbf{0}$ and thus, $ft = gt = \mathbf{0}$. By Theorem 2.8 we get that $f, g \in P$ and so $P \in h(f) \cap h(g)$. By Theorem 2.8, $h(Ann(f^2 + g^2)) = \mathscr{P}(C(X)_{\mathcal{P}}) \setminus h(f^2 + g^2)$ where $h(f^2 + g^2) = \emptyset$. So we have $h(Ann(f^2 + g^2)) = \mathscr{P}(C(X)_{\mathcal{P}})$. Thus, $Ann(f^2 + g^2)$ is contained in the intersection of all minimal prime ideals of R. It then follows from Lemma 2.5 and the fact that $C(X)_{\mathcal{P}}$ is a reduced ring that $Ann(f^2 + g^2) = \{\mathbf{0}\}$. It is then clear that $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f^2 + g^2) = \emptyset$ which directly implies that $int_{X_{\mathcal{P}}}(Z_{\mathcal{P}}(f) \cap Z_{\mathcal{P}}(g)) = \emptyset$.

Conversely let $f \in C(X)_{\mathcal{P}}$. Then there exists $g \in C(X)_{\mathcal{P}}$ such that $Z_{\mathcal{P}}(f) \cup Z_{\mathcal{P}}(g) = X$ and $int_{X_{\mathcal{P}}}(Z_{\mathcal{P}}(f) \cap Z_{\mathcal{P}}(g)) = \emptyset$. It is sufficient to show that $h(g) = h(Ann(f))$ and the rest will follow from Theorem 2.12. Indeed, $Z_{\mathcal{P}}(f) \cup Z_{\mathcal{P}}(g) = X$ implies that $fg = \mathbf{0}$, that is $g \in Ann(f)$ and so $h(Ann(f)) \subseteq h(g)$. Next see that $int_{X_{\mathcal{P}}}(Z_{\mathcal{P}}(f) \cap Z_{\mathcal{P}}(g)) = \emptyset$ implies that $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f^2 + g^2) = \emptyset$ and so $Ann(f^2 + g^2) = \{\mathbf{0}\}$. It follows that $Ann(f^2 + g^2)$ is contained in all minimal prime ideals of R and hence $h(Ann(f^2 + g^2)) = \mathscr{P}(C(X)_{\mathcal{P}})$. It is now evident from Theorem 2.8 that $h(f^2 + g^2) = \emptyset$. Now, if $P \in h(g)$, then $P \notin h(f)$. Indeed, if $P \in h(f) \cap h(g)$, then $f, g \in P$ and so $f^2 + g^2 \in P$ which contradicts that $h(f^2 + g^2) = \emptyset$. Finally, it follows from Theorem 2.8 that $P \in h(Ann(f))$. Thus, $h(g) \subseteq h(Ann(f))$. This completes the proof. □

The next corollary follows immediately using Observation 2.2.

Corollary 2.13. *If $C(X)_{\mathcal{P}}$ contains the collection $\{\chi_{\{p\}} : p \in X\}$, then the space $\mathscr{P}(C(X)_{\mathcal{P}})$ is compact if and only if for each $f \in C(X)_{\mathcal{P}}$ there exists $g \in C(X)_{\mathcal{P}}$ such that $Z_{\mathcal{P}}(f) = X \setminus Z_{\mathcal{P}}(g)$.*

3. Zero-divisor graph of $C(X)_{\mathcal{P}}$

The objective of this section is to discuss the zero-divisor graph $\Gamma(C(X)_{\mathcal{P}})$ of the ring $C(X)_{\mathcal{P}}$. We first characterise the zero-divisors of the ring $C(X)_{\mathcal{P}}$.

Theorem 3.1. *An element $f \in C(X)_{\mathcal{P}} \setminus \{\mathbf{0}\}$ is a divisor of zero in the ring $C(X)_{\mathcal{P}}$ if and only if $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \neq \emptyset$.*

Proof. Suppose there exists $g \in C(X)_{\mathcal{P}} \setminus \{\mathbf{0}\}$ such that $fg = \mathbf{0}$. Then $\emptyset \neq X \setminus Z_{\mathcal{P}}(g) \subseteq Z_{\mathcal{P}}(f)$ and so $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \neq \emptyset$. Conversely let $x \in int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f)$. Then there exists $h \in C(X)_{\mathcal{P}}$ such that $x \in X \setminus Z_{\mathcal{P}}(h) \subseteq int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f)$ and so $fh = \mathbf{0}$ where $h \neq \mathbf{0}$. □

The above theorem ensures that the vertex set V of the three graphs which are under study in this article is given by:

$$V = \{f \in C(X)_{\mathcal{P}} : \emptyset \neq int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \subsetneq X\}.$$

We aim to shed light on the distance between two vertices in Γ. To achieve this, we first establish a lemma.

Lemma 3.2. *For $f, g \in V$, there exists a vertex adjacent to both f and g if and only if $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \cap int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g) \neq \emptyset$.*

Proof. Let $x \in int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \cap int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g)$. Then there exists $h \in C(X)_{\mathcal{P}}$ such that $x \in X \setminus Z_{\mathcal{P}}(h) \subseteq int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \cap int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g)$. This ensures that $h \in V$ and h is adjacent to both f and g. Conversely let there exist $h \in V$ adjacent to both f and g. Then it follows that $\emptyset \subsetneq X \setminus Z_{\mathcal{P}}(h) \subseteq int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \cap int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g)$. □

The next theorem can be proven with the help of the above lemma and by following the steps in the proof of Lemma 1.2 in [4].

Theorem 3.3. *For $f, g \in V$,*

(i) $d(f,g) = 1$ *if and only if* $Z_{\mathcal{P}}(f) \cup Z_{\mathcal{P}}(g) = X$,
(ii) $d(f,g) = 2$ *if and only if* $Z_{\mathcal{P}}(f) \cup Z_{\mathcal{P}}(g) \neq X$ *and* $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \cap int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g) \neq \emptyset$,
(iii) $d(f,g) = 3$ *if and only if* $Z_{\mathcal{P}}(f) \cup Z_{\mathcal{P}}(g) \neq X$ *and* $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \cap int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g) = \emptyset$.

In order to calculate the diameter and girth of Γ, we need another lemma.

Lemma 3.4. *If X contains (at least) three distinct elements x, y and z, then there exists $f \in C(X)_{\mathcal{P}}$ such that* $0 = f(x) < f(y) < f(z)$.

Proof. Let $A = \{x, y\}$, $B = \{y, z\}$ and $C = \{z, x\}$. Then $x \in X \setminus B$, which is open in X, and hence in $X_{\mathcal{P}}$. Thus, there exists $g \in C(X)_{\mathcal{P}}$ such that $x \in X \setminus Z_{\mathcal{P}}(g) \subseteq X \setminus B$. Choose $f_x = \frac{1}{g(x)} g$. Then $f_x(x) = 1$ and $f_x(B) = \{0\}$. Similarly, there exist $f_y, f_z \in C(X)_{\mathcal{P}}$ such that $f_y(y) = 2$, $f_z(z) = 3$ and $f_y(C) = 0 = f_z(A)$. Finally, take $f = f_x + f_y + f_z - \mathbf{1}$. □

Theorem 3.5. *If X is a topological space which contains at least three points, then the* $diam(\Gamma) = 3$ *and* $gr(\Gamma) = 3$.

Proof. Let X contain three distinct elements a, b and c. Then by Lemma 3.4, there exists $f \in C(X)_{\mathcal{P}}$ such that $0 = f(a) < f(b) < f(c)$. Choose $r, s \in \mathbb{R}$ such that $0 = f(a) < r < f(b) < s < f(c)$. We then have $a \in \{x \in X \colon f(x) < r\} \subseteq \{x \in X \colon f(x) \leq r\}$ and $c \in \{x \in X \colon f(x) > s\} \subseteq \{x \in X \colon f(x) \geq s\}$ where $\{x \in X \colon f(x) \leq r\}$ and $\{x \in X \colon f(x) \geq s\}$ (resp. $\{x \in X \colon f(x) < r\}$ and $\{x \in X \colon f(x) > s\}$) are zero-sets (resp. cozero-sets) of functions in $C(X)_{\mathcal{P}}$. Therefore, there exist $f_1, g_1 \in C(X)_{\mathcal{P}}$ such that $Z_{\mathcal{P}}(f_1) = \{x \in X \colon f(x) \leq r\}$ and $Z_{\mathcal{P}}(g_1) = \{x \in X \colon f(x) \geq s\}$ and we have $Z_{\mathcal{P}}(f_1) \cap Z_{\mathcal{P}}(g_1) = \emptyset$ and $b \notin Z_{\mathcal{P}}(f_1) \cup Z_{\mathcal{P}}(g_1)$. It follows from Theorem 3.3 that $d(f_1, g_1) = 3$ and hence $diam(\Gamma) = 3$.

Employing the steps used in the proof of Lemma 3.4, there exists $h \in C(X)_{\mathcal{P}}$ such that $h(x) = -1$, $h(y) = 0$ and $h(z) = 1$. Set $Z_{\mathcal{P}}(f_2) = \{t \in X : h(t) \leq \frac{1}{2}\}$ and $Z_{\mathcal{P}}(g_2) = \{t \in X : h(t) \geq \frac{1}{2}\}$ for some $f_2, g_2 \in C(X)_{\mathcal{P}}$. Then $y \in Z_{\mathcal{P}}(f_2) \cap Z_{\mathcal{P}}(g_2)$ which ensures that $Z_{\mathcal{P}}(f_2^2 + g_2^2) \neq \emptyset$. Thus, $f_2, g_2, f_2^2, g_2^2, f_2^2 + g_2^2$ are all non-zero zero-divisors in $C(X)_{\mathcal{P}}$. Thus, there exists $k \in C(X)_{\mathcal{P}}$ such that $k(f_2^2 + g_2^2) = \mathbf{0}$ and so $kf_2 = \mathbf{0} = kg_2$. Also see that $X \setminus Z_{\mathcal{P}}(f_2) \subseteq Z_{\mathcal{P}}(g_2)$ and so $f_2 g_2 = \mathbf{0}$. Therefore, k, f_2, g_2 constitutes a circle in Γ. Thus, $gr(\Gamma) = 3$. □

When $|X| = 2$, the graph Γ is of a particular form.

Theorem 3.6. *The following statements are equivalent.*

1. $|X| = 2$.
2. Γ *is a bipartite graph.*
3. Γ *is a complete bipartite graph.*

Proof. We note that if $X = \{p, q\}$, then the vertex set $V = P \sqcup Q$, where $P = \{c\chi_{\{p\}} : c \in \mathbb{R} \setminus \{0\}\}$ and $Q = \{c\chi_{\{q\}} : c \in \mathbb{R} \setminus \{0\}\}$. It follows that P and Q are stable sets and each vertex in P is adjacent to each vertex in Q. This ensures that Γ is a complete bipartite graph.

Moreover, if $|X| \geq 3$, then it follows from Theorem 3.5 that Γ contains a triangle and thus it cannot be a bipartite graph. □

The following corollary is therefore immediate.

Corollary 3.7. *If X is a space with exactly two points, then* $diam(\Gamma) = 2$ *and* $gr(\Gamma) = \infty$.

The associated number of each vertex is crucial data when calculating the radius of a graph. Keeping this in mind, we calculate the associated number of each vertex of Γ.

Theorem 3.8. *For $f \in V$,* $e(f) = \begin{cases} 2 & if\ X \setminus Z_{\mathcal{P}}(f)\ is\ a\ singleton\ set \\ 3 & otherwise \end{cases}$.

Proof. Let $f \in V$ and $X \setminus Z_{\mathcal{P}}(f) = \{p\}$. Let $g \in V \setminus \{f\}$. If $x \in Z_{\mathcal{P}}(g)$, then $fg = \mathbf{0}$ and hence $d(f, g) = 1$. Again if $x \notin Z_{\mathcal{P}}(g)$, then $Z_{\mathcal{P}}(g) \subseteq Z_{\mathcal{P}}(f)$. This shows that $x \notin Z_{\mathcal{P}}(f) \cup Z_{\mathcal{P}}(g)$ and $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \cap int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g) = int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g) \neq \emptyset$. It follows from Theorem 3.3 that $d(f, g) = 2$. Therefore, $e(f) = 2$.

Now, let there exist distinct points x and y in $X \setminus Z_{\mathcal{P}}(f)$. Then $Z_{\mathcal{P}}(f) \cup \{x\}$ is a closed set in $X_{\mathcal{P}}$ and $y \notin Z_{\mathcal{P}}(f) \cup \{x\}$. Thus, by Theorem 2.1, there exists $h \in C(X)_{\mathcal{P}}$ such that $y \in int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(h) \subseteq Z_{\mathcal{P}}(h) \subseteq X \setminus (Z_{\mathcal{P}}(f) \cup \{x\})$. This ensures that $x \notin Z_{\mathcal{P}}(f) \cup Z_{\mathcal{P}}(h)$ and $Z_{\mathcal{P}}(f) \cap Z_{\mathcal{P}}(h) = \emptyset$. Therefore, we have from Theorem 3.3 that $d(f, g) = 3$ and so $e(f) = 3$. □

Remark 3.9. We realise that if $C(X)_{\mathcal{P}}$ contains at least one f such that it is zero everywhere except exactly a single point, then $\rho(\Gamma) = 2$. Recall that if $\mathcal{P}$ contains all singleton subsets of X (e.g.: $\mathcal{P}_f$ and $\mathcal{K}$), then $C(X)_{\mathcal{P}}$ contains functions of the form $\chi_{\{p\}}$. In such cases, $\rho(\Gamma) = 2$. In particular, $\rho(\Gamma(C(X)_F)) = 2 = \rho(\Gamma(C(X)_K))$ Moreover, recall that the ring $T'(X) = C(X)_{\mathcal{P}_{nd}}$ contains functions of the form $\chi_{\{p\}}$ for each $p \in X$, even though $\mathcal{P}_{nd}$ may not contain all singleton subsets of X. In conclusion, we have $\rho(\Gamma(T'(X))) = 2$.

The above remark prompts us to realise that the existence of $\chi_{\{p\}}$ in $C(X)_{\mathcal{P}}$, for even a single $p \in X$ induces the radius of Γ to be 2. The following corollary is an immediate consequence of this apprehension and can be deduced using Lemma 2.3.

Corollary 3.10. *If $C(X) \subsetneq C(X)_{\mathcal{P}}$, then $\rho(\Gamma) = 2$.*

In the context of the zero-divisor graph being triangulated (resp. hypertriangulated), we first recall that following result established by Azarpanah and Motamedi in [4].

Theorem 3.11 ([4, Proposition 2.1])**.** *Suppose that $|X| > 1$.*

- *(i) $\Gamma(C(X))$ is a triangulated graph if and only if X has no isolated points.*
- *(ii) $\Gamma(C(X))$ is a hypertriangulated graph if and only if X is a connected middle P-space, i.e., if every non-empty zero set $Z \in Z[X]$ of the form $Z = E \cap F$, where $E, F \in Z[X]$ with $E \cup F = X$, has a non-empty interior.*

We now realise when is a vertex in Γ a part of a triangle.

Theorem 3.12. *Let $f \in V$. Then f is a vertex in a triangle if and only if $|X| \geq 3$ and $|int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f)| \geq 2$.*

Proof. Suppose that $|X| \geq 3$ and x and y are two distinct points in $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f)$. Then by Theorem 2.1, there exists $g \in C(X)_{\mathcal{P}}$ such that $x \in X \setminus Z_{\mathcal{P}}(g) \subseteq X \setminus int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g) \subseteq int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \setminus \{y\}$. It is evident that $g \in V$ and $fg = \mathbf{0}$. Also, $y \in int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \cap int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g)$ and so by Lemma 3.2, there exists a vertex $h \in V$ such that h is adjacent to both f and g. Therefore, f, g and h form a triangle.

Conversely, let f be a vertex of a triangle. Then it follows from Theorem 3.6 that $|X| \geq 3$. Now let $g, h \in V$ be such that $fg = \mathbf{0} = gh = hf$. Then $(X \setminus Z_{\mathcal{P}}(g)) \cup (X \setminus Z_{\mathcal{P}}(h)) \subseteq int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f)$. Since $gh = \mathbf{0}$, $(X \setminus Z_{\mathcal{P}}(g)) \cap (X \setminus Z_{\mathcal{P}}(h)) = \emptyset$ and hence $|int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f)| \geq 2$. □

We achieve something more general and concrete when $C(X)_{\mathcal{P}}$ contains $C(X)$ properly.

Theorem 3.13. *If $C(X) \subsetneq C(X)_{\mathcal{P}}$, then Γ is neither triangulated nor hypertriangulated.*

Proof. Since $C(X) \subsetneq C(X)_{\mathcal{P}}$, by Lemma 2.3, there exists a non-isolated $p \in X$ such that $\{p\} \in \mathcal{P}$ and hence $f = \chi_{\{p\}}$ and $g = \chi_{X \setminus \{p\}}$ are in V. It follows from Theorem 3.12 that g is not a vertex of a triangle and the edge joining f and g is not an edge of a triangle. □

In this context, we must make the following remark.

Remark 3.14. It is immediate from Theorem 3.13 that for Γ to be triangulated and/or hypertriangulated, $C(X)_{\mathcal{P}}$ must be equal to $C(X)$. Moreover, we can make the following conclusions by implementing Theorem 3.11.

(i) Γ is triangulated if and only if $C(X)_{\mathcal{P}} = C(X)$ and X has no isolated points.
(ii) Γ is hypertriangulated if and only if $C(X)_{\mathcal{P}} = C(X)$ and X is a connected middle P-space.

In particular, if X is a connected middle P-space with no isolated points, then note that $\Gamma(C(X))$ is both triangulated and hypertriangulated but Γ is neither triangulated nor hypertriangulated, when $C(X)_{\mathcal{P}} \supsetneqq C(X)$. We state an example to support our remark.

Example 3.15. Let s denote the set of all $\{0,1\}$-valued transfinite sequences and the set of all upper elements of s is denoted by Q. Then the Dedekind completion, R of Q (see [10]) is a connected almost P-space (and hence a middle P-space) (see [13]). Thus, $\Gamma(C(R))$ is both triangulated and hypertriangulated; and if $\mathcal{P} \supsetneqq \{\emptyset\}$, then $\Gamma(C(R)_{\mathcal{P}})$ is neither triangulated nor hypertriangulated.

In particular, $\Gamma(C(R)_F)$ (See [14]) (resp. $\Gamma(T'(R))$ and $\Gamma(C(R)_K)$) is neither triangulated nor hypertriangulated.

Next we discuss the length of a cycle in Γ containing two specific vertices $f, g \in V$, which can be proved by following closely the proof of Proposition 2.2 in [4].

Theorem 3.16. *For a topological space X and $f, g \in V$, the following assertions are true.*

(i) $c(f,g) = 3$ *if and only if* $Z_{\mathcal{P}}(f) \cup Z_{\mathcal{P}}(g) = X$ *and* $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \cap int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g) \neq \emptyset$.
(ii) $c(f,g) = 4$ *if and only if* $Z_{\mathcal{P}}(f) \cup Z_{\mathcal{P}}(g) \neq X$ *and* $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \cap int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g) \neq \emptyset$ *or* $Z_{\mathcal{P}}(f) \cup Z_{\mathcal{P}}(g) = X$ *and* $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \cap int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g) = \emptyset$.
(iii) $c(f,g) = 6$ *if and only if* $Z_{\mathcal{P}}(f) \cup Z_{\mathcal{P}}(g) \neq X$ *and* $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \cap int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g) = \emptyset$.

At this point we recall Observation 2.2 to obtain the following remark.

Remark 3.17. If $\{\chi_{\{p\}} : p \in X\} \subseteq C(X)_{\mathcal{P}}$ and $f, g \in C(X)_{\mathcal{P}}$, we have

(i) $c(f,g) = 3$ if and only if $Z_{\mathcal{P}}(f) \cup Z_{\mathcal{P}}(g) = X$ and $Z_{\mathcal{P}}(f) \cap Z_{\mathcal{P}}(g) \neq \emptyset$.
(ii) $c(f,g) = 4$ if and only if either $Z_{\mathcal{P}}(f) \cup Z_{\mathcal{P}}(g) \neq X$ and $Z_{\mathcal{P}}(f) \cap Z_{\mathcal{P}}(g) \neq \emptyset$ or $Z_{\mathcal{P}}(f) \cup Z_{\mathcal{P}}(g) = X$ and $Z_{\mathcal{P}}(f) \cap Z_{\mathcal{P}}(g) = \emptyset$. In particular, if $X \setminus Z_{\mathcal{P}}(f) = Z_{\mathcal{P}}(g)$, then $c(f,g) = 4$.
(iii) $c(f,g) = 6$ if and only if $Z_{\mathcal{P}}(f) \subsetneqq X \setminus Z_{\mathcal{P}}(g)$ or $Z_{\mathcal{P}}(g) \subsetneqq X \setminus Z_{\mathcal{P}}(f)$.

We now realise that clique number of Γ coincides with the cellularity of $X_{\mathcal{P}}$, and hence also with $dim(C(X)_{\mathcal{P}})$.

Theorem 3.18. *For a $\tau\mathcal{P}$-space $(X, \tau, \mathcal{P})$, $c(X_{\mathcal{P}}) = \omega\Gamma = dim(C(X)_{\mathcal{P}})$.*

Proof. Suppose H is a complete subgraph of Γ. Then for $f, g \in H$, $fg = 0$ and so $X \setminus Z_{\mathcal{P}}(f) \cap X \setminus Z_{\mathcal{P}}(g) = \emptyset$. Hence the family, $\{X \setminus Z_{\mathcal{P}}(f) : f \in H\}$ is a collection of pairwise disjoint open sets in $X_{\mathcal{P}}$. It follows that $|H| \leq c(X_{\mathcal{P}})$ and since H is an arbitrary complete subgraph of Γ, $\omega\Gamma \leq c(X_{\mathcal{P}})$

Conversely let $\mathcal{H}$ be a collection of pairwise disjoint non-empty open subsets of $X_{\mathcal{P}}$. For $A \in \mathcal{H}$ and $p \in A$, there exists $h_1 \in C(X)_{\mathcal{P}}$ such that $p \in X \setminus Z_{\mathcal{P}}(h_1) \subseteq A$. Fix $h_A = \frac{1}{h_1(p)} h_1$. Then $h_A(p) = 1$ and $h_A(X \setminus A) = \{0\}$. Let $H = \{h_A : A \in \mathcal{H}\}$. Then each member of H is a non-zero zero-divisor in $C(X)_{\mathcal{P}}$ and for each $A, B \in \mathcal{H}$, $f_A f_B = \mathbf{0}$ as $A \cap B = \emptyset$. Therefore, H is a complete subgraph of Γ. Thus, we have $|\mathcal{H}| = |H| \leq \omega\Gamma$ and as $\mathcal{H}$ is an arbitrary collection of pairwise disjoint non-empty open subsets of $X_{\mathcal{P}}$, $c(X_{\mathcal{P}}) \leq \omega\Gamma$. The last equality follows directly from Theorem 2.4. □

The following remark is obvious from Observation 2.2.

Remark 3.19. If $C(X)_{\mathcal{P}}$ contains the collection $\{\chi_{\{p\}} : p \in X\}$, then $dim(C(X)_{\mathcal{P}}) = \omega\Gamma$ is nothing but the cardinality of X, $|X|$.

In particular, $dim(C(X)_F) = |X| = dim(C(X)_K) = dim(T'(X))$ and $\omega\Gamma(C(X)_F) = |X| = \omega\Gamma(C(X)_K) = \omega\Gamma(T'(X))$.

We now proceed to discuss when is Γ a complemented graph. Recall that two vertices $f, g \in V$ are orthogonal if and only if they are adjacent and there is no vertex $h \in V$ which is adjacent to both f and g. It follows now from Lemma 3.2 and Theorem 3.3(i) that $f, g \in V$ are orthogonal if and only if $Z_{\mathcal{P}}(f) \cup Z_{\mathcal{P}}(g) = X$ and $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \cap int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g) = \emptyset$. The following theorem is now immediate.

Theorem 3.20. *Γ is a complemented graph if and only if for each $f \in V$, there exists $g \in \Gamma$ such that $Z_{\mathcal{P}}(f) \cup Z_{\mathcal{P}}(g) = X$ and $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \cap int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g) = \emptyset$.*

The following corollary is now immediate with the assistance of Theorem 2.12.

Corollary 3.21. *The zero-divisor graph Γ of the ring $C(X)_{\mathcal{P}}$ is a complemented graph if and only if the space of minimal prime ideals $\mathscr{P}(C(X)_{\mathcal{P}})$, of $C(X)_{\mathcal{P}}$ is compact.*

An interesting characterisation of $C(X)_{\mathcal{P}}$ as a Von-Neumann regular ring can be achieved using the above theories, provided $C(X)_{\mathcal{P}}$ contains the collection $\{\chi_{\{p\}} : p \in X\}$. A ring R is said to be Von-Neumann regular if for each $a \in R$, there exists $x \in R$ such that $a = axa$. We first recall the following result.

Theorem 3.22 ([7, Theorem 5.7])**.** *A $\tau\mathcal{P}$-space, $(X, \tau, \mathcal{P})$ is a $\mathcal{P}P$-space (i.e., $C(X)_{\mathcal{P}}$ is a Von-Neumann regular ring) if and only if for $f \in C(X)_{\mathcal{P}}$, there exists $g \in C(X)_{\mathcal{P}}$ with $Z_{\mathcal{P}}(f) = X \setminus Z_{\mathcal{P}}(g)$.*

The next observation follows directly by using the above discussions and Observation 2.2.

Theorem 3.23. *Consider a $\tau_{\mathcal{P}}$-space $(X, \tau, \mathcal{P})$ such that $C(X)_{\mathcal{P}} \supseteq \{\chi_{\{p\}} : p \in X\}$. Then the following statements are equivalent.*

(1) The zero-divisor graph Γ of $C(X)_{\mathcal{P}}$ is a complemented graph.
(2) The space of minimal prime ideals $\mathscr{P}(C(X)_{\mathcal{P}})$ of $C(X)_{\mathcal{P}}$ is compact.
(3) $C(X)_{\mathcal{P}}$ is a Von-Neumann regular ring.

Remark 3.24. Since the rings $C(X)_F$ [9], $C(X)_K$ [7] and $T'(X)$ [2] all contain the collection $\{\chi_{\{p\}} : p \in X\}$, Theorem 3.23 is valid for each of the aforementioned rings. Moreover, the ring $T'(X)$ is a Von-Neumann regular ring, for any topological space X and hence it follows that $\Gamma(T'(X))$ is always a complemented graph and the space of minimal prime ideals of $T'(X)$ is compact, for any topological space X.

4. Annihilator graph on $C(X)_{\mathcal{P}}$

In this section, we discuss the annihilator graph on the ring $C(X)_{\mathcal{P}}$. For simplicity, we write AG instead of $AG(C(X)_{\mathcal{P}})$ when there is no ambiguity. Note that Γ is a spanning subgraph of AG. We begin with a lemma, which was established by us in [8] and sketch its proof to make this article self-contained.

Lemma 4.1. *Let $f, g \in C(X)_{\mathcal{P}}$. Then $int_{X_{\mathcal{P}}}(Z_{\mathcal{P}}(f)) \subseteq int_{X_{\mathcal{P}}}(Z_{\mathcal{P}}(g))$ if and only if $Ann(f) \subseteq Ann(g)$.*

Proof. Suppose $int_{X_{\mathcal{P}}}(Z_{\mathcal{P}}(f)) \subseteq int_{X_{\mathcal{P}}}(Z_{\mathcal{P}}(g))$ and $h \in Ann(f)$. Then we have $X \setminus Z_{\mathcal{P}}(h) \subseteq int_{X_{\mathcal{P}}}(Z_{\mathcal{P}}(f)) \subseteq int_{X_{\mathcal{P}}}(Z_{\mathcal{P}}(g)) \subseteq Z_{\mathcal{P}}(g)$ and so $h \in Ann(g)$. Conversely let $x \in int_{X_{\mathcal{P}}}(Z_{\mathcal{P}}(f))$. Then there exists $h \in C(X)_{\mathcal{P}}$ such that $x \in X \setminus Z_{\mathcal{P}}(h) \subseteq int_{X_{\mathcal{P}}}(Z_{\mathcal{P}}(f))$ which implies that $h \in Ann(f) \subseteq Ann(g)$ and so $x \in X \setminus Z_{\mathcal{P}}(h) \subseteq int_{X_{\mathcal{P}}}(Z_{\mathcal{P}}(g))$. □

Now, we provide a way to check when two vertices in AG are adjacent to each other.

Theorem 4.2. *Let $f, g \in V$, then f and g are adjacent in AG if and only if $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \nsubseteq int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g)$ and $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g) \nsubseteq int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f)$.*

Proof. Let $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \subseteq int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g)$. Then it can be easily observed that $Ann(fg) = Ann(f) \cup Ann(g)$. Indeed, if $h \in Ann(fg)$, then $X \setminus Z(gh) \subseteq int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \subseteq int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g) \subseteq Z_{\mathcal{P}}(g) \subseteq Z_{\mathcal{P}}(fg)$, which is possible only if $Z_{\mathcal{P}}(gh) = X$, that is $gh = \mathbf{0}$ and $h \in Ann(g) \subseteq Ann(f) \cup Ann(g)$. Therefore, f and g are non-adjacent. Analogously, it can be shown that if $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g) \subseteq int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f)$, then also f and g are non-adjacent.

Conversely let f and g be non-adjacent. Then $Ann(f) \cup Ann(g) = Ann(fg)$. Since the union of two ideals is an ideal if and only if one contains the other, it follows that $Ann(f) \subseteq Ann(g)$ or $Ann(g) \subseteq Ann(f)$. The conclusion then follows from Lemma 4.1. □

The next question that we address is when the graph AG equals Γ.

Theorem 4.3. *The graphs AG and Γ coincide if and only if $|X| \leq 2$.*

Proof. We first aim to show that if X contains three distinct points, then AG and Γ are unequal as graphs. Let x, y, z be three distinct points in X. Then, by Theorem 2.1, there exist $f, g \in C(X)_{\mathcal{P}}$ such that $\{x, y\} \subseteq X \setminus Z_{\mathcal{P}}(f) \subseteq cl_{X_{\mathcal{P}}}(X \setminus Z_{\mathcal{P}}(f)) \subseteq X \setminus \{z\}$ and $\{x, z\} \subseteq X \setminus Z_{\mathcal{P}}(g) \subseteq cl_{X_{\mathcal{P}}}(X \setminus Z_{\mathcal{P}}(g)) \subseteq X \setminus \{y\}$. It then follows that $y \in int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g) \setminus int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f)$ and $z \in int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \setminus int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g)$. Therefore, f and g are adjacent in AG, but non-adjacent in Γ, as $x \notin (Z_{\mathcal{P}}(f) \cup Z_{\mathcal{P}}(g))$. Now if $X = \{p, q\}$, then $V = P \sqcup Q$, where $P = \{c\chi_{\{p\}} : c \in \mathbb{R} \setminus \{0\}\}$ and $Q = \{d\chi_{\{q\}} : d \in \mathbb{R} \setminus \{0\}\}$. See that $Ann(c\chi_{\{p\}}) = Ann(\chi_{\{p\}})$ for any $c \in \mathbb{R} \setminus \{0\}$ and hence P is a stable set in AG. Analogously, Q is a stable set in AG. The rest follows from Theorem 3.6 and the fact that Γ is a spanning subgraph of AG. $\square$

The next corollary follows immediately using Theorem 3.6.

Corollary 4.4. *AG is a complete bipartite graph if and only if $|X| = 2$.*

Our next aim is to compute the diameter of AG. We first establish a result describing the existence of a vertex adjacent to two distinct vertices in AG.

Theorem 4.5. *Let $f, g \in V$. Then there exists a vertex adjacent to both f and g in AG if and only if either $fg \neq \mathbf{0}$ on X or $|int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f)| \geq 2$ and $|int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g)| \geq 2$.*

Proof. First suppose that $fg \neq \mathbf{0}$ and choose $x \in X \setminus Z_{\mathcal{P}}(fg)$. By Theorem 2.1, there exists $h \in C(X)_{\mathcal{P}}$ such that $x \in int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(h) \subseteq X \setminus Z_{\mathcal{P}}(fg) \subsetneq X$. It is clear that $h \in V$ and is adjacent to both f and g in AG. Now, let $fg = \mathbf{0}$ and $|int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f)| \geq 2$ and $|int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g)| \geq 2$. If $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \cap int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g) \neq \emptyset$, then there exists a vertex adjacent to f and g in Γ, and hence in AG. If not, then there exist distinct points $x, y \in int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f)$ and $z, w \in int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g)$. By Theorem 2.1, there exists $h \in C(X)_{\mathcal{P}}$ such that $\{x, z\} \subseteq int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(h) \subseteq X \setminus \{y, w\}$. It follows that $h \in V$ and is adjacent to both f and g.

Finally, assume that $fg = \mathbf{0}$ and $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) = \{x\}$. Then $Z_{\mathcal{P}}(g) = X \setminus \{x\}$, which is open in $X_{\mathcal{P}}$ and so $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g) = X \setminus \{x\}$. Let $h \in V$ be adjacent to g in AG. Then $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g) \not\subseteq int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(h)$ which implies that $x \in int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(h)$. But $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) = \{x\} \subseteq int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(h)$, which implies that f and h are non-adjacent. $\square$

We obtain the following result as a corollary.

Corollary 4.6. *The distance between two non-adjacent vertices in AG is 2.*

Proof. Let $f, g \in V$ be non-adjacent in AG. Then they are also non-adjacent in Γ. Therefore, $fg \neq \mathbf{0}$ on X. It follows from Theorem 4.5 that there exists $h \in V$, which is adjacent to both f and g. It follows that $d(f, g) = 2$. $\square$

It is also evident from Theorem 4.5 that if $f \in V$, then $d(f, 2f) = 2$ and it follows from Corollary 4.6 that $e(f) \leq 2$ for any $f \in V$. Consequently, the following conclusions can be drawn about the graph AG.

Theorem 4.7. *In the graph AG, the following assertions hold.*

1. $e(f) = 2$ *for any* $f \in V$.
2. $\rho(AG) = 2$
3. *Each vertex* $f \in V$ *is a center.*
4. $diam(AG) = 2$

Again, if f and g are vertices of an edge in AG which is not part of a triangle, then by Theorem 4.5, $fg = \mathbf{0}$ on X and either $int_{X_\mathcal{P}} Z_\mathcal{P}(f)$ or $int_{X_\mathcal{P}} Z_\mathcal{P}(g)$ is a singleton set. In light of these arguments, we establish the following result.

Theorem 4.8. *AG is hypertriangulated if and only if* $C(X)_\mathcal{P} = C(X)$ *and X has no isolated points.*

Proof. Suppose that $C(X) \subsetneq C(X)_\mathcal{P}$. Then there exists a point p in X such that $\{p\} \in \mathcal{P}$. Therefore, $\chi_{\{p\}}$ and $\chi_{X\setminus\{p\}}$ constitutes an edge in AG and by Theorem 4.5, this edge is not part of a triangle. Now, let p be an isolated point in X. Then $\chi_{\{p\}}$ and $\chi_{X\setminus\{p\}}$ are in $C(X)$ and again $\chi_{\{p\}}$ and $\chi_{X\setminus\{p\}}$ constitutes an edge which is not a part of a triangle.

Conversely let AG be not hypertriangulated. Then there exists an edge joining vertices f and g which is not part of a triangle. Then by Theorem 4.5, $fg = \mathbf{0}$ and either $int_{X_\mathcal{P}} Z_\mathcal{P}(f)$ or $int_{X_\mathcal{P}} Z_\mathcal{P}(g)$ is a singleton set. Without loss of generality let $int_{X_\mathcal{P}} Z_\mathcal{P}(f) = \{p\}$. Then there exists a basic open set $X \setminus Z_\mathcal{P}(h)$ in $X_\mathcal{P}$ such that $p \in X \setminus Z_\mathcal{P}(h) \subseteq int_{X_\mathcal{P}} Z_\mathcal{P}(f) = \{p\}$ and so $X \setminus Z_\mathcal{P}(h) = \{p\}$. But $h = h(p)\chi_{\{p\}}$ and so $\chi_{\{p\}} \in C(X)_\mathcal{P}$. Therefore, either p is an isolated point in X or $\chi_{\{p\}} \in C(X)_\mathcal{P} \setminus C(X)$. □

The next result, which was already established in [15], also follows from Theorem 4.8.

Corollary 4.9. $AG(C(X))$ *is hypertriangulated if and only if X has no isolated points.*

We now address when is AG a triangulated graph.

Theorem 4.10. *The graph AG is triangulated if and only if* $|X| \geq 3$.

Proof. Suppose that $|X| \geq 3$ and let $f \in V$. If $int_{X_\mathcal{P}} Z_\mathcal{P}(f)$ is a singleton set, $\{x\}$, then there exist distinct points $y, z \in X \setminus \{x\}$. By Theorem 2.1, there exist $g, h \in C(X)_\mathcal{P}$ such that $z \in int_{X_\mathcal{P}} Z_\mathcal{P}(g) \subseteq X \setminus \{x, y\}$ and so $g \in V$ with f and g adjacent in AG. Analogously, there exists $h \in V$ with $y \in int_{X_\mathcal{P}} Z_\mathcal{P}(h) \subseteq X \setminus \{x, z\}$ which is adjacent to f as well. Moreover, $y \in int_{X_\mathcal{P}} Z_\mathcal{P}(h) \setminus int_{X_\mathcal{P}} Z_\mathcal{P}(g)$ and $z \in int_{X_\mathcal{P}} Z_\mathcal{P}(g) \setminus int_{X_\mathcal{P}} Z_\mathcal{P}(h)$, which ensures that g and h are also adjacent. Thus, f, g and h constitutes a triangle in AG. Now, if

$|int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(f)| \geq 2$, then by Theorem 3.12 f is a vertex of a triangle in Γ and the rest follows from the fact that Γ is a spanning subgraph of AG. The converse follows from Corollary 4.4. □

We next address the question of complementation of the graph AG.

Theorem 4.11. *The following assertions are true for the graph AG.*

1. *A pair of vertices $f, g \in V$ are orthogonal if and only if $fg = \mathbf{0}$ on X and either $int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(f)$ or $int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(g)$ is singleton.*
2. *AG is a complemented graph if and only if $|X| \leq 3$.*

Proof.

1. If $f \perp g$, then f and g are adjacent and there does not exist any vertex adjacent to both. By Theorem 4.5, $fg = \mathbf{0}$ and either $int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(f)$ or $int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(g)$ is singleton. Conversely let $f, g \in V$ be such that $fg = \mathbf{0}$ and either $int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(f)$ or $int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(g)$ is singleton. Therefore, f and g are adjacent in Γ, and hence in AG and by Theorem 4.5, there is no vertex adjacent to both f and g.
2. By Corollary 4.4, AG is complemented if $|X| = 2$. Let $|X| = 3$ and $f \in V$. Then either $Z_{\mathcal{P}}(f)$ or $X \setminus Z_{\mathcal{P}}(f)$ is singleton. It follows that $\chi_{Z_{\mathcal{P}}(f)} \in V$ and by the above result, $f \perp \chi_{Z_{\mathcal{P}}(f)}$. Finally, let $|X| \geq 4$ and x, y, z, w be four distinct points in X. By Theorem 2.1, there exists $f \in V$ such that $\{x, y\} \subseteq int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(f) \subseteq Z_{\mathcal{P}}(f) \subseteq X \setminus \{z, w\}$. If possible let there exist $g \in V$ such that $f \perp g$. Then $X \setminus Z_{\mathcal{P}}(f) \subseteq int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(g)$ which implies that $\{z, w\} \subseteq int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(g)$. This contradicts that $f \perp g$.

□

Theorem 4.12. *$\{f, g\} \subseteq V$ is a dominating set in AG if and only if $fg = \mathbf{0}$ on X and $int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(f) \cap int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(g) = \emptyset$.*

Proof. Suppose $\{f, g\}$ constitutes a dominating set in AG. If $fg \neq \mathbf{0}$, then $fg \in V$ and is non-adjacent to both f and g. If $int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(f) \cap int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(g) \neq \emptyset$, then $f^2 + g^2 \in V$ and is non-adjacent to both f and g.

Conversely let there exist $h \in V$ which is non-adjacent to both f and g in AG. Then there are four possibilities:

(i) $int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(f) \subsetneq int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(h) \subsetneq int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(g)$
(ii) $int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(g) \subsetneq int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(h) \subsetneq int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(f)$
(iii) $int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(h) \subsetneq int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(f)$ and $int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(h) \subsetneq int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(g)$
(iv) $int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(f) \subsetneq int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(h)$ and $int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(g) \subsetneq int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(h)$

The first three possibilities contradict the fact that $int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(f) \cap int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(g) = \emptyset$. Now, assume the fourth possibility. Since $fg = \mathbf{0}$, $X \setminus Z_{\mathcal{P}}(f) \subseteq int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(g) \subsetneq int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(h)$ and $X \setminus Z_{\mathcal{P}}(g) \subseteq int_{X_{\mathcal{P}}}Z_{\mathcal{P}}(f) \subsetneq$

$int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(h)$. Thus, $X \setminus Z_{\mathcal{P}}(f) \cup X \setminus Z_{\mathcal{P}}(g) \subseteq int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(h)$ and so, $X \setminus Z_{\mathcal{P}}(h) \subseteq X \setminus int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(h) \subseteq Z_{\mathcal{P}}(f) \cap Z_{\mathcal{P}}(g)$. Therefore, $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \cap int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g) \neq \emptyset$, which contradicts the hypothesis. □

We see that if $C(X) \subsetneq C(X)_{\mathcal{P}}$, then there exists a point $p \in X$ such that $\chi_{\{p\}}, \chi_{X \setminus \{p\}} \in C(X)_{\mathcal{P}}$ and the same thing happens when p is an isolated point in X. This observation leads to the following corollary.

Corollary 4.13. *$dt(AG) = 2$ if either $C(X) \subsetneq C(X)_{\mathcal{P}}$ or X has an isolated point.*

The following corollary follows from Lemma 3.2, Theorem 4.12 and Corollary 4.13.

Corollary 4.14. *$dt(AG) = 2$ if and only if Γ is not hypertriangulated.*

5. Weakly zero-divisor graph on $C(X)_{\mathcal{P}}$

The aim of this section is to study the weakly zero-divisor graph $W\Gamma$ of $C(X)_{\mathcal{P}}$. The following observation has been discussed in the introduction, in a more general setting. We state it explicitly for ease of reference.

Observation 5.1. If $\leq$ denotes the spanning subgraph relation between two graphs over the same vertex set, then we have:

$$\Gamma \leq AG \leq W\Gamma.$$

We next realise that the weakly zero-divisor graph and zero-divisor graph of $C(X)_{\mathcal{P}}$ coincide if and only if X contains exactly two points. To see this, we first check what happens when $|X| = 2$.

Theorem 5.2. *If $|X| = 2$, $W\Gamma$ is a complete bipartite graph and in this case, the graphs $W\Gamma$, AG and Γ are equal.*

Proof. If $X = \{p, q\}$, then $V = P \sqcup Q$, where $P = \{c\chi_{\{p\}} : c \in \mathbb{R} \setminus \{0\}\}$ and $Q = \{c\chi_{\{q\}} : c \in \mathbb{R} \setminus \{0\}\}$. Note that for any $f \in P$, $Ann(f) = Q \cup \{\mathbf{0}\}$ and for $g \in Q$, $Ann(g) = P \cup \{\mathbf{0}\}$ and thus P and Q are stable sets in $W\Gamma$. Therefore, $W\Gamma$ is a complete bipartite graph. Moreover, it has been noted in Theorem 3.6 and Corollary 4.4 that the same structure is exhibited by Γ and AG as well. Therefore, the three graphs coincide. □

Note that a complete bipartite graph is uniquely complemented. This gives us the following corollary.

Corollary 5.3. *If $|X| = 2$, $W\Gamma$ is a uniquely complemented graph.*

The following corollary follows from Observation 5.1, Theorem 5.2 and Theorem 4.3.

Corollary 5.4. *The graphs $W\Gamma$ and Γ coincide if and only if $|X| = 2$.*

In order to explore the condition under which AG coincides with $W\Gamma$, we establish the adjacency relation between two vertices in $W\Gamma$.

Theorem 5.5. *Two distinct vertices $f, g \in V$ are adjacent in $W\Gamma$ if and only if $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \cup int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g)$ contains at least two points.*

Proof. First note that since f and g are zero-divisors, $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \neq \emptyset$ and $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g) \neq \emptyset$. Now, assume that $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \cup int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g)$ contains exactly one point p. Then $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) = int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g) = \{p\}$. If possible let there exist $h \in Ann(f) \setminus \{\mathbf{0}\}$ and $k \in Ann(g) \setminus \{\mathbf{0}\}$ such that $hk = \mathbf{0}$. Then $\emptyset \neq X \setminus Z_{\mathcal{P}}(h) \subseteq int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f)$ and $\emptyset \neq X \setminus Z_{\mathcal{P}}(k) \subseteq int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g)$ where $X \setminus Z_{\mathcal{P}}(h) \cap X \setminus Z_{\mathcal{P}}(k) = \emptyset$; which is impossible as $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) = int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g)$ is a singleton set. Therefore, no such h and k can exist and so f and g are not adjacent.

Let us now suppose that $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \cup int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g)$ contains at least two distinct points. Choose $x \in int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f)$ and $y \in int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g)$ such that $x \neq y$. By Theorem 2.1, there exist $h, k \in C(X)_{\mathcal{P}}$ such that $x \in X \setminus Z_{\mathcal{P}}(h) \subseteq cl_{X_{\mathcal{P}}}(X \setminus Z_{\mathcal{P}}(h)) \subseteq int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \setminus \{y\}$. Again $y \in int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g) \setminus cl_{X_{\mathcal{P}}}(X \setminus Z_{\mathcal{P}}(h))$ and so there exists $k \in C(X)_{\mathcal{P}}$ such that $y \in X \setminus Z_{\mathcal{P}}(k) \subseteq int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g) \setminus cl_{X_{\mathcal{P}}}(X \setminus Z_{\mathcal{P}}(h))$. Therefore, we have $h \in Ann(f) \setminus \{\mathbf{0}\}$, $k \in Ann(g) \setminus \{\mathbf{0}\}$ and $hk = \mathbf{0}$. Hence, f and g are adjacent vertices. □

Corollary 5.6. *If $f \in V$ is such that $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f)$ contains at least two distinct points, then f is adjacent to all other vertices in $W\Gamma$. That is, f is a universal vertex.*

By using Theorem 5.5, we realise when the graph $W\Gamma$ coincides with the graph AG.

Theorem 5.7. *The graphs AG and $W\Gamma$ coincide if and only if $|X| \leq 2$.*

Proof. Suppose x, y, z are three distinct points in X. By Theorem 2.1, there exists $f \in C(X)_{\mathcal{P}}$ such that $\{x, y\} \subseteq X \setminus Z_{\mathcal{P}}(f) \subseteq cl_{X_{\mathcal{P}}}(X \setminus Z_{\mathcal{P}}(f)) \subseteq X \setminus \{z\}$. Moreover, $Z_{\mathcal{P}}(f) \cup \{y\}$ is a closed set in $X_{\mathcal{P}}$, which misses x. Again, by Theorem 2.1, there exists $g \in C(X)_{\mathcal{P}}$ such that $x \in X \setminus Z_{\mathcal{P}}(g) \subseteq cl_{X_{\mathcal{P}}}(X \setminus Z_{\mathcal{P}}(g)) \subseteq X \setminus (Z_{\mathcal{P}}(f) \cup \{y\})$. Therefore, $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \subseteq Z_{\mathcal{P}}(f) \subseteq int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g)$ and so f and g are not adjacent in AG. However, $\{y, z\} \subseteq int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \cup int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g)$, where $y \neq z$ and so f and g adjacent in $W\Gamma$. Thus, AG is a proper (spanning) subgraph of $W\Gamma$. The rest follows from Observation 5.1 and Corollary 5.4.

□

Observe that if $f, g \in V$ are such that $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) = int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g) = \{p\}$ for some $p \in X$, then p is an isolated point in $X_{\mathcal{P}}$. Therefore, there exists a basic open set $X \setminus Z_{\mathcal{P}}(h)$ for some $h \in C(X)_{\mathcal{P}}$ such that $X \setminus Z_{\mathcal{P}}(h) = \{p\}$ and hence $fh = gh = \mathbf{0}$ on X. Therefore, h is adjacent to both f and g in $W\Gamma$. In this case, $d(f, g) = 2$. Consequently, we have the following result.

Theorem 5.8. *For two distinct vertices $f, g \in V$,*

$$d(f,g) = \begin{cases} 1 & if\ |int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \cup int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g)| \geq 2 \\ 2 & if\ |int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \cup int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g)| = 1 \end{cases},$$

Corollary 5.9. $diam(W\Gamma) = \begin{cases} 1 & \text{if } X_{\mathcal{P}} \text{ has no isolated points} \\ 2 & otherwise \end{cases}.$

Moreover, it follows from Corollary 5.6 that if there exists $f \in C(X)_{\mathcal{P}}$ with $|int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f)| \geq 2$, then $\{f\}$ is a dominating set. If no such f exists, then $|X| = 2$ and hence $W\Gamma$ is a complete bipartite graph. Thus, we have the following result.

Theorem 5.10. *The dominating number of $W\Gamma$ is given by:*

$$dt(W\Gamma) = \begin{cases} 1 & if\ |X| \geq 3 \\ 2 & otherwise \end{cases}.$$

We now establish when this graph is a complete graph.

Theorem 5.11. *$W\Gamma$ is a complete graph if and only if X has no isolated points and $C(X)_{\mathcal{P}} = C(X)$.*

Proof. If $p \in X$ is an isolated point in X, then $f = \chi_{X\setminus\{p\}}$ and $2f$ are in V and are non-adjacent. Now, let $f \in C(X)_{\mathcal{P}} \setminus C(X)$. Then by Lemma 2.3, $\{p\} \in \mathcal{P}$. This implies that the functions $f = \chi_{X\setminus\{p\}}$ and $2f$ are in V and are non-adjacent. Therefore, $W\Gamma$ is not a complete graph. If X has no isolated points and $C(X)_{\mathcal{P}} = C(X)$, then for each $f \in V$, $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f)$ consists of at least two points and so is adjacent to all other vertices. Therefore, $W\Gamma$ is a complete graph. □

Examples 5.12. The weakly zero-divisor graphs of the well-known rings $C(X)_F$, $C(X)_K$ and $T'(X)$ are not complete. On the other hand, if X is a connected topological space, then $W\Gamma(C(X))$ is a complete graph.

Moreover, the following corollary is immediate from Remark 3.14 and Theorem 4.8.

Corollary 5.13. *The following statements are equivalent.*

1. *X has no isolated points and $C(X)_{\mathcal{P}} = C(X)$.*
2. *Γ is triangulated.*
3. *AG is a hypertriangulated graph.*
4. *$W\Gamma$ is a complete graph.*

Next we proceed to discuss when is $W\Gamma$ a triangulated graph.

Theorem 5.14. *The following assertions are equivalent.*

(a) $|X| \geq 3$

(b) $W\Gamma$ *is triangulated.*

(c) $W\Gamma$ *is hypertriangulated.*

Proof. If $|X| = 2$, then we have seen that $W\Gamma$ is a complete bipartite graph, and so is neither triangulated nor hypertriangulated.

Let $|X| \geq 3$ and $f \in V$. If $|int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f)| \geq 2$, then f, $2f$ and $3f$ constitute a triangle. Suppose $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) = \{x\}$, for some $x \in X$. Choose $y, z \in X$ such that x, y, z are distinct. Then there exist $g, h \in C(X)_{\mathcal{P}}$ such that $y \in int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g)$ and $z \in int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(h)$, by Theorem 2.1 and Theorem 5.5, f, g and h constitute a triangle. Again see that if $f, g \in V$ form an edge, then $int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f) \cup int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g)$ contains atleast two points. If $x \in int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(f)$ and $y \in int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(g)$ with $x \neq y$, then choose $z \in X \setminus \{x, y\}$ and $h \in C(X)_{\mathcal{P}}$ such that $z \in int_{X_{\mathcal{P}}} Z_{\mathcal{P}}(h)$. It then follows that f, g and h are vertices of a triangle. □

As a hypertriangulated graph cannot be complemented, the following result follows from Corollary 5.3.

Corollary 5.15. $W\Gamma$ *is (uniquely) complemented if and only if* $|X| \leq 2$.

The following conclusions can be drawn about the weakly zero-divisor graph $W\Gamma(C(X))$ of the ring $C(X)$.

Theorem 5.16. *The following assertions hold for a Tychonoff space X.*

1. *Two vertices* f, g *in* $W\Gamma(C(X))$ *are adjacent if and only if* $int\ Z(f) \cup int\ Z(g)$ *contains at least two distinct points.*
2. *A vertex* $f \in W\Gamma(C(X))$ *is a universal vertex if and only if* $int\ Z(f)$ *contains at least two distinct points.*
3. *For* $f, g \in W\Gamma(C(X))$, $d(f, g) = \begin{cases} 1 & if\ |int\ Z(f) \cup int\ Z(g)| \geq 2 \\ 2 & if\ |int\ Z(f) \cup int\ Z(g)| = 1 \end{cases}$.
4. $diam(W\Gamma(C(X))) = \begin{cases} 1 & \text{if } X \text{ has no isolated points} \\ 2 & \text{otherwise} \end{cases}$.
5. $W\Gamma(C(X))$ *is a complete graph if and only if X has no isolated points.*
6. *The following statements are equivalent.*
   (i) $W\Gamma(C(X))$ *is triangulated.*
   (ii) $W\Gamma(C(X))$ *is hypertriangulated.*
   (iii) $W\Gamma(C(X))$ *is not complemented.*
   (iv) $|X| \geq 3$.

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