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T_i-reflections and the Hartman–Mycielski construction in semitopological and paratopological groups

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Abstract

We show that, given a semitopological group G, the semitopological groups $T_i(G^\bullet)$ and $(T_i(G))^\bullet$, for each $i \in \{0, 1, 2\}$, and $(G_{sr})^\bullet$ and $(G^\bullet)_{sr}$, are topologically isomorphic. Moreover, we prove that, given a paratopological group G, the paratopological groups $Reg(G^\bullet)$ and $(Reg(G))^\bullet$ are topologically isomorphic.

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1. Introduction

Given a topological (paratopological, semitopological) group G, we can construct other topological (paratopological, semitopological) groups associated with G, possibly with some improvements in their topological properties. In [2], S. Hartman and J. Mycielski presented a functorial construction that permits to embed a Hausdorff topological group into a path connected, locally path connected Hausdorff topological group. A brief outline of their construction is given below.

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Let G be a topological group with identity e. Consider the set $G^\bullet$ of all functions f on $J = [0, 1)$ with values in G such that, for some sequence $0 = a_0 < a_1 < \cdots < a_n = 1$, the function f is constant on $[a_k, a_{k+1})$ for each $k = 0, 1, \ldots, n-1$.

Define a binary operation $*$ on $G^\bullet$ by $(f * g)(x) = f(x)g(x)$, for all $f, g \in G^\bullet$ and $x \in J$. Each element $f \in G^\bullet$ has a unique inverse $f^{-1} \in G^\bullet$ defined by $(f^{-1})(x) = (f(x))^{-1}$, for each $x \in J$. Then $(G^\bullet, *)$ is a group with identity $e^\bullet$, where $e^\bullet(x) = e$ for each $x \in J$.

Let V be an open neighborhood of e in G and ϵ a positive real number. Consider the subset

$$O(V, \epsilon) = \{f \in G^\bullet : \mu(\{x \in J : f(x) \notin V\}) < \epsilon\}$$

of $G^\bullet$, where μ is the usual Lebesgue measure on J. One can verify that the subsets $O(V, \epsilon)$ form a base of a Hausdorff topological group topology at the identity of $G^\bullet$, thus making $G^\bullet$ into a pathwise connected and locally pathwise connected topological group. For details, the reader is referred to [2].

Let $f \in G^\bullet$. Then there exist real numbers $a_0, a_1, \ldots, a_n$ with $0 = a_0 < a_1 < \cdots < a_n = 1$ such that f is constant on $[a_k, a_{k+1})$ for each $k = 0, 1, \ldots, n-1$. Assuming that $f(a_k) \neq f(a_{k+1})$ for each $k \leq n-1$, we put $P(f) = \{a_0, a_1, \ldots, a_{n-1}\}$, $\Delta(f) = \min\{a_{k+1} - a_k : 0 \leq k < n\}$ and $V(f) = \{f(a_0), \ldots, f(a_{n-1})\}$. For the identity $e_G^\bullet$ of the group $G^\bullet$ we put $P(e_G^\bullet) = \{a_0\}$.

It is easy to verify that in the Hartman–Mycielski construction, if G is a paratopological (semitopological) group, then $G^\bullet$ is also a paratopological (semitopological) group. For more information on the Hartman–Mycielski construction, see [1, Section 3.8] and [4]. In what follows, $G^\bullet$ refers to the Hartman–Mycielski group associated to G and $\mathcal{N}(e)$ to the family of all open neighborhoods of the identity in G.

In [8], M. Tkachenko defines what we will call the *T_i-reflection*, $T_i(G)$, of an arbitrary semitopological group G for $i \in \{0, 1, 2, 3, 3.5\}$, as well as the *regular reflection*, $Reg(G)$, of G. All these objects are new semitopological groups, and their names indicate that $T_i(G)$ satisfies the T_i separation axiom for $i \in \{0, 1, 2, 3, 3.5\}$, whereas $Reg(G)$ is regular, i.e., satisfies the T_1 and T_3 separation axioms.

In this paper we show that, in the category of semitopological groups, the functors $^\bullet$ and T_i for $i \in \{0, 1, 2\}$ commute, that is, the semitopological groups $T_i(G^\bullet)$ and $(T_i(G))^\bullet$ for $i \in \{0, 1, 2\}$ are topologically isomorphic. Likewise, we prove that $(G_{sr})^\bullet$ and $(G^\bullet)_{sr}$ are topologically isomorphic. Finally, we show that in the category of paratopological groups, the functors Reg and $^\bullet$ commute, that is, the paratopological groups $Reg(G^\bullet)$ and $(Reg(G))^\bullet$ are topologically isomorphic.

First, we present some simple but very important properties of the Hartman–Mycielski group $G^\bullet$ associated to a semitopological group G. We omit their proofs, since they are identical to those found in [1, Theorem 3.8.3], [1, Proposition 3.8.6], and [1, Proposition 3.8.7], respectively.

For each $g \in G$, denote by $g^\bullet$ the element of $G^\bullet$ defined by $g^\bullet(r) = g$, for all $r \in J$. Let $i_G\colon G \to G^\bullet$ be the function defined by $i_G(g) = g^\bullet$ for each $g \in G$.

Theorem 1.1. *Let G be a semitopological group. Then $i_G\colon G \to G^\bullet$ is a topological monomorphism. If G is Hausdorff, then the image $i_G(G)$ is a closed subgroup of $G^\bullet$.*

Proposition 1.2. *Let $\varphi\colon G \to H$ be a continuous homomorphism of semitopological groups. Then φ admits a natural extension to a continuous homomorphism $\varphi^\bullet\colon G^\bullet \to H^\bullet$. If φ is open and onto, then so is $\varphi^\bullet$.*

The correspondences $G \mapsto G^\bullet$ and $\varphi \mapsto \varphi^\bullet$ are functorial since the equality $(\psi \circ \varphi)^\bullet = \psi^\bullet \circ \varphi^\bullet$ holds for any continuous homomorphisms $\varphi\colon G \to H$ and $\psi\colon H \to K$. This defines the covariant functor $^\bullet$ in the category of semitopological groups and continuous homomorphisms. It turns out that this functor preserves subgroups and quotient groups:

Proposition 1.3. *Let G be a semitopological group and H a subgroup of G.*

a) *If φ is the identity embedding of H to G, then the natural homomorphic extension $\varphi^\bullet\colon H^\bullet \to G^\bullet$ is a topological monomorphism.*
b) *If H is closed or invariant in G, then $H^\bullet$ is also closed or invariant in $G^\bullet$.*
c) *If H is a closed invariant subgroup of G, then the groups $(G/H)^\bullet$ and $G^\bullet/H^\bullet$ are naturally topologically isomorphic.*

Following [8], we say that a class C of spaces is a *PS-class* if it contains arbitrary products of its elements, is hereditary with respect to taking subspaces, and contains a one-point space. It is clear that the class $\mathcal{T}_i$ of T_i-spaces is a PS-class for each $i \in \{0, 1, 2, 3, 3.5\}$, and so are the classes of regular spaces (i.e., T_1 and T_3) and Tychonoff spaces (i.e., T_1 and $T_{3.5}$).

Given a semitopological group G and a number $i \in \{0, 1, 2, 3, 3.5\}$, the *$T_i$-reflection* of G, denoted by $T_i(G)$, is a pair $(H, \varphi_{G,i})$, where H is a semitopological group satisfying the T_i separation axiom and $\varphi_{G,i}$ is a continuous homomorphism from G onto H with the following property: For every continuous mapping $f\colon G \to X$, where X is a space satisfying the T_i separation axiom, there exists a continuous mapping $h : H \to X$ such that $f = h \circ \varphi_{G,i}$.

$$\begin{tikzcd} G \arrow[r, "\varphi_{G,i}"] \arrow[d, "f"'] & H \arrow[dl, "h"] \\ X & \end{tikzcd}$$

In [8], M. Tkachenko provides an internal characterization of the kernel of the canonical homomorphism $\varphi_{G,i}$ for $i \in \{0, 1\}$. I. Sánchez, in [6] describes the kernel of the canonical homomorphism $\varphi_{G,2}$, for any semitopological group G.

The following example shows that the functors T_1 and T_3 do not commute in the category of paratopological groups.

Example 1.4 ([8, Example 3.10])**.** Let $\mathbb{Z}$ be the additive group of integers. For every integer $n \geq 0$, let $U_n = \{0\} \cup \{m \in \mathbb{Z} : m > n\}$. It is easy to verify that the family $\{U_n : n \in \mathbb{N}\}$ forms a neighborhood base at zero for a paratopological group topology τ_1 on $\mathbb{Z}$, and that the paratopological group $G = (\mathbb{Z}, \tau_1)$ satisfies the T_1 separation axiom. It is also clear that any two nonempty open subsets of G have a nonempty intersection. Since G is a T_1-space, we have $T_1(G) = G$ and hence $T_3(T_1(G)) = T_3(G)$. By [8, Lemma 3.9], the canonical homomorphism $\varphi_{G,3}$ of G onto $T_3(G)$ is a bijection. Therefore, $T_3(G)$ is an infinite group (endowed with the anti-discrete topology, see [7, Theorem 2.6]).

On the other hand, $T_1(T_3(G))$ is the trivial group, for [8, Theorem 3.4]. Consequently, the groups $T_1(T_3(G))$ and $T_3(T_1(G))$ are quite different.

2. T_i-reflection and the Hartman–Mycielski construction in semitopological groups for $i \in \{0, 1, 2\}$

In this section we deal with the T_i-reflection of $G^\bullet$ for $i \in \{0, 1, 2\}$, where G is a semitopological group. We will prove that $T_i(G^\bullet)$ and $(T_i(G))^\bullet$ are topologically isomorphic.

The following Lemmas 2.1 and 2.2 are almost immediate from the definition of subsets $O(U, \epsilon)$ of $G^\bullet$.

Lemma 2.1. *Let G be a semitopological group, $U \in \mathcal{N}(e)$ and $\epsilon > 0$. Then $[O(U, \epsilon)]^{-1} = O(U^{-1}, \epsilon)$.*

Proof. Let $f \in [O(U, \epsilon)]^{-1}$. If $g = f^{-1}$, then $g \in O(U, \epsilon) \iff \mu(\{r \in J : g(r) \notin U\}) < \epsilon \iff \mu(\{r \in J : f(r) \notin U^{-1}\}) < \epsilon \iff f \in O(U^{-1}, \epsilon)$. □

Lemma 2.2. *Let G be a semitopological group, and U and V be nonempty subsets of G. Then $O(U, \epsilon)O(V, \delta) \subset O(UV, \epsilon + \delta)$ for all $\epsilon, \delta > 0$.*

Proof. If $h \in O(U, \epsilon)O(V, \delta)$, then $h = fg$ for some $f \in O(U, \epsilon)$ and $g \in O(V, \delta)$. To see that $h \in O(UV, \epsilon + \delta)$ note that

$$\begin{aligned}\{r \in J : h(r) \notin UV\} &= \{r \in J : f(r)g(r) \notin UV\} \\ &\subset \{r \in J : f(r) \notin U\} \cup \{r \in J : g(r) \notin V\}.\end{aligned}$$

This proves the lemma. □

In Lemmas 2.3 and 2.4, we present some properties of the closures of basic open sets in $G^\bullet$.

Lemma 2.3. *Let G be a semitopological group, $U \in \mathcal{N}(e)$, and $\epsilon > 0$. Then the equality $O(\overline{U}, \epsilon) = \bigcap_{V \in \mathcal{N}(e)} O(UV^{-1}, \epsilon)$ holds.*

Proof. We have that $\overline{U} = \bigcap_{V\in\mathcal{N}(e)} UV^{-1}$. Hence, $O(\overline{U},\epsilon) \subset O(UV^{-1},\epsilon)$ for each $V \in \mathcal{N}(e)$, whence it follows that $O(\overline{U},\epsilon) \subset \bigcap_{V\in\mathcal{N}(e)} O(UV^{-1},\epsilon)$.

If $f \in \bigcap_{V\in\mathcal{N}(e)} O(UV^{-1},\epsilon)$, then $f \in O(UV^{-1},\epsilon)$ for each $V \in \mathcal{N}(e)$. Let us verify that the Lebesgue measure of the set $\{r \in J : f(r) \notin \overline{U}\}$ is less than ϵ. Since $f \in G^\bullet$, there exist real numbers $a_1, a_2, \dots, a_n$ with $0 = a_0 < a_1 < \cdots < a_n < a_{n+1} = 1$, and elements $x_0, x_1, \dots, x_n \in G$, such that $f(r) = x_i$ for each $r \in [a_i, a_{i+1})$, where $0 \le i \le n$.

If $V(f) \subset \overline{U}$, then $f \in O(\overline{U},\epsilon)$. If $V(f) \not\subseteq \overline{U}$ and $f(a_i) \notin \overline{U}$ for some $i \le n$, we can find an element $W_i \in \mathcal{N}(e)$ such that $f(a_i)W_i \cap U = \emptyset$, so $f(a_i) \notin UW_i^{-1}$. Let $M = \{i \le n : f(a_i) \notin \overline{U}\}$ and $W = \bigcap_{i\in M} W_i$. Since $f(a_i) \notin UW^{-1}$ for each $i \in M$, we have

$$\{r \in J : f(r) \notin \overline{U}\} = \{r \in J : f(r) \notin UW^{-1}\}.$$

Therefore, $\mu(\{r \in J : f(r) \notin \overline{U}\}) < \epsilon$, and we conclude that $f \in O(\overline{U},\epsilon)$. This completes the proof. □

Lemma 2.4. *Let G be a semitopological group, $U \in \mathcal{N}(e)$ and $\epsilon > 0$. Then $O(\overline{U},\epsilon) \subset \overline{O(U,\epsilon)}$.*

Proof. Let $f \in O(\overline{U},\epsilon)$, $W \in \mathcal{N}(e)$ and $\delta > 0$. Suppose that $P(f) = \{a_0, \dots, a_n\}$. For each $i \le n$ such that $f(a_i) \in \overline{U}$, we have $f(a_i)W \cap U \neq \emptyset$, that is, there exists $w_i \in W$ such that $f(a_i)w_i \in U$. Now, defining $g(r) = w_i$ for each $r \in [a_i, a_{i+1})$ if $f(a_i) \in \overline{U}$, and $g(r) = e$ otherwise, we obtain an element $g \in G^\bullet$ such that $fg \in fO(W,\delta) \cap O(U,\epsilon)$. Therefore, $f \in \overline{O(U,\epsilon)}$, that is, $O(\overline{U},\epsilon) \subset \overline{O(U,\epsilon)}$. □

In the following example we show that the inclusion in Lemma 2.4 can be proper.

Example 2.5. Let $G = \mathbb{Z}_p$ be the topological group of p-adic integers, for any integer $p \ge 2$. Suppose that the equality $\overline{O(U,\epsilon)} = O(\overline{U},\epsilon)$ holds in $G^\bullet$ for all $U \in \mathcal{N}(e)$ and $\epsilon > 0$. Since G is zero-dimensional, our assumption implies that

$$O(V,\epsilon) = O(\overline{V},\epsilon) = \overline{O(V,\epsilon)},$$

for each open subgroup V of G. This is a contradiction, since $G^\bullet$ is a connected semitopological group by [1, Theorem 3.8.3]. Therefore, the equality is not true.

It is known (see [7]) that the T_2-reflection of an arbitrary topological group G is the quotient topological group G/N, where N is the closure of the identity element of G, that is, $T_2(G) = G/\overline{\{e\}}$. Furthermore, since in topological groups the separation axiom T_0 is equivalent to complete regularity, it follows that for any topological group G, the T_i-reflections of G, for $i = 0, 1, 2, 3, 3.5$, are equivalent.

Theorem 2.6. *Let G be a topological group. Then $T_2(G^\bullet)$ and $(T_2(G))^\bullet$ are topologically isomorphic.*

Proof. Let $N = \overline{\{e\}}$. Since $T_2(G) = G/N$, we have $(T_2(G))^\bullet = (G/N)^\bullet$ and, by Proposition 1.3, the groups $(G/N)^\bullet$ and $G^\bullet/N^\bullet$ are topologically isomorphic. We also know that $T_2(G^\bullet) = G^\bullet/K$, where $K = \overline{\{e^\bullet\}}$. Hence, it suffices to show that $N^\bullet = K$, where

$$K = \{f \in G^\bullet : e^\bullet \in fO(U, \epsilon) \text{ for all } U \in \mathcal{N}(e) \text{ and } \epsilon > 0\}$$

and $N^\bullet = \{f \in G^\bullet : V(f) \subset N\}$. Let $f \in N^\bullet$ and assume that $P(f) = \{a_0, \ldots, a_n\}$. Since $V(f) \subset N$ we have $f(a_k) \in N$ for each $k \leq n$, so $e \in f(a_k)U$ for all $U \in \mathcal{N}(e)$ and $k \leq n$. Therefore, for each k, there exists $u_k \in U$ such that $f(a_k)u_k = e$. We can then define $g \in G^\bullet$ with $P(f) = P(g)$ and $g(a_k) = u_k$ for $k = 0, \ldots, n$. Since $g \in O(U, \epsilon)$ for all $\epsilon > 0$ and the equality $fg = e^\bullet$ holds, we see that $e^\bullet \in fO(U, \epsilon)$. Hence $f \in K$, that is, $N^\bullet \subset K$.

Conversely, if $f \notin N^\bullet$, then there exists $x \in V(f)$ with $x \notin N$. Since N is closed in G, there is $V \in \mathcal{N}(e)$ such that $e \notin xV$. Taking the basic open set $W = O(V, \Delta(f)/2)$, we clearly have $e^\bullet \notin fW$, that is, $f \notin K$. Therefore, $K \subset N^\bullet$. This completes the proof. □

The subsequent Lemmas 2.7, 2.8 and 2.9 follow from [6, Lemma 2.1], [6, Lemma 2.2] and [6, Theorem 2.3], respectively.

Lemma 2.7. *Let G be a semitopological group with identity e and $\mathcal{N}(e)$ be the family of all open neighborhoods of the identity in G. Then*

$$K = \bigcap \{\overline{O(U, \epsilon)} : U \in \mathcal{N}(e),\ \epsilon > 0\}$$

is a closed invariant subgroup of $G^\bullet$.

The subgroup K of $G^\bullet$ defined in Lemma 2.7 is used in the following two results.

Lemma 2.8. *Let G be a semitopological group and $U, V \in \mathcal{N}(e)$. Then*

$$O(U, \epsilon)\, K\, O(V, \delta)^{-1} = O(U, \epsilon)\, O(V, \delta)^{-1},$$

for all positive real numbers ϵ, δ. Moreover, the equality $\overline{O(U, \epsilon)}\, K = \overline{O(U, \epsilon)}$ holds for all $U \in \mathcal{N}(e)$ and $\epsilon > 0$.

Lemma 2.9. *Let G be a semitopological group. Then, $T_2(G^\bullet) = G^\bullet/K$, so K is the kernel of the canonical homomorphism $\varphi_{G^\bullet,2} : G^\bullet \to T_2(G^\bullet)$. Moreover, $\varphi_{G^\bullet,2}^{-1}(\varphi_{G^\bullet,2}(\overline{O(U, \epsilon)})) = \overline{O(U, \epsilon)}$, for all $U \in \mathcal{N}(e)$ and $\epsilon > 0$.*

In the following theorem we show that the functors T_2 and $^\bullet$ commute in the category of semitopological groups.

Theorem 2.10. *Let G be an arbitrary semitopological group. Then $T_2(G^\bullet)$ and $(T_2(G))^\bullet$ are topologically isomorphic.*

Proof. If G is a Hausdorff semitopological group, then $T_2(G) = G$ and $T_2(G^\bullet) = G^\bullet$, hence $T_2(G^\bullet) = (T_2(G))^\bullet$. Suppose that G is not Hausdorff. By [6, Theorem 2.3], we have $T_2(G) = G/K$, where $K = \bigcap\{\overline{U} : U \in \mathcal{N}(e)\}$. Then $T_2(G)^\bullet = (G/K)^\bullet$ and by Proposition 1.3, the groups $(G/K)^\bullet$ and $G^\bullet/K^\bullet$ are topologically isomorphic. Also, by Lemma 2.7, we have the equality $T_2(G^\bullet) = G^\bullet/H$, where $H = \bigcap\{\overline{O(U,\epsilon)} : U \in \mathcal{N}(e),\ \epsilon > 0\}$. It is therefore suffices to prove that the groups $K^\bullet$ and H coincide. Note that

$$K^\bullet = \{f \in G^\bullet : V(f) \subset \bigcap_{U\in\mathcal{N}(e)} \overline{U}\}.$$

If $f \in K^\bullet$, then $V(f) \subset \bigcap_{U\in\mathcal{N}(e)} \overline{U}$, so Lemma 2.4 implies that

$$f \in \bigcap\{O(\overline{U},\epsilon) : U \in \mathcal{N}(e),\ \epsilon > 0\} \subset H.$$

Thus, $K^\bullet \subset H$.

Conversely, if $f \notin K^\bullet$, then we can find $x \in V(f)$ and $V \in \mathcal{N}(e)$ such that $x \notin \overline{V}$. Then there exists $W \in \mathcal{N}(e)$ such that $x \notin VW^{-1}$, which implies $f \notin O(VW^{-1}, \Delta(f)/2)$. By Lemma 2.2, we have

$$f \notin O(V,\Delta(f)/4)O(W^{-1},\Delta(f)/4) = O(V,\Delta(f)/4)O(W,\Delta(f)/4)^{-1}.$$

It follows that $f \notin \overline{O(V,\Delta(f)/4)}$, thus implying that $f \notin H$. Hence, $H \subset K^\bullet$. This completes the proof. $\square$

Proposition 2.11. *Let H be a subgroup of a semitopological group G. Then H is closed in G if and only if $H^\bullet = \{f \in G^\bullet : V(f) \subset H\}$ is a closed subgroup of $G^\bullet$.*

Proof. By item b) of Proposition 1.3, it suffices to show that if $H^\bullet$ is closed in $G^\bullet$, then H is closed in G. Suppose that $H^\bullet$ is closed in $G^\bullet$ and take $x \in \overline{H}$. Then, for any $U \in \mathcal{N}(e)$, we have $xU \cap H \neq \varnothing$, which clearly implies that $x^\bullet O(U,\epsilon) \cap H^\bullet \neq \varnothing$ for each $\epsilon > 0$. Hence, $x^\bullet \in \overline{H^\bullet}$, and so $x^\bullet \in H^\bullet$. We conclude that $x \in H$ and that H is closed in G. $\square$

In Lemma 2.12, we describe the T_1-reflection of $G^\bullet$, for an arbitrary semitopological group G.

Lemma 2.12. *Let G be a semitopological group, K be the smallest closed subgroup of G, and F the smallest closed subgroup of $G^\bullet$. Then $K^\bullet = F$, that is, $K^\bullet$ is the smallest closed subgroup of $G^\bullet$. Moreover, $T_1(G^\bullet) = G^\bullet/K^\bullet$.*

Proof. Let $I = [a, b)$, where $0 \leq a < b \leq 1$. Define a mapping $\varphi_I \colon G \to G^\bullet$ by

$$\varphi_I(x)(r) = \begin{cases} x, & \text{if} \quad r \in I; \\ e, & \text{if} \quad r \notin I. \end{cases}$$

Clearly, φ_I is a topological monomorphism. Let $p \colon G^\bullet \to G^\bullet/F$ be the quotient homomorphism. Then $p \circ \varphi_I \colon G \to G^\bullet/F$ is a continuous homomorphism. Since $G^\bullet/F$ is a T_1-space, the kernel $\ker(p \circ \varphi_I)$ of $p \circ \varphi_I$ is a closed subgroup of G. Therefore, $K \subset \ker(p \circ \varphi_I)$, which implies that $\varphi_I(K) \subset F$.

For every $f \in K^\bullet$, there exist real numbers $a_1, a_2, \ldots, a_n$ with $0 = a_0 < a_1 < \cdots < a_n < a_{n+1} = 1$ such that $f(r) = x_m \in K$ for each $r \in J_m = [a_m, a_{m+1})$, where $0 \leq m \leq n$. Taking $I = J_m$, we have $\varphi_{J_m}(x_m) \in F$ for all $0 \leq m \leq n$. Therefore,

$$f = \prod_{m=0}^{n} \varphi_{J_m}(x_m) \in F.$$

This shows that $K^\bullet \subset F$. Since, by Proposition 2.11, $K^\bullet$ is a closed subgroup of $G^\bullet$ and F is the smallest closed subgroup of $G^\bullet$, it follows that $K^\bullet = F$. Finally, by [8, Theorem 3.4], we have the equality $T_1(G^\bullet) = G^\bullet/K^\bullet$. □

From [8, Theorem 3.4], Proposition 1.3 and Lemma 2.12 we obtain the following theorem:

Theorem 2.13. *Let G be a semitopological group. Then $T_1(G^\bullet)$ and $(T_1(G))^\bullet$ are topologically isomorphic.*

The next two facts follow from [8, Theorem 3.1].

Lemma 2.14. *Let G be a semitopological group with identity e. Let also*

$$P = \bigcap \{O(U, \epsilon) : U \in \mathcal{N}(e),\ \epsilon > 0\} \subset G^\bullet.$$

Then $K = P \cap P^{-1}$ is an invariant subgroup of $G^\bullet$. Furthermore, the equality $KO(U, \epsilon) = O(U, \epsilon)$ holds for all $U \in \mathcal{N}(e)$ and $\epsilon > 0$.

We use the subgroup K of $G^\bullet$ defined in Lemma 2.14 in the subsequent result.

Lemma 2.15. *Let G be a semitopological group. Then $T_0(G^\bullet) = G^\bullet/K$.*

Theorem 2.16. *Let G be an arbitrary semitopological group. Then the groups $T_0(G^\bullet)$ and $(T_0(G))^\bullet$ are topologically isomorphic.*

Proof. By [8, Theorem 3.1], the equality $T_0(G) = G/K$ holds, where $K = \bigcap \mathcal{N}(e) \cap (\bigcap \mathcal{N}(e))^{-1}$. Let $\pi\colon G \to G/K$ be the quotient homomorphism. Then, by Proposition 1.2, π admits an extension to a continuous open homomorphism $\pi^\bullet\colon G^\bullet \to (G/K)^\bullet$. Note that

$$\ker(\pi^\bullet) = \{f \in G^\bullet : \pi^\bullet(f) = e'\} = \{g \in G^\bullet : V(g) \subset K\} = K^\bullet,$$

where e' is the identity of the group $(G/K)^\bullet$. Then, by [1, Theorem 1.5.13], $G^\bullet/K^\bullet$ is topologically isomorphic to $(G/K)^\bullet$. Therefore, the groups $T_0(G^\bullet)$ and $(T_0(G))^\bullet$ are topologically isomorphic. □

3. Semiregularization and the Hartman–Mycielski construction in semitopological groups

The operation of semiregularization of a topological space was defined by M. Katetov in [3]. A. Ravsky [5] considered this operation in the classes of semitopological and paratopological groups and proved that the semiregularization of a semitopological (paratopological) group is again a semitopological (paratopological) group. Furthermore, the semiregularization of a Hausdorff paratopological group is a regular space.

In this section we deal with semiregularizations of semitopological groups. Let G be a semitopological group with identity e. The family

$$\beta_{ro} = \{\mathrm{Int}_G\, \overline{U} : U \in \mathcal{N}(e)\}$$

serves as a base at the identity e for a topology $\mathcal{T}_{sr}$ on G, which is coarser than the original topology of G. The group G with the topology $\mathcal{T}_{sr}$ is called the semiregularization of G and is usually denoted by G_{sr}. We say that G is semiregular if $G = G_{sr}$.

Our aim is to prove that the semitopological groups $(G^\bullet)_{sr}$ and $(G_{sr})^\bullet$ are topologically isomorphic. This requires two auxiliary results.

Lemma 3.1. *Let G be a semitopological group, $0 < \epsilon < 1$, and $U \in \mathcal{N}(e)$. Then*

$$\mathrm{Int}\, O(\overline{U}, \epsilon) = \mathrm{Int}\, \overline{O(U, \epsilon)}.$$

Proof. By Lemma 2.4, it suffices to prove that $\mathrm{Int}\, \overline{O(U, \epsilon)} \subset \mathrm{Int}\, O(\overline{U}, \epsilon)$. Take an arbitrary element $f \in G^\bullet \setminus O(\overline{U}, \epsilon)$. Then $\mu(\{r \in J : f(r) \notin \overline{U}\}) \geq \epsilon$. Let $P(f) = \{a_0, \ldots, a_n\}$ and $M = \{0 \leq i \leq n : f(a_i) \notin \overline{U}\}$.

Case I. $\mu(\{r \in J : f(r) \notin \overline{U}\}) > \epsilon$. For each $m \in M$, there exists $W_m \in \mathcal{N}(e)$ such that $f(a_m)W_m \cap U = \varnothing$. Let $W = \bigcap_{m\in M} W_m$. Then $f(a_m)W \cap U = \varnothing$ for each $m \in M$. We claim that $fO(W, \delta/2) \cap O(U, \epsilon) = \varnothing$, where $\delta = \mu(\{r \in J : f(r) \notin \overline{U}\}) - \epsilon > 0$. Indeed, suppose that there exists $h \in fO(W, \delta/2) \cap O(U, \epsilon)$. Since $h \in fO(W, \delta/2)$, we have $\mu(\{r \in J : h(r) \notin f(r)W\}) < \delta/2$. For each $m \in M$,

$$\{r \in [a_m, a_{m+1}) : h(r) \in f(a_m)W\} \subset \{r \in J : h(r) \notin U\},$$

whence it follows that

$$\bigcup_{m\in M}\{r\in[a_m,a_{m+1}) : h(r)\in f(a_m)W\}\subset\{r\in J : h(r)\notin U\}.$$

By the choice of δ, we have

$$\mu\Big(\bigcup_{m\in M}\{r\in[a_m,a_{m+1}) : h(r)\in f(a_m)W\}\Big)\geq\epsilon+\delta/2>\epsilon,$$

thus $\mu(\{r\in J : h(r)\notin U\})>\epsilon$, which contradicts the assumption. Therefore, $fO(W,\delta/2)\cap O(U,\epsilon)=\varnothing$, that is, $f\notin\overline{O(U,\epsilon)}$.

Case II. $\mu(\{r\in J : f(r)\notin\overline{U}\})=\epsilon$. Let $V\in\mathcal{N}(e)$, $m\in M$ and $\delta>0$. Choose $a_k\in P(f)$ and $b\in J$ such that $f(a_k)\in\overline{U}$, $a_k<b\leq a_{k+1}$ and $b-a_k<\delta$. Define an element $g\in G^\bullet$ by the rule

$$g(r)=\begin{cases} f(a_k)^{-1}f(a_m) & \text{if} \quad r\in[a_k,b),\\ e & \text{if} \quad r\notin[a_k,b).\end{cases}$$

Clearly, $g\in O(V,\delta)$. Then $fg\in\{h\in G^\bullet : \mu(\{r\in J : h(r)\notin\overline{U}\})>\epsilon\}$, so by Case I, we have $fg\notin\overline{O(U,\epsilon)}$. Thus $f\notin\operatorname{Int}\overline{O(U,\epsilon)}$.

We have shown that $\operatorname{Int}\overline{O(U,\epsilon)}\subset O(\overline{U},\epsilon)$, which implies the required inclusion $\operatorname{Int}\overline{O(U,\epsilon)}\subset\operatorname{Int}O(\overline{U},\epsilon)$.

□

Lemma 3.2. *Let G be a semitopological group, $0<\epsilon<1$ and $U\in\mathcal{N}(e)$. Then*

$$O(\operatorname{Int}\overline{U},\epsilon)=\operatorname{Int}\overline{O(U,\epsilon)}.$$

Proof. Since $\operatorname{Int}\overline{U}\subset\overline{U}$ for all $U\in\mathcal{N}(e)$, we have $O(\operatorname{Int}\overline{U},\epsilon)\subset O(\overline{U},\epsilon)$. Then, by Lemma 3.1, we deduce that $O(\operatorname{Int}\overline{U},\epsilon)\subset\operatorname{Int}\overline{O(U,\epsilon)}$. Conversely, let $f\in G^\bullet\setminus O(\operatorname{Int}\overline{U},\epsilon)$, and suppose that $P(f)=\{a_0,a_1,\dots,a_n\}$. Then $\mu(\{r\in J : f(r)\notin\operatorname{Int}\overline{U}\})\geq\epsilon$. If $\mu(\{r\in J : f(r)\notin\overline{U}\})\geq\epsilon$, then $f\notin O(\overline{U},\epsilon)$, hence $f\notin\operatorname{Int}\overline{O(U,\epsilon)}$. Otherwise, let

$$M=\{0\leq k\leq n : f(a_k)\in\overline{U}\setminus\operatorname{Int}\overline{U}\}\neq\varnothing.$$

It follows that

$$\mu(\{r\in J : f(r)\notin\operatorname{Int}\overline{U}\})=\sum_{k\in M}(a_{k+1}-a_k)+\mu(\{r\in J : f(r)\notin\overline{U}\})\geq\epsilon.$$

For each $k\in M$ and each $W\in\mathcal{N}(e)$, we have $f(a_k)W\cap U\neq\varnothing$ and $f(a_k)W\cap(G\setminus\overline{U})\neq\varnothing$. Take $y_k\in W$ such that $f(a_k)y_k\notin\overline{U}$, where $k\in M$. Choosing $g\in G^\bullet$ with $P(g)=P(f)$ and $g(a_k)=y_k$ for each $k\in M$, and $g(r)=e$ otherwise, we have that $g\in O(W,\delta)$ for all $\delta>0$ and

$$\mu(\{r\in J : (fg)(r)\notin\overline{U}\})=1.$$

Thus $fg \in fO(W,\delta) \nsubseteq O(\overline{U},\epsilon)$, that is, $f \notin \operatorname{Int} O(\overline{U},\epsilon)$. Hence, $\operatorname{Int} \overline{O(U,\epsilon)} \subset O(\operatorname{Int} \overline{U},\epsilon)$. This completes the proof. □

The subsequent theorem is the main result of this section, it follows from the combination of Lemmas 3.1 and 3.2.

Theorem 3.3. *Let G be a semitopological group. Then the local bases*

$$(\beta_{ro})^{\bullet} = \{O(\operatorname{Int} \overline{U},\epsilon) : U \in \mathcal{N}(e),\ \epsilon > 0\}$$

and

$$(\beta^{\bullet})_{ro} = \{\operatorname{Int} \overline{O(V,\epsilon)} : V \in \mathcal{N}(e),\ \epsilon > 0\}$$

at $e^{\bullet}$ generate the same topology on $G^{\bullet}$, that is, $(G^{\bullet})_{sr}$ and $(G_{sr})^{\bullet}$ are topologically isomorphic.

In [7], M. Tkachenko proved that in paratopological groups, the T_3-reflection and semiregularization coincide. Therefore, combining Theorems 3.3, 2.13 and [7, Theorem 3.8], we obtain the following result.

Theorem 3.4. *Let G be a paratopological group. Then the groups $Reg(G^{\bullet})$ and $Reg(G)^{\bullet}$ are topologically isomorphic.*

4. Conclusions

From this work we can conclude the following:

(1) In the category of paratopological groups, the functors T_i and $^{\bullet}$ commute for $i \in \{0, 1, 2, 3, 3.5\}$.
(2) In the category of semitopological groups, the functors T_i and $^{\bullet}$ commute for $i \in \{0, 1, 2\}$. Moreover, the semiregularization and the functor $^{\bullet}$ also commute in this category.

5. Open Problems

Since the semiregularization of a semitopological group does not, in general, coincide with its T_3-reflection, it is natural to consider the following problems.

Problem 5.1. *Describe in internal terms the kernel of the canonical homomorphism $\varphi_{G,3}$ of a semitopological group G onto $T_3(G)$.*

Problem 5.2. *Let G be a semitopological group. Are the semitopological groups $T_3(G^{\bullet})$ and $(T_3(G))^{\bullet}$ topologically isomorphic?*

We recall that a class of spaces is a *PS-class* if it is closed under arbitrary products, taking subspaces, and contains a singleton.

Let G be a topological group and C be a *PS*-class. By [7, Theorem 2.3], the C-reflection $C(G)$ of G exists and is a topological group.

Problem 5.3. *Is* $(C(G))^{\bullet}$ *an element of the PS-class* C*?*

Problem 5.4. *Are the topological groups* $C(G^{\bullet})$ *and* $(C(G))^{\bullet}$ *topologically isomorphic?*

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