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L-Fuzzy rough point-wise proximity spaces

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Abstract

In this paper, we study the concept of nearness between L-fuzzy rough sets and L-fuzzy points. The nearness between two L-fuzzy rough sets are viewed in terms of this new concept. Also, we investigate the relationship of an L-fuzzy rough point-wise proximity space with L-fuzzy rough grills, L-fuzzy rough $\delta_{\mathcal{R}}$-clan and L-fuzzy rough $\delta_{\mathcal{R}}$-clusters. Moreover, we introduce the L-fuzzy rough point-wise LO-proximity spaces and study a few of its fundamental properties. The research is supported by a sufficient number of examples.

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1. Introduction

The fundamental concept of fuzzy set theory was first presented by Zadeh [32], in 1965, and it then found use in topology as well as other areas of mathematics. After this Chang [3] proceeded to examine fuzzy topology, Lowen [18] worked on fuzzy uniformity, and Katsaras [8] investigated the area of fuzzy proximity [9, 10]. Over time, an overall number of other scholars expanded on these investigations.

In 1982, Pawlak [20] introduced rough set theory. The idea of "rough fuzzy sets", first presented by Dubois and Prade [4] in 1990, includes approximation both crisp and fuzzy sets using fuzzy and crisp approximation spaces, respectively. Additionally, they underlined that fuzzy

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rough sets specifically take the form of rough fuzzy sets. A fuzzy relation represents the relationship between items in a set in the context of fuzzy rough set theory. Numerous applications in fields including data mining, machine learning, pattern recognition, granular representation, and magnetic resonance brain image segmentation have been addressed using fuzzy rough set theory. It proved effective in improving segmentation results, especially in difficult areas of the image. In [7], the authors explore the real-world difficulties that arise in a variety of applications, including stock price prediction, medical time series analysis, neural networks, and descriptive dimensionality reduction, with an emphasis on the use of fuzzy approximation spaces. Previous research has shown that fuzzy rough set theory is a very useful method for solving feature selection difficulties. Fuzzy lower approximation operators and fuzzy upper approximation operators are the essential parts of fuzzy approximation spaces in the framework of rough set theory. A topological framework known as a proximity structure has shown to be useful in a number of fields, such as computational biology, pattern recognition, feature selection, digital image classification, data analysis, cluster analysis, multidimensional scaling, and data analysis (see [6, 14, 21, 28]). Images may occasionally need to have some degree of fuzziness or degeneration added to them. Using fuzzy approximation spaces and establishing a closeness relationship between fuzzy rough sets are required when dealing with such situations. Consequently, this motivates our study of proximity structures in the framework of fuzzy rough sets [12, 13, 27], where we introduced L-fuzzy rough proximity spaces to study the nearness between two rough sets. By estimating the separation between each fuzzy upper approximation, two fuzzy rough sets may be compared to see how close they are. A Čech closure spaces [2] and proximity spaces [11, 19] are substantially related since they come from the same field of study. Each of these spaces is an expansion of the notion of topological spaces. Crucially, every basic proximity structure can provide a Čech closure operator [17, 19]. One study examined the relationship between Čech closure spaces and Čech rough proximity spaces in [29, 30]. Additionally, proximity spaces have a connection to the notion of grills and filters (see [5, 25, 26]). In [23], point-wise proximity spaces were introduced to study the nearness of fuzzy sets and fuzzy points. All these studies motivate us to study the nearness between L-fuzzy sets and L-fuzzy points in terms of approximation operator and find their relationships with the L-fuzzy grill operators.

The organization of this paper is as follows: In Section 2, we collect some basic definitions and results is necessary for the development of other sections of the paper. In Section 3, we define the L-fuzzy rough point-wise proximity spaces and study their properties. Examples are given to support the theory. In Section 4, we define L-fuzzy rough $\delta_{\mathcal{R}}$-clusters, L-fuzzy rough

$\delta_{\mathcal{R}}$-clans and L-fuzzy rough grills on L-fuzzy rough point-wise proximity spaces and L-fuzzy rough point-wise LO-proximity spaces and prove some important results.

2. Preliminaries

In this paper, X is a non-empty set, called the universal set and X^c denotes the complement of set X. Let $\mathcal{R}$ denotes an L-fuzzy relation on X. Then the pair $(X, \mathcal{R})$ will be called an L-fuzzy approximation space. In this paper, we consider $\mathcal{R}$ as an L-fuzzy equivalence (L-fuzzy reflexive, L-fuzzy symmetric and L-fuzzy transitive) relation on X. Suppose that $(L, \wedge, \vee, c, 0, 1)$ is a complete lattice with order reversing inclusion 'c' where, 0 and 1 denotes the smallest and largest elements of lattice L. Further, **0, 1** denote the smallest and the largest element of L^X, respectively. In this section, we collect some basic definitions and results on Čech L-fuzzy rough proximity spaces, L-fuzzy closure rough operators and L-fuzzy rough sets.

2.1. **An L-fuzzy set.** Let X be a non-empty set and L be a lattice with smallest element 0 and largest element 1. A non-empty *L-fuzzy set* (A, μ_A) is a pair consisting of a non-empty set A and a membership function μ_A. The membership function μ_A maps each element in A to a value of L representing the degree of membership of that element in the L-fuzzy set. A value of 0 indicates no membership, while a value of 1 indicates full membership. For convenience, we write an L-fuzzy set A on a set X is a function $A : X \to L$, or $A \in L^X$. The set L^X of all L-fuzzy sets of X forms a lattice with smallest element **0** and largest element **1**. In particular, if L is taken as the unit closed interval $[0, 1]$, then we call A as a *fuzzy set.*

2.2. **An L-fuzzy relation and an L-fuzzy point.** Let X be a non-empty set. A mapping $\mathcal{R} : X \times X \to L$ is called an *L-fuzzy relation* on X. Here, $\mathcal{R}(x, y)$ is referred to as the degree of relation between x and y, where $(x, y) \in X \times X$. For all $x, y, z \in X$, the relation $\mathcal{R}$ is said to be

(1) *Reflexive* if $\mathcal{R}(x, x) = 1$.
(2) *Symmetric* if $\mathcal{R}(x, y) = \mathcal{R}(y, x)$.
(3) *Transitive* if $\bigvee_{y \in X} (\mathcal{R}(x, y) \wedge \mathcal{R}(y, z)) \leq \mathcal{R}(x, z)$.

An L-fuzzy relation on X is called an *L-fuzzy equivalence relation* if it is reflexive, symmetric and transitive. In this paper, we will take $\mathcal{R}$ to be an L-fuzzy equivalence relation on X.

An *L-fuzzy point* is defined by $x_p : X \to L$ such that

$$x_p(y) = \begin{cases} p, & \text{if} \quad y = x \\ 0, & \text{if} \quad y \neq x. \end{cases}$$

2.3. **An L-fuzzy approximation space.** Let X be a non-empty universal set and $\mathcal{R}$ an L-fuzzy relation on X. The pair $(X, \mathcal{R})$ is called an *L-fuzzy approximation space.* The *L-fuzzy upper* and *L-fuzzy lower approximations* of L-fuzzy set $A \in L^X$ with respect to $(X, \mathcal{R})$, denoted by $\overline{\mathcal{R}}(A)$ and $\underline{\mathcal{R}}(A)$, are respectively, defined as follows:

$$\overline{\mathcal{R}}(A)(x) = \bigvee_{y \in X} (\mathcal{R}(x,y) \wedge A(y)),\ \forall\ x \in X,$$
$$\underline{\mathcal{R}}(A)(x) = \bigwedge_{y \in X} ((\mathcal{R}(x,y))^c \vee A(y)),\ \forall\ x \in X.$$

The pair $(\underline{\mathcal{R}}(A), \overline{\mathcal{R}}(A))$ is called an *L-fuzzy rough set* of A with respect to $(X, \mathcal{R})$ (see [16]).

2.4. **A Čech L-fuzzy rough proximity space.** Let $\delta_{\mathcal{R}}$ be a relation on L^X satisfying the following axioms, for non-empty sets $A, B, C \in L^X$:

(P.1) $\overline{\mathcal{R}}(A)\delta_{\mathcal{R}}\overline{\mathcal{R}}(B) \Leftrightarrow \overline{\mathcal{R}}(B)\delta_{\mathcal{R}}\overline{\mathcal{R}}(A)$.
(P.2) $(\overline{\mathcal{R}}(A) \vee \overline{\mathcal{R}}(B))\delta_{\mathcal{R}}\overline{\mathcal{R}}(C) \Leftrightarrow \overline{\mathcal{R}}(A)\delta_{\mathcal{R}}\overline{\mathcal{R}}(C)$ or $\overline{\mathcal{R}}(B)\delta_{\mathcal{R}}\overline{\mathcal{R}}(C)$.
(P.3) $\overline{\mathcal{R}}(A)\delta_{\mathcal{R}}\overline{\mathcal{R}}(B) \implies A \neq \mathbf{0}$ and $B \neq \mathbf{0}$.
(P.4) $\overline{\mathcal{R}}(A) \wedge \overline{\mathcal{R}}(B) \neq \mathbf{0} \implies A\delta_{\mathcal{R}}B$.

Then $\delta_{\mathcal{R}}$ is called a *Čech L-fuzzy rough proximity* on X and the pair $(X, \delta_{\mathcal{R}})$ is known as a *Čech L-fuzzy rough proximity space* (see [12]).

Definition 2.1. Let $(X, \mathcal{R})$ be an L-fuzzy approximation space. A function $\mathcal{C}l_{\mathcal{R}} : L^X \to L^X$ is said to be a *Čech L-fuzzy rough closure operator* on X if it satisfies the following axioms for $A, B \in L^X$:

(1) $\mathcal{C}l_{\mathcal{R}}(\mathbf{0}) = \mathbf{0}$.
(2) $\overline{\mathcal{R}}(A) \leq \mathcal{C}l_{\mathcal{R}}(A)$.
(3) $\mathcal{C}l_{\mathcal{R}}(A \vee B) = \mathcal{C}l_{\mathcal{R}}(A) \vee \mathcal{C}l_{\mathcal{R}}(B)$.

The pair $(X, \mathcal{C}l_{\mathcal{R}})$ is said to be a *Čech L-fuzzy rough closure space.* Further, if $\mathcal{C}l_{\mathcal{R}}$ satisfies

(4) $\mathcal{C}l_{\mathcal{R}}(\mathcal{C}l_{\mathcal{R}}(A)) = \mathcal{C}l_{\mathcal{R}}(A)$.

Then $\mathcal{C}l_{\mathcal{R}}$ is a *Kuratowski L-fuzzy rough closure operator* on X (see [12]).

2.5. **An L-fuzzy rough filter and an L-fuzzy rough grill.**

Definition 2.2. An *L-fuzzy rough stack* $\mathbb{S}$ on X is a subset of L^X such that $\overline{\mathcal{R}}(A) \leq \overline{\mathcal{R}}(B)$ and $A \in \mathbb{S} \implies B \in \mathbb{S}$.

Definition 2.3. Let $\mathbf{F} \subset L^X$. Then $\mathbf{F}$ is called a *base* for an L-fuzzy rough filter in X, if the following two conditions are satisfied:

(1) $\mathbf{0} \notin \mathbf{F}$.
(2) If $A, B \in \mathbf{F}$, then $\overline{\mathcal{R}}(A) \wedge \overline{\mathcal{R}}(B) \in \mathbf{F}$.
If $\mathbf{F}$ also has the property
(3) $A \in \mathbf{F}$ and $\overline{\mathcal{R}}(A) \leq \overline{\mathcal{R}}(B) \implies B \in \mathbf{F}$.

Then $\mathbf{F}$ is called an *L-fuzzy rough filter.* A maximal L-fuzzy rough filter in X with respect to '$\leq$' is called an *L-fuzzy rough ultrafilter* in X.

Definition 2.4. A collection $\mathbf{G}$ of subsets of L^X is called an *L-fuzzy rough grill* [12] if it satisfies the following conditions, for non-empty sets $A, B \in L^X$:

(1) $\mathbf{0} \notin \mathbf{G}$.
(2) $A \in \mathbf{G}$ and $\overline{\mathcal{R}}(A) \leq \overline{\mathcal{R}}(B) \implies B \in \mathbf{G}$.
(3) $\overline{\mathcal{R}}(A) \vee \overline{\mathcal{R}}(B) \in \mathbf{G} \implies \overline{\mathcal{R}}(A) \in \mathbf{G}$ or $\overline{\mathcal{R}}(B) \in \mathbf{G}$.

Notation: For a set X, let $\Sigma(X), \Omega(X)$, and $\Gamma(X)$ denote respectively the set of all L-fuzzy rough stacks, L-fuzzy rough ultrafilters and L-fuzzy rough grills in X.

Remark 2.5. Every L-fuzzy rough ultrafilter on X is an L-fuzzy rough grill on X.

3. L-fuzzy rough point-wise proximity spaces

In this section, we study L-fuzzy rough point-wise proximity spaces and prove some basic results on them. Examples are constructed to support the theory developed. Throughout this paper, let $(X, \mathcal{R})$ be an L-fuzzy approximation space, where $\mathcal{R}$ is an L-fuzzy equivalence relation on the set X.

Definition 3.1. Let $\delta_{\mathcal{R}}$ be a relation on L^X is called an *L-fuzzy rough point-wise proximity* if satisfies the following axioms, for non-empty sets $A, B \in L^X$:

(F.1) $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(A) \Leftrightarrow \overline{\mathcal{R}}(A)\delta_{\mathcal{R}}\overline{\mathcal{R}}(x_p)$, for all $x_p \in L^X$.
(F.2) $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}(\overline{\mathcal{R}}(A) \vee \overline{\mathcal{R}}(B)) \Leftrightarrow \overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(A)$ or $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(B)$, for all $x_p \in L^X$.
(F.3) $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(A) \implies A \neq \mathbf{0}$, for all $x_p \in L^X$.
(F.4) $\overline{\mathcal{R}}(x_p) \wedge \overline{\mathcal{R}}(A) \neq \mathbf{0} \implies x_p\delta_{\mathcal{R}}A$, for all $x_p \in L^X$.

The pair $(X, \delta_{\mathcal{R}})$ is called as an *L-fuzzy rough point-wise proximity space.*

Definition 3.2. Let $\delta_{\mathcal{R}}$ be an L-fuzzy rough point-wise proximity on X, for non-empty fuzzy sets $A, B \in L^X$. Define $A\delta_{\mathcal{R}}B \Leftrightarrow \overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(B)$, for some $x_p \in A$.

We will use $\overline{\delta}_{\mathcal{R}}$ to represent the relation "not near", i.e., $A\overline{\delta}_{\mathcal{R}}B$ means A is not near B or A is far from B, for $A, B \in L^X$.
Now, we proceed to illustrate some examples of L-fuzzy rough point-wise proximity spaces.

Example 3.3. Let $(X, \mathcal{R})$ be an L-fuzzy approximation space. Define a relation $\delta_{\mathcal{R}}$ on L^X such that $A\delta_{\mathcal{R}}B \Leftrightarrow \overline{\mathcal{R}}(x_p) \wedge \overline{\mathcal{R}}(B) \neq \mathbf{0}$, for some $x_p \in \overline{R}(A)$ and $A, B \in L^X$. Then $\delta_{\mathcal{R}}$ is an L-fuzzy rough point-wise proximity on X.

Example 3.4. Let $(X, \mathcal{C}l_{\mathcal{R}})$ be an L-fuzzy rough closure space. Define a relation $\delta_{\mathcal{R}}$ on L^X such that $A\delta_{\mathcal{R}}B \Leftrightarrow x_p \wedge \mathcal{C}l_{\mathcal{R}}(\overline{\mathcal{R}}(A)) \neq \mathbf{0}$ and $x_p \wedge \mathcal{C}l_{\mathcal{R}}(\overline{\mathcal{R}}(B)) \neq \mathbf{0}$, for some $x_p \in L^X$. Then $(X, \delta_{\mathcal{R}})$ is an L-fuzzy rough point-wise proximity space.

Example 3.5. Let $(X, \mathcal{R})$ be an L-fuzzy approximation space. Define a relation $\delta_{\mathcal{R}}$ on L^X such that $A\delta_{\mathcal{R}}B \Leftrightarrow \overline{\mathcal{R}}(x_p) \wedge \overline{\mathcal{R}}(A) \neq \mathbf{0}$ and $\overline{\mathcal{R}}(x_p) \wedge \overline{\mathcal{R}}(B) \neq \mathbf{0}$, for some $x_p \in L^X$. Then $\delta_{\mathcal{R}}$ is an L-fuzzy rough point-wise proximity on X.

Remark 3.6. Every Čech L-fuzzy rough proximity on L^X is an L-fuzzy rough point-wise proximity on L^X. The converse need not be true.

Theorem 3.7. *Let $(X, \delta_{\mathcal{R}})$ be an L-fuzzy rough point-wise proximity space. Define the operator $\mathcal{C}l_{\mathcal{R}} : L^X \to L^X$ by $\mathcal{C}l_{\mathcal{R}}(A) = \bigvee\{x_p \in L^X : \overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(A)\}$. Then $\mathcal{C}l_{\mathcal{R}}$ is a Čech L-fuzzy rough closure operator on L^X.*

Proof. Let $(X, \delta_{\mathcal{R}})$ be an L-fuzzy rough point-wise proximity space, and let $A, B \in L^X$. Then $\mathcal{C}l_{\mathcal{R}}(\mathbf{0}) = \mathbf{0}$, clearly. To show that $\overline{\mathcal{R}}(A) \leq \mathcal{C}l_{\mathcal{R}}(A)$, let $x_p \leq \overline{\mathcal{R}}(A)$ implies that $\overline{\mathcal{R}}(x_p) \leq \overline{\mathcal{R}}(\overline{\mathcal{R}}(A))$. Then $\overline{\mathcal{R}}(\overline{\mathcal{R}}(x_p)) \wedge \overline{\mathcal{R}}(\overline{\mathcal{R}}(A)) \neq \mathbf{0}$ which gives $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(A)$. Hence, $x_p \leq \mathcal{C}l_{\mathcal{R}}(A)$. Thus $\overline{\mathcal{R}}(A) \leq \mathcal{C}l_{\mathcal{R}}(A)$. Next,

$$\begin{aligned}
\mathcal{C}l_{\mathcal{R}}(A \vee B) &= \bigvee\{x_p \in L^X : \overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(A \vee B)\} \\
&= \bigvee \left[\{x_p \in L^X : \overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(A)\} \cup \{x_p \in L^X : \overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(B)\}\right] \\
&= \left(\bigvee\{x_p \in L^X : \overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(A)\}\right) \vee \left(\bigvee\{x_p \in L^X : \overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(B)\}\right) \\
&= \mathcal{C}l_{\mathcal{R}}(A) \vee \mathcal{C}l_{\mathcal{R}}(B).
\end{aligned}$$

Hence $\mathcal{C}l_{\mathcal{R}}$ is a Čech L-fuzzy rough closure operator on X.

□

Lemma 3.8. *Let $(X, \delta_{\mathcal{R}})$ be an L-fuzzy rough point-wise proximity space. Then $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(A)$ and $\overline{\mathcal{R}}(A) \leq \overline{\mathcal{R}}(B)$ implies $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(B)$.*

Proof. Let $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(A)$ and $\overline{\mathcal{R}}(A) \leq \overline{\mathcal{R}}(B)$. Then $\overline{\mathcal{R}}(B) = \overline{\mathcal{R}}(A) \vee \overline{\mathcal{R}}(B)$. Suppose, if possible $\overline{\mathcal{R}}(x_p)\overline{\delta}_{\mathcal{R}}\overline{\mathcal{R}}(B)$, then $\overline{\mathcal{R}}(x_p)\overline{\delta}_{\mathcal{R}}(\overline{\mathcal{R}}(A) \vee \overline{\mathcal{R}}(B))$. Thus $\overline{\mathcal{R}}(x_p)\overline{\delta}_{\mathcal{R}}\overline{\mathcal{R}}(A)$ and $\overline{\mathcal{R}}(x_p)\overline{\delta}_{\mathcal{R}}\overline{\mathcal{R}}(B)$, which is a contradiction to $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(A)$. Thus $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(B)$. □

Theorem 3.9. *If $\delta_{\mathcal{R}}$ is an L-fuzzy rough point-wise proximity on X and $\overline{\mathcal{R}}(A) \leq \overline{\mathcal{R}}(B)$, then $\mathcal{C}l_{\mathcal{R}}(A) \leq \mathcal{C}l_{\mathcal{R}}(B)$, for non-empty sets $A, B \in L^X$, where $\mathcal{C}l_{\mathcal{R}}$ is the operator defined in Theorem 3.7.*

Proof. Let $\overline{\mathcal{R}}(A) \leq \overline{\mathcal{R}}(B)$. Then $\overline{\mathcal{R}}(x_p) \wedge \overline{\mathcal{R}}(A) \neq \mathbf{0}$ which implies $\overline{\mathcal{R}}(x_p) \wedge \overline{\mathcal{R}}(B) \neq \mathbf{0}$ which gives $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(A)$ implies that $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(B)$. Thus $\{x_p \in L^X : \overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(A)\} \subset \{x_p \in L^X : \overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(B)\}$ which implies that $\bigvee\{x_p \in L^X : \overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(A)\} \leq \bigvee\{x_p \in L^X : \overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(B)\}$. Hence $\mathcal{C}l_{\mathcal{R}}(A) \leq \mathcal{C}l_{\mathcal{R}}(B)$. □

Definition 3.10. Let $\delta_{\mathcal{R}}$ be an L-fuzzy rough point-wise proximity on X. Then a non-empty fuzzy set $A \in L^X$ is called an *L-fuzzy rough proximal neighborhood* of fuzzy point $x_p \in L^X$ if and only if $\overline{\mathcal{R}}(x_p)\overline{\delta}_{\mathcal{R}}(\overline{\mathcal{R}}(A))^c$. The set of all L-fuzzy rough proximal neighborhoods of x_p with respect to $\delta_{\mathcal{R}}$ is denoted by $\mathbb{N}_{\delta_{\mathcal{R}}}(x_p)$, i.e.,

$$\mathbb{N}_{\delta_{\mathcal{R}}}(x_p) = \{A \in L^X : \overline{\mathcal{R}}(x_p)\overline{\delta}_{\mathcal{R}}(\overline{\mathcal{R}}(A))^c\}.$$

Definition 3.11. Let $\delta_{\mathcal{R}}$ be an L-fuzzy rough point-wise proximity on X. Then for an L-fuzzy point $x_p \in L^X$, define $\Phi(x_p) = \{A \in L^X : \overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(A)\}$.

Definition 3.12. For $\mathbb{S} \in \Sigma(X)$, we define

(1) $\mathcal{K}(\mathbb{S}) = \{A \in L^X : (\overline{\mathcal{R}}(A))^c \notin \mathbb{S}\}$.
(2) $\mathcal{L}(\mathbb{S}) = \{A \in L^X : (\overline{\mathcal{R}}(x_p))^c \notin \mathbb{S},$ for some $x_p \in A\}$.
(3) $\mathcal{M}(\mathbb{S}) = \{A \in L^X : \overline{\mathcal{R}}(x_p) \wedge \overline{\mathcal{R}}(B) \neq \mathbf{0},$ for some $x_p \in A$ and for all $B \in \mathbb{S}\}$.

Definition 3.13. Let $\delta_{\mathcal{R}}$ be an L-fuzzy rough point-wise proximity on X, for non-empty fuzzy sets $A, B \in L^X$. Define

(1) $\Phi(A) = \{B \in L^X : \overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(B)$, for some $x_p \in A\}$.
(2) $\Phi(\mathfrak{U}) = \bigcap\{\Phi(A) : A \in \mathfrak{U}\}$, where $\mathfrak{U} \in \Omega(X)$.

Theorem 3.14. *For all L-fuzzy rough stacks $\mathbb{S} \in \Sigma(X)$, $\mathcal{L}(\mathbb{S}) \subset \mathcal{M}(\mathbb{S})$.*

Proof. Let us assume that $\mathbb{S} \neq \varnothing$ and $\mathbf{0} \notin \mathbb{S}$. Suppose that $A \in \mathcal{L}(\mathbb{S})$ and $\overline{\mathcal{R}}(x_p) \wedge \overline{\mathcal{R}}(B) = \mathbf{0}$, for some $B \in \mathbb{S}$ and for all $x_p \in A$. Then $(\overline{\mathcal{R}}(x_p) \wedge \overline{\mathcal{R}}(B))(y) = 0,$ for all $y \in X$ which implies $\overline{\mathcal{R}}(x_p)(y) = 0$ or $\overline{\mathcal{R}}(B)(y) = 0,$ for all $y \in X$, for some $B \in \mathbb{S}$ and for all $x_p \in A$. If

$\overline{\mathcal{R}}(x_p)(y) = 0$, then $\overline{\mathcal{R}}(B)(y) \leq 1 = (\overline{\mathcal{R}}(x_p))^c(y)$. On other hand, if $\overline{\mathcal{R}}(B)(y) = 0$, then $\overline{\mathcal{R}}(B)(y) \leq (\overline{\mathcal{R}}(x_p))^c(y)$, for all $x_p \in A$. Thus, $\overline{\mathcal{R}}(B) \leq (\overline{\mathcal{R}}(x_p))^c \leq \overline{\mathcal{R}}((\overline{\mathcal{R}}(x_p))^c)$. Hence $(\overline{\mathcal{R}}(x_p))^c \in \mathbb{S}$, which is a contradiction. Thus $\mathcal{L}(\mathbb{S}) \subset \mathcal{M}(\mathbb{S})$. □

Theorem 3.15. *A relation $\delta_{\mathcal{R}}$ on L^X is an L-fuzzy rough point-wise proximity on X if and only if it satisfies the following conditions:*

(1) $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(A) \Leftrightarrow \overline{\mathcal{R}}(A)\delta_{\mathcal{R}}\overline{\mathcal{R}}(x_p)$.
(2) $\Phi(x_p) \in \Gamma(X)$, *for all* $x_p \in L^X$.
(3) $\overline{\mathcal{R}}(x_p) \wedge \overline{\mathcal{R}}(A) \neq \boldsymbol{0} \implies x_p\delta_{\mathcal{R}}A$.

Proof. Let $\delta_{\mathcal{R}}$ be an L-fuzzy rough point-wise proximity on X. If $p = 0$, then $\Phi(\mathbf{0}) = \varnothing \in \Gamma(X)$. Let $p \neq 0$. Suppose that $A \in \Phi(x_p)$ and $\overline{\mathcal{R}}(A) \leq \overline{\mathcal{R}}(B)$ implies that $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(A)$ and $\overline{\mathcal{R}}(A) \leq \overline{\mathcal{R}}(B)$ which gives $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(B)$ by Lemma 3.8. Hence $B \in \Phi(x_p)$. Next, let $\overline{\mathcal{R}}(A) \vee \overline{\mathcal{R}}(B) \in \Phi(x_p)$. Then $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(A \vee B)$ which implies that $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(A)$ or $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(B)$, i.e., $\overline{\mathcal{R}}(A) \in \Phi(x_p)$ or $\overline{\mathcal{R}}(B) \in \Phi(x_p)$. Hence $\Phi(x_p)$ is an L-fuzzy rough grill in X. Conversely, let $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(A \vee B)$. Then $A \vee B \in \Phi(x_p)$ which implies $A \in \Phi(x_p)$ or $B \in \Phi(x_p)$. Consequently, $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(A)$ or $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(B)$. Clearly, $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(A)$ implies $A \in \Phi(x_p)$ which implies $p \neq 0$. Thus $\delta_{\mathcal{R}}$ is an L-fuzzy rough point-wise proximity on X. □

Our next result gives the relationship between ultrafilters and the relation Φ. For this, we need to define an atom of the lattice: An atom in L is a minimal element in $L\backslash\{0\}$ (see [15]).

Proposition 3.16. *If $\delta_{\mathcal{R}}$ is an L-fuzzy rough point-wise proximity on X, then*

$$\bigcap\{\mathfrak{U} \in \Omega(X) : A \in \mathfrak{U}\} \subset \Phi(x_p),\ \textit{for all } x_p \in A,\ p \textit{ is an atom of } L.$$

Proof. Let $\delta_{\mathcal{R}}$ be an L-fuzzy rough point-wise proximity on X and $B \in \bigcap\{\mathfrak{U} \in \Omega(X) : A \in \mathfrak{U}\}$. Then $B \in \mathfrak{U}$, for all $\mathfrak{U} \in \Omega(X)$ such that $A \in \mathfrak{U}$. Since $[x_p] = \{A \in L^X : x_p \in A,\ p \text{ is an atom of } L\}$ is an ultrafilter, therefore $B \in [x_p]$ for all $x_p \in A,\ p$ is an atom of L, which gives $\overline{\mathcal{R}}(x_p) \wedge \overline{\mathcal{R}}(B) \neq \mathbf{0}$, that is, $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(B)$, for all $x_p \in A,\ p$ is an atom of L. Hence $B \in \Phi(x_p)$. Thus, $\bigcap\{\mathfrak{U} \in \Omega(X) : A \in \mathfrak{U}\} \subset \Phi(x_p)$, for all $x_p \in A,\ p$ is an atom of L. □

Theorem 3.17. *Let $(X, \delta_{\mathcal{R}})$ be an L-fuzzy rough point wise proximity space. Then for $x_p \in L^X$,*

(1) $\mathbb{N}_{\delta_{\mathcal{R}}}(x_p) = \mathcal{K}(\Phi(x_p))$.
(2) $\mathbb{N}_{\delta_{\mathcal{R}}}(x_p) \subset \{A \in L^X : \overline{\mathcal{R}}(x_p) \leq \overline{\mathcal{R}}(A)\}$.

Proof. 1. Let $A \in \mathcal{K}(\Phi(x_p))$. Then $(\overline{\mathcal{R}}(A))^c \notin \Phi(x_p),\ x_p \in L^X$, that is, $\overline{\mathcal{R}}(x_p)\bar{\delta}_{\mathcal{R}}\overline{\mathcal{R}}((\overline{\mathcal{R}}(A))^c)$ which gives $\overline{\mathcal{R}}(x_p)\bar{\delta}_{\mathcal{R}}(\overline{\mathcal{R}}(A))^c$. Hence $A \in \mathbb{N}_{\delta_{\mathcal{R}}}(x_p)$. Conversely, let $A \in \mathbb{N}_{\delta_{\mathcal{R}}}(x_p)$. Then

$\overline{\mathcal{R}}(x_p)\overline{\delta}_{\mathcal{R}}(\overline{\mathcal{R}}(A))^c$, which implies that $(\overline{\mathcal{R}}(A))^c \notin \Phi(x_p)$. Thus $A \in \mathcal{K}(\Phi(x_p))$. Hence $\mathbb{N}_{\delta_{\mathcal{R}}}(x_p) = \mathcal{K}(\Phi(x_p)) = \{A \in L^X : (\overline{\mathcal{R}}(A))^c \notin \Phi(x_p)\}$.

2. Let $A \in \mathbb{N}_{\delta_{\mathcal{R}}}(x_p)$. Then $\overline{\mathcal{R}}(x_p)\overline{\delta}_{\mathcal{R}}(\overline{\mathcal{R}}(A))^c$ which implies that $\overline{\mathcal{R}}(x_p) \wedge (\overline{\mathcal{R}}(A))^c = \mathbf{0}$. $[\overline{\mathcal{R}}(x_p) \wedge (\overline{\mathcal{R}}(A))^c](y) = \mathbf{0}$, for all $y \in X$. Consequently, $\overline{\mathcal{R}}(x_p)(y) = 0$ or $(\overline{\mathcal{R}}(A))^c(y) = 0,\ y \in X$. If $\overline{\mathcal{R}}(x_p)(y) = 0$, then $\overline{\mathcal{R}}(x_p)(y) \leq \overline{\mathcal{R}}(A)(y),\ y \in X$. If $(\overline{\mathcal{R}}(A))^c(y) = 0$, then $\overline{\mathcal{R}}(x_p)(y) \leq 1 = \overline{\mathcal{R}}(A)(y),\ y \in X$. Thus $\mathbb{N}_{\delta_{\mathcal{R}}}(x_p) \subset \{A \in L^X : \overline{\mathcal{R}}(x_p) \leq \overline{\mathcal{R}}(A)\}$. □

Theorem 3.18. *Let $\delta_{\mathcal{R}}$ be an L-fuzzy rough point-wise proximity on X. Then for an L-fuzzy rough proximal neighborhood of the fuzzy points $x_p, x_q \in L^X$,*

$$\mathbb{N}_{\delta_{\mathcal{R}}}(x_p \vee x_q) = \mathbb{N}_{\delta_{\mathcal{R}}}(x_p) \cap \mathbb{N}_{\delta_{\mathcal{R}}}(x_q).$$

Proof. By definition of an L-fuzzy rough proximal neighborhood of fuzzy point x_p, $A \in \mathbb{N}_{\delta_{\mathcal{R}}}(x_p \vee x_q) = \mathbb{N}_{\delta_{\mathcal{R}}}(x_{p\vee q})$ if and only if $\overline{\mathcal{R}}(x_p \vee x_q)\overline{\delta}_{\mathcal{R}}(\overline{\mathcal{R}}(A))^c$ if and only if $\overline{\mathcal{R}}(x_p)\overline{\delta}_{\mathcal{R}}(\overline{\mathcal{R}}(A))^c$ and $\overline{\mathcal{R}}(x_q)\overline{\delta}_{\mathcal{R}}(\overline{\mathcal{R}}(A))^c$ if and only if $A \in \mathbb{N}_{\delta_{\mathcal{R}}}(x_p)$ and $A \in \mathbb{N}_{\delta_{\mathcal{R}}}(x_q)$ if and only if $A \in [\mathbb{N}_{\delta_{\mathcal{R}}}(x_p) \bigcap \mathbb{N}_{\delta_{\mathcal{R}}}(x_q)]$. Thus $\mathbb{N}_{\delta_{\mathcal{R}}}(x_p \vee x_q) = \mathbb{N}_{\delta_{\mathcal{R}}}(x_p) \cap \mathbb{N}_{\delta_{\mathcal{R}}}(x_q)$. □

Theorem 3.19. *Let $\delta_{\mathcal{R}}$ be an L-fuzzy rough point-wise proximity on X. For non-empty fuzzy sets $A, B \in L^X$, we have $\Phi(x_p \vee x_q) = \Phi(x_p) \cup \Phi(x_q)$.*

Proof. Let $A \in \Phi(x_{p\vee q})$ if and only if $x_{p\vee q} \in \Phi(A)$ if and only if $x_p \vee x_q \in \Phi(A)$ if and only if $x_p \in \Phi(A)$ or $x_q \in \Phi(A)$ if and only if $A \in \Phi(x_p)$ or $A \in \Phi(x_q)$ if and only if $A \in \Phi(x_p) \bigcup \Phi(x_q)$. Thus $\Phi(x_p \vee x_q) = \Phi(x_p) \cup \Phi(x_q)$. □

Theorem 3.20. *Let $\delta_{\mathcal{R}}$ be an L-fuzzy rough point-wise proximity on X and $\mathbf{G}$ be an L-fuzzy rough grill on X. Define $\mathfrak{S}_{\delta_{\mathcal{R}}}(\mathbf{G}) = \{A \in L^X : \mathbb{N}_{\delta_{\mathcal{R}}}(x_p) \subset \mathbf{G},$ for some $x_p \in \overline{\mathcal{R}}(A)\}$, where p is an atom of L. Then $\mathfrak{S}_{\delta_{\mathcal{R}}}(\mathbf{G})$ is an L-fuzzy rough grill on X.*

Proof. Clearly, $\mathbf{0} \notin \mathfrak{S}_{\delta_{\mathcal{R}}}(\mathbf{G})$. Next, let $B \in \mathfrak{S}_{\delta_{\mathcal{R}}}(\mathbf{G})$ and $\overline{\mathcal{R}}(B) \leq \overline{\mathcal{R}}(A)$. Then for some $x_p \in \overline{\mathcal{R}}(A)$, p is an atom of L, $\mathbb{N}_{\delta_{\mathcal{R}}}(x_p) \subset \mathbf{G}$. Hence $A \in \mathfrak{S}_{\delta_{\mathcal{R}}}(\mathbf{G})$. Finally, let $\mathbb{N}_{\delta_{\mathcal{R}}}(x_p) \subset \mathbf{G}$, for some $x_p \in \overline{\mathcal{R}}(A \vee B)$, p is an atom of L. Then $\mathbb{N}_{\delta_{\mathcal{R}}}(x_p) \subset \mathbf{G}$, for some $x_p \in \overline{\mathcal{R}}(A)$ or $x_p \in \overline{\mathcal{R}}(B)$ which gives $\overline{\mathcal{R}}(A) \in \mathfrak{S}_{\delta_{\mathcal{R}}}(\mathbf{G})$ or $\overline{\mathcal{R}}(B) \in \mathfrak{S}_{\delta_{\mathcal{R}}}(\mathbf{G})$. Thus $\mathfrak{S}_{\delta_{\mathcal{R}}}(\mathbf{G})$ is an L-fuzzy rough grill on X. □

4. L-fuzzy rough grills on an L-fuzzy rough point-wise proximity

In this section, we study the concepts of L-fuzzy rough grills, L-fuzzy rough $\delta_{\mathcal{R}}$-clusters and L-fuzzy rough $\delta_{\mathcal{R}}$-clans on an L-fuzzy rough point-wise proximity space and explore their relationships.

Definition 4.1. Let $\delta_{\mathcal{R}}$ be an L-fuzzy rough point-wise proximity on X. A subset Θ of L^X is called an L-fuzzy rough $\delta_{\mathcal{R}}$-cluster if the following conditions are satisfied:

(C.1) If $A, B \in \Theta$, then $\overline{\mathcal{R}}(A)\delta_{\mathcal{R}}\overline{\mathcal{R}}(B)$.
(C.2) If $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(B)$ for all $B \in \Theta$ and for some $x_p \in A$, then $A \in \Theta$.
(C.3) $\overline{\mathcal{R}}(A) \vee \overline{\mathcal{R}}(B) \in \Theta \implies \overline{\mathcal{R}}(A) \in \Theta$ or $\overline{\mathcal{R}}(B) \in \Theta$.

Definition 4.2. Let $\delta_{\mathcal{R}}$ be an L-fuzzy rough point-wise proximity on X. An L-fuzzy rough grill $\mathbf{G}$ on X is called an *L-fuzzy rough $\delta_{\mathcal{R}}$-clan* on X if $A, B \in \mathbf{G}$ implies that $\overline{\mathcal{R}}(A)\delta_{\mathcal{R}}\overline{\mathcal{R}}(B)$ i.e., $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(B)$, for some $x_p \in A$, i.e., $B \in \Phi(A)$.

Definition 4.3. An L-fuzzy rough $\delta_{\mathcal{R}}$-clan Θ on X is an *L-fuzzy rough $\delta_{\mathcal{R}}$-cluster* if and only if it satisfies the additional condition, $\Theta \subset \Phi(A)$ implies $A \in \Theta$.

Definition 4.4. Let $\delta_{\mathcal{R}}$ be a Čech L-fuzzy rough proximity on X. For $\mathbf{G} \subset L^X$, define

$$\mathfrak{B}_{\delta_{\mathcal{R}}}(\mathbf{G}) = \{A \in L^X : \mathcal{C}l_{\mathcal{R}}(\overline{\mathcal{R}}(A)) \in \mathbf{G}\}.$$

An L-fuzzy rough $\delta_{\mathcal{R}}$-clan $\mathbf{G}$ which satisfies the additional condition $\mathfrak{B}_{\delta_{\mathcal{R}}}(\mathbf{G}) = \mathbf{G}$ is called an *L-fuzzy rough $\delta_{\mathcal{R}}$-bunch.*

Theorem 4.5. *Let $\delta_{\mathcal{R}}$ is an L-fuzzy rough point-wise proximity on X, $\mathfrak{U} \in \Omega(X)$ and $A, B \in L^X$. Then $\mathfrak{U} \subset \Phi(\mathfrak{U})$.*

Proof. We know that $\Phi(\mathfrak{U}) = \bigcap\{\Phi(A) : A \in \mathfrak{U}\}$. Suppose $B \in \mathfrak{U}$. For all $A \in \mathfrak{U}$, $\overline{\mathcal{R}}(A) \wedge \overline{\mathcal{R}}(B) \neq \mathbf{0}$. Thus, there is an $x_p \in \overline{\mathcal{R}}(A)$ and $x_p \in \overline{\mathcal{R}}(B)$. That is, $B \in \Phi(A)$, for all $A \in \mathfrak{U}$. Hence $B \in \Phi(\mathfrak{U})$. Thus $\mathfrak{U} \subset \Phi(\mathfrak{U})$. □

Proposition 4.6. *Let $(X, \delta_{\mathcal{R}})$ be an L-fuzzy rough point-wise proximity space and $\mathbf{G}$ be an L-fuzzy rough $\delta_{\mathcal{R}}$-bunch. If $\mathfrak{U} \in \Omega(X)$ and $\mathfrak{U} \subset \mathbf{G}$, then $\mathbf{G} \subset \Phi(\mathfrak{U})$.*

Proof. Let $\mathbf{G}$ be an L-fuzzy rough $\delta_{\mathcal{R}}$-bunch, $\mathfrak{U} \in \Omega(X)$ and $\mathfrak{U} \subset \mathbf{G}$. Then, for every $A \in \mathfrak{U}$, $\mathbf{G} \subset \Phi(A)$ which implies that $\mathbf{G} \subset \Phi(\mathfrak{U})$ (equivalently). □

Theorem 4.7. *Let $(X, \delta_{\mathcal{R}})$ be an L-fuzzy rough point-wise proximity space. Then for an L-fuzzy rough $\delta_{\mathcal{R}}$-clan $\mathbf{G}$, we have $\mathbf{G} \subset \bigcap\{\Phi(A) : A \in \mathbf{G}\}$.*

Proof. Suppose that $B \notin \bigcap\{\Phi(A) : A \in \mathbf{G}\}$. Then $\overline{\mathcal{R}}(x_p)\bar{\delta}_{\mathcal{R}}\overline{\mathcal{R}}(B)$, for some $A \in \mathbf{G}$ and for all $x_p \in A$ which gives $\overline{\mathcal{R}}(A)\bar{\delta}_{\mathcal{R}}\overline{\mathcal{R}}(B)$. Since $\mathbf{G}$ is an L-fuzzy rough $\delta_{\mathcal{R}}$-clan, therefore $B \notin \mathbf{G}$. Hence, $\mathbf{G} \subset \bigcap\{\Phi(A) : A \in \mathbf{G}\}$. □

Theorem 4.8. *Let $\delta_{\mathcal{R}}$ is an L-fuzzy rough point-wise proximity on X, $\mathfrak{U} \in \Omega(X)$ and $A, B, C \in L^X$. Then $\Phi(\mathfrak{U})$ is an L-fuzzy rough grill, i.e., $\Phi(\mathfrak{U}) \in \Gamma(X)$.*

Proof. Clearly, $\mathbf{0} \notin \Phi(\mathfrak{U})$. Let $B \in \Phi(\mathfrak{U})$ and $\overline{\mathcal{R}}(B) \leq \overline{\mathcal{R}}(C)$. Then $B \in \Phi(A)$ for all $A \in \mathfrak{U}$ which implies that $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(B)$ for some $x_p \in A$. Consequently, $C \in \Phi(A)$ for all $A \in \mathfrak{U}$ which gives $C \in \Phi(\mathfrak{U})$. Finally, let $\overline{\mathcal{R}}(B) \notin \Phi(\mathfrak{U})$ and $\overline{\mathcal{R}}(C) \notin \Phi(\mathfrak{U})$. Then there exist $A_1, A_2 \in \mathfrak{U}$ such that $\overline{\mathcal{R}}(B) \notin \Phi(A_1)$ and $\overline{\mathcal{R}}(C) \notin \Phi(A_2)$ which implies that $\overline{\mathcal{R}}(A_1)\overline{\delta}_{\mathcal{R}}\overline{\mathcal{R}}(\overline{\mathcal{R}}(B))$ and $\overline{\mathcal{R}}(A_2)\overline{\delta}_{\mathcal{R}}\overline{\mathcal{R}}(\overline{\mathcal{R}}(C))$ which implies $\overline{\mathcal{R}}(x_p)\overline{\delta}_{\mathcal{R}}\overline{\mathcal{R}}(B)$ for all $x_p \in \overline{\mathcal{R}}(A_1)$ and $\overline{\mathcal{R}}(y_q)\overline{\delta}_{\mathcal{R}}\overline{\mathcal{R}}(C)$ for all $y_q \in \overline{\mathcal{R}}(A_2)$ implies that $\overline{\mathcal{R}}(x_p)\overline{\delta}_{\mathcal{R}}\overline{\mathcal{R}}(B)$ and $\overline{\mathcal{R}}(x_p)\overline{\delta}_{\mathcal{R}}\overline{\mathcal{R}}(C)$ for all $x_p \in \overline{\mathcal{R}}(A_1) \wedge \overline{\mathcal{R}}(A_2)$. Consequently, $\overline{\mathcal{R}}(x_p)\overline{\delta}_{\mathcal{R}}\overline{\mathcal{R}}(\overline{\mathcal{R}}(B)) \vee \overline{\mathcal{R}}(C))$ for all $x_p \in \overline{\mathcal{R}}(A_1) \wedge \overline{\mathcal{R}}(A_2)$ which gives $\overline{\mathcal{R}}(B)) \vee \overline{\mathcal{R}}(C) \notin \Phi(\overline{\mathcal{R}}(A_1) \wedge \overline{\mathcal{R}}(A_2))$ which implies that $\overline{\mathcal{R}}(B) \vee \overline{\mathcal{R}}(C) \notin \Phi(\mathfrak{U})$. Thus $\Phi(\mathfrak{U}) \in \Gamma(X)$, i.e., $\Phi(\mathfrak{U})$ is an L-fuzzy rough grill. □

Theorem 4.9. *Let $(X, \delta_{\mathcal{R}})$ be an L-fuzzy rough point-wise proximity space. If Θ is an L-fuzzy rough $\delta_{\mathcal{R}}$-cluster, then $\Theta = \bigcap\{\Phi(A) : A \in \Theta\}$.*

Proof. Let $B \in \bigcap\{\Phi(A) : A \in \Theta\}$. Then for every $A \in \Theta$, $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(B)$ for some $x_p \in A$. Since Θ is an L-fuzzy rough $\delta_{\mathcal{R}}$-cluster, therefore for every $A \in \Theta$, $\overline{\mathcal{R}}(A)\delta_{\mathcal{R}}\overline{\mathcal{R}}(B)$ which implies $B \in \Theta$. So, $\bigcap\{\Phi(A) : A \in \Theta\} \subset \Theta$. Let $B \in \Theta$ and Θ is an L-fuzzy rough $\delta_{\mathcal{R}}$-cluster. Then $B \in \Phi(A)$, for all $A \in \Theta$ which gives $B \in \bigcap\{\Phi(A) : A \in \Theta\}$. Thus $\Theta \subset \bigcap\{\Phi(A) : A \in \Theta\}$. Hence $\Theta = \bigcap\{\Phi(A) : A \in \Theta\}$. □

Theorem 4.10. *Let $(X, \delta_{\mathcal{R}})$ be an L-fuzzy rough point-wise proximity space. For an L-fuzzy rough $\delta_{\mathcal{R}}$-cluster Θ, we have $\Theta = \bigcap\{\Phi(A) : \Theta \subset \Phi(A)\}$.*

Proof. Let Θ be an L-fuzzy rough $\delta_{\mathcal{R}}$-cluster. Define a set $\mathfrak{P} = \{A \in L^X : \Theta \subset \Phi(A)\}$, where $\Phi(A) = \{B \in L^X : \overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(B)$, for some $x_p \in A\}$. Clearly, $\Theta \subset \bigcap\{\Phi(A) : A \in \mathfrak{P}\}$. Since Θ is an L-fuzzy rough $\delta_{\mathcal{R}}$-cluster, for every $A \in \Theta$, $\Theta \subset \Phi(A)$, implies that $\Theta \subset \mathfrak{P}$. Hence $\bigcap\{\Phi(A) : A \in \mathfrak{P}\} \subset \bigcap\{\Phi(A) : A \in \Theta\}$ which implies $\bigcap\{\Phi(A) : A \in \mathfrak{P}\} \subset \Theta$ by Theorem 4.9, which gives $\Theta = \bigcap\{\Phi(A) : A \in \mathfrak{P}\}$. Thus $\Theta = \bigcap\{\Phi(A) : \Theta \subset \Phi(A)\}$. □

Definition 4.11. Let $\delta_{\mathcal{R}}$ be an L-fuzzy rough point-wise proximity on X which satisfies the following axiom, for non-empty sets $A, B, C \in L^X$:

(F.5) $\overline{\mathcal{R}}(A)\delta_{\mathcal{R}}\overline{\mathcal{R}}(B)$ and $\overline{\mathcal{R}}(C)\delta_{\mathcal{R}}\overline{\mathcal{R}}(x_p)$ for every $x_p \leq \overline{\mathcal{R}}(A)$ together implies $\overline{\mathcal{R}}(B)\delta_{\mathcal{R}}\overline{\mathcal{R}}(C)$.

Then $\delta_{\mathcal{R}}$ is called an *L-fuzzy rough point-wise LO-proximity* on X and the pair $(X, \delta_{\mathcal{R}})$ is known as an *L-fuzzy rough point-wise LO-proximity space.* It may be verified that an L-fuzzy rough point-wise LO-proximity $\delta_{\mathcal{R}}$ generates a Kuratowski L-fuzzy rough closure operator $\mathcal{C}l_{\mathcal{R}} : L^X \to L^X$ as defined in Theorem 3.7, which is a topological closure operator on X (see [1]).

Theorem 4.12. *Let $(X, \delta_{\mathcal{R}})$ be an L-fuzzy rough point-wise LO-proximity space and $\mathcal{C}l_{\mathcal{R}}$ be the L-fuzzy rough closure operator generated by $\delta_{\mathcal{R}}$ as defined in Theorem 3.7. Then $\mathcal{C}l_{\mathcal{R}}(B) \in \Phi(A)$ if and only if $B \in \Phi(A)$.*

Proof. Let $B \in \Phi(A)$. Then $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(B)$ for some $x_p \in A$ which implies $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(\mathcal{C}l_{\mathcal{R}}(B))$ for some $x_p \in A$. Consequently, $\mathcal{C}l_{\mathcal{R}}(B) \in \Phi(A)$. Conversely, let $\mathcal{C}l_{\mathcal{R}}(B) \in \Phi(A)$. Then $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(\mathcal{C}l_{\mathcal{R}}(B))$ for some $x_p \in A$. Consequently, $\overline{\mathcal{R}}(\mathcal{C}l_{\mathcal{R}}(B))\delta_{\mathcal{R}}\overline{\mathcal{R}}(x_p)$ for some $x_p \in A$. By (F.5), $\overline{\mathcal{R}}(\mathcal{C}l_{\mathcal{R}}(B))\delta_{\mathcal{R}}\overline{\mathcal{R}}(x_p)$ and $\overline{\mathcal{R}}(B)\delta_{\mathcal{R}}\overline{\mathcal{R}}(y_q)$ for every $y_q \leq \overline{\mathcal{R}}(\mathcal{C}l_{\mathcal{R}}(B))$ implies that $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(B)$ for some $x_p \in A$. Hence $B \in \Phi(A)$. $\square$

Theorem 4.13.

(1) *If $\mathbf{G}$ is an L-fuzzy rough grill, then $\mathfrak{B}_{\delta_{\mathcal{R}}}(\mathbf{G})$ is an L-fuzzy rough grill and $\mathbf{G} \subset \mathfrak{B}_{\delta_{\mathcal{R}}}(\mathbf{G})$.*

(2) *If $\mathbf{G}_1$ and $\mathbf{G}_2$ are two L-fuzzy rough grills such that $\mathbf{G}_2 \subset \mathbf{G}_1$, then $\mathfrak{B}_{\delta_{\mathcal{R}}}(\mathbf{G}_2) \subset \mathfrak{B}_{\delta_{\mathcal{R}}}(\mathbf{G}_1)$.*

Proof. 1. Let $\mathbf{G}$ be an L-fuzzy rough grill. Then $\mathbf{0} \notin \mathfrak{B}_{\delta_{\mathcal{R}}}(\mathbf{G})$ Let $A \in \mathfrak{B}_{\delta_{\mathcal{R}}}(\mathbf{G})$ and $\overline{\mathcal{R}}(A) \leq \overline{\mathcal{R}}(B)$. Then $\mathcal{C}l_{\delta_{\mathcal{R}}}(\overline{\mathcal{R}}(A)) \leq \mathcal{C}l_{\delta_{\mathcal{R}}}(\overline{\mathcal{R}}(B))$ which implies $\overline{\mathcal{R}}(\mathcal{C}l_{\delta_{\mathcal{R}}}(\overline{\mathcal{R}}(A))) \leq \overline{\mathcal{R}}(\mathcal{C}l_{\delta_{\mathcal{R}}}(\overline{\mathcal{R}}(B)))$. Consequently, $\mathcal{C}l_{\delta_{\mathcal{R}}}(\overline{\mathcal{R}}(B)) \in \mathbf{G}$. Thus $B \in \mathfrak{B}_{\delta_{\mathcal{R}}}(\mathbf{G})$. Next, let $\overline{\mathcal{R}}(A) \vee \overline{\mathcal{R}}(B) \in \mathfrak{B}_{\delta_{\mathcal{R}}}(\mathbf{G})$. Then $\mathcal{C}l_{\delta_{\mathcal{R}}}(\overline{\mathcal{R}}(\overline{\mathcal{R}}(A)) \vee \overline{\mathcal{R}}(\overline{\mathcal{R}}(B))) \in \mathbf{G}$ which implies that $\mathcal{C}l_{\delta_{\mathcal{R}}}(\overline{\mathcal{R}}(\overline{\mathcal{R}}(A))) \in \mathbf{G}$ or $\mathcal{C}l_{\delta_{\mathcal{R}}}(\overline{\mathcal{R}}(\overline{\mathcal{R}}(B))) \in \mathbf{G}$. Consequently, $\overline{\mathcal{R}}(A) \in \mathfrak{B}_{\delta_{\mathcal{R}}}(\mathbf{G})$ or $\overline{\mathcal{R}}(B) \in \mathfrak{B}_{\delta_{\mathcal{R}}}(\mathbf{G})$. Hence $\mathfrak{B}_{\delta_{\mathcal{R}}}(\mathbf{G})$ is an L-fuzzy rough grill. Next, let $A \in \mathbf{G}$. Then $\overline{\mathcal{R}}(A) \in \mathbf{G}$ which implies $\mathcal{C}l_{\delta_{\mathcal{R}}}(\overline{\mathcal{R}}(A)) \in \mathbf{G}$. Consequently, $A \in \mathfrak{B}_{\delta_{\mathcal{R}}}(\mathbf{G})$. Hence $\mathbf{G} \subset \mathfrak{B}_{\delta_{\mathcal{R}}}(\mathbf{G})$.

2. Suppose that $\mathbf{G}_1$ and $\mathbf{G}_2$ are two L-fuzzy rough sets such that $\mathbf{G}_2 \subset \mathbf{G}_1$. Let $A \in \mathfrak{B}_{\delta_{\mathcal{R}}}(\mathbf{G}_2)$. Then $\mathcal{C}l_{\delta_{\mathcal{R}}}(\overline{\mathcal{R}}(A)) \in \mathbf{G}_2$ which gives $\mathcal{C}l_{\delta_{\mathcal{R}}}(\overline{\mathcal{R}}(A)) \in \mathbf{G}_1$. Consequently, $A \in \mathfrak{B}_{\delta_{\mathcal{R}}}(\mathbf{G}_1)$. Hence $\mathfrak{B}_{\delta_{\mathcal{R}}}(\mathbf{G}_2) \subset \mathfrak{B}_{\delta_{\mathcal{R}}}(\mathbf{G}_1)$.

$\square$

Theorem 4.14. *Let $(X, \delta_{\mathcal{R}})$ be an L-fuzzy rough point-wise LO-proximity space. Then every L-fuzzy rough $\delta_{\mathcal{R}}$-cluster is an L-fuzzy rough $\delta_{\mathcal{R}}$-bunch.*

Proof. Let $\mathbf{G}$ be an L-fuzzy rough $\delta_{\mathcal{R}}$-cluster. Then by Theorem 4.13, $\mathfrak{B}_{\delta_{\mathcal{R}}}(\mathbf{G})$ is an L-fuzzy rough grill and $\mathbf{G} \subset \mathfrak{B}_{\delta_{\mathcal{R}}}(\mathbf{G})$. Let $A \in \mathfrak{B}_{\delta_{\mathcal{R}}}(\mathbf{G})$ and $B \in \mathbf{G}$. Since $\mathbf{G}$ is an L-fuzzy rough $\delta_{\mathcal{R}}$-clan, therefore $\mathcal{C}l_{\delta_{\mathcal{R}}}(\overline{\mathcal{R}}(A)) \in \mathbf{G}$, or $\mathcal{C}l_{\mathcal{R}}(\overline{\mathcal{R}}(A)) \in \Phi(B)$ which gives $A \in \Phi(B)$ which implies that $\overline{\mathcal{R}}(A)\delta_{\mathcal{R}}\overline{\mathcal{R}}(B)$. Consequently, $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(B)$ for some $x_p \in A$. Thus $B \in \Phi(A)$. Hence

$\mathbf{G} \subset \Phi(A)$. Since $\mathbf{G}$ is an L-fuzzy rough $\delta_{\mathcal{R}}$-cluster, it follows that $A \in \mathbf{G}$. Hence $\mathfrak{B}_{\delta_{\mathcal{R}}}(\mathbf{G}) \subset \mathbf{G}$. Thus $\mathfrak{B}_{\delta_{\mathcal{R}}}(\mathbf{G}) = \mathbf{G}$. □

Theorem 4.15. *Let (X, δ_R) be an L-fuzzy rough point-wise LO-proximity space, $\mathfrak{U}$ be an L-fuzzy rough ultrafilter in X. Then $\mathfrak{B}_{\delta_{\mathcal{R}}}(\mathfrak{U})$ is an L-fuzzy rough $\delta_{\mathcal{R}}$-clan.*

Proof. Since every L-fuzzy rough ultrafilter is an L-fuzzy rough grill, therefore $\mathfrak{U}$ is an L-fuzzy rough grill. Also $\mathfrak{B}_{\delta_{\mathcal{R}}}(\mathfrak{U})$ is an L-fuzzy rough grill and $\mathfrak{U} \subset \mathfrak{B}_{\delta_{\mathcal{R}}}(\mathfrak{U})$. Suppose that $A, B \in \mathfrak{B}_{\delta_{\mathcal{R}}}(\mathfrak{U})$. Then $\mathcal{C}l_{\mathcal{R}}(\overline{\mathcal{R}}(A)) \in \mathfrak{U}$ and $\mathcal{C}l_{\mathcal{R}}(\overline{\mathcal{R}}(B)) \in \mathfrak{U}$. Hence $\overline{\mathcal{R}}(\mathcal{C}l_{\mathcal{R}}(\overline{\mathcal{R}}(A))) \wedge \overline{\mathcal{R}}(\mathcal{C}l_{\mathcal{R}}(\overline{\mathcal{R}}(B))) \neq \mathbf{0}$, therefore there exist $x_p \in \mathcal{C}l_{\mathcal{R}}(\overline{\mathcal{R}}(A))$ and $x_p \in \mathcal{C}l_{\mathcal{R}}(\overline{\mathcal{R}}(B))$ which gives $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(\mathcal{C}l_{\mathcal{R}}(\overline{\mathcal{R}}(A)))$ for some $x_p \in \mathcal{C}l_{\mathcal{R}}(\overline{\mathcal{R}}(B))$. Consequently, $\overline{\mathcal{R}}(\mathcal{C}l_{\mathcal{R}}(\overline{\mathcal{R}}(A)))\delta_{\mathcal{R}}\overline{\mathcal{R}}(\mathcal{C}l_{\mathcal{R}}(\overline{\mathcal{R}}(B)))$ which implies $\mathcal{C}l_{\mathcal{R}}(\overline{\mathcal{R}}(A)) \in \Phi(\mathcal{C}l_{\mathcal{R}}(\overline{\mathcal{R}}(B)))$ implies $\overline{\mathcal{R}}(B)\delta_{\mathcal{R}}\overline{\mathcal{R}}(A)$, by Theorem 4.12. Hence $\mathfrak{B}_{\delta_{\mathcal{R}}}(\mathfrak{U})$ is an L-fuzzy rough $\delta_{\mathcal{R}}$-clan. □

Theorem 4.16. *Let $\delta_{\mathcal{R}}$ be an L-fuzzy rough point-wise LO-proximity on X and $\mathbf{G}$ is an L-fuzzy rough $\delta_{\mathcal{R}}$-clan. Then $\mathfrak{B}_{\delta_{\mathcal{R}}}(\mathbf{G})$ is an L-fuzzy rough $\delta_{\mathcal{R}}$-clan.*

Proof. Let $B, C \in \mathbf{G}$. Then $\overline{\mathcal{R}}(B)\delta_{\mathcal{R}}\overline{\mathcal{R}}(C)$, which implies $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(C)$, for some $x_p \in B$ which gives $C \in \Phi(B)$. Consequently, $\mathbf{G} \subset \Phi(B)$. Let $B, C \in \mathfrak{B}_{\delta_{\mathcal{R}}}(\mathbf{G})$. Then $\mathcal{C}l_{\mathcal{R}}(\overline{\mathcal{R}}(B)) \in \mathbf{G}$ and $\mathcal{C}l_{\mathcal{R}}(\overline{\mathcal{R}}(C)) \in \mathbf{G}$. Thus, $\mathcal{C}l_{\mathcal{R}}(\overline{\mathcal{R}}(B)) \in \Phi(\mathcal{C}l_{\mathcal{R}}(\overline{\mathcal{R}}(C)))$ which gives $B \in \Phi(\mathcal{C}l_{\mathcal{R}}(\overline{\mathcal{R}}(C)))$. Consequently, $\mathcal{C}l_{\mathcal{R}}(\overline{\mathcal{R}}(C)) \in \Phi(B)$ which implies that $C \in \Phi(B)$. That is, $\overline{\mathcal{R}}(C)\delta_{\mathcal{R}}\overline{\mathcal{R}}(B)$. Hence $\mathfrak{B}_{\delta_{\mathcal{R}}}(\mathbf{G})$ is an L-fuzzy rough $\delta_{\mathcal{R}}$-clan. □

Theorem 4.17. *An L-fuzzy rough point-wise proximity $\delta_{\mathcal{R}}$ on X is an L-fuzzy rough point-wise LO-proximity if and only if $\mathfrak{B}_{\delta_{\mathcal{R}}}(\Phi(A)) = \Phi(A)$, for every $A \in L^X$.*

Proof. Let $\delta_{\mathcal{R}}$ be an L-fuzzy rough point-wise LO-proximity on X and $B \in \mathfrak{B}_{\delta_{\mathcal{R}}}(\Phi(A))$. Then $\mathcal{C}l_{\mathcal{R}}(\overline{\mathcal{R}}(B)) \in \Phi(A)$ which implies that $B \in \Phi(A)$. Thus $\mathfrak{B}_{\delta_{\mathcal{R}}}(\Phi(A)) \subset \Phi(A)$. Next, let $B \in \Phi(A)$. Then $\overline{\mathcal{R}}(B) \in \Phi(A)$ which implies $\mathcal{C}l_{\mathcal{R}}(\overline{\mathcal{R}}(B)) \in \Phi(A)$ which gives $B \in \mathfrak{B}_{\delta_{\mathcal{R}}}(\Phi(A))$. Thus $\Phi(A) \subset \mathfrak{B}_{\delta_{\mathcal{R}}}(\Phi(A))$. Hence $\mathfrak{B}_{\delta_{\mathcal{R}}}(\Phi(A)) = \Phi(A)$.

Conversely, suppose that $\delta_{\mathcal{R}}$ is an L-fuzzy rough point-wise proximity on X and $\mathfrak{B}_{\delta_{\mathcal{R}}}(\Phi(A)) = \Phi(A)$, for all $A \in L^X$. Suppose $\overline{\mathcal{R}}(B)\delta_{\mathcal{R}}\overline{\mathcal{R}}(D)$ and $\overline{\mathcal{R}}(C)\delta_{\mathcal{R}}\overline{\mathcal{R}}(x_p)$ for all $x_p \leq \overline{\mathcal{R}}(B)$. By Theorem 3.7, we have $B \leq \bigvee\{x_p \in L^X : \overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(B)\} \leq \bigvee\{x_p \in L^X : \overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(C)\}$ which implies that $B \leq \mathcal{C}l_{\mathcal{R}}(B) \leq \mathcal{C}l_{\mathcal{R}}(C)$. Consequently, $B \leq \mathcal{C}l_{\mathcal{R}}(\overline{\mathcal{R}}(C))$. Since $\overline{\mathcal{R}}(B)\delta_{\mathcal{R}}\overline{\mathcal{R}}(D)$ and $\overline{\mathcal{R}}(B) \leq \overline{\mathcal{R}}(\mathcal{C}l_{\mathcal{R}}(\overline{\mathcal{R}}(C)))$, therefore $\overline{\mathcal{R}}(D)\delta_{\mathcal{R}}\overline{\mathcal{R}}(\mathcal{C}l_{\mathcal{R}}(\overline{\mathcal{R}}(C)))$ which gives $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(\mathcal{C}l_{\mathcal{R}}(\overline{\mathcal{R}}(C)))$ for some $x_p \in \overline{\mathcal{R}}(D)$. Consequently, $\mathcal{C}l_{\mathcal{R}}(\overline{\mathcal{R}}(C)) \in \Phi(\overline{\mathcal{R}}(D)) = \Phi(D) = \mathfrak{B}_{\delta_{\mathcal{R}}}(\Phi(D))$. Hence $C \in \Phi(D)$. Consequently, $\overline{\mathcal{R}}(x_p)\delta_{\mathcal{R}}\overline{\mathcal{R}}(D)$, for some $x_p \in C$. Thus $\overline{\mathcal{R}}(C)\delta_{\mathcal{R}}\overline{\mathcal{R}}(D)$. Hence $\delta_{\mathcal{R}}$ is an L-fuzzy rough point-wise LO-proximity on X. □

Theorem 4.18. *Let $(X, \delta_{\mathcal{R}})$ be an L-fuzzy rough point-wise LO-proximity space. Then for $A \in L^X$ and $\mathfrak{U} \in \Omega(X)$, we have $\Phi(A) = \bigcup\{\Phi(\mathfrak{U}) : A \in \mathfrak{U}\}$.*

Proof. Let $B \notin \Phi(A)$. Then $\overline{\mathcal{R}}(x_p)\bar{\delta}_{\mathcal{R}}\overline{\mathcal{R}}(B)$, for all $x_p \in A$. That is, $B \notin \Phi(\mathfrak{U})$ for every $\mathfrak{U} \in \Omega(X)$ such that $A \in \mathfrak{U}$ which gives $B \notin \bigcup\{\Phi(\mathfrak{U}) : A \in \mathfrak{U}\}$. So, $\bigcup\{\Phi(\mathfrak{U}) : A \in \mathfrak{U}\} \subset \Phi(A)$. Next, let $B \notin \bigcup\{\Phi(\mathfrak{U}) : A \in \mathfrak{U}\}$. Then for all $\mathfrak{U} \in \Omega(X)$ such that $A \in \mathfrak{U}$ there exists $C \in \mathfrak{U}$ such that $B \notin \Phi(C)$ which implies that $\overline{\mathcal{R}}(x_p)\bar{\delta}_{\mathcal{R}}\overline{\mathcal{R}}(B)$, for all $x_p \in C$. Since $A, C \in \mathfrak{U}$, therefore $\overline{\mathcal{R}}(A) \wedge \overline{\mathcal{R}}(C) \neq \mathbf{0}$. Hence there exists $x_p \leq \overline{\mathcal{R}}(A) \wedge \overline{\mathcal{R}}(C) \leq \overline{\mathcal{R}}(A)$ such that $\overline{\mathcal{R}}(C)\delta_{\mathcal{R}}\overline{\mathcal{R}}(x_p)$ which gives $B \notin \Phi(A)$, by (F.5). Thus $\Phi(A) = \bigcup\{\Phi(\mathfrak{U}) : A \in \mathfrak{U}\}$. □

5. Conclusion

The topological structure on an L-fuzzy approximation space that the current theory provides us essentially depends upon the proximity of an L-fuzzy upper approximations. The main work done in this paper is the study of an L-fuzzy rough point-wise proximity spaces, L-fuzzy rough grills, L-fuzzy rough $\delta_{\mathcal{R}}$-clusters and L-fuzzy rough $\delta_{\mathcal{R}}$-clans and their relationships. The study of rough point-wise proximity spaces is missing in the literature, in the framework of rough set theory. So, this study is a unified study of rough set theory and L-fuzzy rough set theory. In this article, we have given a brief account of the discussion on an L-fuzzy closure space and an L-fuzzy rough point-wise proximity space in the framework of an L-fuzzy rough set theory.

It is well-known that the theory of grills and clusters are foundations to study compactness and completeness properties of proximity spaces [19, 30]. In the field of digital image analysis, proximity structures have various applications such as image comparison, image contraction, image classification, etc. (see [21, 22, 24, 29, 31]). The future work employing this paper can be to find various real-life applications of L-fuzzy rough point-wise proximity spaces. Moreover, the study done in this paper opens the door to study compactness and completeness properties of L-fuzzy rough point-wise proximity spaces.

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