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Mappings contracting transverse axis of hyperbola

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Abstract

In this paper a new contractive type mapping known as mapping contracting transverse axis of hyperbola is introduced. This mapping can reduce the length of transverse axis of a hyperbola. This is a geometric technique that is connected to the study of geometric characteristics of certain curves defined over metric spaces. The paper is decorated by some suitable examples that support our proven results and showing the distinctness of our mapping from the usual contractive type mappings. Our proposed mapping also admits discontinuity at fixed point, thus gives a new solution to an open problem posed by B.E. Rhoades. Finally a geometric figure is illustrated to describe the speciality of our mapping.

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1. Introduction and Preliminaries

If fixed point set of a self mapping is not singleton then it can include some geometrical figures like a circle, disc or different types of conics. In the year 2017, Özgür and Taş [5, 6] first initiated the concept of fixed circle theory on arbitrary metric spaces. This topic has been generalized in many ways nowadays by several researchers (see [10]). Some new types of activation functions are there which may fix a circle for a complex valued neural network (CVNN). In the year 2021, Joshi et al. [4] introduced the notion of a fixed ellipse in metric spaces as a continuation of work on fixed geometric

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shapes. Two dimensional simple geometric figures like circles and conics have different applications in physics, astronomy, biology, neural networks, economics, artificial intelligence, and so on.
It is well known that the famous Banach contraction [1] reduces the radius of a circle with center as it's unique fixed point i.e. it reduces perimeter of this circle also. In the year 2023, E. Petrov have introduced mapping contracting perimeters of triangles [9], by which he showed that it is possible to shrink the lengths of sides of a given triangle using such mappings. Recently, K. Roy has given a concept of mapping (see [11]) which can decrease the lengths of axes of an ellipse. Look at the following definitions.

Definition 1.1 (Mapping contracting perimeters of triangles [9])**.** A self mapping $\mathcal{T}$ defined on a metric space $(\mathcal{B}, D)$ with $|\mathcal{B}| \geq 3$ is called a mapping contracting perimeters of triangles if there exists $k_1 \in [0, 1)$ such that the inequality

$$D(\mathcal{T}x, \mathcal{T}y) + D(\mathcal{T}y, \mathcal{T}z) + D(\mathcal{T}z, \mathcal{T}x) \leq k_1[D(x, y) + D(y, z) + D(z, x)] \tag{1.1}$$

holds for all three pairwise distinct points $x, y, z \in \mathcal{B}$.

Definition 1.2 (Mapping contracting axes of ellipse [11])**.** Let $(\mathcal{B}, D)$ be a metric space and $\alpha, \beta \in \mathcal{B}$ be two distinct elements. $\mathcal{T} : \mathcal{B} \to \mathcal{B}$ is said to be a mapping contracting axes of ellipse if for all $\xi \in \mathcal{B} \setminus \{\alpha, \beta\}$ with $\xi \neq \mathcal{T}\xi$,

$$D(\mathcal{T}\xi, \alpha) + D(\mathcal{T}\xi, \beta) \leq k(D(\xi, \alpha) + D(\xi, \beta)),\ 0 \leq k < 1. \tag{1.2}$$

Inspired from the above, in this manuscript we introduce a mapping $\mathcal{T}$ which can contract the transverse axis of a hyperbola.

2. Mappings contracting transverse axis of hyperbola

Definition 2.1. Let $(\mathcal{B}, D)$ be a metric space and $\lambda, \mu \in \mathcal{B}$ be two distinct elements. $\mathcal{T} : \mathcal{B} \to \mathcal{B}$ is said to be a mapping contracting transverse axis of hyperbola if for all $\xi(\neq \mathcal{T}\xi) \in \mathcal{B} \setminus \{\lambda, \mu\}$,

$$D(\xi, \mathcal{T}\xi) + |D(\mathcal{T}\xi, \lambda) - D(\mathcal{T}\xi, \mu)| \leq r|D(\xi, \lambda) - D(\xi, \mu)|,\ 0 \leq r < 1. \tag{2.1}$$

Example 2.2. Let us consider $\mathcal{B} = \{-2, -1, 0, 1, 2\}$ with the usual real metric d_u. Also let $\mathcal{T} : \mathcal{B} \to \mathcal{B}$ be defined as $\mathcal{T}(\xi) = \begin{cases} 0 & \text{if } \xi = -1 \text{ or } 1, \\ \xi & \text{otherwise.} \end{cases}$ and $\lambda = -2$, $\mu = 2$. Clearly $\mathcal{B} \setminus \{\lambda, \mu\} = \{-1, 0, 1\}$ and $\{-1, 1\} \nsubseteq Fix(\mathcal{T})$.

For $\xi = -1$,

$$\begin{aligned}
&d_u(\xi, \mathcal{T}\xi) + |d_u(\mathcal{T}\xi, \lambda) - d_u(\mathcal{T}\xi, \mu)| \\
=&d_u(-1, 0) + |d_u(0, -2) - d_u(0, 2)| = 1 \text{ and} \\
&|d_u(\xi, \lambda) - d_u(\xi, \mu)| \\
=&|d_u(-1, -2) - d_u(-1, 2)| = |1 - 3| = 2.
\end{aligned}$$

For $\xi = 1$,

$$\begin{aligned}
&d_u(\xi, \mathcal{T}\xi) + |d_u(\mathcal{T}\xi, \lambda) - d_u(\mathcal{T}\xi, \mu)| \\
=&d_u(1, 0) + |d_u(0, -2) - d_u(0, 2)| = 1 \text{ and} \\
&|d_u(\xi, \lambda) - d_u(\xi, \mu)| \\
=&|d_u(1, -2) - d_u(1, 2)| = |3 - 1| = 2.
\end{aligned}$$

Then it can be verified that $\mathcal{T}$ is a mapping contracting transverse axis of hyperbola for $\lambda = -2$, $\mu = 2$ and $r = \frac{2}{3}$.

Before going to our main theorem, first we recall the definitions of orbital continuity and p-continuity; $p \geq 1$ of a mapping. For a self mapping $\mathcal{T}$ over a metric space $(\mathcal{B}, D)$ the set $O(\xi, \mathcal{T}) := \{\mathcal{T}^n\xi : n = 0, 1, 2 \cdots\}$, $\xi \in \mathcal{B}$, is called an orbit of the mapping $\mathcal{T}$.

Definition 2.3. Let $(\mathcal{B}, D)$ be a metric space. A mapping $\mathcal{T} : \mathcal{B} \to \mathcal{B}$ is called

(1) *Orbitally continuous* [3] at a point $\mu \in \mathcal{B}$ if for any sequence $\{\xi_n\} \subset O(\xi, \mathcal{T})$ for some $\xi \in \mathcal{B}$, $\xi_n \to \mu$ implies $\mathcal{T}\xi_n \to \mathcal{T}\mu$ as $n \to \infty$.

(2) *p-continuous* [7], for $p = 1, 2, 3, \ldots$, if $\lim_{n\to\infty} \mathcal{T}^p\xi_n = \mathcal{T}\mu$, whenever $\{\xi_n\}$ is a sequence in $\mathcal{B}$ such that $\lim_{n\to\infty} \mathcal{T}^{p-1}\xi_n = \mu$ for $\mu \in \mathcal{B}$.

For more informations on such weaker forms of continuity one can see the paper [2]. In the aforesaid paper the author compares various weaker forms of continuity and explores their importance in fixed point theory.

Theorem 2.4. *Let $(\mathcal{B}, D)$ be a complete metric space, $\lambda, \mu \in \mathcal{B}$ be two distinct elements and $\mathcal{T} : \mathcal{B} \to \mathcal{B}$ be a mapping contracting transverse axis of hyperbola (see (2.1)). Also let $\mathcal{T}$ satisfy the following two conditions:*

(a) $\mathcal{T}(\mathcal{B} \setminus \{\lambda, \mu\}) \subset \mathcal{B} \setminus \{\lambda, \mu\}$;

(b) $\mathcal{T}$ is orbitally continuous or, p-continuous for some $p \geq 1$ in $\mathcal{B}$.

Then $\mathcal{T}$ has atleast one fixed point in $\mathcal{B}$.

Proof. Let us choose $\xi_0 \in \mathcal{B} \setminus \{\lambda, \mu\}$ and construct the Picard iterating sequence $\{\xi_n\}_{n\geq 1} = \{\mathcal{T}^n \xi_0\}_{n\in\mathbb{N}}$. If for some , $\xi_n = \xi_{n+1} = \mathcal{T}\xi_n$ then $\mathcal{T}$ has a fixed point in $\mathcal{B}$.

So without loss of generality we assume that $\xi_n \neq \xi_{n+1}$ for all $n \in \mathbb{N} \cup \{0\}$. Due to condition (a) it is clear that $\{\xi_n\} \subset \mathcal{B} \setminus \{\lambda, \mu\}$. Therefore from the contractive condition (2.1) for all $n \geq 1$ we get

$$\begin{aligned}
|D(\xi_n, \lambda) - D(\xi_n, \mu)| &\leq D(\xi_{n-1}, \xi_n) + |D(\xi_n, \lambda) - D(\xi_n, \mu)| \\
&\leq r|D(\xi_{n-1}, \lambda) - D(\xi_{n-1}, \mu)| \\
&\cdots \\
&\leq r^n|D(\xi_0, \lambda) - D(\xi_0, \mu)| = r^n A, \\
&\text{where } A = |D(\xi_0, \lambda) - D(\xi_0, \mu)|. \qquad (2.2)
\end{aligned}$$

Now for any $n \geq 1$,

$$\begin{aligned}
D(\xi_n, \xi_{n+1}) &\leq D(\xi_n, \xi_{n+1}) + |D(\xi_{n+1}, \lambda) - D(\xi_{n+1}, \mu)| \\
&\leq r|D(\xi_n, \lambda) - D(\xi_n, \mu)| \leq r^{n+1}A. \qquad (2.3)
\end{aligned}$$

For any $p = 1, 2, \cdots,$

$$\begin{aligned}
D(\xi_n, \xi_{n+p}) &\leq D(\xi_n, \xi_{n+1}) + D(\xi_{n+1}, \xi_{n+2}) + \cdots + D(\xi_{n+p-1}, \xi_{n+p}) \\
&\leq [r^{n+1} + r^{n+2} + \cdots + r^{n+p}]A \\
&= r^{n+1}[1 + r + \cdots + r^{p-1}]A \\
&= r^{n+1}\frac{1 - r^p}{1 - r}A = r^{n+1}\frac{A}{1 - r} \text{ tending to 0 as } n \to \infty. \qquad (2.4)
\end{aligned}$$

Therefore $\{\xi_n\}$ is Cauchy in $\mathcal{B}$. Since $\mathcal{B}$ is complete it follows that $\{\xi_n\}$ is convergent in $\mathcal{B}$ and converges to some $\zeta \in \mathcal{B}$.

Now if $\mathcal{T}$ is orbitally continuous then for the convergent sequence $\{\xi_n\}$ in the orbit $O(\xi_0, \mathcal{T}) := \{\mathcal{T}^n \xi_0 : n = 0, 1, 2 \cdots\}$, $\xi_0 \in \mathcal{B}$, it is obvious that $\mathcal{T}\xi_n = \mathcal{T}^{n+1}\xi_0 = \xi_{n+1} \to \mathcal{T}\zeta$. Which proves that $\mathcal{T}\zeta = \zeta$.
If $\mathcal{T}$ is p-continuous for some $p \geq 1$ then $\mathcal{T}^p \xi_n = \mathcal{T}^{n+p}\xi_0 = \xi_{n+p} \to \mathcal{T}\zeta$, since $\mathcal{T}^{p-1}\xi_n = \mathcal{T}^{n+p-1}\xi_0 = \xi_{n+p-1} \to \zeta$. Which shows that $\mathcal{T}\zeta = \zeta$. Hence in either case $\mathcal{T}$ has a fixed point in $\mathcal{B}$. □

Example 2.5. In Example 2.2 the mapping $\mathcal{T}$ satisfies all the conditions of Theorem 2.4 and possesses three fixed points $-2, 0$ and 2 in $\mathcal{B}$.

Example 2.6. Let $\mathcal{B} = [0, 1]$ endowed with the usual metric d_u. Define $\mathcal{T} : [0, 1] \to [0, 1]$ as follows:

$$\mathcal{T}(\xi) = \begin{cases} \frac{1}{2} & \text{if } 0 < \xi < \frac{1}{3}, \\ \xi & \text{otherwise.} \end{cases}$$

Then $\mathcal{T}$ is a mapping contracting transverse axis of hyperbola for $\lambda = 0, \mu = 1$ and $r = \frac{1}{2}$. Moreover all the conditions of Theorem 2.4 are satisfied by $\mathcal{T}$ and $\mathcal{T}$ has infinitely many fixed points in $\mathcal{B}$.

Corollary 2.7. *Let $(\mathcal{B}, D)$ be a complete metric space, $\lambda, \mu \in \mathcal{B}$ be two distinct elements and $\mathcal{T} : \mathcal{B} \to \mathcal{B}$ be a mapping contracting transverse axis of hyperbola (see (2.1)). Also let $\mathcal{T}$ satisfy the following two conditions:*

(a) $\mathcal{T}(\mathcal{B} \setminus \{\lambda, \mu\}) \subset \mathcal{B} \setminus \{\lambda, \mu\}$;

(b) $\mathcal{T}$ is continuous in $\mathcal{B}$.

Then $\mathcal{T}$ has atleast one fixed point in $\mathcal{B}$.

A mapping which is mapping contracting axes of ellipse not necessarily mapping contracting transverse axis of hyperbola and conversely. See the following examples.

Example 2.8. In Example 2.5 of [11], $\mathcal{T}$ is a mapping contracting axes of ellipse but not a mapping contracting transverse axis of hyperbola since

$$d_u(6, \mathcal{T}6) + |d_u(\mathcal{T}6, 0) - d_u(\mathcal{T}6, 2)| = 5 \nleq r|d_u(6, 0) - d_u(6, 2)| = 2r$$

for any $r \in [0, 1)$.

Example 2.9. In Example 2.6 for $\xi = \frac{1}{4}$ we see that

$$d_u(\mathcal{T}\frac{1}{4}, 0) + d_u(\mathcal{T}\frac{1}{4}, 1) = 1 \nleq r(d_u(\frac{1}{4}, 0) + d_u(\frac{1}{4}, 1)) = r$$

for any $r \in [0, 1)$. Therefore $\mathcal{T}$ can not be a mapping contracting axes of ellipse.

Remark 2.10. Example 2.6 shows that there are mappings contracting transverse axis of hyperbola which do not satisfy any type of contractive conditions.

The continuity of contractive definitions at fixed points in metric spaces was initially investigated by Rhoades [14], who showed that the usual contractive type mappings are continuous at their fixed points. Rhoades [14] also posed an open question regarding the existence of a contractive condition that permits discontinuity of a mapping at its fixed point. In 1999, Pant [12] was the first researcher to provide an affirmative solution to this interesting problem within the framework of metric spaces. Following which different solutions together with applications can be found in [8, 13].

Remark 2.11. Our proposed mapping gives a new solution to Rhoades open problem. The mapping $\mathcal{T}$ in Example 2.6 is discontinuous at $\xi = \frac{1}{3}$, which is a fixed point of $\mathcal{T}$.

3. Geometrical aspect of mappings contracting transverse axis of hyperbola

In this section we discuss the effect of our proposed mapping to the geometrical structure of any hyperbola.

Definition 3.1. A hyperbola with foci at the points c_1 and c_2 and transverse axis of length $2a$ in a metric space $(\mathcal{B}, D)$ is defined as

$$\mathfrak{H}(c_1, c_2, a) = \{\xi \in \mathcal{B} : |D(\xi, c_1) - D(\xi, c_2)| = 2a\} \tag{3.1}$$

for $c_1, c_2 (c_1 \neq c_2) \in \mathcal{B}$, $a \in [0, \infty)$ and $2a < D(c_1, c_2)$.

Ecentricity of this hyperbola $\mathfrak{H}(c_1, c_2, a)$ is $e > 1$, where $e = \frac{D(c_1,c_2)}{2a}$. Then the length of its conjugate axis is $2a\sqrt{e^2 - 1}$. Let us choose some $k \in (0, 1)$ and consider the image of the hyperbola $\mathfrak{H}(c_1, c_2, a)$ under the mapping $\mathcal{T}$ given in (2.1) with Lipschitz constant k, then it will be a subset of union of hyperbolas given by

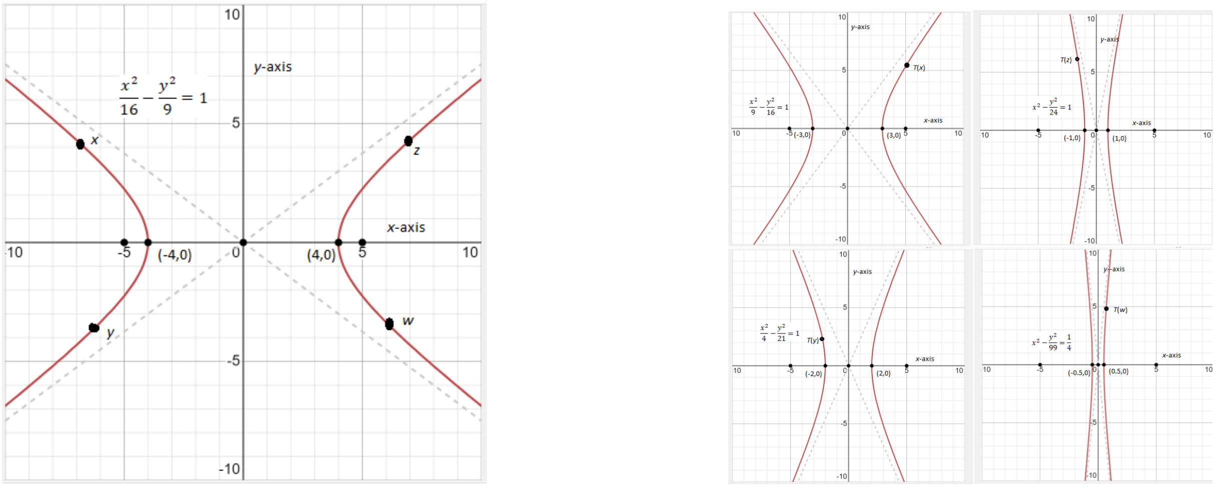
$$\begin{aligned}\mathcal{T}(\mathfrak{H}(c_1, c_2, a)) &= \{\mathcal{T}(\xi) : \xi \in \mathfrak{H}(c_1, c_2, a)\}\\ &\subset \cup_{0<t\leq ak<a<\frac{D(c_1,c_2)}{2}} \{\eta \in \mathcal{B} : |D(\eta, c_1) - D(\eta, c_2)| = 2t\}.\end{aligned}$$

It is seen that an arbitrary point of $\mathcal{T}(\mathfrak{H}(c_1, c_2, a))$ belongs to some hyperbola with semi-transverse axis of length t, ecentricity $e' = \frac{D(c_1,c_2)}{2t} \geq \frac{D(\alpha,\beta)}{2ak} = \frac{e}{k} > e > 1$ and whose semi-conjugate axis is of length $t\sqrt{e'^2 - 1}$ (For clear understanding see FIGURE 1).

Here we give an example of a mapping contracting transverse axis of hyperbola on the Euclidean plane $\mathbb{R}^2$ endowed with the usual metric d_u.

Example 3.2. Consider $\mathcal{B} = \mathbb{R}^2$ and $d_u : X \times X \to [0, \infty)$ is the Euclidean metric defined by $d_u(x, y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$ for all $x \equiv (x_1, x_2), y \equiv (y_1, y_2) \in \mathcal{B}$. Let $\mathcal{T} : \mathcal{B} \to \mathcal{B}$ be defined by $\mathcal{T}((\xi, \eta)) = \begin{cases}(0,0) & \text{if } -3 < \xi < 3, \eta = 0;\\ (\xi,\eta) & \text{otherwise.}\end{cases}$
Then for $(\xi, 0) \in \mathcal{B}$ with $-3 < \xi < 3$ we see that

$$\begin{aligned}&d_u((\xi, 0), T((\xi, 0))) + |d_u(T((\xi, 0)), (-3, 0)) - d_u(T((\xi, 0)), (3, 0))|\\ &= d_u((0, 0), T((\xi, 0))) + |d_u((0, 0), (-3, 0)) - d_u((0, 0), (3, 0))|\\ &= |\xi|\end{aligned}$$



(A) (B)

Figure 1. Image of an arbitrary hyperbola under mapping contracting transverse axis of hyperbola

and

$$\begin{aligned}
&|d_u(T((\xi,0)),(-3,0)) - d_u(T((\xi,0)),(3,0))| \\
&= ||\xi+3| - |\xi-3|| \\
&\leq |(\xi+3)-(3-\xi)| = 2|\xi|.
\end{aligned}$$

Which implies that

$$\begin{aligned}
&d_u((\xi,0),T((\xi,0))) + |d_u(T((\xi,0)),(-3,0)) - d_u(T((\xi,0)),(3,0))| \\
&\leq \frac{5}{6}|d_u(T((\xi,0)),(-3,0)) - d_u(T((\xi,0)),(3,0))| \text{ for any } -3 < \xi < 3.
\end{aligned}$$

Therefore it is a mapping contracting transverse axis of hyperbola.
If we take $c_1 = (3,0)$, $c_2 = (-3,0)$ and length of the semi transverse axis as $a = 2$, then the hyperbola $\mathfrak{H}((3,0),(-3,0),2)$ in $\mathcal{B}$ is given by

$$\mathfrak{H}((3,0),(-3,0),2) = \{(x,y) \in \mathcal{B} : |d_u((3,0),(x,y)) - d_u((-3,0),(x,y))| = 4\}, \tag{3.2}$$

which can be represented as the set $\{(x,y) \in \mathcal{B} : \frac{x^2}{4} - \frac{y^2}{5} = 1\}$.
This mapping can contract the transverse axis of the hyperbola $\mathfrak{H}((3,0),(-3,0),2)$.

4. Conclusion

This paper deals with a mapping with a unique characteristic that it can shrink the transverse axis of a hyperbola. Though mapping contracting axes of ellipse can contract both the major and minor axes of an ellipse, our proposed mapping can't guarantee whether the length of conjugate axis of a hyperbola can be reduced or not. This happens, since ellipse is a closed curve but hyperbola is not.

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