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# Remarks on Suzuki-type fixed point results in relational metric spaces

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Abstract

*In this paper, we identify some gaps in the proofs of certain recent results obtained by some authors in relational metric spaces. We also provide corrected versions of these proofs, thereby improving the accuracy and reliability of the corresponding results.*

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## 1. Introduction

In 2008, Suzuki [18] proved an interesting refinement of the well-known and celebrated Banach fixed point theorem.

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**Theorem 1.1.** *Consider a non-decreasing mapping $\vartheta : [0,1) \longrightarrow (1/2, 1]$ defined by*

$$\vartheta(s) = \begin{cases} 1, & \text{if } 0 \le s \le (\sqrt{5}-1)/2, \\ (1-s)s^{-2}, & \text{if } (\sqrt{5}-1)/2 \le s \le 2^{-1/2}, \\ (1+s)^{-1}, & \text{if } 2^{-1/2} \le s < 1. \end{cases}$$

*If $T$ is a self-mapping on a complete metric space $(X, d)$ satisfying*

$$\vartheta(s) d(\omega, T(\omega)) \le d(\omega, \upsilon) \quad \Longrightarrow \quad d(T(x), T(\upsilon)) \le s d(\omega, \upsilon), \tag{1.1}$$

*for some $s \in [0,1)$ and all $\omega, \upsilon \in X$ then $T$ has a unique fixed point.*

The significance of the above result lies in the fact that the contraction condition need not hold for every pair of elements in the space but only for specific pairs. This refinement has been widely studied, extended, and generalized by several mathematicians in various directions, see [5, 7, 8, 9, 11, 12].

In 2015, Alam and Imdad [1] introduced the notion of relation-theoretic metric spaces and presented relational analogs of classical concepts such as contraction, completeness, and continuity. These analogs reduce to their standard counterparts under the universal relation. Moreover, they established a refinement of the Banach contraction principle (BCP) within the relation-theoretic framework, which is equivalent to the classical BCP under the universal relation. Their work inspired subsequent investigations, including those by Antal [3], Arif [4], Hossain [5], Sawangsup and Kumam [15], Senapati and Dey [16], and Shil and Dey [17].

In [13], Prasad et al. established a relation-theoretic analog of Theorem 1.1 by employing the $T$-transitivity assumption on the relation. Subsequently, Arif and Imdad [4] and Hossain et al. [5] presented further results concerning Suzuki-type contractive mappings in relational metric spaces. These contributions not only generalize the results of Alam and Imdad [1] but also expand the scope of Suzuki-type fixed point theorems to metric spaces equipped with a binary relation.

In this paper, we identify certain gaps that appear in the proofs of some results presented in [4, 5, 13], and provide corrected and improved versions of these proofs. We show that the assumption of $T$-transitivity is redundant from the main results of Prasad et al. [13]. Moreover, we present some examples in support of our claims.

## 2. Preliminaries

In this paper, $\mathcal{R}$ denotes a non-empty binary relation, $\mathbb{N}$ represents the set of natural numbers, and $\mathbb{R}$ denotes the set of all real numbers. We adopt the following definitions and results from [1, 2, 6, 13].

**Definition 2.1.** Let $X$ be a non-empty set and $\mathcal{R}$ be a binary relation on $X$.

- $(a_1)$ If $(\omega, \upsilon) \in \mathcal{R}$ or $(\upsilon, \omega) \in \mathcal{R}$, we say elements $\omega, \upsilon$ are $\mathcal{R}$-comparable and $[\omega, \upsilon] \in \mathcal{R}$.
- $(a_2)$ We denote $\mathcal{R}^{-1} = \{(\omega, \upsilon) \in X \times X : (\upsilon, \omega) \in \mathcal{R}\}$.
- $(a_3)$ $\mathcal{R}^s = \mathcal{R} \cup \mathcal{R}^{-1}$ represents the smallest symmetric relation on $X$ containing $\mathcal{R}$.
- $(a_4)$ If $(\omega_n, \omega_{n+1}) \in \mathcal{R}$ for $n \in \mathbb{N}$ then $\{\omega_n\}$ is a $\mathcal{R}$-preserving sequence ($\mathcal{R}$-PS, in short) in $X$.
- $(a_5)$ A metric space $(X, d)$ is called $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence converges in $X$.
- $(a_6)$ $(X, d, \mathcal{R})$ is called regular if for every $\mathcal{R}$-PS $\{\omega_n\} \subseteq X$ such that $\omega_n \to \omega$, we have $(\omega_n, \omega) \in \mathcal{R}$ for $n \in \mathbb{N}$.
- $(a_7)$ $(X, d, \mathcal{R})$ has the $\mathcal{R}$-sequential limit property ($\mathcal{R}$-SLP, in short) if, for any two $\mathcal{R}$-PSs $\{\omega_n\}, \{\upsilon_n\} \subset X$ such that $\omega_n \xrightarrow{d} \omega$, $\upsilon_n \xrightarrow{d} \upsilon$, and $(\omega_n, \upsilon_n) \in \mathcal{R},\ n \in \mathbb{N}$, we have $(\omega, \upsilon) \in \mathcal{R}$.
- $(a_8)$ A path of length $k \in \mathbb{N}$ from $\omega$ to $\upsilon$ in $\mathcal{R}$ is a finite sequence $\{z_0, z_1, \dots, z_k\} \subseteq X$ satisfying: (i) $z_0 = \omega$ and $z_k = \upsilon$, and (ii) $(z_i, z_{i+1}) \in \mathcal{R}$ for all $i \in \{0, 1, 2, \dots, k-1\}$.
- $(a_9)$ The family of all paths from $\omega$ to $\upsilon$ in $\mathcal{R}$ is denoted by $\gamma(\omega, \upsilon, \mathcal{R})$, and the set of points $\omega \in X$ satisfying $(\omega, T(\omega)) \in \mathcal{R}$ is denoted by $X(T, \mathcal{R})$.
- $(a_{10})$ If $(\omega, \upsilon), (\upsilon, z) \in \mathcal{R}$ imply $(\omega, z) \in \mathcal{R}$ then $\mathcal{R}$ is called a transitive relation.

**Definition 2.2.** Let $T$ be a self-mapping on a metric space $(X, d)$ equipped with a binary relation $\mathcal{R}$. Then,

- $(b_1)$ $\mathcal{R}$ is $T$-transitive if, for any $\omega, \upsilon, z \in X$ such that $(T(\omega), T(z)), (T(z), T(\upsilon)) \in \mathcal{R}$ imply $(T(\omega), T(\upsilon)) \in \mathcal{R}$.
- $(b_2)$ $\mathcal{R}$ is $T$-closed if $(\omega, \upsilon) \in \mathcal{R}$ implies $(T(\omega), T(\upsilon)) \in \mathcal{R}$.
- $(b_3)$ $T$ is $\mathcal{R}$-continuous at $\omega \in X$ if for any $\mathcal{R}$-PS $\{\omega_n\}$ such that $\omega_n \xrightarrow{d} \omega$, the sequence $\{T(\omega_n)\}$ converges to $T(\omega)$. If $T$ is $\mathcal{R}$-continuous at every point of $X$, it is called $\mathcal{R}$-continuous.
- $(b_4)$ $\mathcal{R}^s$ is $T$-closed, provided $\mathcal{R}$ is $T$-closed.
- $(b_5)$ $\mathcal{R}$ is $T^n$-closed for $n \in \mathbb{N} \cup \{0\}$, provided $\mathcal{R}$ is $T$-closed, where $T^n$ denotes the $n^{\text{th}}$ iterate of $T$.

Wardowski [19] introduced the notions of $\mathcal{F}$-class and $F$-contraction as described below:

**Definition 2.3.** A self-mapping $T$ on a metric space $(X, d)$ is said to be $F$-contraction if $\exists\, \tau > 0$ such that for all $x, y \in X$,

$$d(Tx, Ty) > 0 \implies \tau + F(d(Tx, Ty)) \leq F(d(x, y)),$$

where $F : \mathbb{R}^+ \longrightarrow \mathbb{R}$ satisfies the following conditions:

- $(F_1)$ $F$ is strictly decreasing,
- $(F_2)$ For any sequence $\{\omega_n\} \subset \mathbb{R}^+$, $\lim\limits_{n\to\infty} \omega_n = 0$ if and only if $\lim\limits_{n\to\infty} F(\omega_n) = -\infty$,

$(F_3)$ There exists a constant $k \in (0,1)$ such that $\lim_{\omega \to 0^+} \omega^k F(\omega) = 0$.

We denote by $\mathcal{F}$, the set of all mappings $F$ satisfies the conditions (F1)-(F3) and let $\Phi = \{\varphi : (0,\infty) \longrightarrow (0,\infty) \mid \liminf_{s \to t^+} \varphi(s) > 0 \text{ for all } t \geq 0\}$.

**Definition 2.4** ([10])**.** Let $X \neq \varnothing$ and $\mathcal{P} : X \times X \to [0,\infty)$ be a mapping satisfying the following for $\omega, \upsilon, z \in X$,

(1) $\mathcal{P}(\omega,\omega) = \mathcal{P}(\omega,\upsilon) = \mathcal{P}(\upsilon,\upsilon)$ iff $\omega = \upsilon$,
(2) $\mathcal{P}(\omega,\omega) \leq \mathcal{P}(\omega,\upsilon)$,
(3) $\mathcal{P}(\omega,\upsilon) = \mathcal{P}(\upsilon,\omega)$,
(4) $\mathcal{P}(\omega,\upsilon) \leq \mathcal{P}(\omega,z) + \mathcal{P}(z,\upsilon) - \mathcal{P}(z,z)$,

then $\mathcal{P}$ is called a partial metric on $X$.

## 3. Results and discussions

Prasad et al. [13] presented the following result:

**Theorem 3.1.** *Let $T$ be a self-mapping on a metric space $(X,d)$ equipped with a binary relation $\mathcal{R}$. If there exists $s \in [0,1)$ such that*

$$\vartheta(s)d(\omega,T(\omega)) \leq d(\omega,\upsilon) \implies d(T(\omega),T(\upsilon)) \leq sd(\omega,\upsilon), \tag{3.1}$$

*holds for all $\omega,\upsilon \in X$ with $(\omega,\upsilon) \in \mathcal{R}$, where $\vartheta(s)$ is defined in Theorem 1.1. Assume that the following hold:*

(a) *$X(T,\mathcal{R}) \neq \varnothing$,*
(b) *$(X,d)$ is $\mathcal{R}$-complete,*
(c) *$\mathcal{R}$ is $T$-closed and $T$-transitive,*
(d) *$(X,d,\mathcal{R})$ is regular and has $\mathcal{R}$-SLP,.*

*Then $T$ has a fixed point. Further, if $T(X)$ is $\mathcal{R}^S$-connected then $T$ has a unique fixed point in $X$.*

In [13, Theorem 3.1], we observe the following issues:

(1) The assumption that $\mathcal{R}$ is $T$-transitive is redundant in Theorem 3.1.
(2) On page 1351, lines 13–15, the authors showed that $\{x_n\}$ is a $\mathcal{R}$-PS in $X$ and that $(x_{n+1},T(z)) \in \mathcal{R}$. Then, by taking the limit as $n \to \infty$ and using the $\mathcal{R}$-SLP, they concluded that $(z,T(z)) \in \mathcal{R}$. However, this conclusion is unclear without first establishing that $\{T(z)\}$ is a $\mathcal{R}$-PS in $X$, that is, $(T(z),T(z)) \in \mathcal{R}$.

(3) On page 1353, lines 17–20, the authors derived the following inequality:

$$d(T^{n+1}(x_0), T^{j+1}(z)) \le sd(T^n(x_0), T^j(z)) \quad \text{for all} \quad j, n \ge \nu, \tag{3.2}$$

and then, by replacing $j$ with $n$, $x_0$ with $z_i$, and $z$ with $z_{i+1}$, they obtained

$$d(T^{n+1}(z_i), T^{n+1}(z_{i+1})) \le sd(T^n(z_i), T^n(z_{i+1})) \quad \text{for all} \quad n \ge \nu. \tag{3.3}$$

However, in [13], the inequality (3.2) is obtained from the contraction condition (3.1), which requires that $\vartheta(s)d(T^n(x_0), T^{n+1}(x_0)) \le d(T^n(x_0), T^j(z))$ holds for $n \in \mathbb{N}$. Without verifying that $\vartheta(s)d(T^n(z_i), T^{n+1}(z_i) \le d(T^n(z_i), T^n(z_{i+1}))$ for $n \in \mathbb{N}$, we cannot apply (3.2) to derive (3.3).

To rectify these issues, we will first restate Theorem 3.1 and then present a corrected proof of it.

**Theorem 3.2.** *Suppose all the hypotheses of Theorem 3.1 hold except that $\mathcal{R}$ is necessarily $T$-transitive. Then $T$ has a unique fixed point in $X$.*

*Proof.* Let $u \in X$ be an arbitrary element such that $(u, T(u)) \in \mathcal{R}$. Then from (3.1), it follows

$$d(T(u), T^2(u)) \le sd(u, T(u)) \quad \text{for } (u, T(u)) \in \mathcal{R}. \tag{3.4}$$

As $X(T; \mathcal{R}) \neq \emptyset$, we take $\omega_0 \in X(T, \mathcal{R})$ and construct a sequence of iterates $\{\omega_n\} \in X$ such that

$$\omega_n = T^n(\omega_0) \quad \text{for } n \in \mathbb{N}.$$

Since $(\omega_0, T(\omega_0)) \in \mathcal{R}$, by $T$-closedness of $\mathcal{R}$, we have $(T^{n-1}(\omega_0), T^n(\omega_0)) = (\omega_{n-1}, \omega_n) \in \mathcal{R}$ for $n \in \mathbb{N}$. It follows that $\{\omega_n\}$ is a $\mathcal{R}$-PS and from (3.4), we have

$$d(\omega_n, \omega_{n+1}) \le sd(\omega_{n-1}, \omega_n) \text{ for all } n \in \mathbb{N}$$

and consequently,

$$d(\omega_n, \omega_{n+1}) \le s^n d(\omega_0, \omega_1) \text{ for all } n \in \mathbb{N}.$$

Therefore, for each $n, k \in \mathbb{N}$,

$$\begin{aligned} d(\omega_n, \omega_{n+k}) &\le d(\omega_n, \omega_{n+1}) + d(\omega_{n+1}, \omega_{n+2}) + \cdots + d(\omega_{n+k-1}, \omega_{n+k}) \\ &\le \left(s^n + \cdots + s^{n+k-1}\right) d(\omega_0, \omega_1) \\ &< \frac{s^n}{1-s} d(\omega_0, \omega_1), \end{aligned}$$

which implies that $\{\omega_n\}$ is a Cauchy sequence in $X$. By the $\mathcal{R}$-completeness of $X$, there exists a point $z \in X$ such that $\omega_n \xrightarrow{d} z$. Since $\{\omega_n\}$ and $\{\omega_{n+1}\}$ are $\mathcal{R}$-PSs in $X$ such that $(\omega_n, \omega_{n+1}) \in \mathcal{R}$, $\omega_n \xrightarrow{d} z$ and $\omega_{n+1} \xrightarrow{d} z$ then $\mathcal{R}$-SLP implies $(z, z) \in \mathcal{R}$ and by $T$-closedness, we have $(T^n(z), T^n(z)) \in \mathcal{R}$ for $n \in \mathbb{N}$.

Thus, for fix $n = j$, $\{T^j(z)\}$ is also $\mathcal{R}$-PS. Since $(X, d, \mathcal{R})$ is regular and $\{\omega_n\}$ is $\mathcal{R}$-PS, we have $(\omega_n, z) \in \mathcal{R}$ for $n \in \mathbb{N}$ and by $T$-closedness,

$$(\omega_{n+j}, T^j(z)) \in \mathcal{R} \qquad \text{for} \quad n,\ j \in \mathbb{N}. \tag{3.5}$$

Now, we have $\{\omega_{n+j}\}$ and $\{T^j(z)\}$, two $\mathcal{R}$-PSs in $X$ such that $(\omega_{n+j}, T^j(z)) \in \mathcal{R}$, $\omega_n \to z$ and $T^j(z) \to T^j(z)$. Applying $\mathcal{R}$-SLP, we get $(z, T^j(z)) \in \mathcal{R}$ for $j \in \mathbb{N}$.

Next, we show that

$$d(z, T^{j+1}(z)) \leq s^j d(z, T(z)) \quad \text{ for } j \in \mathbb{N}. \tag{3.6}$$

Since $\omega_n \to z$, so for each $\upsilon \in X - \{z\}$ there exists $\nu_\omega \in \mathbb{N}$ such that $d(\omega_n, z) \leq d(\upsilon, z)/3$, for $n \in \mathbb{N}$ with $n \geq \nu_\omega$, and

$$\begin{aligned} \vartheta(s)d(\omega_n, T(\omega_n)) \leq d(\omega_n, T(\omega_n)) \leq d(\omega_n, z) + d(\omega_{n+1}, z) \leq \frac{2}{3}d(\upsilon, z) \\ \leq d(\upsilon, z) - d(\omega_n, z) \leq d(\omega_n, \upsilon). \end{aligned} \tag{3.7}$$

Take $n := n + j$ and $\upsilon := T^j(z)$ in (3.7), we get

$$\vartheta(s)d(\omega_{n+j}, T(\omega_{n+j})) \leq d(\omega_{n+j}, T^j(z)).$$

In view of (3.5) and from (3.1), we have

$$d(T(\omega_{n+j}), T^{j+1}(z)) \leq sd(\omega_{n+j}, T^j(z)) \quad \text{for} \quad j \in \mathbb{N}.$$

Making $n \to \infty$, we get

$$d(z, T^{j+1}(z)) \leq sd(z, T^j(z)) \quad \text{ for } j \in \mathbb{N}. \tag{3.8}$$

Thus, inequality (3.6) holds for all $j \in \mathbb{N}$, and the sequence $\{T^n(z)\}$ converges to $z$.

We will now prove that $z$ is a fixed point of $T$. Let $z \neq T(z)$. Then it follows from (3.8) that $T^j(z) \neq z$ for $j \in \mathbb{N}$ and $s \neq 0$. By the induction method, we first show that

$$d(T^n(z), T(z)) \leq sd(z, T(z))$$

holds for $n \in \mathbb{N}$ with $n \geq 2$. For $n = 2$, it is straightforward to see from (3.4). We assume that $d(T^n(z), T(z)) \leq sd(z, T(z))$ holds for some $n \in \mathbb{N}$ with $n \geq 2$. Then by the triangle inequality,

$$d(z, T(z)) \leq d(z, T^n(z)) + d(T^n(z), T(z)) \leq d(z, T^n(z)) + sd(z, T(z))$$

implies

$$d(z, T(z)) \leq \frac{1}{1 - s}d(z, T^n(z)).$$

We consider two cases: In the first case, let $0 < s < \dfrac{1}{\sqrt{2}}$. Then $\vartheta(s) \leq \dfrac{1-s}{s^2}$ and in view of (3.4), we have

$$\begin{aligned}\vartheta(s)d(T^n(z), T^{n+1}(z)) &\leq \frac{1-s}{s^2}d(T^n(z), T^{n+1}(z)) < \frac{1-s}{s^n}d(T^n(z), T^{n+1}(z)) \leq (1-s)d(z, T(z))\\ &\leq d(z, T^n(z)).\end{aligned}$$

In second case, let $\dfrac{1}{\sqrt{2}} \leq s < 1$. Then $\vartheta(s) = \dfrac{1}{1+s}$ and by triangle inequality and the assumption, we have

$$\begin{aligned}\vartheta(s)d(T^n(z), T^{n+1}(z)) &\leq \vartheta(s)\left[d(T^n(z), z) + d(z, T^{n+1}(z))\right] \leq \vartheta(s)(1+s)d(z, T^n(z))\\ &= d(z, T^n(z)).\end{aligned}$$

We get, in both cases, $\vartheta(s)d(T^n(z), T^{n+1}(z)) \leq d(z, T^n(z))$ for $n \in \mathbb{N}$, so by (3.1), we have

$$d(T(z), T^{n+1}(z)) \leq sd(z, T^n(z)) \leq \cdots \leq s^n d(z, T(z)).$$

Making $n \to \infty$, we get $d(z, T(z)) = 0$, hence $T(z) = z$.

To prove the uniqueness of the fixed point, assume that $\upsilon$ is another fixed point of $T$, that is, $T(\upsilon) = \upsilon$ and $z \neq \upsilon$. Since $z, \upsilon \in T(X)$ and $T(X)$ is $\mathcal{R}^S$-connected by assumption, there exists a path $\{z_0, z_1, z_2, \ldots, z_k\}$ of finite length $k$ in $\mathcal{R}$ such that $z_0 = z$, $z_k = \upsilon$, and $[z_i, z_{i+1}] \in \mathcal{R}$, $i = 0, 1, 2, \ldots, k-1$. By the $T$-closedness property, we get

$$[T^n(z_i), T^n(z_{i+1})] \in \mathcal{R} \quad \text{for } i = 0, 1, 2, \ldots, k-1, \text{ and } n \in \mathbb{N}.$$

Next, we prove that $T^n(z_i) \to z$ for $i = 0, 1, \ldots, k$. Clearly, for $i = 0$, $T^n(z_0) = z \to z$. Since $\vartheta(s)d(T^n(z_0), T^{n+1}(z_0)) = 0 \leq d(T^n(z_0), T^{n+1}(z_1))$ holds for $n \in \mathbb{N}$, from (3.1) we have

$$d(T^{n+1}(z_0), T^{n+1}(z_1) \leq sd(T^n(z_0), T^n(z_1) \leq \cdots \leq s^n d(z_0, z_1).$$

Making $n \to \infty$, we get $d(T^n(z_0), T^n(z_1)) \to 0$, hence $T^n(z_i) \to z$ for $i = 1$.

Assume that $T^n(z_i) \to z$ for some $i = \ell < k$. If $T^N(z_{\ell+1}) = z$ for some positive integer $N$ then $T^n(z_{\ell+1}) = z$ for all $n \geq N$ implies $T^n(z_{\ell+1}) \to z$. So, we may assume $T^n(z_{\ell+1}) \neq z$ for all $n \in \mathbb{N}$. Since $T^n(z_\ell) \to z$ and $T^n(z_{\ell+1}) \neq z$ for $n \in \mathbb{N}$, thus in light of (3.7), for every $T^n(z_{\ell+1}) \neq z$ there exists $\nu_n \in \mathbb{N}$ such that

$$\vartheta(s)d(T^n(z_\ell), T^{n+1}(z_\ell)) \leq d(T^n(z_\ell), T^n(z_{\ell+1})) \text{ for } n \geq \nu_n$$

and by (3.1), we have

$$d(T^{n+1}(z_\ell), T^{n+1}(z_{\ell+1})) \leq sd(T^n(z_\ell), T^n(z_{\ell+1})) \text{ for } n \geq \nu_n,$$

which follows that

$$d(T^{n+1}(z_\ell), T^{n+1}(z_{\ell+1})) \leq s^n d(z_\ell, z_{\ell+1}) \text{ for } n \geq v = \max\{v_1, v_2, \ldots, v_n\}.$$

Making $n \to \infty$, we get $d(T^n(z_\ell), T^n(z_{\ell+1})) \to 0$, that is, $T^n(z_i) \to z$ for $i = \ell + 1$. Hence by induction method, $T^n(z_i) \to z$ for $i = 3, 4, \ldots, k$. But $T^n(z_k) = \upsilon \to \upsilon$, so by the uniqueness of the limit, we have $\upsilon = z$. □

The following example demonstrates that $T$-transitivity is not a necessary assumption for the existence of fixed points for mappings satisfying condition (3.1).

**Example 3.3.** Let $X = \{(0,0), (4,5), (5,4), (4,0), (0,4)\}$ be endowed with the metric $d$ defined by

$$d((\omega_1, \omega_2), (\upsilon_1, \upsilon_2)) = |\omega_1 - \upsilon_1| + |\omega_2 - \upsilon_2|.$$

Define a mapping $T : X \to X$ by

$$T(\omega_1, \omega_2) = \begin{cases} (\omega_1, 0), & \text{if } \omega_1 \leq \omega_2, \\ (0, \omega_2), & \text{if } \omega_1 > \omega_2, \end{cases}$$

and a relation $\mathcal{R}$ as

$$\mathcal{R} = \{((0,0),(0,0)),\ ((4,5),(5,4)),\ ((4,0),(0,4)),\ ((5,4),(0,4)),\ ((0,4),(0,0))\}.$$

Then $T$ and $\mathcal{R}$ satisfy all the assumptions of Theorem 3.2, and $T$ has a unique fixed point at $(0,0)$.

*Proof.* Since $((0,0),(0,0)) \in \mathcal{R}$, it follows that $X(T, \mathcal{R}) \neq \emptyset$. Consider $\omega = (4,5)$, $\upsilon = (5,4)$, and $z = (0,0)$. We have $(T(\omega), T(\upsilon)), (T(\upsilon), T(z)) \in \mathcal{R}$ but $(T(\omega), T(z)) \notin \mathcal{R}$, showing that $\mathcal{R}$ is not $T$-transitive. For $(\omega, \upsilon) = ((4,5),(5,4)) \in \mathcal{R}$,

$$\vartheta(s) d((4,5), T(4,5)) > d((4,5),(5,4)),$$

and for $(\omega, \upsilon) \in \mathcal{R}$ with $(\omega, \upsilon) \neq ((4,5),(5,4))$, we have

$$d(T(\omega), T(\upsilon)) \leq \frac{4}{5} d(\omega, \upsilon).$$

Thus, $T$ satisfies condition (3.1) relative to $\mathcal{R}$. Moreover, one can easily verify that $(X, T, \mathcal{R})$ is regular and has the $\mathcal{R}$-SLP. Hence, all the assumptions of Theorem 3.2 are satisfied, and $T$ has a unique fixed point at $(0,0)$. □

In the following result, we replace the assumptions that $(X, d, \mathcal{R})$ is regular and has the $\mathcal{R}$-SLP with a weaker form of continuity.

**Theorem 3.4.** *Theorem 3.2 remains valid if the assumptions that $(X, d, \mathcal{R})$ is regular and has the $\mathcal{R}$-SLP are replaced with the assumption that $T$ is $\mathcal{R}$-continuous.*

*Proof.* Following the proof of Theorem 3.1, we show that for each $\omega \in X(T, \mathcal{R})$, the sequence of iterates $\{T^n(\omega)\}$ is $\mathcal{R}$-PS and $T^n(\omega) \xrightarrow{d} z \in X$. Since $T$ is $\mathcal{R}$-continuous, it follows that $T^{n+1}(\omega) \xrightarrow{d} T(z)$, and by the uniqueness of limits, we obtain $T(z) = z$. Thus, $z$ is a fixed point of $T$. The uniqueness of $z$ follows directly from the arguments in Theorem 3.2. □

In the following example, we illustrate the assumption that $(X, d, \mathcal{R})$ is regular, which is not essential to ensure fixed points for mappings satisfying condition (3.1).

**Example 3.5.** Assume that $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$ with the usual metric $d$. Define a binary relation $\mathcal{R}$ and a self-mapping $T$ on $X$ by

$$\mathcal{R} = \{(5,0), (0,1), (1,1), (5,7), (0,2), (7,5), (2,0)\} \text{ and } T(\omega) = \begin{cases} 0, & \text{if } \omega = 5, \\ 2, & \text{if } \omega = 7 \\ 1, & \text{otherwise .} \end{cases}$$

Then all the assumptions of Theorem 3.4 hold and $T$ has a unique fixed point at $\omega = 1$.

*Proof.* We observe that $X(T, \mathcal{R}) = \{5, 0, 1\} \neq \emptyset$. It is easy to verify that $\mathcal{R}$ is $T$-closed and $T$ is $\mathcal{R}$-continuous. For $(\omega, \upsilon) = (5, 7)$ or $(7, 5) \in \mathcal{R}$, we have

$$\vartheta(s)d(5, T5) = \vartheta(s)d(5, 0) = 5\vartheta(s) > 2 = d(5, 7),$$

and

$$\vartheta(s)d(7, T7) = \vartheta(s)d(7, 2) = 5\vartheta(s) > 2 = d(5, 7).$$

Also, for $(\omega, \upsilon) \in \mathcal{R} \setminus \{(5, 7), (7, 5)\}$, we have

$$d(T(\omega), T(\upsilon)) \leq \frac{1}{5} d(\omega, \upsilon).$$

Thus, $T$ satisfies condition (3.1). By Theorem 3.4, the mapping $T$ has a unique fixed point at $\omega = 1$.

Moreover, $\{5, 0, 1\}$ is a $\mathcal{R}$-PS such that $\{5, 0, 1\} \xrightarrow{d} 1$ but $(5, 1) \notin \mathcal{R}$. It follows that $(X, T, \mathcal{R})$ is not a regular. Also, for $\omega = 7$, $\upsilon = 5$, and $z = 0$ in $X$, we have $(T(\omega), T(\upsilon)), (T(\upsilon), T(z)) \in \mathcal{R}$ but $(T(\omega), T(z)) \notin \mathcal{R}$ implies $\mathcal{R}$ is not $T$-transitive. □

*Remark* 3.6. We observe that in Examples 3.3 and 3.5, the relation $\mathcal{R}$ fails to be $T$-transitive. Consequently, Theorems 3.1 and 3.3 of Prasad et al. [13] are not applicable in these cases.

In [4], the authors established the following existence result (see [4, Theorem 4.1]) in relational metric spaces.

**Theorem 3.7.** *Let $T$ be a self-mapping on a metric space $(X, d)$ equipped with a binary relation $\mathcal{R}$. If there exist $\varphi \in \Phi$ and $F \in \mathcal{F}$ such that*

$$\frac{1}{2}d(\omega, T(\omega)) < d(\omega, \upsilon) \implies \varphi(d(\omega, \upsilon)) + F(d(T(\omega), T(\upsilon))) \leq F(d(\omega, \upsilon)) \tag{e}$$

*holds for all $\omega, \upsilon \in X$ with $(\omega, \upsilon) \in \mathcal{R}$, then $T$ has a fixed point under the following conditions:*

(*i*) *$X(T, \mathcal{R}) \neq \emptyset$;*
(*ii*) *$\mathcal{R}$ is $T$-closed;*
(*iii*) *$(X, d)$ is $\mathcal{R}$-complete;*
(*iv*) *either $T$ is $\mathcal{R}$-continuous or $\mathcal{R}$ is $d$-self closed.*

In the proof of Theorem 3.7, the authors of [4] did not apply the contraction condition (e) properly; see [4, page 3160, line 18]. They established that $\frac{1}{2}d(u, u_{n_k}) < d(u, u_{n_k})$ to employ condition (e) on the pair $(u, u_{n_k})$. However, condition (e) is applicable only for those pairs $(\omega, \upsilon)$ for which either $\frac{1}{2}d(\omega, T(\omega)) < d(\omega, \upsilon)$ or $\frac{1}{2}d(\upsilon, T(\upsilon)) < d(\omega, \upsilon)$ holds. Therefore, to apply condition (e) to the pair $(u, u_{n_k})$, we must show that either $\frac{1}{2}d(u_{n_k}, T(u_{n_k})) < d(u, u_{n_k})$ or $\frac{1}{2}d(u, T(u)) < d(u, u_{n_k})$. To rectify this, we now present a corrected proof of Theorem 3.7.

*Proof.* Take $u_0 \in X(T, \mathcal{R})$ such that $u_1 = T(u_0)$ and define a sequence $\{u_n\}$ in $X$ by $u_n = T^n u_0$ for each $n \in \mathbb{N}$. Following the proof of Theorem 4.1 in [4], $\{u_n\}$ is $\mathcal{R}$-PS and converges to a point $u \in X$.

Suppose $T$ is $\mathcal{R}$-continuous. Then $u_{n+1} = T(u_n) \xrightarrow{d} T(u)$. By the uniqueness of the limit, $T(u) = u$, meaning $u$ is a fixed point of $T$.

Alternatively, assume that $\mathcal{R}$ is $d$-self-closed. Then there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $[u_{n_k}, u] \in \mathcal{R}$ for all $k \in \mathbb{N} \cup \{0\}$. Since condition (e) implies that $d(T(\omega), T(\upsilon)) < d(\omega, \upsilon)$ whenever $\omega \neq \upsilon$, we assert that either

$$\frac{1}{2}d(u_{n_k}, u_{n_k+1}) < d(u_{n_k}, u) \quad \text{or} \quad \frac{1}{2}d(u_{n_k+1}, u_{n_k+2}) < d(u_{n_k+1}, u)$$

holds for each $k \in \mathbb{N} \cup \{0\}$.

Assume, on the contrary, that for some $k$,

$$\frac{1}{2}d(u_{n_k}, u_{n_k+1}) \geq d(u_{n_k}, u) \quad \text{and} \quad \frac{1}{2}d(u_{n_k+1}, u_{n_k+2}) \geq d(u_{n_k+1}, u).$$

Then, by the triangle inequality,

$$\begin{aligned}
d(u_{n_k}, u_{n_k+1}) &\leq d(u_{n_k}, u) + d(u, u_{n_k+1}) \\
&\leq \frac{1}{2}d(u_{n_k}, u_{n_k+1}) + \frac{1}{2}d(u_{n_k+1}, u_{n_k+2}) \\
&< \frac{1}{2}d(u_{n_k}, u_{n_k+1}) + \frac{1}{2}d(u_{n_k}, u_{n_k+1}) = d(u_{n_k}, u_{n_k+1}),
\end{aligned}$$

which is a contradiction. Hence, by a standard argument, there exists a subsequence $\{n_{k_j}\}$ of $\{n_k\}$ such that

$$\frac{1}{2}d(u_{n_{k_j}}, u_{n_{k_j}+1}) < d(u_{n_{k_j}}, u)$$

and applying condition (e), we get

$$F(d(T(u_{n_{k_j}}), T(u))) \leq F(d(u_{n_{k_j}}, u)) - \varphi(d(u_{n_{k_j}}, u)) < F(d(u_{n_{k_j}}, u)).$$

In view of property $(F_1)$, we conclude that

$$d(u_{n_{k_j}+1}, T(u)) < d(u_{n_{k_j}}, u) \quad \text{for all} \quad j \in \mathbb{N}.$$

Letting $j \to \infty$, we deduce that $d(u, T(u)) = 0$, and hence $T(u) = u$. Thus, $u$ is a fixed point of $T$. □

To demonstrate the uniqueness of the fixed point, the authors of [4] stated the following result.

**Theorem 3.8.** *If $T(X)$ is $\mathcal{R}^S$-connected and all the assumptions of Theorem 3.7 hold, then $T$ has a unique fixed point.*

A similar kind of mathematical oversight has been observed in [4, page 3161, line 9]. They showed that $\frac{1}{2}d(T^n(\xi_i), T^n(\xi_{i+1})) < d(T^n(\xi_i), T^n(\xi_{i+1}))$ to apply condition (e) to the pair $(T^n(\xi_i), T^n(\xi_{i+1}))$. However, we must instead show that either $\frac{1}{2}d(T^n(\xi_i), T^{n+1}(\xi_i)) < d(T^n(\xi_i), T^n(\xi_{i+1}))$ or $\frac{1}{2}d(T^n(\xi_{i+1}), T^{n+1}(\xi_{i+1})) < d(T^n(\xi_i), T^n(\xi_{i+1}))$. We now present a corrected proof of Theorem 3.8.

*Proof.* Suppose $u$ and $v$ are two fixed points of $T$, i.e., $T(u) = u$, $T(v) = v$, and $u \neq v$. Since $T(X)$ is $\mathcal{R}$-connected, there exists a finite sequence $\{\xi_0, \xi_1, \ldots, \xi_k\}$ with $\xi_0 = u$, $\xi_k = v$, and $[\xi_i, \xi_{i+1}] \in \mathcal{R}$ for all $i = 0, 1, \ldots, k-1$. By the $T$-closedness of $\mathcal{R}$, we have

$$[T^n(\xi_i), T^n(\xi_{i+1})] \in \mathcal{R} \quad \text{for all} \quad n \in \mathbb{N},\ i = 0, 1, \ldots, k-1.$$

We will show that $T^n(\xi_i) \to u$ for each $i = 0, 1, \ldots, k$. Clearly, $T^n(\xi_0) = u$ for all $n$, hence $T^n(\xi_0) \to u$. Assume by induction that $T^n(\xi_\ell) \to u$ for some $\ell < k$. We aim to show $T^n(\xi_{\ell+1}) \to u$. Suppose $T^n(\xi_{\ell+1}) \neq u$ for all $n$. Then, there exists $\nu_n \in \mathbb{N}$ such that for all $n \geq \nu_n$,

$$\frac{1}{2}d(T^n(\xi_\ell), T^{n+1}(\xi_\ell)) < d(T^n(\xi_\ell), T^n(\xi_{\ell+1})).$$

Applying condition (e), we get

$$F(d(T^{n+1}(\xi_\ell), T^{n+1}(\xi_{\ell+1}))) \leq F(d(T^n(\xi_\ell), T^n(\xi_{\ell+1}))) - \varphi(d(T^n(\xi_\ell), T^n(\xi_{\ell+1}))). \tag{3.9}$$

Set $\Delta_n := d(T^n(\xi_\ell), T^n(\xi_{\ell+1}))$. Then (3.9) becomes

$$F(\Delta_{n+1}) \leq F(\Delta_n) - \varphi(\Delta_n). \tag{3.10}$$

Define

$$\varphi(\Delta_n^p) := \min\{\varphi(\Delta_n), \varphi(\Delta_{n-1}), \ldots, \varphi(\Delta_0)\}.$$

Then from (3.10), it follows

$$F(\Delta_n) \leq F(\Delta_0) - n\varphi(\Delta_n^p).$$

Since $\{\Delta_n\}$ is decreasing and bounded below, there exists $\Delta \geq 0$ such that $\lim_{n\to\infty} \Delta_n = \Delta$. If $\Delta > 0$, then $\liminf_{n\to\infty} \varphi(\Delta_n^p) > 0$, implying $F(\Delta_n) \to -\infty$ as $n \to \infty$, a contradiction. Therefore, $\Delta = 0$, hence $T^n(\xi_{\ell+1}) \to u$. By induction, $T^n(\xi_i) \to u$ for each $i = 0, 1, \ldots, k$. But $T^n(\xi_k) = v$ for all $n$, so $v = u$ by uniqueness of the limit. Thus, the fixed point is unique. □

In 2022, Hossain et al. established the following results in [5].

**Theorem 3.9.** *Let $(X, \mathcal{P})$ be a partial metric space endowed with a binary relation $\mathcal{R}$. Then a self-mapping $T$ on $X$ has a fixed point under the following hypothesis:*

(1) *$X(T, \mathcal{R}) \neq \emptyset$;*

(2) *$\mathcal{R}$ is $T$-closed;*

(3) *$(X, \mathcal{P})$ is $\mathcal{R}$-complete;*

(4) *there exists $\varphi \in \Phi$ such that for all $\omega, \upsilon \in X$ with $(\omega, \upsilon) \in \mathcal{R}$,*

$$\frac{1}{2}\mathcal{P}(\omega, T(\omega)) \leq \mathcal{P}(\omega, \upsilon) \implies \mathcal{P}(T(\omega), T(\upsilon)) \leq \varphi\left(m(\omega, \upsilon)\right), \tag{d}$$

*where* $m(\omega, \upsilon) = \max\left\{ \mathcal{P}(\omega, \upsilon), \mathcal{P}(\omega, T(\omega)), \mathcal{P}(\upsilon, T(\upsilon)), \dfrac{\mathcal{P}(\omega, T(\omega)) + \mathcal{P}(\upsilon, T(\upsilon))}{2} \right\}$;

(5) *either $T$ is $\mathcal{R}$-continuous or $\mathcal{R}$ is $\mathcal{P}$-self-closed.*

To prove the uniqueness of fixed points, the authors in [5] presented the following result.

**Theorem 3.10.** *If all the assertions of Theorem 3.9 hold together with the following one:*

(6) *$\gamma(\omega, \upsilon, \mathcal{R}^S) \neq \emptyset$,*

*then fixed point of $T$ is unique.*

In the proof of Theorem 3.10 (see [5, page 7, lines 18–22]), the authors stated that "the sequence $\{T^n(\xi_i)\}$ is $\mathcal{R}$-preserving," which is not the case. To claim this, it is necessary to demonstrate that $(T^n(\xi_i), T^{n+1}(\xi_i)) \in \mathcal{R}$ for all $n \in \mathbb{N}$. Secondly, they showed that

$$\frac{1}{2}\mathcal{P}(T^n(\xi_i), T^n(\xi_i)) \leq \mathcal{P}(T^n(\xi_i), T^n(\xi_{i+1})),$$

instead of showing

$$\frac{1}{2}\mathcal{P}(T^n(\xi_i), T^{n+1}(\xi_i)) \leq \mathcal{P}(T^n(\xi_i), T^n(\xi_{i+1}))$$

in order to apply the contractive condition (d) to the pair $(T^n(\xi_i), T^n(\xi_{i+1}))$, which is not correct.

At this stage, we are not certain whether assumption (6) alone is sufficient to guarantee the uniqueness of fixed points for mappings satisfying condition (d). However, by imposing an additional assumption on $\mathcal{R}$, we ensure the uniqueness of the fixed point.

**Theorem 3.11.** *In addition to all the hypotheses of Theorem 3.9, suppose further that:*

(7) $\gamma(\omega, , \mathcal{R}^S)$ *is nonempty and* $\mathcal{R}$ *is* $T$*-transitive,*

*then* $T$ *has a unique fixed point.*

*Proof.* Let $u$ and $v$ be two distinct fixed points of $T$, that is, $T(u) = u$, $T(v) = v$, and $u \neq v$. Since $\gamma(u, v, \mathcal{R}^S)$ is nonempty, there exists a finite path $\{\xi_0, \xi_1, \xi_2, \ldots, \xi_k\}$ of finite length $k$ such that $\xi_0 = u$, $\xi_k = v$, and $[\xi_i, \xi_{i+1}] \in \mathcal{R}$ for each $i = 0, 1, \ldots, k-1$. By the $T$-closedness property, we have

$$[T^n(\xi_i), T^n(\xi_{i+1})] \in \mathcal{R} \quad \text{for all } i = 0, 1, \ldots, k-1 \text{ and } n \in \mathbb{N}.$$

Since $\mathcal{R}$ is $T$-transitive, it follows that $(u, v) \in \mathcal{R}$. Then, by applying condition (d), we get

$$d(u, v) = d(T(u), T(v)) < d(u, v),$$

a contradiction unless $u = v$. Thus, $T$ has a unique fixed point. □

## 4. Conclusion

In this paper, we have addressed and corrected some oversights/gaps present in the papers [4, 13, 5]. Theorem 3.2 shows that the assumption of $T$-transitivity is not necessarily for the existence of fixed points for a mapping satisfying condition (1.1). Furthermore, relation-theoretic contractive mappings do not force mapping to be continuous at their fixed point, thereby providing an affirmative answer to an open question of Rhodes [14] regarding the existence of contractive mapping that do not force a mapping to be continuous at its fixed point.

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## References


[1] A Alam and M. Imdad, Relation-theoretic contraction principle, Journal of Fixed Point Theory and Applications 17, no. 4 (2015), 693–702.

[2] A. Alam and M. Imdad, Nonlinear contractions in metric spaces under locally $t$-transitive binary relations, Fixed Point Theory 19), no. 1 (2018, 13–24.

[3] S. Antal, D. Khantwal, S. Negi, and U. C. Gairola, Fixed points theorems for $(\phi, \psi, p)$-weakly contractive mappings via $w$-distance in relational metric spaces with applications, Filomat 37, no. 21 (2023), 7319–7328.

[4] M. Arif and M. Imdad, Fixed point results under nonlinear Suzuki $(F, \mathcal{R})$-contractions with an application, Filomat 36, no. 9 (2022), 3155–3165.

[5] A. Hossain, M. Arif, S. Sessa, and Q. H. Khan, Nonlinear relation-theoretic Suzuki-generalized Ciric-type contractions and application to fractal spaces, Fractal and Fractional 6, no. 12 (2022), 711.

[6] E. Karapınar, A. Roldan, N. Shahzad, and W. Sintunavarat, Discussion of coupled and tripled coincidence point theorems for $\phi$-contractive mappings without the mixed $g$-monotone property, Fixed Point Theory Appl. 2014 (2014), 92.

[7] D. Khantwal, S. Aneja, and U. C. Gairola, A generalization of Matkowski's and Suzuki's fixed point theorems, Asian-Eur. J. Math. 15, no. 9 (2022), 2250169.

[8] D. Khantwal, S. Aneja, G. Prasad, and U. C. Gairola, A generalization of relation-theoretic contraction principle, TWMS J. App. and Eng. Math. 13, no. 1 (2023), 166–174.

[9] D. Khantwal, I. K. Letlhage, and R. Pant, Fixed point results for Suzuki type contractions in relational metric spaces with applications, Indian J. Math. 64, no. 3 (2022), 279–304.

[10] S. G. Matthews, Partial metric topology, Papers on general topology and applications (Flushing, NY, 1992), Ann. New York Acad. Sci., vol. 728, New York Acad. Sci., New York, 1994, pp. 183–197.

[11] R. Pant, Fixed point theorems for generalized semi-quasi contractions, Journal of Fixed Point Theory and Applications 19, no. 2 (2016), 1581–1590.

[12] R. Pant, S. L. Singh, and S. N. Mishra, A coincidence and fixed point theorems for semi-quasi contractions, Fixed Point Theory 17 (2016), no. 2, 449–456.

[13] G. Prasad, R. C. Dimri, and A. Bartwal, Fixed points of Suzuki contractive mappings in relational metric spaces, Rend. Circ. Mat. Palermo (2) 69, no. 3 (2020), 1347–1358.

[14] B. E. Rhoades, Contractive definitions and continuity, Fixed point theory and its applications (Berkeley, CA, 1986), Contemp. Math., vol. 72, Amer. Math. Soc., Providence, RI, 1988, pp. 233–245.

[15] K. Sawangsup, W. Sintunavarat, and A. F. Roldán López de Hierro, Fixed point theorems for $F_R$-contractions with applications to solution of nonlinear matrix equations, J. Fixed Point Theory Appl. 19, no. 3 (2017), 1711–1725.

[16] T. Senapati and L. K. Dey, Relation-theoretic metrical fixed-point results via $w$-distance with applications, J. Fixed Point Theory Appl. 19, no. 4 (2017), 2945–2961.

[17] S. Shil and H. K. Nashine, Unique positive definite solution of non-linear matrix equation on relational metric spaces, Fixed Point Theory 24, no. 1 (2023), 367–382.

[18] T. Suzuki, A generalized Banach contraction principle that characterizes metric completeness, Proceedings of the American Mathematical Society 136, no. 5 (2007), 1861–1870.

[19] D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, Fixed Point Theory and Applications 2012 (2012), 94.