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Eccentric p -Summing Lipschitz Operators and Integral Inequalities on Metric Spaces and Graphs

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Abstract: The extension of the concept of p -summability for linear operators to the context of Lipschitz operators on metric spaces has been extensively studied in recent years. This research primarily uses the linearization of the metric space M afforded by the associated Arens–Eells space, along with the duality between M and the metric dual space $M^\#$ defined by the real-valued Lipschitz functions on M . However, alternative approaches to measuring distances between sequences of elements of metric spaces (essentially involved in the definition of p -summability) exist. One approach involves considering specific subsets of the unit ball of $M^\#$ for computing the distances between sequences, such as the real Lipschitz functions derived from evaluating the difference in the values of the metric from two points to a fixed point. We introduce new notions of summability for Lipschitz operators involving such functions, which are characterized by integral dominations for those operators. To show the applicability of our results, in the last part of this paper, we use the theoretical tools obtained in the first part to analyze metric graphs. In particular, we show new results on the behavior of numerical indices defined on these graphs satisfying certain conditions of summability and symmetry.

Keywords: Lipschitz; metric; summability; integral inequalities; domination

MSC: 26A16; 47B10; 05C12



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1. Introduction

The triangular inequality that satisfies every metric allows us to write another expression associated with any distance: if (M, d) is a metric space (see the definition below), consider the expression

$$\widehat{d}(x_1, x_2) = \sup_{y \in M} |d(x_1, y) - d(x_2, y)|, \quad x_1, x_2 \in M.$$

Clearly, $\widehat{d} = d$; indeed, by the triangular inequality $\widehat{d} \leq d$, and for the other inequality, just take $y = x_2$. However, the situation changes if we compute the supremum on a smaller set $S \subseteq M$, perhaps with some extra properties (e.g., compactness). In this case, one can only be sure that the formula

$$\widehat{d}_S(x_1, x_2) = \sup_{y \in S} |d(x_1, y) - d(x_2, y)|$$

is a semi-norm, since it could happen that $\widehat{d}(x_1, x_2) = 0$ while $x_1 \neq x_2$. For example, if (M, d) is the Euclidean real line, $S = \{0\}$, $x_1 = 1/2$ and $x_2 = -1/2$; in this case, $\widehat{d}_S(x_1, x_2) = 0$, so $\widehat{d}_S$ cannot be a distance. We will call such an expression $\widehat{d}_S$ an eccentric pseudometric associated to the distance d . Although this notion can be found in many contexts of the theory of metric spaces, functional analysis and its applications, we have not

found any specific notation for it in the literature. We have chosen this name because of the relation to the notion of eccentricity in a bounded metric space. The eccentricity or Koenig number of a point x in the metric space (M, d) is given by $e(x) = \max_{y \in M} d(x, y)$ (see [§1.3] of [1]), and the interested reader can find a lot of applications (see, for example, [2–4] and the references therein for concrete applications in graph theory). Our definition comes from this notion when we try to use it to define a symmetric metric-like function of two variables, taking into account that it is equal to the original metric when $S = M$ is chosen.

Eccentric pseudometrics as $\hat{d}_S$ have been used in [5] in the context of Lipschitz map representation to provide new definitions of summability for these maps. Some interesting questions could be raised about their own definition, such as the characterization of the subsets S for which such a pseudometric $\hat{d}_S$ is a metric, and when it is equivalent to d . This pseudometry appears in the mathematical literature mainly in two contexts. The first is related to the isometric representation of metric spaces as subspaces of spaces of bounded continuous functions. Indeed, if (M, d) is a metric space, consider the function $\theta_x : M \rightarrow \mathbb{R}$, $x \in M$, given by $\theta_x(y) = d(y, x)$ for all $y \in M$. It can easily be seen (see Embedding Lemma [§3.5] of [6]) that the map $x \mapsto \theta_x$ defines an isometry $M \hookrightarrow C_b(M)$ sometimes referred to as Kuratowski mapping, where $C_b(M)$ is the space of bounded uniformly continuous real functions endowed with the usual supremum norm, since

$$\|\theta_x(\cdot) - \theta_y(\cdot)\|_\infty = \sup_{z \in S} |\theta_x(z) - \theta_y(z)| = d(x, y).$$

If we change M by any subset $S \subseteq M$, we find that our definition satisfies $\hat{d}_S(x, y) = \sup_{z \in S} |\theta_x(z) - \theta_y(z)| \leq d(x, y)$. The second context in which similar ideas can be found is that defined by the so-called Hausdorff distance between sets. It is well known that the Hausdorff distance h between non-empty subsets A, B of a metric space (M, d) can be computed by the formula

$$h(A, B) = \sup_{z \in M} |d(z, A) - d(z, B)|,$$

where $d(z, A) = \inf_{y \in A} d(z, y)$ [§3.14] of [6]. Recall that

$$h(A, B) = \max\{\sup_{x \in A} d(x, B), \sup_{y \in B} d(y, A)\}.$$

Thus, if we consider singletons $A = \{x\}$ and $B = \{y\}$, we find that $h(\{x\}, \{y\}) = d(x, y)$, and so we can consider the local Hausdorff metric formula for $S \subset M$

$$h_S(A, B) := \sup_{z \in S} |d(z, A) - d(z, B)|$$

which again gives $h_S(\{x\}, \{y\}) = \hat{d}_S(x, y)$. Applications of the Hausdorff metric for subset hyperspaces can be found in many fields, from set theory to computer science (see, e.g., [7–9].)

The definition of eccentric metric can also be used to generate new tools for the study of the summability of Lipschitz operators, both in general metric spaces and in specific cases of special interest. The main idea is that eccentric metrics can be used to define pseudodistances for sequences in metric spaces, providing the notion of weakly p -summable sequence in the non-linear context that allows a similar definition of what are weakly p -summable sequences and strongly p -summable sequences in the setting of Banach spaces. We deal with this type of application in this paper, in which, after showing some new general results on summability of Lipschitz maps, we focus attention on the case of metric graphs.

Let us fix some concepts and notation. All the definitions and results needed on graph theory are elementary; the reader can find information about this in [10]. Throughout this paper, we use standard concepts of Banach spaces, metric spaces and measure theory.

Recall that a metric space (M, d) is a set M endowed with a function $d : M \times M \rightarrow \mathbb{R}^+$ satisfying that $d(x, y) = 0$ if and only if $x = y$, and for all $x, y, z \in M$, $d(x, y) = d(y, x)$ and $d(x, z) \leq d(x, y) + d(y, z)$ [1]. A pseudometric is defined in the same way, but $d(x, y) = 0$ does not imply $x = y$. A Lipschitz function $f : M \rightarrow N$ between metric spaces (M, d_M) and (N, d_N) is a function that satisfies, for a certain constant $K > 0$, that $d_N(f(x), f(y)) \leq K d_M(x, y)$ for all $x, y \in M$, and $Lip(f)$ is the infimum of all such constants K [11,12]. If X is a Banach space, we write $\|\cdot\|_X$ for its norm (or $\|\cdot\|$ if the space is fixed in the context), B_X for its unit ball and X^* for its dual space. For example, if μ is a regular Borel probability measure on the σ -algebra of Borel subsets of a metric space M , we write $L^p(\mu)$ for the classical Banach lattice of all the p -integrable functions with respect to μ .

Remark 1. Let us place the pseudometrics $\hat{d}_S$ defined above in the context of the classical linearization of metric spaces. For this purpose, we will introduce new tools along the lines of the Kuratowski mapping explained earlier in this section, with the ultimate goal of obtaining the linearization of metric spaces and Lipschitz functions as elements of the Banach dual of this linear normed space. Recall that the elementary functions $x \mapsto \theta_x(\cdot) = d(\cdot, x)$ that we use are Lipschitz maps, so we will consider them to define a subset of the space of Lipschitz functions. As we shall see, a summability property associated with such a set of functions by duality will become equivalent to an integral inequality, extending the classical situation of equivalence between summability and integral domination. We will explain below how this turns out concretely.

Although we are working in the general framework of the metric spaces, for the case that $S = M$, we can understand the formula $\hat{d}$ for the computation of d from the classical point of view of duality in Banach spaces. Indeed, if we fix a point $z \in M$ in the metric space (M, d) , we can consider an isometric embedding $M \hookrightarrow AE(M)$ into the Arens–Eells function space (also called the free space) [13,14]. The space is generated by real functions defined as the difference in characteristic functions $\chi_{\{x\}} - \chi_{\{z\}}$. The dual space of this function space can be identified with the Banach space $M^\#$, which consists of all real Lipschitz functions that are zero at z . The reader can find a complete explanation of such spaces in [14]. Now, take the set of all Lipschitz functions as $x \mapsto f_y(x) = d(x, y) - d(y, z)$ (a slightly different definition from those appearing in the Kuratowski mapping) that are clearly in the unit ball and are 0 in z . Then, if $x_1, x_2 \in M$, $\hat{d}$ can be computed by duality as

$$d(x_1, x_2) = \sup_{y \in M} |\langle \chi_{x_1} - \chi_z - \chi_{x_2} + \chi_z, f_y \rangle| = \sup_{y \in M} |d(x_1, y) - d(x_2, y)| = \hat{d}(x_1, x_2).$$

Indeed, note that it is enough to take $y = x_2$ to attain the value, and by the triangular inequality, we always have that $|d(x_1, y) - d(x_2, y)| \leq d(x_1, x_2)$ for every $y \in M$.

In other words, the functions $\{f_y : y \in M\} \subset M^\# = (AE(M))^*$ are norming for the elements $\chi_{\{x_1\}} - \chi_{\{x_2\}}$ (but not necessarily for all the elements of $AE(M)$). To see a counterexample, just consider the metric space $\{X_1, X_2, X_3, X_4\}$ endowed with the discrete metric. Then, a direct computation shows that

$$\sup_{f_{X_i} : i=1,2,3,4} \left\langle \frac{1}{2}(\chi_{\{X_1\}} - \chi_{\{X_2\}}) + \frac{1}{2}(\chi_{\{X_3\}} - \chi_{\{X_4\}}), f_{X_i} \right\rangle = 1/2,$$

while

$$\left\| \frac{1}{2}(\chi_{\{X_1\}} - \chi_{\{X_2\}}) + \frac{1}{2}(\chi_{\{X_3\}} - \chi_{\{X_4\}}) \right\| = 1.$$

Let us explain our approach. To do so, let us recall some fundamental issues about summability in Banach spaces. After the introduction of the powerful machinery of so-called operator ideals [15,16], the central notions of summability in Banach spaces are often characterized by means of integral inequalities for operators. For example, the relation between weak convergence and norm convergence of sequences is approached from the point of view of the properties of the operators that transform weakly summable sequences

into strongly summable sequences by defining the so-called 1-summing operators [15–17]. Recall that, given $p \geq 1$ and a sequence $(x_i)_i$ in a Banach space $(X, \|\cdot\|)$, two sequence norms can be naturally defined for it, which are the so-called strong p -norm $\|\cdot\|_{p,s}$ and the weak p -norm $\|\cdot\|_{p,w}$, and are given by

$$\|(x_i)_i\|_{p,s} = \left(\sum_{i=1}^{\infty} \|x_i\|^p \right)^{1/p}, \quad \text{and} \quad \|(x_i)_i\|_{p,w} = \sup_{x' \in B_{X^*}} \left(\sum_{i=1}^{\infty} |\langle x_i, x' \rangle|^p \right)^{1/p},$$

respectively. So, a sequence is absolutely p -summable (also called strongly p -summable) when its strong p -norm is finite, and weakly p -summable if its weak norm is. A linear operator between Banach spaces $T : X \rightarrow Y$ is absolutely p -summing if and only if it transforms weakly p -summable sequences into strongly p -summable sequences, which is equivalent to the existence of a constant $K > 0$ such that the inequality

$$\|(T(x_i))_{i=1}^n\|_{p,s} \leq K \|(x_i)_{i=1}^n\|_{p,w}$$

holds for every finite sequence in X . Pietsch's factorization theorem [Ch.1] of [17] establishes that this is equivalent to the existence of a Borel regular measure μ on B_{X^*} such that the following integral domination

$$\|T(x)\|_Y \leq K \int_{B_{X^*}} |\langle x, x' \rangle| d\mu(x')$$

holds for all $x \in X$. For example, using the associated factorization properties for these ideal operators, it can be easily shown that a Banach space in which weakly summable sequences and strongly summable sequences coincide is necessarily finite dimensional (see [Ch.1] of [17]). This is the type of integral domination and associated factorization that we also want to obtain for our classes of non-linear operators.

Mimicking (to some extent) the situation in Banach spaces, in this paper, we also face the summability problem based on integral inequalities as above but for the relation between the summability associated with the distance and the summability associated with the eccentric distance, which characterize some relevant properties regarding the summability of Lipschitz functions in general metric spaces. Let us first clarify that the translation of the operator-ideal way of understanding summability from the linear context to the Lipschitz setting has already been studied in depth in recent years [18–22]. This research has even been extended to a broader class of non-linear operators [23]. The main definitions and results have already been adapted to this context, and some relevant theorems have been proven according to this Lipschitz version of ideal operator theory (see, e.g., [18,22,24]). Our purpose in the present paper does not follow this research trend, but the comparison of the sequence summability properties associated with the original d -metric and the associated $\hat{d}$ -metric corresponding to the adapted versions of the strong and weak norms in the new context. Let us now introduce these notions; we have not found similar definitions in the vast literature on metric spaces, so we present them as new.

Fix $p \geq 1$. We say that two sequences $(x_i^1)_i$ and $(x_i^2)_i$ in a metric space (M, d) are p -absolutely close to each other if

$$D_{p,ac}((x_i^1)_i, (x_i^2)_i) := \left(\sum_{i=1}^{\infty} d(x_i^1, x_i^2)^p \right)^{1/p} < \infty.$$

In relation to the action of the elements of the dual space, we say that these two sequences are p -weakly close to each other if

$$D_{p,wc}((x_i^1)_i, (x_i^2)_i) = \sup_{f \in B_{M^\#}} \left(\sum_{i=1}^{\infty} |f(x_i^1) - f(x_i^2)|^p \right)^{1/p} < \infty.$$

and that they are p -eccentrically close to each other if

$$D_{p,cc}((x_i^1)_i, (x_i^2)_i) = \sup_{y \in M} \left(\sum_{i=1}^{\infty} |d(x_i^1, y) - d(x_i^2, y)|^p \right)^{1/p} < \infty.$$

Note that the last definition can be directly adapted to the case in which the supremum is computed over a subset $S \subseteq M$; we will write $D_{p,cc}^S$ in this case. Note also that, after Remark 1, the definition of $D_{p,cc}^S$ can also be given by restricting the set in which the supremum in $D_{p,wc}$ is computed, that is, $D_{p,cc}^S \leq D_{p,wc}$ for every $S \subseteq M$. However, we prefer to keep the different notations $D_{p,cc}$ and $D_{p,wc}$ for clarity, due to the relevance of the sets defined by functions as the $f_y \in M^\#$ appearing in this remark.

We will analyze the fundamental relations between these three ways of understanding proximity between sequences in the next section, as well as the properties of the following two classes of operators, the study of which will be the main objective of this work.

Extending the definition of the metric summing operator given in [Def.4] of [5], if we fix $p \geq 1$ and a subset $S \subseteq M$, we say that a map $T : M \rightarrow N$ between metric spaces (M, d_M) and (N, d_N) is eccentric p -summing if there is a constant $K > 0$ such that for every couple of finite sequences $(x_i^1)_{i=1}^n$ and $(x_i^2)_{i=1}^n$ in the metric space (M, d_M) , we have

$$\left(\sum_{i=1}^n d_N(T(x_i^1), T(x_i^2))^p \right)^{1/p} \leq K \sup_{y \in S} \left(\sum_{i=1}^n |d_M(x_i^1, y) - d_M(x_i^2, y)|^p \right)^{1/p},$$

that is, $D_{p,ac}((T(x_i^1))_{i=1}^n, (T(x_i^2))_{i=1}^n) \leq K D_{p,cc}((x_i^1)_{i=1}^n, (x_i^2)_{i=1}^n)$. This definition was made in [5] for the case that S is a metric generating system, but it makes sense for any subset $S \subseteq M$. The infimum of such constants K is denoted by $MLip_p(T)$. The reader who is familiar with the ideals of linear operators and their adaptation to the case of Lipschitz maps can easily understand that this definition is inspired by the case of 1-summing operators (see, for example, [Ch.1] of [17]). In fact, as said in [5], all eccentric 1-summing operators are Lipschitz 1-summing in the sense of Farmer and Johnson [22], as can be easily seen taking into account that for any fixed $y \in M$, the function $x \mapsto d(x, y)$ is a map belonging to the unit ball of the Banach space $M^\#$ of all real Lipschitz functions, following the notation used in [22].

The next definition is inspired to some extent by other classical operator ideals, the one of the (p, q) -mixing operators (see [Ch.21] of [15]), which has also found its analogue in the case of the Lipschitz operators [25]. Thus, in a similar way, we say that an operator $T : M \rightarrow N$ is eccentrically p -approximating if for every couple of finite sequences $(x_i^1)_{i=1}^n$ and $(x_i^2)_{i=1}^n$ in the metric space (M, d_M) , we have

$$\sup_{z \in N} \left(\sum_{i=1}^n |d_N(T(x_i^1), z) - d_N(T(x_i^2), z)|^p \right)^{1/p} \leq K \sup_{y \in M} \left(\sum_{i=1}^n |d_M(x_i^1, y) - d_M(x_i^2, y)|^p \right)^{1/p},$$

that is, $D_{p,cc}((T(x_i^1))_{i=1}^n, (T(x_i^2))_{i=1}^n) \leq K D_{p,cc}((x_i^1)_{i=1}^n, (x_i^2)_{i=1}^n)$. The infimum of such constants K is $ELip_p(T)$.

Both definitions can be directly adapted if we use eccentric metrics defined by subsets S of M and B of N instead of the direct ones, that is, with the supremum in the formulas computed over S and B instead of M and N , respectively.

2. Absolutely Close and Eccentrically Close Sequences in Metric Spaces

In this section, we show the fundamental relationships between the various definitions of p -summation of sequences in metric spaces that we have introduced above. As we will see in the next section, the operators we define fulfill that they convert pairs of sequences that are similar with respect to one summation method into sequences that are similar with respect to another, stronger summation method. To understand this, we first need to know

what the natural order between these sums is, to distinguish between stronger and weaker summation methods.

Given two sequences $(x_i^1)_{i=1}^n$ and $(x_i^2)_{i=1}^n$ in a metric space (M, d) , the definitions of absolutely close and eccentrically close are directly inspired by those of absolutely and weakly summable sequences. However, while the former leads to the corresponding notion in the theory of Banach spaces (just change $d(x^1, x^2)$ to $\|x^1 - x^2\|$ in the definition), the concept of eccentrically close sequences does not coincide with the notion of weakly summable sequences. Let us fix the basic relations in the following result, and the relation to the case of Banach space in the following.

Example 1. Let us show some easy examples of p -eccentrically close sequences.

- Consider the infinite discrete metric space $(\{0, 1\}, c)$, where $c(x, y) = 0$ if $x = y$ and $c(x, y) = 1$ if $x \neq y$. Then, two infinite sequences (x_i^1) and (x_i^2) in it are p -eccentrically close if and only if the set $N = \{i \in \mathbb{N} : x_i^1 \neq x_i^2\}$ is finite. Indeed, note that if $x_i^1 = x_i^2$, we have that $d(x_i^1, y) = d(x_i^2, y)$, and if $x_i^1 \neq x_i^2$, then $|d(x_i^1, y) - d(x_i^2, y)| = 1$ independently of the fact that $y = 0$ or $y = 1$. So, if the set N is not finite, we have that the corresponding p -sum is infinite.
- In the case of normed spaces $(X, \|\cdot\|)$, two sequences are p -eccentrically close

$$\sup_{y \in X} \left(\sum_{i=1}^{\infty} \left| \|x_i^1 - y\| - \|x_i^2 - y\| \right|^p \right)^{1/p} < \infty.$$

Consider the space ℓ_∞ with the supremum norm and sequence of sequences $(x_i^1) = (\frac{1}{i}e_i)$ of elements in ℓ_∞ , where e_i is the unit vector $e_i = (0, 0, \dots, 0, 1, 0, \dots)$ and 1 appears at the i -th position. Take also the null sequence of sequences $(x_i^2) = (0)$. Then, if we choose $y = 0 \in \ell_\infty$,

$$\sum_{i=1}^{\infty} \left| \|x_i^1 - 0\|_{\ell_\infty} - \|x_i^2 - 0\|_{\ell_\infty} \right| = \sum_{i=1}^{\infty} \frac{1}{i} = \infty,$$

and so $\sum_{i=1}^{\infty} \left| \|x_i^1 - 0\|_{\ell_\infty} - \|x_i^2 - 0\|_{\ell_\infty} \right| \leq \sup_{y \in \ell_\infty} \left(\sum_{i=1}^{\infty} \left| \|x_i^1 - y\|_{\ell_\infty} - \|x_i^2 - y\|_{\ell_\infty} \right| \right)$. Thus, the sequences are not 1-eccentrically close to each other.

- However, note that if we take $p > 1$ in the previous example,

$$\sup_{y \in \ell_\infty} \left(\sum_{i=1}^{\infty} \left| \|x_i^1 - y\|_{\ell_\infty} - \|x_i^2 - y\|_{\ell_\infty} \right|^p \right)^{1/p} \leq \left(\sum_{i=1}^{\infty} \|x_i^1 - x_i^2\|_{\ell_\infty}^p \right)^{1/p} = \left(\sum_{i=1}^{\infty} \frac{1}{i^p} \right)^{1/p} < \infty.$$

Thus, the sequences are p -absolutely close to each other, and also p -eccentrically close.

Proposition 1. Let (M, d) be a metric space and $p \geq 1$, and consider two sequences $(x_i^1)_{i=1}^\infty$ and $(x_i^2)_{i=1}^\infty$ in it. Then, for every $S \subseteq M$,

$$D_{p,cc}^S((x_i^1)_i, (x_i^2)_i) \leq D_{p,cc}((x_i^1)_i, (x_i^2)_i) \leq D_{p,wc}((x_i^1)_i, (x_i^2)_i) \leq D_{p,ac}((x_i^1)_i, (x_i^2)_i).$$

Proof. The first inequality is obvious. After Remark 1, we know that all the functions of the set $\{f_y : y \in S\}$ defined there belong to the unit ball $B_{M^\#}$, which makes the inequality in the middle of the expression obvious. The inequality on the right-hand side is just a consequence of the duality of $AE(M)$ and $M^\# = Lip_0(M)$. Indeed, for every $x^1, x^2, y \in M$ and $f \in B_{M^\#}$, we have that $|f(x_i^1) - f(x_i^2)| \leq d(x_i^1, x_i^2)$, $i \in \mathbb{N}$, and so for all $y \in M^\#$,

$$\sum_{i=1}^{\infty} |d(x_i^1, y) - d(x_i^2, y)|^p \leq \sum_{i=1}^{\infty} d(x_i^1, x_i^2)^p,$$

which gives the result. $\square$

After these obvious relations, we show in what follows conditions under which the converse inequalities could hold. This is relevant because we will then determine metric spaces for which the identity map has a certain summing property, which will provide some equivalent integral average formulas for the original metric. Let us first see what happens for the linear case. In what follows, we will assume that p is a fixed parameter $1 \leq p < \infty$.

Remark 2. Let $(X, \|\cdot\|)$ be a Banach space. Recall that a subset $S \subseteq B_{X^*}$ is said to be norming if it is enough to compute the norm for all the elements of X by duality. In other words, the formula

$$\sup_{x' \in S} |\langle x, x' \rangle| = \|x\|$$

holds for every $x \in X$. It is well known (in fact, it is a consequence of a direct computation) that for any sequence $(x_i)_i$, its weak p -norm can be computed just using a norming subset S instead of all B_{X^*} (see, for example, [Ch.2, §3, p. 36] of [17]). This fact has also consequences in the Lipschitz case, since in the computation of $D_{p,wc}((x_i^1)_i, (x_i^2)_i)$, the set $B_{M^\#}$ can be substituted by any norming subset: this is just a consequence of understanding this expression as a dual action between the space $AE(M)$ and its dual space $M^\#$. However, the set of all the functions defined in Remark 1 $\{f_y : y \in M\} \subseteq B_{M^\#}$ is in principle not norming for the whole space $AE(M)$ as shown there, so this argument cannot be used, although of course, it is norming for the functions $\chi_{\{x_1\}} - \chi_{\{x_2\}}$.

In order to facilitate the finding of applications (for example, for finite graphs, see Section 4), we extend the notion of norming to the following: we say that a subset $S \subseteq B_{X^*}$ of a Banach space $(X, \|\cdot\|)$ is k -norming for short if there is a constant $k > 0$ such that

$$\|x\| \leq k \sup_{x' \in S} |\langle x, x' \rangle|, \quad x \in X.$$

(Of course, the inequality $\sup_{x' \in S} |\langle x, x' \rangle| \leq \|x\|$ always holds.)

Proposition 2. Let (M, d) be a pointed metric space with distinguished point 0, and consider two sequences $(x_i^1)_{i=1}^n$ and $(x_i^2)_{i=1}^n$ in it. Then, if the set $\mathcal{F}_S = \{f_y = d(\cdot, y) - d(y, 0) : y \in S \subset M\} \subseteq B_{M^\#}$ is k -norming for a certain constant k , we have

$$D_{p,cc}^S((x_i^1)_{i=1}^n, (x_i^2)_{i=1}^n) \leq D_{p,wc}((x_i^1)_{i=1}^n, (x_i^2)_{i=1}^n) \leq k D_{p,cc}^S((x_i^1)_{i=1}^n, (x_i^2)_{i=1}^n).$$

Proof. It is just a consequence of writing the summation expressions in terms of duality. Indeed, note that the first inequality is obvious. For the second one, just write p' for the extended real number that satisfies $1/p + 1/p' = 1$, $\ell_{p'}$, the classical space of sequences of real numbers with finite p' with the corresponding norm, and notice that

$$\begin{aligned} D_{p,wc}((x_i^1)_{i=1}^n, (x_i^2)_{i=1}^n) &= \sup_{f \in B_{M^\#}} \sup_{(\lambda_i)_{i=1}^n \in B_{\ell_{p'}}} \left\langle \sum_{i=1}^n \lambda_i (\chi_{\{x_i^1\}} - \chi_{\{x_i^2\}}), f \right\rangle \\ &\leq k \sup_{f_y \in \mathcal{F}_S} \sup_{(\lambda_i)_{i=1}^n \in B_{\ell_{p'}}} \left\langle \sum_{i=1}^n \lambda_i (\chi_{\{x_i^1\}} - \chi_{\{x_i^2\}}), f_y \right\rangle = k D_{p,cc}^S((x_i^1)_{i=1}^n, (x_i^2)_{i=1}^n). \end{aligned}$$

□

3. Factorization of Eccentric p -Summing and Eccentrically p -Approximating Lipschitz Operators

In this section, we show the main characterizations of the classes of operators that we defined in the introduction, as well as the properties that can be derived from them. As will be seen, the definition of these operators involves the uniform transformation of pairs of sequences that are summable with respect to a method explained in Section 2 to a pair of

sequences that are summable with respect to a stronger method of those also appearing in Section 2. The characterizations of the resulting operators are given mainly by integral inequalities, which can be translated into factorization theorems for the Lipschitz operators involved.

The techniques used here are in some sense related to those employed in [26], which analyze suitable formulas to represent metrics as integral averages as the ones involved in the so-called Wasserstein space [§2.1] of [27]. Also, a particular case of eccentric 1-summing operators was studied in [5] (metric summing operators in this reference). For the case of Banach spaces, one of the first questions that was faced in the original paper on Lipschitz p -summing operators [22] was connected to the problem of what happens if the Lipschitz map involved is also linear. It was proven there that in this case, to be Lipschitz p -summing and to be absolutely p -summing were the same [Th.2] of [22]. In this section, we also compare different types of summability for Lipschitz operators, and we show some structural applications for general metric spaces. In the next section, we will analyze the concrete case of infinite metric graphs as an example of application, and also as a proposal of a new tool for studying symmetry notions for graphs. The next theorem, which is the consequence of a classical separation argument that often appears in results on the factorization of operators, is the main result of this section.

Remark 3. Note that the requirements on the operators that appear in all the results of this section imply that they are in particular Lipschitz. This is the reason why we do not ask in the statements of the propositions that the operators are Lipschitz; we simply write that they are maps. The proofs that they are all Lipschitz are always consequences of direct computations.

Let us now write the main characterization of the domination of Lipschitz functionals in terms of the summability properties that they satisfy. In other words, the class of functionals that are characterized below (eccentric p -summing functionals) carry p -eccentrically close couples of (finite) sequences in a metric space to p -absolutely close couples of sequences in a uniform way, and the uniform constant relating the corresponding summations is the operator eccentric p -summing norm $L_p(T)$.

Let us remark that, to pass from a summability property of a map involving a eccentric p -pseudometric to an integral domination, we need the set used to compute it (the set that we denoted by $S \subseteq M$ in the previous sections) to be compact for the metric topology of (M, d) . To underline this fact, in the following results, we use the letter K (which usually denotes compact sets) for S .

Theorem 1. Let (M, d) be a metric space and K a compact subset of M . Let $p \geq 1$, $f : M \rightarrow \mathbb{R}$ be a mapping and $C > 0$. Then, the following statements are equivalent.

(i) For any $n \in \mathbb{N}$, $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in M$,

$$\left(\sum_{i=1}^n |f(x_i) - f(y_i)|^p \right)^{1/p} \leq C \sup_{w \in K} \left(\sum_{i=1}^n |d(x_i, w) - d(y_i, w)|^p \right)^{1/p}.$$

(ii) There exists a Borel regular probability measure μ on K such that for any $x, y \in M$

$$|f(x) - f(y)| \leq C \left(\int_K |d(x, w) - d(y, w)|^p d\mu(w) \right)^{1/p}.$$

A real function satisfying (i) or (ii) is Lipschitz with constant $\leq C$. Following the general notation introduced before, we call such a function an eccentric p -summing functional.

Proof. The proof is an application of a classical separation argument that is found in many factorization theorems. Here, we will use Ky Fan's Lemma instead of the argument based on the Hahn–Banach Theorem that is usual.

(i) $\Rightarrow$ (ii) First, note that, taking into account that the same elements x_i, y_i could appear several times in the inequalities in (i), using rational number approximation, we can see that these inequalities can be extended to a larger class of inequalities that also involve positive constants such as the following, where $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in M$ are general finite sequences of elements of M and $a_1, a_2, \dots, a_n > 0$,

$$\sum_{i=1}^n a_i |f(x_i) - f(y_i)|^p \leq C \sup_{w \in K} \sum_{i=1}^n a_i |d(x_i, w) - d(y_i, w)|^p. \quad (1)$$

The interested reader can find an explanation of this argument in [22].

Let us recall Ky Fan's minimax Lemma (the one used in analysis; there is another famous Ky Fan's Lemma on labelings of triangulations that generalizes Tucher's Lemma). In particular, a family $\mathcal{F}$ of real functions is concave if every (finite) convex combination of elements of the family gives again an element of $\mathcal{F}$. Ky Fan's Lemma (see for example [Ch.9] of [17]) states that,

- if W is a compact convex subset of a Hausdorff topological vector space and
- Ψ is a concave family of lower semi-continuous, convex, $\mathbb{R}$ -valued functions on W ,
- and there is $c \in \mathbb{R}$ satisfying that for every $\psi \in \Psi$, there exists $x_\psi \in W$ such that $\psi(x_\psi) \leq c$,

then there exists $x \in W$ such that $\psi(x) \leq c$ for every $\psi \in \Psi$.

Let us show the proof. Fix f be a real function as the one in the statement of the theorem. Since the set K is compact in M , we can define the space of continuous functions on it, $C(K)$, whose dual space is the space of Borel regular measures $\mathcal{M}(K)$. Define any subset of $M \times M \times [0, +\infty[$ as $A = \{(x_1, y_1, a_1), (x_2, y_2, a_2), \dots, (x_n, y_n, a_n)\} \subseteq M \times M \times [0, +\infty[$, $n \in \mathbb{N}$; the function $\psi_A : \mathcal{M}(K) \rightarrow \mathbb{R}$ is given by

$$\psi_A(\mu) = \sum_{i=1}^n a_i |f(x_i) - f(y_i)|^p - \int_K \sum_{i=1}^n a_i C^p |d(x_i, w) - d(y_i, w)|^p d\mu(w).$$

Consider the dual pair $\langle C(K), \mathcal{M}(K) \rangle$, and consider the space $\mathcal{M}(K)$ endowed with its weak* topology. The function ψ_A is clearly continuous, since for every fixed $x \in M$, the function $w \mapsto d(x, w)$ is Lipschitz continuous, so in particular, the p -th power appearing in the function belongs to $C(K)$. The integral represents the dual action, so it works by duality and is continuous for the weak* topology of $\mathcal{M}(K)$. Due to the multiplication by arbitrary scalars that appears in the definition, the family of all these functions is concave, and each of them is also convex.

On the other hand, there is always a measure μ_A that satisfies that $\psi_A(\mu_A) \leq 0$. Indeed, by compactness, the supremum appearing in the corresponding inequality given in Equation (1) is always attained at a certain point w_A , so it is enough to consider $\mu_A = \delta_{\{w_A\}}$ to achieve this requirement. Now, we apply Ky Fan's Lemma for $c = 0$ to obtain a fixed measure μ such that

$$\sum_{i=1}^n a_i |f(x_i) - f(y_i)|^p \leq \int_K \sum_{i=1}^n a_i C^p |d(x_i, w) - d(y_i, w)|^p d\mu(w)$$

for all the functions of the family. In particular, for every $x, y \in M$,

$$|f(x) - f(y)|^p \leq C^p \int_K |d(x_i, w) - d(y_i, w)|^p d\mu(w).$$

(ii) $\Rightarrow$ (i) If there is probability measure μ as in (ii), taking into account that μ is a probability measure and computing the power p in the inequality, we obtain

$$\begin{aligned} \sum_{i=1}^n |f(x_i) - f(y_i)|^p &\leq C^p \int_K \left(\sum_{i=1}^n |d(x_i, w) - d(y_i, w)|^p \right) d\mu(w) \\ &\leq C^p \sup_{t \in K} \sum_{i=1}^n |d(x_i, t) - d(y_i, t)|^p, \end{aligned}$$

so (i) is obtained. $\square$

This result can also be written as a commutative factorization for the Lipschitz functional f through a subset of $L^p(\mu)$, as

$$\begin{array}{ccc} M & \xrightarrow{f} & \mathbb{R} \\ i_d \downarrow & \nearrow \hat{f} & \\ L^p(\mu) \supset S & & \end{array}$$

where i_d is a Lipschitz map given by $i_d(x)(\cdot) = d(x, \cdot) \in L^p(\mu)$, S is the set of all the classes of functions as $d(x, \cdot)$ that are equal μ -a.e. and $\hat{f}$ is also a real-valued Lipschitz functional on S given by $\hat{f}(i_d(x)) = f(x)$. Indeed, note that for $x, y \in M$, we have that the difference of the functions $w \mapsto i_d(x)$ and $w \mapsto i_d(y)$ satisfies

$$\begin{aligned} &\left(\int_K |d(x, w) - d(y, w)|^p d\mu(w) \right)^{1/p} \\ &\leq \left(\int_K \sup_{w \in K} |d(x, w) - d(y, w)|^p d\mu(w) \right)^{1/p} \leq \mu(K) \sup_{w \in K} |i_d(x)(w) - i_d(y)(w)| \leq d(x, y), \end{aligned}$$

and the continuity of $\hat{f}$ is assured by the inequality in Theorem 1 (2). The functions i_d and $\hat{f}$ are Lipschitz.

Example 2. Consider the space $\mathbb{R}^2$ endowed with the ℓ_1 -norm, that is $\|(a, b)\|_{\ell_1} = |a| + |b|$, and take the following metric subspace,

$$M = A \cup \{(n, n) : 2 \leq n \in \mathbb{N}\} \quad \text{where} \quad A = \{(0, 0), (1, 0), (0, 1), (1, 1)\}.$$

Consider the functional $f((n, n)) = 1/n$ for $n \geq 2$ and $f((a, b)) = 0$ for $(a, b) \in A$. Note that A is a compact set, since it is finite. Take any couple of finite sequences $(x_i) = ((a_i, b_i))$ and $(y_i) = ((r_i, s_i))$ and note that

$$\begin{aligned} |f(x_i) - f(y_i)| &= 0 \text{ if } x_i, y_i \in A, \\ |f(x_i) - f(y_i)| &\leq \frac{1}{a_i} \text{ if } x_i \notin A, y_i \in A, \\ |f(x_i) - f(y_i)| &\leq \frac{1}{r_i} \text{ if } x_i \in A, y_i \notin A, \text{ and} \\ |f(x_i) - f(y_i)| &\leq \frac{1}{\min\{a_i, r_i\}} \text{ if } x_i, y_i \notin A. \end{aligned}$$

Note also that

$$\begin{aligned} &\max_{w \in A} \sum_{i=1}^n |d(x_i, w) - d(y_i, w)|^p \\ &\geq \sum_{x_i, y_i \notin A} 2^p |a_i - r_i|^p + \sum_{x_i \in A, y_i \notin A} 2^p |a_i|^p + \sum_{x_i \notin A, y_i \in A} 2^p |r_i|^p + \sum_{x_i \in A, y_i \in A} 1. \end{aligned}$$

Therefore, we have that the inequality (i) in Theorem 1 holds for the function f , for every $p \geq 1$, for the compact set A and for a constant $C \leq 1$. This means that f is eccentric p -summing for all $p \geq 1$.

According to the theorem, there has to exist a probability measure such that the corresponding integral dominates f . In fact, it can also be easily seen that any suitable convex combination of the Dirac deltas δ_{x_i} for $x_i \in A$ gives a measure on A for which an integral domination for f can be obtained. We will explain more examples in Section 4.

Remark 4. It can be easily seen that the same proof of Theorem 1 works for eccentric p -summing operators if the differences $|f(x_i) - f(y_i)|$ are substituted by $d(x_i, y_i)$ on the left-hand side of the inequality in the statement of this theorem. Recall that an operator $T : M \rightarrow N$ is eccentric p -summing for a compact set $K \subset M$ if there is a constant $R > 0$ such that for every couple of finite sequences $(x_i^1)_{i=1}^n$ and $(x_i^2)_{i=1}^n$ in the metric space (M, d_M) , we have

$$\left(\sum_{i=1}^n d_N(T(x_i^1), T(x_i^2))^p \right)^{1/p} \leq R \sup_{w \in K} \left(\sum_{i=1}^n |d_M(x_i^1, w) - d_M(x_i^2, w)|^p \right)^{1/p},$$

or, if we write it in terms of sequence metrics,

$$D_{p,ac}((T(x_i^1))_{i=1}^n, (T(x_i^2))_{i=1}^n) \leq R \cdot D_{p,cc}((x_i^1)_{i=1}^n, (x_i^2)_{i=1}^n).$$

Following the arguments in the proof of Theorem 1, we find that this is equivalent to the integral domination

$$d(x, y) \leq R \left(\int_K |d(x, w) - d(y, w)|^p d\mu_f(w) \right)^{1/p}$$

for all $x, y \in K$. This provides an extension of Theorem 2 in [5] for the p -th power case.

The previous results can also be written when a weak-type metric for sequences is put in the left-hand side of the inequality. For example, if we consider a mapping $T : M \rightarrow N$ and f belongs to a certain subset D of Lipschitz functions on N , the set of compositions $\{f \circ T : f \in D\}$ gives the following application of Theorem 1. Note that the supremum in the left can also be read as follows: for every $f \in D$, the inequality in Theorem 1 works for $f \circ T$.

Corollary 1. Let (M, d) , (N, ρ) be two metric spaces and K a compact subset of M . Let $T : M \rightarrow N$ a mapping and $C > 0$, and let $D \subseteq \text{Lip}_0(N)$. The following statements are equivalent.

1. For any $n \in \mathbb{N}$, $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n \in M$,

$$\sup_{f \in D} \left(\sum_{i=1}^n |f \circ T(x_i) - f \circ T(y_i)|^p \right)^{1/p} \leq C \sup_{w \in K} \left(\sum_{i=1}^n |d(x_i, w) - d(y_i, w)|^p \right)^{1/p}.$$

2. For every $f \in D$, there exists a Borel regular probability measure μ_f on K such that for any $x, y \in M$,

$$|f \circ T(x) - f \circ T(y)| \leq C \left(\int_K |d(x, w) - d(y, w)|^p d\mu_f(w) \right)^{1/p}.$$

That is, if T satisfies any of these statements, then it is eccentrically p -approximating.

This result can also be written in terms of factorizations. If $T : M \rightarrow N$ is a mapping, we can draw the following scheme for a given real functional f :

$$\begin{array}{ccc} M & \xrightarrow{T} & N \\ i_d \downarrow & & \downarrow f \\ S & \xrightarrow{\hat{f}} & \mathbb{R} \\ i \downarrow & & \\ L^p(\mu) & & \end{array}$$

where i is a inclusion map and i_d is the Lipschitz map given by $i_d(x)(\cdot) = d(x, \cdot) \in L^p(\mu)$ that was explained in the factorization associated to Theorem 1; also the set S was explained there, as well as $\hat{f}$. It can be easily seen that all the maps in this commutative factorization diagram are Lipschitz.

Remark 5. In the case that we use the adaptation of this notion for general Lipschitz maps $T : M \rightarrow N$, we find the definition of the eccentrically p -approximating operator. Assume that (M, d_M) and (N, d_N) are metric spaces, and $T : M \rightarrow N$ is a map. Let us recall the definition of eccentrically p -approximating operators associated with a pair of compact subsets K_1 and K_2 . Suppose that the following inequalities are satisfied for a constant $Q > 0$ and every finite sequence of elements $x_1^1, \dots, x_n^1$ and $x_1^2, \dots, x_n^2$ in M

$$\sup_{y \in K_2} \left(\sum_{i=1}^n |d_N(T(x_i^1), y) - d_N(T(x_i^2), y)|^p \right)^{1/p} \leq Q \sup_{x \in K_1} \left(\sum_{i=1}^n |d_M(x_i^1, x) - d_M(x_i^2, x)|^p \right)^{1/p},$$

where $K_1 \subset M$ and $K_2 \subset N$ are compact subsets.

Note now that for every probability measure $\mu_N \in \mathcal{M}(K_2)$, the inequality above clearly gives the domination

$$\sum_{i=1}^n \int_N |d_N(T(x_i^1), y) - d_N(T(x_i^2), y)|^p d\mu_N(y) \leq Q^p \sup_{x \in K_1} \left(\sum_{i=1}^n |d_M(x_i^1, x) - d_M(x_i^2, x)|^p \right)^{1/p}.$$

Using the same argument that proves Theorem 1, writing

$$\left(\int_N |d_N(T(x_1), y) - d_N(T(x_2), y)|^p d\mu_N(y) \right)^{1/p}$$

instead of $|f(x_1^1) - f(x_1^2)|$ shows that there is a probability measure $\mu_M \in \mathcal{M}(K_1)$ such that

$$\int_N |d_N(T(x_1), y) - d_N(T(x_2), y)|^p d\mu_N(y) \leq Q^p \int_M |d_M(x_1, x) - d_M(x_2, x)|^p d\mu_M(x)$$

for all $x_1, x_2 \in M$. This gives the desired characterization for eccentrically p -approximating operators by means of integral inequalities.

4. Applications: Distance Summing Inequalities in Metric Graphs

Having developed the core theoretical framework for eccentric p -summing domination, we now apply these results within the context of metric undirected graphs. All the notions and results that we use on graphs are elementary. The reader can find all the information needed about graphs and distances in graphs in [14] and [1,28], respectively.

Consider a weighted connected undirected graph with a countable set of vertices endowed with the weighted shortest path distance q . We first need to clarify the type of graphs we will deal with. We will consider the case in which, although there is an infinite set of vertices in the graph, all of them are connected by a finite path. This finiteness of the paths is relevant, since given a certain $p \geq 1$, we can use the usual formula for the

p -weighted shortest path distance (positive weights) without the need to consider infinite paths. In other words, given a couple of vertices v_1 and v_2 in the graphs, there is a finite number of paths connecting them, so the shortest path p -distance can be defined as usual by

$$q_p(v_1, v_2) = \inf \left\{ \left(\sum_{i=1}^{n-1} w(v^i, v^{i+1})^p \right)^{1/p} : \text{the sequence } (v^i)_i \text{ connects } v^1 = v_1 \text{ and } v^n = v_2 \right\},$$

where $w(v^i, v^{i+1})$ denotes the weight of the edge connecting both vertices.

Example 3. A canonical example of the type of structure we have fixed is a graph composed of vertices that are identified with the elements of an increasing convergent sequence in $\mathbb{R}$ with its boundary s (which does not belong to the sequence), in which the connectivity is given by the adjacency on the real line and the distance (weight) between consecutive vertices is the size of the segment connecting them. Furthermore, the boundary vertex s is assumed to be connected to all other vertices, and the weights for the distance are again the size of the segment connecting it to the other points in the sequence.

It can be easily seen that the distance of the shortest path between any two vertices is in this case again the size of the segment connecting the two vertices. This (infinite) graph is clearly compact, since it is isomorphic and isometric to the convergent sequence with its limit (Figure 1).

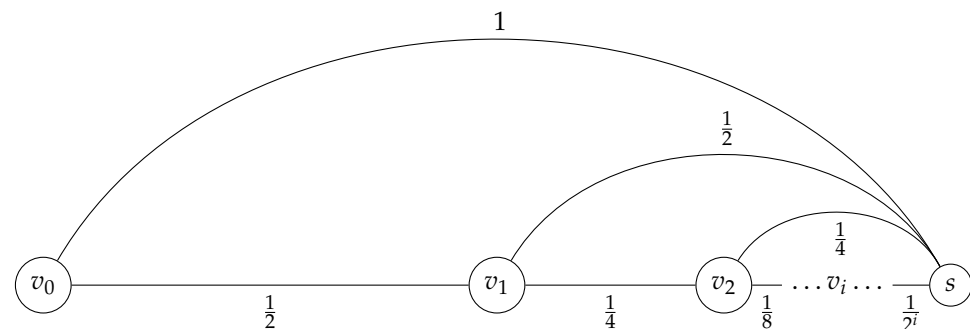


Figure 1. A canonical example using the sequence $(1 - 1/2^i)_i$ with its limit, 1. The vertices v_i represent the element $\{1 - 1/2^i\}$ of the convergent sequence, and s represents the boundary vertex, that is, $s = \lim_i v_i$. The edge values are weights between the vertices.

4.1. Weighted p -Shortest Path Metrics and Integral p -Averages on Graphs

Following the idea that leads to the definition of general path metrics, we can use the integrals appearing in the characterization theorems explained in Section 3 to introduce a new family of pseudometrics that are closely related to the metric symmetry on graphs. Broadly speaking, we will compute the shortest paths between two vertices in a graph considering the distance between two adjacent vertices given by an integral p -average as the ones appearing in previous section. We will focus the attention on the weighted p -shortest path metrics, which will be used in the next section, although some of the results presented here hold for general metrics on graphs.

Our main motivation is that in metric modeling, indices on graphs are often defined as real Lipschitz functions acting on the set of vertices of a given graph. The interested reader can find many such applications, for example, in subjects as different as chemistry [29] and network analysis in the social sciences [30].

Let us first fix some definitions.

Definition 1. Two points x_1, x_2 in a metric space (M, D) are metrically symmetric with respect to a certain set S if $d(x_1, y) = d(x_2, y)$ for every $y \in S$.

This property will be characterized in what follows in terms of domination by an integral p -average. Let us show first some other metric definitions and the relation between them and some already defined functions.

Let $G = (V, E, d)$, be a(n) (undirected connected) metric graph $G = (V, E, d)$. Recall that we assume that all the vertices of the graph can be connected by a finite path. Let us define below a generalized version of the p -weighted shortest path metric for a graph starting from any distance d .

Definition 2. For $p \geq 1$, we define the p -shortest path pseudodistance between two vertices $x_1, x_2 \in V$ associated with the metric d by

$$d_p(v_1, v_2) := \inf \left\{ \left(\sum_{i=1}^{n-1} d(v^i, v^{i+1})^p d\mu \right)^{1/p} : \text{path } v_1 = v^1 \rightarrow v^2, \dots, v^{n-1} \rightarrow v^n = v_2 \right\},$$

where $v^i \rightarrow v^{i+1}$ means that v^i and v^{i+1} are connected.

It can be easily seen that, in general, we cannot ensure that $d_p \leq d$ for any distance d . Indeed, consider the (countable infinite) graph in which V is identified with the set $\mathbb{N}$ of natural numbers, and the adjacent vertices are only the consecutive natural numbers. Consider the discrete distance in it ($d(v_1, v_2) = 1$ whenever $v_1 \neq v_2$.) If we compute the pseudometric d_p between $1 \in \mathbb{N}$ and $n \in \mathbb{N}$, we have that $d_p(1, n) = (n-1)^{1/p}$, while $d(1, n) = 1$. Since n is arbitrary, we find that this inequality cannot follow in general.

However, for the case when d is a weighted shortest path metric q_r , $r \geq 1$, we always obtain this inequality for the adequate values of p . We assume that the weights $W = \{w(v_i, v_j) : v_i \text{ adjacent to } v_j\}$ between adjacent vertices of the graph make q_r a metric, and not just a pseudometric.

Lemma 1. Fix $1 \leq r \leq p$. For every graph $G = (V, E)$, endowed with a weighted path metric q_r , we have

$$(q_r)_p(v_1, v_2) \leq q_r(v_1, v_2), \quad v_1, v_2 \in V.$$

Proof. Take $\varepsilon > 0$ and a path $v_1 = v^1 \rightarrow v^2 \rightarrow \dots \rightarrow v^{n-1} \rightarrow v^n = v_2$ such that

$$\left(\sum_{i=1}^{n-1} w(v^i, v^{i+1})^r \right)^{1/r} \leq q_r(v_1, v_2) + \varepsilon,$$

and note that, since by definition, for adjacent vertices v^i and v^{i+1} , we have $q_r(v^i, v^{i+1}) \leq w(v^i, v^{i+1})$, by the inequality above, we obtain

$$\begin{aligned} (q_r)_p(v_1, v_2) &:= \inf \left\{ \left(\sum_{i=1}^{n-1} q_r(v^i, v^{i+1})^p d\mu \right)^{1/p} : \text{path } v_1 = v^1 \rightarrow v^2, \dots, v^{n-1} \rightarrow v^n = v_2 \right\} \\ &\leq \left(\sum_{i=1}^{n-1} w(v^i, v^{i+1})^p \right)^{1/p}. \end{aligned}$$

Thus, taking into account that $r \leq p$, we obtain

$$(q_r)_p(v_1, v_2) \leq \left(\sum_{i=1}^{n-1} w(v^i, v^{i+1})^p \right)^{1/p} \leq \left(\sum_{i=1}^{n-1} w(v^i, v^{i+1})^r \right)^{1/r} \leq q_r(v_1, v_2) + \varepsilon.$$

Since this happens for every $\varepsilon > 0$, we obtain the result. $\square$

It can be easily seen that for $p = r$, we obtain $(q_p)_p = q_p$, and so the inequality in Lemma 1 becomes an equality.

Let us define below the associated p -average notion of the weighted shortest path distance for a graph starting from any distance d . Let $S \subset V$ be a compact subset of vertices of a(n) (undirected connected) metric graph $G = (V, E, d)$.

Definition 3. For $p \geq 1$ and a probability measure $\mu \in \mathcal{M}(S)$, we define the p -shortest path pseudodistance between two vertices $v_1, v_2 \in V$ by

$$\begin{aligned} d_{p,\mu}(v_1, v_2) &:= \\ \inf \left\{ \left(\sum_{i=1}^{n-1} \int_S |d(v^i, w) - d(v^{i+1}, w)|^p d\mu \right)^{1/p} : \text{path } v_1 = v^1 \rightarrow v^2, \dots, v^{n-1} \rightarrow v^n = v_2 \right\} \\ &= \inf_{\text{paths}} \left\| \left(\sum_{i=1}^{n-1} |d(v^i, \cdot) - d(v^{i+1}, \cdot)|^p \right)^{1/p} \right\|_{L^p(\mu)}, \end{aligned}$$

where $v^i \rightarrow v^{i+1}$ means that v^i and v^{i+1} are connected.

Lemma 2. For every metric graph $G = (V, E, d)$, every $p \geq 1$ and every probability measure $\mu \in \mathcal{M}(S)$,

$$d_{p,\mu}(v_1, v_2) \leq d_p(v_1, v_2), \quad v_1, v_2 \in V.$$

Proof. This is a consequence of a direct calculation. Indeed, note that by the triangular inequality for d , and taking into account that μ is a probability measure, we obtain

$$\begin{aligned} &\inf \left\{ \left(\sum_{i=1}^{n-1} \int_S |d(v^i, w) - d(v^{i+1}, w)|^p d\mu \right)^{1/p} : \text{path } v_1 = v^1 \rightarrow v^2, \dots, v^{n-1} \rightarrow v^n = v_2 \right\} \\ &\leq \inf \left\{ \left(\sum_{i=1}^{n-1} d(v^i, v^{i+1})^p d\mu \right)^{1/p} : \text{path } v_1 = v^1 \rightarrow v^2, \dots, v^{n-1} \rightarrow v^n = v_2 \right\} = d_p(v_1, v_2). \end{aligned}$$

□

A straightforward consequence of Lemmas 1 and 2 is the following:

Proposition 3. Let $S \subset V$ be a compact subset of a weighted undirected connected graph $G = (V, E)$. Fix $1 \leq r \leq p$. Then, for every probability measure, $\mu \in \mathcal{M}(S)$,

$$(q_r)_{p,\mu}(v_1, v_2) \leq (q_r)_p(v_1, v_2) \leq q_r(v_1, v_2), \quad v_1, v_2 \in V.$$

4.2. Eccentric p -Summing Domination for Functionals on Graphs and Metric Symmetry

In this section, we show how domination of particular real Lipschitz functions can be used for the characterization of indices of metric symmetry on graphs. Let us motivate this application with two examples.

Example 4. Consider a graph defined by the following points of $\mathbb{R}^2$ as vertices. Take $v_1 = (-1, 0)$, $v_2 = (1, 0)$ and $v_k = a_{k-2}$ for $k \geq 2$, where $(a_k)_{k=1}^\infty$ is a convergent sequence formed by elements as $(0, s_k)$ together with its limit $(0, s_\infty) \in \mathbb{R}^2$. Suppose that the connectivity is given as in Example 3 for the elements of the sequence and its limit, and the elements v_1 and v_2 are connected to all the elements of the sequence, and with its limit too. The weights for the metric are again the size of the segments connecting every couple of adjacent vertices. It is clearly a compact graph with the weighted shortest path metric (Figure 2).

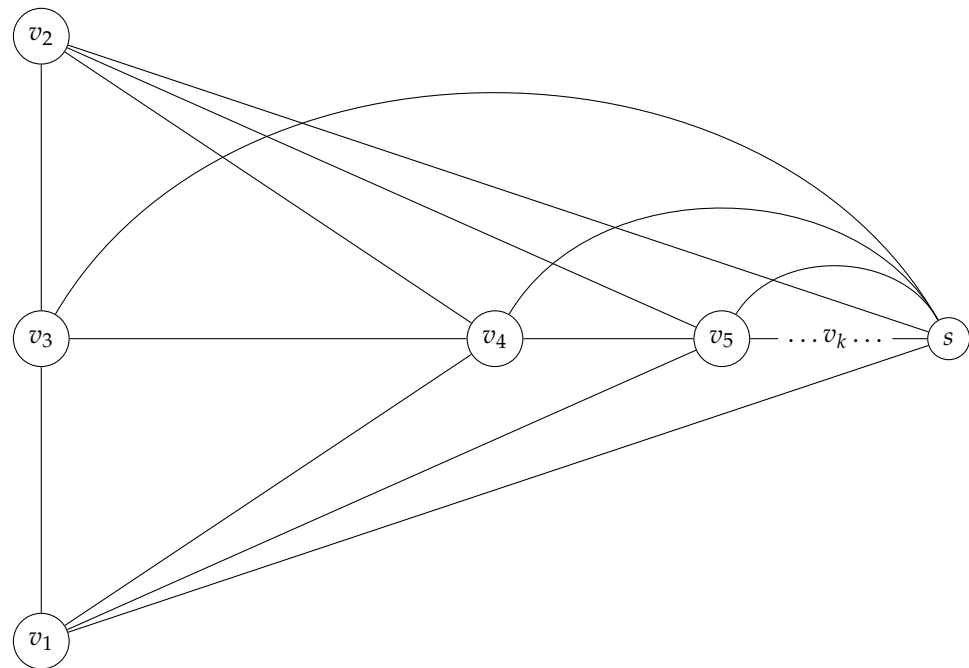


Figure 2. As in Figure 1 of Example 3, in this graph, the convergent sequence $(a_k)_{k=1}^{\infty}$ is formed by the elements $(0, 1 - 1/2^k)_k$ together with their limit $s = (0, 1)$. In this figure, the weights of the graph are not represented.

Take the Borel regular measure μ on the whole set V of vertices of the graph that is written as a series

$$\mu = \sum_{i=3}^{\infty} \alpha_i \delta_{\{v_i\}},$$

where δ_{v_i} is the Dirac measure on the vertex v_i and α_i are non-negative real numbers such that $\sum_{i=3}^{\infty} \alpha_i = 1$.

Suppose that a real-valued Lipschitz function $f : V \rightarrow \mathbb{R}$ satisfies a domination as

$$|f(v_i) - f(v_j)| \leq K \int |d(v_i, v) - d(v_j, v)| d\mu(v).$$

Note that this implies that $f(v_1) = f(v_2)$, since by symmetry $d(v_1, v) = d(v_2, v)$ for all $v = v_i$ for $i = 3, 4, \dots$, and for $v = v_1$ or $v = v_2$, $|d(v_i, v) - d(v_j, v)| = 1$, but $\mu(\{v\}) = 0$ for these points. This means that all Lipschitz functionals dominated by such an integral are symmetric, in the sense that $f(v_1) = f(v_2)$. This example can be considered in the context of functions that model transportation costs on both sides of a border between two regions that must be treated equitably. Cities located at the same distance from the border should have the same transportation costs.

Example 5. A noncountable graph (Figure 3). Consider all the points of the circle of radius 1 together with its center, which is supposed to be the point $v_0 = (0, 0)$, as a subset of $\mathbb{R}^2$. Suppose that all the points of the circle are connected, and also with the center v_0 . Consider the sets of weights $w(v_i, v_j)$ given by the size of the arc connecting two points of the circle, and $w(v_0, v_i) = 1$ for all the points v_i in the circle. This is a metric graph with the shortest path metric for which the distance $d(v_i, v_j)$ equals again the size of the arc connecting them if they belong to the circle, and 1 if one of the points involved is v_0 .

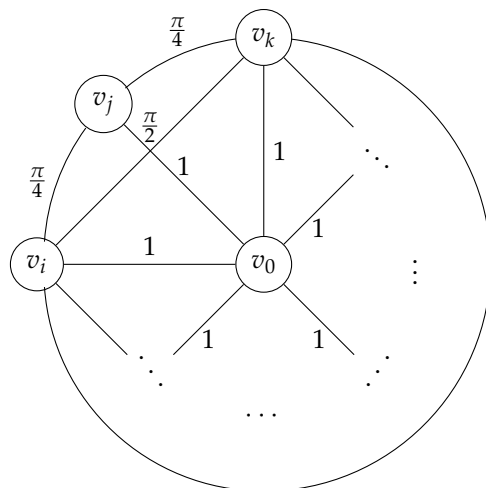


Figure 3. Graph on the circle with center v_0 and arc weights.

Take the measure $\mu = \delta_{\{v_1\}}$. Then, every Lipschitz functional dominated as

$$|f(v_i) - f(v_j)| \leq K \int |d(v_i, v) - d(v_j, v)| d\delta_{\{v_0\}}(v)$$

for any $K > 0$ satisfies that $f(v_i) = f(v_j) = Q$ for all v_i, v_j in the unit circle (that is, it takes a constant value Q), and $|f(v_0) - f(v_i)| \leq K$, that is, $Q - K \leq f(v_0) \leq Q + K$. This shows a requirement for the circular symmetry of functions under integral domination: for a dominated real function, all points that are radially symmetric with respect to the center must take the same value. This concept could be applied, for instance, in modeling resource distribution in peripheral areas of a large city.

These examples show that the integral domination gives information about the properties of the functionals on certain paths connecting concrete vertices in a graph. The problem we face in this section is to give conditions under which a Lipschitz functional satisfies that for given two vertices, there is a path connecting them that satisfies that the functional is almost constant, or at least that its p -th variation is controlled.

Let $G = (V, E)$ be a weighted graph with weights W . Given a functional $f : V \rightarrow \mathbb{R}$, we define its p -best path estimate as

$$E_p(f)(v_1, v_2) :=$$

$$\inf \left\{ \left(\sum_{i=1}^{n-1} |f(v^i) - f(v^{i+1})|^p d\mu \right)^{1/p} : \text{path } v_1 = v^1 \rightarrow v^2, \dots, v^{n-1} \rightarrow v^n = v_2 \right\}$$

for every $v_1, v_2 \in V$.

Definition 4. Let $G = (V, E)$ be a weighted graph with weights W . We say that a functional $f : V \rightarrow \mathbb{R}$ satisfies a p -th power integral estimate if there is a constant $Q > 0$, a compact subset $S \subset V$ and a probability measure $\mu \in \mathcal{M}(S)$ such that for every $v_1, v_2 \in V$,

$$E_p(f)(v_1, v_2) \leq Q d_{p,\mu}(v_1, v_2).$$

Using the tools developed in other sections, the next result shows the functionals that satisfy this definition.

Theorem 2. Let $f : V \rightarrow \mathbb{R}$. Let $G = (V, E)$ be a weighted graph with weights W . Let $f : V \rightarrow \mathbb{R}$ be a Lipschitz map. Then,

$$1 \quad E_p(f)(v_1, v_2) \leq \text{Lip}(f) d_p(v_1, v_2) \text{ for all } v_1, v_2 \in V.$$

- 2 If f is an eccentric p -summing functional, then there is a compact subset S and a probability measure $\mu \in \mathcal{M}(S)$ such that f satisfies a p -th power integral estimate

$$E_p(f)(v_1, v_2) \leq C(f) d_{p,\mu}(v_1, v_2).$$

That is, for every $v_1, v_2 \in V$,

$$\inf \left\{ \left(\sum_{i=1}^{n-1} |f(v^i) - f(v^{i+1})|^p d\mu \right)^{1/p} : \text{path } v_1 = v^1 \rightarrow v^2, \dots, v^{n-1} \rightarrow v^n = v_2 \right\}$$

$$\leq C(f) \inf \left\{ \left(\sum_{i=1}^{n-1} \int_S |d(v^i, w) - d(v^{i+1}, w)|^p d\mu(w) \right)^{1/p} : \text{path } v_1 \rightarrow v^2, \dots, \rightarrow v_2 \right\},$$

where $C(f)$ is the eccentric p -summing constant of f .

Proof. 1. Consider a pair $v_1, v_2 \in V$ and take a path connecting them $v_1 = v^1 \rightarrow v^2, \dots, v^{n-1} \rightarrow v^n = v_2$. Then,

$$\left(\sum_{i=1}^{n-1} |f(v^i) - f(v^{i+1})|^p d\mu \right)^{1/p} \leq \left(\sum_{i=1}^{n-1} \text{Lip}(f) d(v^i, v^{i+1})^p d\mu \right)^{1/p},$$

and so $E_p(f)(v_1, v_2) \leq \text{Lip}(f) d \left(\sum_{i=1}^{n-1} (v^i, v^{i+1})^p d\mu \right)^{1/p}$. Since this happens for every path connecting v_1 and v_2 , we obtain the result.

2. Fix $v_1, v_2 \in V$. Consider a path connecting them $v_1 = v^1 \rightarrow v^2, \dots, v^{n-1} \rightarrow v^n = v_2$. Since f is eccentric p -summing, by Theorem 1, we find a set S , a constant $C > 0$ and a measure $\mu \in \mathcal{M}(S)$ such that

$$|f(v) - f(s)| \leq C(f) \left(\int_S |d(v, w) - d(s, w)|^p d\mu(w) \right)^{1/p}, \quad v, s \in V.$$

Writing the p -sum in both sides, we obtain

$$\left(\sum_{i=1}^{n-1} |f(v^i) - f(v^{i+1})|^p d\mu \right)^{1/p} \leq \left(C(f)^p \sum_{i=1}^{n-1} \int_S |d(v^i, w) - d(v^{i+1}, w)|^p d\mu(w) \right)^{1/p},$$

and so

$$E_p(f)(v_1, v_2) \leq C(f) \left(\sum_{i=1}^{n-1} \int_S |d(v^i, w) - d(v^{i+1}, w)|^p d\mu(w) \right)^{1/p}.$$

Now, we only need to consider the infimum on the right-hand side to obtain the desired inequality. $\square$

Let us explicitly explain the application of the above results to graph analysis. Following the standard tool definition for metric graph analysis in applied science, we say that a function $I : V \rightarrow \mathbb{R}$ that quantitatively represents a certain property of the graph vertices is an index [29]. It is usually assumed to be a Lipschitz function for a certain distance defined in the graph. In the metric modeling of graphs (and in the modeling of metric spaces in general), it is common to use this type of functions to represent relevant properties of the graphs: for example, if one considers the graph of the cities of a region with the road connections between them, an index of interest could be given by the real function that maps the number of inhabitants of each city. Thus, such an index is defined as a Lipschitz functional on V .

For a given measure μ , let us consider the pseudometric $d_{p,\mu}$. Two vertices v_1 and v_2 in V are symmetric with respect to μ if $d_{p,\mu}(v_1, v_2) = 0$.

The meaning of this relation is clear: two vertices v_1 and v_2 are symmetric with respect to μ if the associate metric functions $w \mapsto d(v_1, w)$ and $w \mapsto d(v_2, w)$ are μ -almost

everywhere equal, that is $\int_S |d(v_1, w) - d(v_2, w)|^p d\mu = 0$. The term symmetric is explained by the fact that, if this holds, then the vertices v_1 and v_2 have a similar distribution of distances except in a set of vertices w that is μ -null.

Recall that, in the context of metric modeling, an index I is a real Lipschitz function $I : V \rightarrow \mathbb{R}$. The next result gives the formal property that characterizes the existence of I -constant paths for a certain index I . The proof is a straightforward application of Theorem 2, which proves the existence of a measure μ and a domination of $E_p(f)$ by $d_{p\mu}(f)$.

Corollary 2. *Let $I : V \rightarrow \mathbb{R}$ be an index on a weighted undirected connected graph $G = (V, E)$. If I is eccentric p -summing, then there is a compact set S and a probability measure $\mu \in \mathcal{M}(S)$ such that if two vertices in V are symmetric with respect to μ , then there is a path connecting v_1 and v_2 such that $I(v_1) = I(v) = I(v_2)$ for all vertices v in the path.*

The approximate version of this result also holds, in the sense that if the vertices v_1 and v_2 are not symmetric with respect to μ , but satisfy that $d_{p,\mu}(v_1, v_2)$ is small, we obtain that the p -variation $E_p(I)$ is small because $E_p(I) \leq C(f) d_{p\mu}(I)$.

To finish, let us return to the example of the graph of the cities of a region with the road connections between them, and give the weights of the edges by ordering the distances between them (e.g., 1 if the distance is 0 to 5 km, 2 if the distance is 6 to 10, and so on). Let us fix an index I that orders the set of vertices (cities) by thousands of inhabitants: $I(v) = 1$ if the city v has from 1 to 1000 inhabitants, $I(v) = 2$ if v has from 1001 to 2000 inhabitants, and so on. Suppose I is controlled by an appropriate measure μ modeled in order to control the routes between two cities to reduce risk by restricting crossing large cities; for example, the support of the measure μ contains only the largest cities, so distances to small cities is irrelevant. Given that there exists a constant R such that $E_p(I) \leq R d_{p,\mu}$, then a small value of $d_{p,\mu}(v_1, v_2)$ for two small cities (populations less than 1000 inhabitants, $I(v_1) = I(v_2) = 1$) means that we can find a path connecting v_1 and v_2 such that $E_p(I)(v_1, v_2)$ is also small (or 0). For instance, if it is 0, it means that there is a path connecting cities v_1 and v_2 that crosses only towns of less than 1000 inhabitants, since for all vertices that the path crosses, $I(v) = I(v_1) = 1$.

5. Conclusions

In this paper, we introduce a new type of p -summability for Lipschitz maps between metric spaces, which we call eccentric p -summing. Through its detailed characterization, we explore the general summation properties of metric spaces, introducing some related integral inequalities for real Lipschitz functionals. These inequalities lead to factorization theorems for general Lipschitz operators. We establish some key definitions related to summing metrics for sequences of metric spaces, laying the groundwork for our new notion of operator summability. Using them, we focus on proving the main results concerning summing Lipschitz operators, providing not only the associated factorization theorems but also discussing general cases with concrete examples to illustrate the breadth of applicability. These results extend classical summability concepts, offering new insights into the behavior of Lipschitz operators across diverse settings.

The final section of this paper addresses the application of these summing Lipschitz operators in the context of metric graphs. Given their potential applications in various fields, we show how the integral inequalities that characterize summing properties can be adapted to compact metric graphs. This extension provides a powerful new tool for investigating the concept of metric symmetry in graphs. Furthermore, we present examples and discuss new results regarding the characterization of approximate metric symmetry, utilizing analytic techniques in contrast to the classical algebraic approaches. By shifting the focus to analytic methods, we offer a new perspective on the study of symmetry in metric spaces, with implications for both theoretical research and practical applications in graph theory.

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