Fractional Calculus and Applied Analysis (2025) 28:1717–1731
https://doi.org/10.1007/s13540-025-00430-4

ORIGINAL PAPER

Dynamics of the Caputo fractional derivative

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Received: 2 August 2024 / Revised: 27 May 2025 / Accepted: 28 May 2025 /
Published online: 16 June 2025

Abstract

In this article we analyse the dynamical behaviour of the Caputo complex fractional derivative. We prove that the Caputo complex fractional derivative operator is Devaney chaotic in the Mittag-Leffler Caputo space. We will also show that a tuple of different iterates of a Caputo derivative multiple is disjoint hypercyclic. Finally, we provide sufficient conditions that ensure chaos for one of the most common numerical approximation of the Caputo derivative, that is, its L1 discretization.

Keywords Caputo fractional derivative · Devaney chaos · Disjoint hypercyclicity · L1 discretization

Mathematics Subject Classification 26A33 (primary) · 47A16 · 33E12 · 47B32 · 65L12

1 Introduction

In the last decades, fractional calculus has seen a great revival due to its application in modeling partial and ordinary differential equations [23]. The main advantage of fractional derivatives is that they are non-local operators and, therefore, more suitable for modeling physical or biological processes involving memory. Fractional calculus has been used for modeling diffusion processes [27], in control theory [1], continuum mechanics [9] or viscoelasticity [24] among other applications [28]. There is a wide variety of fractional derivatives such as the Grünwald-Letnikov [22], the Riemann-Liouville or the sequential fractional derivatives. However, the most widely used is the

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Caputo derivative. This is mainly due to two reasons. The Caputo fractional derivative behaves well with constant functions (the Caputo derivative of a constant function anihilates). The other reason is based on the fact that fractional order differential equations using Caputo derivatives require integer order initial conditions that have physical meaning, in contrast with differential equations using Riemann-Liouville derivatives that require fractional initial conditions.

Recently, many researchers have been attracted by the possibilities of fractional calculus and there has been a growing interest in deeply understanding the behaviour of these operators from different points of view [13, 14, 21]. The study of dynamical behaviour is an interesting perspective inspired by some of these recent works. By their definition, fractional operators seem to be associated with unstable dynamics, hence the question arises as to whether fractional operators are hypercyclic or chaotic. It is also a natural question because fractional derivatives are extensions of the usual derivative, which is both hypercyclic and Devaney chaotic on $H(\mathbb{C})$. The phenomenon of chaos is usually identified with nonlinear phenomena, but chaos also appears in linear dynamical systems as long as the underlying space is infinite-dimensional. This theory has led to important applications in physics, chemistry, biology, and engineering. Further information on linear dynamics can be found in the books [5, 19].

In this article, we examine the dynamical behaviour of the Caputo complex fractional derivative. Based on the Mittag-Leffler reproducing kernel space [30], we will present the Mittag-Leffler-Caputo space, and prove that the Caputo complex fractional derivative is well-defined and Devaney chaotic in such space. We will also show that a tuple $(\lambda T, \lambda^2 T^2, \dots, \lambda^n T^n)$, where T is the Caputo complex fractional derivative and $\lambda \neq 0$, is disjoint hypercyclic. Finally, and following the study started in [21] where Devaney chaos for some discretizations of fractional derivatives was obtained, we will prove that the L1 discretization of the Caputo fractional derivative is Devaney chaotic under certain assumptions on the step size τ of the discretization.

The article is organized as follows: Section 2 is dedicated to recall some notions about linear dynamics, fractional calculus and reproducing kernel Hilbert spaces (RKHS). In Section 3 we will study Devaney chaos for the Caputo complex fractional derivative and the disjoint hypercyclicity of tuples involving different iterates of a non-zero multiple of the Caputo derivative. Finally in Section 4 we will study Devaney chaos for the L1 discretization.

2 Preliminaries

In this section we recall some basic notions and results about dynamics of linear operators and some necessary definitions of fractional derivatives.

2.1 Dynamics of linear operators

We recall that an operator T on a topological vector space X is called hypercyclic if there is a vector $x \in X$ such that its orbit $\mathrm{Orb}(x, T) = \{x, Tx, T^2x, \ldots\}$ is dense in X. The operator T is topologically transitive (mixing) if for any pair U, V of nonempty open subsets of X, there exists some $n \geqslant 0$ ($n_0 \geqslant 0$) such that $T^n(U) \cap V \neq \emptyset$ (for all $n \geqslant n_0$). For complete, separable and metrizable X, hypercyclicity of T is equivalent to topological transitivity.

According to [12], a continuous map T on a metric space (X, d) is Devaney chaotic if it is transitive, the set of periodic points $\mathrm{Per}(T)$ is dense in X, and it is sensitive i.e., there exists $\delta > 0$ such that for each $x \in X$ and each $\epsilon > 0$, there are $y \in X$ and $n \in \mathbb{N}$ with $d(x, y) < \epsilon$ and $d(T^n(x), T^n(y)) > \delta$. It is well-known that if the first two conditions hold for T then sensitivity is granted [2].

Given an operator $T : X \to X$ on a complex Banach space X, a function $E : A \to X$ for certain $A \subset \mathbb{C}$ is an eigenvector field if $E(\lambda) \in \ker(\lambda I - T)$ for any $\lambda \in A$. It is weakly holomorphic on an open set $U \subset \mathbb{C}$ if each map $\langle x^*, E(-)\rangle : U \to \mathbb{C}$ is holomorphic for any x^* in the dual X^* of X. Let us also denote $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. The following is a classical criterion to ensure Devaney chaos. It was used by Bayart and Grivaux [4] for frequent hypercyclicity under more general assumptions, but for our purposes we recall it in the following form.

Theorem 1 *[4](**Eigenvector Field Criterion**). Given an operator $T : X \to X$, if there exists a nonempty connected open set $U \subset \mathbb{C}$ such that $U \cap \mathbb{T} \neq \emptyset$ and $G : U \to X$ a weakly holomorphic eigenvector field of T such that*

$$\overline{span\{G(u) : u \in U\}} = X$$

then T is Devaney chaotic.

The concept of disjointness for dynamical systems is due to Furstenberg [15]. This notion inspired the corresponding one for operators (see [6, 17]).

Definition 1 We say that a tuple $(T_1, ..., T_N)$ of operators acting on a topological vector space X is disjoint hypercyclic (in short d-hypercyclic), if there is a vector $(z, z, ..., z) \in X^N$ such that

$$\left\{(z, z, ..., z), (T_1 z, T_2 z, ..., T_N z), (T_1^2 z, T_2^2 z, ..., T_N^2 z), ...\right\}$$

is dense in X^N. We call $z \in X$ a d-hypercyclic vector associated to $(T_1, ..., T_N)$.

Two disjoint hypercyclic operators must be substantially different, and in particular an operator can never be d-hypercyclic with a scalar multiple of itself. To see this and more information about d-hypercyclicity we refer the reader to [6, 17]. The corresponding topological notions that we will need are the following.

Definition 2 [6] We say that a tuple of $N \geqslant 2$ operators $(T_1, \ldots, T_N)$ is d-topologically transitive (respectively, d-mixing) provided that, for every non-empty open subsets $V_0, ..., V_N$ of X, there exists $m \in \mathbb{N}$ (resp., $m_0 \in \mathbb{N}$) so that $\emptyset \neq V_0 \cap T_1^{-m}(V_1) \cap ... \cap T_N^{-m}(V_N)$ (respect., for each $m \geqslant m_0$).

2.2 Fractional calculus

Definition 3 (**Integral of arbitrary order**). For $f \in C(\mathbb{R}_+)$, $t > 0$, and $\alpha > 0$ then

$$I^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau$$

is called the fractional integral of order α.

We also recall that the semigroup property for the fractional integral holds, i.e.,

$$I^\alpha \circ I^\beta f(t) = I^{\alpha+\beta} f(t)$$

for any $\alpha, \beta > 0$.

Definition 4 [28] (**Caputo fractional derivative**). Let $\alpha > 0$, $n - 1 < \alpha < n$ and $f \in C^n(\mathbb{R}_+)$. The Caputo fractional derivative of f is given by

$${}_0^C D_t^\alpha f(t) := \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \quad t > 0.$$

It is well-known that the operator I^α is the right inverse of ${}_0^C D_t^\alpha$, that is, the following identity holds:

$${}_0^C D_t^\alpha \circ I^\alpha f(t) = f(t).$$

Now let us consider $H(\Omega)$ the space of holomorphic functions on a complex open set Ω endowed with the topology of uniform convergence on compact sets. Hardy and Littlewood analysed in [20] a complex extension of the fractional real one.

Definition 5 [20] Let $\alpha > 0$, $n \in \mathbb{N}$ such that $n - 1 < \alpha < n$, the complex fractional integral I_*^α is given by

$$\left(I_*^\alpha f\right)(z) := \frac{1}{\Gamma(\alpha)} \int_0^z \frac{f(\tau)}{(z-\tau)^{1-\alpha}} d\tau,$$

where the integration is over the line-segment from 0 to z.

Based on the Hardy-Littlewood complex integral, the Caputo complex fractional derivative can be defined as follows.

Definition 6 [30] Let $\alpha > 0$, $n \in \mathbb{N}$ such that $n - 1 < \alpha < n$, the Caputo complex fractional derivative D_*^α is defined as

$$\left(D_*^\alpha f\right)(z) := \frac{1}{\Gamma(n-\alpha)} \int_0^z \frac{f^{(n)}(\tau)}{(z-\tau)^{\alpha+1-n}} d\tau,$$

where the integration is over the line-segment from 0 to z.

It is easy to verify that the previous definitions coincide with the Caputo integral and derivative for real values. The following is a well-known application of the fractional integral for which we include a proof for the sake of completeness. In what follows, let us consider for $p \geqslant 0$, $z^p := e^{p\log(z)}$ taking the principal branch of the logarithm defined in $\mathbb{C} \setminus \mathbb{R}_-$.

Proposition 1 *integralpotencias Let $p > 0$ and $f(z) := z^p$ then*

$$\left(I_*^\alpha f\right)(z) = \frac{\Gamma(p+1)}{\Gamma(p+\alpha+1)} z^{p+\alpha}.$$

Proof We will prove the identity for $z \in \mathbb{T} \setminus \{-1\}$ and by the Identity Theorem for holomorphic functions the conclusion follows for all $z \in \mathbb{C} \setminus \mathbb{R}_-$. Computing the fractional integral we have

$$\left(I_*^\alpha f\right)(z) = \frac{1}{\Gamma(\alpha)} \int_0^z \frac{\tau^p}{(z-\tau)^{1-\alpha}} d\tau. \tag{2.1}$$

Now since $z = e^{\theta i}$ for some $\theta \in (-\pi, \pi)$ we can also parameterize the path as $\tau = (1-r)e^{i\theta}$, $r \in [0, 1]$. The integral (2.1) now becomes

$$\left(I_*^\alpha f\right)(z) = \frac{e^{i\theta(p+\alpha)}}{\Gamma(\alpha)} \int_0^1 r^{\alpha-1}(1-r)^p dr. \tag{2.2}$$

Let us recall the following identity for the Beta function

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Applying the previous identity to (2.2) we get

$$\left(I_*^\alpha f\right)(z) = \frac{e^{i\theta(p+\alpha)}}{\Gamma(\alpha)} \frac{\Gamma(\alpha)\Gamma(p+1)}{\Gamma(p+\alpha+1)} = \frac{\Gamma(p+1)}{\Gamma(p+\alpha+1)} z^{p+\alpha}$$

for all $z \in \mathbb{T} \setminus \{-1\}$. Consequently, by the Identity Theorem for holomorphic functions the conclusion holds. □

In [30] the authors proved the corresponding result for the Caputo complex fractional derivative.

Proposition 2 *derivadacaputop Let $p \geqslant \alpha$ and $f(z) := z^p$ then*

$$\left(D_*^\alpha f\right)(z) = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} z^{p-\alpha}.$$

Remark 1 It is easy to observe that the Caputo complex fractional derivative of a constant annihilates as happens in the usual derivative. However this is not always the case for fractional derivatives and the Riemann-Liouville is an example of fractional derivative that does not annihilates for constant functions.

We can easily notice that, as in the real valued case, the operator I_*^α is the right inverse of the Caputo complex fractional derivative. The Caputo complex fractional integral also satisfies the semigroup property [31].

One of the fundamental functions in fractional calculus is the Mittag-Leffler function.

Definition 7 For $\alpha, \beta > 0$ the Mittag-Leffler function $E_{\alpha,\beta} : \mathbb{C} \to \mathbb{C}$ is defined as follows

$$E_{\alpha,\beta}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}.$$

In what follows let denote $E_\alpha(z) := E_{\alpha,1}(z)$, and notice that E_α is an entire function. We also point out the following identity that shows that certain Mittag-Leffler functions are eigenvectors of the Caputo complex fractional derivative [18, Ch. E.2]:

$$D_*^\alpha \left(E_\alpha(\lambda z^\alpha)\right) = \lambda E_\alpha(\lambda z^\alpha) \quad \text{for all } \lambda \in \mathbb{C}. \tag{2.3}$$

2.3 Reproducing Kernel Hilbert Spaces (RKHS)

A RKHS over a set X is a Hilbert space of functions $f : X \to \mathbb{R}$ for which given each $x \in X$ the evaluation functional $E_x f = f(x)$ is bounded. It is a usual space in machine learning and approximation theory [10]. As a consequence of the Riesz representation theorem, for each $x \in X$ there exists a function $k_x \in H$ for which $\langle f, k_x \rangle_H = f(x)$ for all $f \in H$. Hence associated with each RKHS there is a kernel function given by $K(x, y) = \langle k_y, k_x \rangle_H$.

In [29] and [30] the authors defined the Mittag-Leffler RKHS in order to approximate the Caputo fractional derivative of a function.

Definition 8 (**Mittag-Leffler space on the slitted plane**). Given $\alpha > 0$, the Mittag-Leffler space on the slitted plane is defined as

$$ML^2(\mathbb{C} \setminus \mathbb{R}_-; \alpha) := \left\{ f(z) = \sum_{n=0}^{\infty} a_n z^{\alpha n} : \sum_{n=0}^{\infty} |a_n|^2 \Gamma(\alpha n + 1) < \infty \right\}.$$

The Mittag-Leffler space on the slitted plane is a RKHS equipped with the inner product

$$\langle f(z), g(z) \rangle_{ML^2(\mathbb{C}\setminus\mathbb{R}_-;\alpha)} = \sum_{n=0}^{\infty} \Gamma(\alpha n + 1) a_n \overline{b_n}$$

and the kernel function $k_\alpha : (\mathbb{C} \setminus \mathbb{R}_-) \times (\mathbb{C} \setminus \mathbb{R}_-) \to \mathbb{C}$ such that

$$k_\alpha(z, \lambda) = E_\alpha\left(\overline{\lambda^\alpha} z^\alpha\right).$$

2.4 Köthe sequence spaces

Definition 9 A matrix $A = (a_{j,k})_{j,k\in\mathbb{N}_0}$ of non-negative numbers is called a Köthe matrix if it satisfies the following conditions:

(i) For each $j \in \mathbb{N}_0$ there exists $k \in \mathbb{N}_0$ such that $a_{j,k} > 0$.
(ii) $a_{j,k} \leqslant a_{j,k+1}$ for all $j, k \in \mathbb{N}_0$.

For $1 \leqslant p < \infty$, the Köthe sequence space $\lambda^p(A)$ is defined as

$$\lambda^p(A) := \left\{ x \in \mathbb{K}^{\mathbb{N}} : \|x\|_k := \left(\sum_{j=0}^{\infty} \left| x_j a_{j,k} \right|^p \right)^{1/p} < \infty \text{ for all } k \in \mathbb{N}_0 \right\}.$$

The Köthe space $\lambda^p(A)$ is a Fréchet space when endowed with the seminorms $\|\cdot\|_k$.

As pointed out in [26], for a Köthe matrix A the dual space of $\lambda^p(A)$ is known.

Theorem 2 *Let A be a Köthe matrix and $1 \leqslant p < \infty$. Then for each $y \in \lambda^p(A)'$ there exists a sequence $(y_j)_{j\in\mathbb{N}_0}$ such that*

$$\sum_{j=0}^{\infty} |x_j||y_j| < \infty$$

and also

$$y(x) = \sum_{j=0} x_j y_j$$

for all $x \in \lambda^p(A)$.

3 Devaney Chaos and disjoint hypercyclicity for the Caputo derivative

We will now study chaos in the sense of Devaney for the Caputo complex fractional derivative $T = D_*^{\alpha}$, as well as the disjoint mixing property of the tuple $(\lambda T, \lambda^2 T^2, \ldots, \lambda^m T^m)$ for arbitrary $m \in \mathbb{N}$ and $\lambda \in \mathbb{C} \setminus \{0\}$. For this purpose, we define the following space.

Definition 10 Let $\alpha > 0$ the Mittag-Leffler Caputo space on the slitted plane is defined as

$$\begin{aligned} &MLC^2(\mathbb{C} \setminus \mathbb{R}_-; \alpha) \\ &:= \left\{ f(z) = \sum_{j=0}^{\infty} b_j z^{\alpha j} : \sum_{j=0}^{\infty} \frac{\Gamma(\alpha(j+k)+1)^2}{\Gamma(\alpha j+1)} |b_{j+k}|^2 < \infty, \forall k \in \mathbb{N}_0 \right\}, \end{aligned} \tag{3.1}$$

provided with the increasing sequence of norms:

$$\|f\|_k := \left(\sum_{j=0}^{k} \frac{\Gamma(\alpha(j+k-1)+1)^2}{\Gamma(\alpha j+1)} |b_j|^2 + \sum_{j=0}^{\infty} \frac{\Gamma(\alpha(j+k)+1)^2}{\Gamma(\alpha j+1)} |b_{j+k}|^2 \right)^{1/2},$$

for every $k \in \mathbb{N}_0$. Let us observe that the Mittag-Leffler Caputo space on the slitted plane is isometrically isomorphic to the Köthe sequence space $\lambda^2(A)$, where the matrix $A = (a_{j,k})_{j,k\in\mathbb{N}_0}$ is given by

$$a_{j,k} = \begin{cases} \frac{\Gamma(\alpha(j+k-1)+1)}{\sqrt{\Gamma(\alpha j+1)}} & \text{if } j < k, \\ \\ \frac{\Gamma(\alpha j+1)}{\sqrt{\Gamma(\alpha(j-k)+1)}} & \text{if } j \geqslant k. \end{cases}$$

Hence the Mittag-Leffler Caputo space on the slitted plane is a Fréchet space with the increasing sequence of norms $\|\cdot\|_k$.

Let us also notice that given $k \in \mathbb{N}_0$ we can define the space

$$MLC_k^2(\mathbb{C} \setminus \mathbb{R}_-; \alpha) := \left\{ f(z) = \sum_{j=0}^{\infty} b_j z^{\alpha j} : \sum_{j=0}^{\infty} \frac{\Gamma(\alpha(j+k)+1)^2}{\Gamma(\alpha j+1)} |b_{j+k}|^2 < \infty \right\},$$

which is a Hilbert space when endowed with the interior product

$$\langle f, g \rangle_{MLC_k^2} = \sum_{j=0}^{k} \frac{\Gamma(\alpha(j+k-1)+1)^2}{\Gamma(\alpha j+1)} b_j \overline{c_j} + \sum_{j=0}^{\infty} \frac{\Gamma(\alpha(j+k)+1)^2}{\Gamma(\alpha j+1)} b_{j+k} \overline{c_{j+k}}$$

for any $f, g \in MLC_k^2(\mathbb{C} \setminus \mathbb{R}_-; \alpha)$ such that $f(z) = \sum_{j=0}^{\infty} b_j z^{\alpha j}$ and $g(z) = \sum_{j=0}^{\infty} c_j z^{\alpha j}$.

For $k_1 < k_2$ we have the inclusion $MLC_{k_2}^2(\mathbb{C} \setminus \mathbb{R}_-) \subset MLC_{k_1}^2(\mathbb{C} \setminus \mathbb{R}_-)$ and, alternatively, the Fréchet space $MLC^2(\mathbb{C} \setminus \mathbb{R}_-)$ can be defined as an intersection of Hilbert spaces

$$MLC^2(\mathbb{C} \setminus \mathbb{R}_-) = \bigcap_{k\in\mathbb{N}_0} MLC_k^2(\mathbb{C} \setminus \mathbb{R}_-),$$

endowed with the increasing sequence of norms associated with the corresponding Hilbert spaces.

Lemma 1 *Let $k \in \mathbb{N}_0$, then the following identity holds:*

$$\overline{span\{E_\alpha(\lambda z^\alpha) : \lambda \in \mathbb{C}\}} = MLC_k^2(\mathbb{C} \setminus \mathbb{R}_-).$$

Proof Let $\Phi \in MLC_k^2(\mathbb{C}\setminus\mathbb{R}_-)^*$ such that $\Phi\,(E_\alpha(\lambda z^\alpha)) = 0$ for all $\lambda \in \mathbb{C}\setminus\mathbb{R}_-$. Then by the Riesz Representation Theorem there exists some $\phi \in MLC_k^2(\mathbb{C}\setminus\mathbb{R}_-)$ such that $\phi(z) = \sum_{j=0}^{\infty} a_j z^{\alpha j}$ and $\Phi(f) = \langle \phi, f\rangle_{MLC_k^2(\mathbb{C}\setminus\mathbb{R}_-)}$ for any $f \in MLC_k^2(\mathbb{C}\setminus\mathbb{R}_-)$. Therefore we have

$$\begin{aligned}\Phi\left(E_\alpha(\lambda z^\alpha)\right) &= \langle \phi, E_\alpha(\lambda z^\alpha\rangle_{MLC_k^2(\mathbb{C}\setminus\mathbb{R}_-)}\\ &= \sum_{j=0}^{k} \frac{\Gamma(\alpha(j+k-1)+1)^2}{\Gamma(\alpha j+1)^2} a_j \overline{\lambda^j} + \sum_{j=0}^{\infty} \frac{\Gamma(\alpha(j+k)+1)}{\Gamma(\alpha j+1)} a_{j+k}\overline{\lambda^{j+k}} \qquad (3.2)\\ &= 0\end{aligned}$$

for all $\lambda \in \mathbb{C}$. Now taking $\lambda = 0$ we have:

$$\Phi(E_\alpha\left(\lambda z^\alpha\right)) = \frac{\Gamma(\alpha(k-1)+1)^2}{\Gamma(1)^2} a_0 = 0,$$

and therefore $a_0 = 0$. Let us now notice that identity (3.2) can be also expressed as

$$\frac{1}{\overline{\lambda}}\left(\sum_{j=1}^{k} \frac{\Gamma(\alpha(j+k-1)+1)^2}{\Gamma(\alpha j+1)^2} a_j \overline{\lambda^{j-1}} + \sum_{j=0}^{\infty} \frac{\Gamma(\alpha(j+k)+1)}{\Gamma(\alpha j+1)} a_{j+k}\overline{\lambda^{j+k-1}}\right) = 0$$

and consequenlty we have that

$$\sum_{j=1}^{k} \frac{\Gamma(\alpha(j+k-1)+1)^2}{\Gamma(\alpha j+1)^2} a_j \overline{\lambda^{j-1}} + \sum_{j=0}^{\infty} \frac{\Gamma(\alpha(j+k)+1)}{\Gamma(\alpha) j+1} a_{j+k}\overline{\lambda^{j+k-1}} = 0,$$

for all $\lambda \in \mathbb{C}$. In particular, for $\lambda = 0$ we have $\frac{\Gamma(\alpha(k)+1)^2}{\Gamma(\alpha+1)^2} a_1 = 0$, so it follows that $a_1 = 0$. Proceeding recursively we see that $(a_n)_n = 0$ and therefore by the Hahn-Banach theorem the conclusion holds. □

Remark 2 For $k = 0$ the space $MLC_0^2(\mathbb{C}\setminus\mathbb{R}_-)$ coincides with the Mittag-Leffler space on the slitted plane $ML^2(\mathbb{C}\setminus\mathbb{R}_-)$ and the functions $E_\alpha(\lambda^\alpha z^\alpha)$ are kernel functions of the RKHS. Consequently, for $k = 0$ an argument for the previous proof using the representation property of the Mittag-Leffler space would be equivalent to the one we have provided.

Proposition 3 *he Caputo complex fractional derivative D_*^α is well-defined in the Mittag-Leffler Caputo space for any $\alpha > 0$.*

Proof Let $f \in MLC^2(\mathbb{C}\setminus\mathbb{R}_-;\alpha)$ be given by $f(z) = \sum_{j=0}^{\infty} a_j z^{\alpha j}$. Then

$$\left(\left(D_*^\alpha\right)^i f\right)(z) = \sum_{j=0}^{\infty} a_{j+i} \frac{\Gamma(\alpha(j+i)+1)}{\Gamma(\alpha j+1)} z^{\alpha j}$$

for any $i \in \mathbb{N}_0$. Now, for any $k \in \mathbb{N}_0$ we have

$$\sum_{j=0}^{\infty} \left| a_{j+i+k} \frac{\Gamma(\alpha(j+i+k)+1)}{\Gamma(\alpha(j+k)+1)} \right|^2 \frac{\Gamma(\alpha(j+k)+1)^2}{\Gamma(\alpha j+1)}$$
$$= \sum_{j=0}^{\infty} \left| a_{j+i+k} \right|^2 \frac{\Gamma(\alpha(j+i+k)+1)^2}{\Gamma(\alpha j+1)} < \infty,$$

so clearly $\left(D_*^\alpha\right)^i f \in MLC^2(\mathbb{C} \setminus \mathbb{R}_-; \alpha)$ for all $i \in \mathbb{N}_0$. □

Theorem 3 *The Caputo complex fractional derivative D_*^α is Devaney chaotic in the Caputo Mittag-Leffler space for any $\alpha > 0$.*

Proof Once observed that D_*^α acts as a unilateral weighted shift on a Köthe sequence space, the result follows from the well known characterization of chaos for weighted shifts on sequence spaces (see, e.g., Theorem 4.8 in [19]). Nevertheless, we will obtain the result through the Eigenvector Field Criterion since it offers stronger dynamical properties than Devaney chaos itself.

Let us first observe that $D_*^\alpha E_\alpha(\lambda z^\alpha) = \lambda E_\alpha(\lambda z^\alpha)$, so that the function $E_\alpha(\lambda z^\alpha)$ is an eigenvector for the operator D_*^α associated with the eigenvalue λ. We will use the Eigenvector Field Criterion 1 with $G : \mathbb{C} \to MLC^2(\mathbb{C} \setminus \mathbb{R}_-; \alpha)$ given by $G(\lambda) := E_\alpha(\lambda z^\alpha)$. Let us first notice that G is a weakly holomorphic eigenvector field. Indeed, given any functional $Y \in \left(MLC^2(\mathbb{C} \setminus \mathbb{R}_-; \alpha)\right)'$ by Theorem 2 there exists some sequence $(y_j)_j$ such that

$$Y(E_\alpha(\lambda z^\alpha)) = \sum_{j=0}^{\infty} \frac{y_j}{\Gamma(\alpha j+1)} \lambda^j,$$

and the convergence is absolute for any $\lambda \in \mathbb{C}$. Hence it follows that G is weakly holomorphic.

We will now prove that $\operatorname{span}\{G(\mathbb{C})\}$ is dense in $MLC^2(\mathbb{C} \setminus \mathbb{R}_-)$. In order to do so let $f \in MLC^2(\mathbb{C} \setminus \mathbb{R}_-)$ and define the sequence $(g_n)_n \subset \operatorname{span}\{G(\mathbb{C})\}$ such that for each fixed n we have $\|g_n - f\|_n < \frac{1}{n}$. Such a sequence exists since $\operatorname{span}\{G(\mathbb{C})\}$ is dense in $MLC_n^2(\mathbb{C} \setminus \mathbb{R}_-)$ for each $n \in \mathbb{N}$ from Lemma 1. Now we claim that $g_n \to f$. Indeed, $g_n \to f$ if and only if for each $k \in \mathbb{N}_0$, $\|g_n - f\|_k \to 0$ as $n \to \infty$. Nevertheless, since $\|\cdot\|_k$ is an increasing sequence of norms we have the following identity

$$\|g_n - f\|_k \leqslant \|g_n - f\|_n < \frac{1}{n} \text{ for all } n \geqslant k,$$

and therefore the claim holds. Finally, by the Eigenvector Field Criterion 1 the conclusion follows. □

In [6] the authors proved that any finite set of iterates of the derivative operator is d-mixing. The result gives us the intuitive idea that different iterates of the derivative operator are substantially different in terms of their dynamical behaviour. This motivates the study of d-mixing for tuples involving different iterates of a non-zero multiple of the Caputo fractional derivative. To obtain this result we will use a key argument given in [7].

Theorem 4 *Let $\alpha > 0$, $\lambda \neq 0$, and $N \in \mathbb{N}$ with $N \geqslant 2$. Then the tuple $\left(\lambda D_*^\alpha, \ldots, \lambda^N (D_*^\alpha)^N\right)$ is d-mixing on the Mittag-Leffler Caputo space $MLC^2(\mathbb{C} \setminus \mathbb{R}_-)$.*

Proof Let $T := \lambda D_*^\alpha$. We first define the set

$$Y_\alpha := \operatorname{span}\{z^{n\alpha} : n \in \mathbb{N}_0\},$$

and observe that Y_α is dense in the Mittag-Leffler Caputo space $MLC^2(\mathbb{C} \setminus \mathbb{R}_-)$. Let us also consider the linear map $S : Y_\alpha \to Y_\alpha$ defined on the monomials by

$$Sz^{\alpha k} := \lambda^{-1} \frac{\Gamma(\alpha k + 1)}{\Gamma(\alpha(k+1) + 1)} z^{\alpha(k+1)}.$$

By taking $X_0 = X_1 = Y_\alpha$ we have that $T^k z^{\alpha k} = 0$ for every $k \in \mathbb{N}$ since the Caputo derivative of a constant annihilates. On the other hand we have that S is a right inverse of T on Y_α and $S^n x \to 0$ for each $x \in Y_\alpha$. This means that T satisfies the so-called Original Kitai's Criterion, and by [7, Theorem 3.4] we conclude the result. □

4 Devaney Chaos for the discretization of the Caputo derivative

In this section, we study the dynamics of the L1 discretization of the Caputo fractional derivative. The aim here is to prove whether the numerical discretization of the Caputo derivative captures its chaotical behaviour. This section continues with the research initiated in [21], where the authors proved chaos for some discretizations of fractional derivatives.

Let $b \in \ell^1(\mathbb{N}_0)$ be a summable sequence and define its Gelfand transform by

$$\delta(z) := \sum_{n=0}^{\infty} b(n) z^n, \quad z \in \mathbb{D}.$$

The following criterion will be important for proving chaos for the L1 discretization of the Caputo fractional derivative, and it is a reformulation of a classical result by Godefroy and Shapiro [16] for chaos of adjoint multipliers on the Hardy space.

Theorem 5 *[16] Let $b \in \ell^1(\mathbb{N}_0)$ be given and $T_b : \ell^2(\mathbb{N}_0) \to \ell^2(\mathbb{N}_0)$ given by*

$$T_b u(n) = \sum_{j=0}^{\infty} b(j) B^j u(n), \quad n \in \mathbb{N}_0,$$

where B denotes the backward shift operator. Then T_b defines a bounded operator on $\ell^2(\mathbb{N}_0)$ and the following assertions are equivalent:

(i) T_b *is chaotic,*
(ii) $\delta(\mathbb{D}) \cap \mathbb{T} \neq \emptyset$.

The L1 discretization was introduced in [27] and it is the most commonly used numerical approximation for the Caputo fractional derivative of order $\alpha \in (0, 1)$. Given $f \in \ell^2(\mathbb{N}_0)$ the L1 discretization can be seen as an operator defined as follows:

$$\mathcal{D}^\alpha f(n) := \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^{n} c_{n-j}(f(j) - f(j-1))$$

where $\tau > 0$ is the step size of the discretization and $c_k = (k+1)^{1-\alpha} - k^{1-\alpha}$.

A computation shows the representation of $\mathcal{D}^\alpha$ in the canonical basis $\{e_l(n)\}_{n,l\in\mathbb{N}_0}$ is given by

$$\mathcal{D}^\alpha e_l(n) = \begin{cases} 0 \quad \text{if } n < l, \\ \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} c_0 \quad \text{if } n = l, \\ \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} (c_{n-l} - c_{n-l-1}) \quad \text{if } n > l, \end{cases}$$

and therefore we can rewrite the L1 discretization as

$$\mathcal{D}^\alpha f(n) = \sum_{j=0}^{\infty} \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} b(j) B^j f(n)$$

where B is the backward shift operator and b is such that $b(0) = c_0 = 1$ and $b(j) = c_j - c_{j-1}$.

Theorem 6 *For* $0 < \alpha < 1$ *and* $0 < \tau < \sqrt[\alpha]{\frac{1}{\Gamma(2-\alpha)}}$ *the L1 discretization of the Caputo fractional derivative is Devaney chaotic in* $\ell^2(\mathbb{N}_0)$.

Proof First we have to show that $b \in \ell^1(\mathbb{N}_0)$. In order to do so let us observe that $b(n) = c_n - c_{n-1} = (n+1)^{1-\alpha} - 2n^{1-\alpha} + (n-1)^{1-\alpha}$. We claim that $b(n) < 0$ for $n \geqslant 1$. Indeed let us define $f(x) := x^{1-\alpha} - (x-1)^{1-\alpha}$. By the Mean Value Theorem we have that for all $x \geqslant 1$ there exists some $c \in (x, x+1)$ such that

$$\begin{aligned} (x+1)^{1-\alpha} - 2x^{1-\alpha} + (x-1)^{1-\alpha} &= f(x+1) - f(x) = f'(c) \\ &= (1-\alpha)\left(\frac{1}{c^\alpha} - \frac{1}{(c-1)^\alpha}\right) < 0 \end{aligned}$$

and therefore the claim follows. Hence we can compute

$$\begin{aligned}\sum_{n=0}^{\infty}|b(n)| &= 1+\sum_{n=1}^{\infty}\left|(n+1)^{1-\alpha}-2n^{1-\alpha}+(n-1)^{1-\alpha}\right| \\ &= 1+\sum_{n=1}^{\infty}-(n+1)^{1-\alpha}+2n^{1-\alpha}-(n-1)^{1-\alpha} \\ &= 1+\lim_{N\to\infty}\left(\sum_{n=1}^{N}\left(n^{1-\alpha}-(n+1)^{1-\alpha}\right)+\sum_{n=1}^{N}\left(n^{1-\alpha}-(n-1)^{1-\alpha}\right)\right) \\ &= 2\end{aligned}$$

where the last equality follows by noticing that the partial sums are telescopic.

Now we will use Theorem 5 with

$$\delta(z)=\sum_{n=0}^{\infty}b(n)z^n=\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left(1+\sum_{n=1}^{\infty}\left((n+1)^{1-\alpha}-2n^{1-\alpha}+(n-1)^{1-\alpha}\right)z^n\right).$$

We claim that $\delta(\mathbb{D})\cap\mathbb{T}\neq\emptyset$. Indeed let us observe that for $z=0$ we have $\delta(0)=\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}>1$ and for $z=1$ we have

$$\begin{aligned}\delta(1) &= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left(1+\sum_{n=1}^{\infty}(n+1)^{1-\alpha}-2n^{1-\alpha}+(n-1)^{1-\alpha}\right) \\ &= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left(1+\lim_{N\to\infty}\left(\sum_{n=1}^{N}\left(n^{1-\alpha}-(n+1)^{1-\alpha}\right)+\sum_{n=1}^{N}\left(n^{1-\alpha}-(n-1)^{1-\alpha}\right)\right)\right) \\ &= 0.\end{aligned}$$

Hence by continuity it is clear that there exists some $z'\in\mathbb{D}$ such that $\delta(z')\in\mathbb{T}$ and the conclusion holds. □

Acknowledgements The first author is supported by Junta de Andalucía, Consejería de Universidad, Investigación e Innovación, Project ProyExcel 00780: "Operator Theory: An interdisciplinary approach". All the authors were supported by MCIN/AEI/10.13039/501100011033/FEDER, UE, Project PID2022-139449NB-I00, and by Generalitat Valenciana, Project PROMETEU/2021/070.

Funding Funding for open access publishing: Universidad de Cádiz/CBUA

Declarations

Conflicts of Interest The authors declare that they have no conflict of interest.

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