APPLIED GENERAL TOPOLOGY

*Appl. Gen. Topol. 27, no. 2 (2026), 25664*
doi:10.4995/agt.25664

edUPV

# Interpolative results for cyclic multivalued and Perov type contractions

Naila Shabir [a], Ali Raza [a] and Mujahid Abbas [b]

[a] Abdus Salam School of Mathematical Sciences, Government College University Lahore, 54600 Pakistan (nailashabir446688@gmail.com,aliasadraza@gmail.com)

[b] Department of Mechanical Engineering Sciences, University of Johannesburg, South Africa and Department of Medical Research, China Medical University, Taichung, Taiwan (mujahida@uj.ac.za)


*Communicated by S. Romaguera*

Abstract

*The aim of this paper is to obtain fixed point results for interpolative Kannan type contraction mappings. The purpose of this paper is two fold: one relates to the fixed point results for multivalued cyclic interpolative Kannan type contractions, and second deals with proving the Perov fixed point result for interpolative Kannan type contraction mappings in the framework of vector valued metric spaces.*


2020 MSC: *47H10; 54H25; 54E50.*

Keywords: *fixed point; cyclic multivalued mappings; interpolative Kannan type contraction; Perov type contraction; vector valued metric spaces.*


## 1. Introduction

The Banach Contraction Principle [2] holds a significant importance across various mathematical disciplines. Many authors have made generalizations based on different metric structures and types of contractions. For example, Perov [10], [11] extended the result to vector-valued metric spaces whearas Nadler [9] extended it to multivalued mappings. Furthermore, Karapınar [5] proved a fixed point result for interpolative Kannan-type contractions which was later extended by Gaba et al., [4], Errai et al., [3],

*Received 17 February 2026 – Accepted 2 June 2026*

and Safeer et al. [6]. Moreover, Kirk et al., [7] established the fixed point results for cyclic type contractions.

The aim of this paper is to present (a) some fixed point results for multivalued cyclic interpolative Kannan type contraction mappings and (b) a fixed point results for Perov interpolative Kannan type contractions in the framework of vector-valued metric spaces.

Let $X$ be a nonempty set equipped with a metric $d$ and $CB(X)$ the family of all nonempty closed and bounded subsets of $X$. For any $A, B \in CB(X)$, define $H : CB(X) \times CB(X) \to R$ as follows:

$$H(A, B) = \max\{\delta(A, B), \delta(B, A)\},$$

where $\delta(A, B) = \sup_{a \in A} d(a, B)$ and $d(a, B) = \inf_{b \in B} d(a, b)$. Note that, $H$ is a metric on $CB(X)$, called the Pompeiu Hausdorff metric induced by $d$. Moreover, a point $x \in X$ is said to be a fixed point of a multivalued mapping $T : X \to CB(X)$ if $x \in Tx$. Equivalently, a point $x \in X$ is fixed point if and only if $d(x, Tx) = 0$.

**Lemma 1.1.** *Let $A, B \in CB(X)$. Then for any $q > 1$ and $x \in A$, there exist $y \in B$ such that*

$$d(x, y) \leq qH(A, B).$$

**Definition 1.2** ([8])**.** A map $T : X \to CB(X)$ is called multivalued interpolative Kannan type contraction if there exist $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that

$$H(Tx, Ty) \leq \lambda d(x, Tx)^{\alpha} d(y, Ty)^{1-\alpha},$$

holds for all $x, y \in X \setminus Fix\{T\}$.

**Theorem 1.3** ([8])**.** *If $(X, d)$ is a complete metric space and $T : X \to CB(X)$ is a multivalued interpolative Kannan type contraction. Then $Fix\{T\} \neq \phi$ provided that $Tx$ is compact for each $x \in X$.*

Let us recall the following definition.
Denote $\mathbf{0} := (0, 0, 0, \cdots, 0) \in \mathbb{R}^n$. Define a partial order on $\mathbb{R}^n$ as follows:
for any $x, y \in \mathbb{R}^n$, we say $x \leq y$ if and only if $x_i \leq y_i$ for all $1 \leq i \leq n$, where $x = (x_1, x_2, x_3, \cdots, x_n)$ and $y = (y_1, y_2, y_3, \cdots y_n)$.

**Definition 1.4.** Let $X$ be a nonempty set. A mapping $D : X \times X \to \mathbb{R}^n$ is called the vector valued metric on $X$ if the followings conditions are satisfied:
$(D_1)$ $\mathbf{0} \leq D(x, y)$ for all $x, y \in X$ and $D(x, y) = \mathbf{0}$ if and only if $x = y$.
$(D_2)$ $D(x, y) = D(y, x)$ for all $x, y \in X$.

$(D_3)$ $D(x,y) \le D(x,z) + D(z,y)$, for all $x, y, z \in X$
and $(X, D)$ is called a vector valued metric space,

We denote, $M_n(\mathbb{R}^+)$, $\theta_n$, and $I_n$ by the sets of $n \times n$ matrices with nonnegative real elements, the $n \times n$ zero matrix, and the $n \times n$ identity matrix, respectively. For any $A \in M_n(\mathbb{R})$, the *spectral radius* of $A$, denoted by $\rho(A)$, is defined as

$$\rho(A) = \max\{|\mu| : \mu \text{ is an eigenvalue of } A\}.$$

Moreover, a matrix $A \in M_n(\mathbb{R})$ is said to converge to $\theta_n$ if and only if its spectral radius $\rho(A)$ is strictly less than one, that is, $\rho(A) < 1$ (see [12]). Also, a matrix $A \in M_n(\mathbb{R})$ converges to zero if $A^n \to \theta_n$ as $n \to \infty$. Furthermore, if $A$ and $B$ are matrices in $M_n(\mathbb{R}^+)$ with $A \le B$ (in the component-wise sense), then $\rho(B) < 1$ implies that $\rho(A) < 1$.

**Lemma 1.5** ([1])**.** *If $A$ is any matrix with spectral radius $\rho(A) < 1$. Then, $(I - A)$ is nonsingular and its inverse is given by*

$$(I - A)^{-1} = I + A + A^2 + A^3 + \ldots = \sum_{i=0}^{\infty} A^i.$$

The following result generalizes the Banach contraction principle for vector valued metric spaces ( [10], [11]).

**Theorem 1.6.** *Let $(X, D)$ be a complete vector-valued metric space and $T : X \to X$. If there exists a matrix $A \in M_n(R^+)$ which converges to $\theta_n$ and for all $x, y \in X$, we have*

$$D(Tx, Ty) \le AD(x, y),$$

*then $T$ possess a unique fixed point.*

1.1. **Multivalued cyclic interpolative Kannan type contraction.**
Let $A$ and $B$ be nonempty subsets of a metric space $(X, d)$. A mapping $T : A \cup B \to A \cup B$ is said to be cyclic if $T(A) \subset B$ and $T(B) \subset A$.
In 2003, Kirk et al. [7] introduced the concept of cyclic contraction and proved the fixed point result for such mappings.
Let $(X, d)$ be a complete metric space and $\{A_1, A_2, \ldots, A_p\}$ a collection of nonempty closed subsets of $X$ with $\cap_{i=1}^{p} A_i \neq \phi$ and $Y = \cup_{i=1}^{p} A_i$.
Suppose that $T : Y \to CB(X)$ is multivalued. For any set $A$ in $Y$, define $TA = \cup\{Tx : x \in A\}$.

**Definition 1.7.** A mapping $T : Y \to CB(X)$ is called multivalued cyclic interpolative Kannan type contraction if

cc by nc sa

(i) $TA_i \subseteq A_{i+1}$, for all $1 \leq i \leq p$ where $A_{p+1} = A_1$ and $TA_p \subseteq A_1$,
(ii) For any $(x, y) \in A_i \times A_{i+1}, i = 1, 2, \ldots p$ and $A_{p+1} = A_1$ with $H(Tx, Ty) > 0$, there exist $\lambda \in (0, 1)$ such that

$$H(Tx, Ty) \leq \lambda d(x, Tx)^{\alpha} d(y, Ty)^{1-\alpha},$$

holds with $x, y \notin \cap_{i=1}^{p} A_i$ and $d(x, Tx) \neq 0, d(y, Ty) \neq 0$.

**Theorem 1.8.** *Let $X$ be a complete metric space. If $T : Y \to CB(X)$ is a multivalued cyclic interpolative Kannan type contraction. Then $T$ has a fixed point in $\cap_{i=1}^{p} A_i$.*

*Proof.* Let $x_1 \in A_1 \subseteq X$ and $x_2 \in Tx_1 \subseteq A_2$. Then, for any $q = \frac{1}{\sqrt{\lambda}} > 1$, it follows from the Lemma 1.1 that there exist $x_3 \in Tx_2 \subseteq A_3$ such that

$$d(x_2, x_3) \leq qH(Tx_1, Tx_2).$$

Also, for $x_3 \in Tx_2 \subseteq A_3$, there exist $x_4 \in Tx_3 \subseteq A_4$, such that

$$d(x_3, x_4) \leq qH(Tx_2, Tx_3),$$

holds with $q > 1$. Continuing the same way, we obtain that $x_{p-1} \in Tx_{p-2} \subseteq A_{p-1}$. For $q > 1$. By Lemma 1.1, there exist $x_p \in Tx_{p-1} \subseteq A_p$ such that

$$d(x_{p-1}, x_p) \leq qH(Tx_{p-2}, Tx_{p-2}).$$

Moreover, for $x_p \in Tx_{p-1} \subseteq A_p$ there exist $x_{p+1} \in Tx_p \subseteq A_1$, such that

$$d(x_p, x_{p+1}) \leq qH(Tx_{p-1}, Tx_p).$$

Following arguments similar to those given above, we can construct a sequence $\{x_{mp+i}\}$ in $X$ where $1 \leq i \leq p, m \in N$ with $x_{mp+i} \in A_i$. Thus for all $m \in N$ and $1 \leq i \leq p, (x_{mp+i}) = (x_n)$ is a sequence in $\cup_{i=1}^{n} A_i$ such that for any $n \in N$ and $r = \sqrt{\lambda}$, we have

$$\begin{aligned}
d(x_n, x_{n+1}) &\leq qH(Tx_{n-1}, Tx_n) \leq \sqrt{\lambda} d(x_{n-1}, Tx_{n-1})^{\alpha} d(x_n, Tx_n)^{1-\alpha} \\
d(x_n, x_{n+1}) &\leq r d(x_{n-1}, x_n)^{\alpha} d(x_n, x_{n+1})^{1-\alpha} \\
d(x_n, x_{n+1}) &\leq r^{1/\alpha} d(x_{n-1}, x_n) \leq r d(x_{n-1}, x_n).
\end{aligned}$$

Thus,

$$d(x_n, x_{n+1}) \leq r d(x_{n-1}, x_n) \leq r^2 d(x_{n-2}, x_{n-1}) \cdots \leq r^{n-1} d(x_1, x_2).$$

Consequently, for any $n \in N$, we have

$$d(x_n, x_{n+1}) \leq r^{n-1} d(x_1, x_2).$$

Hence for any $m, n \in N$ with $m \geq n$, by triangle inequality we obtain

$$\begin{aligned}
d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\
&\leq r^{n-1} d(x_1, x_2) + r^{n-2} d(x_1, x_2) + \cdots + r^{m-2} d(x_1, x_2) \\
&= (r^{n-1} + r^{n-2} + \cdots + r^{m-2}) d(x_1, x_2) \\
&= (\sum_{i=n-1}^{m-2} r^i) d(x_1, x_2).
\end{aligned}$$

On taking limit as $n, m \to \infty$ we get $d(x_n, x_m) \to 0$. Hence $(x_n)$ is a Cauchy sequence in $X$. As $X$ is complete, it converges to a point $x \in X$, that is, $x_n \to x$. It follows from construction of $(x_n)$ that $x_n = x_{mp+i}$, where $n = mp + i$, with $m \in N$ and $i = 1, 2, \ldots p$. Thus $(x_{mp+i})$ is a sequence in $A_i$, for each fixed $1 \leq i \leq p$, and all these sequences $(x_{mp+i}) \subseteq A_i, 1 \leq i \leq p$ are subsequences of $(x_n)$, and converges to $x$. Since each $A_i$ is closed so for each fixed $1 \leq i \leq p$, $x_{mp+i} \to x \in A_i$. Thus $x \in \cap_{i=1}^{p} A_i$. Now, we show that $x$ is the fixed point of $T$, that is, $d(x, Tx) = 0$. Since for any fixed $1 \leq i \leq p$,, we have

$$\begin{aligned}
0 \leq d(x_{mp+i+1}, Tx) \leq& H(Tx_{mp+i}, Tx) \\
\leq& \lambda d(x_{mp+i}, Tx_{mp+i})^{\alpha} d(x, Tx)^{1-\alpha} \\
\leq& \lambda d(x_{mp+i}, x_{mp+i+1})^{\alpha} d(x, Tx)^{1-\alpha}.
\end{aligned}$$

On taking limit as $m \to \infty$, we have

$$\begin{aligned}
&0 \leq \lim_{m\to\infty} d(x_{mp+i+1}, Tx) \leq \lim_{n\to\infty} \lambda d(x_{mp+i}, x_{mp+i+1})^{\alpha} d(x, Tx)^{\alpha} \\
&0 \leq \lim_{m\to\infty} d(x_{mp+i+1}, Tx) \leq 0.
\end{aligned}$$

Hence $\lim_{m\to\infty} d(x_{mp+i+1}, Tx) = d(x, Tx) = 0$. Thus $x \in Tx$. □

**Definition 1.9.** A mapping $T : Y \to CB(X)$ is called a multivalued cyclic $(\lambda, \alpha + \beta < 1)$-interpolative Kannan type contraction if
(i) $TA_i \subseteq A_{i+1}$, for all $1 \leq i \leq p$ where $A_{p+1} = A_1$ and $TA_p \subseteq A_1$.
(ii) For any $(x, y) \in A_i \times A_{i+1}, i = 1, 2, \ldots p$ and $A_{p+1} = A_1$ with $H(Tx, Ty) > 0$, there exist $\lambda \in (0, 1)$ such that

$$H(Tx, Ty) \leq \lambda d(x, Tx)^{\alpha} d(y, Ty)^{\beta},$$

holds with $x, y \notin \cap_{i=1}^{p} A_i$, $x, y \notin Fix(T)$ and $d(x, Tx) \geq 1, d(y, Ty) \neq 0$.

**Theorem 1.10.** *Let $X$ be a complete metric space. If $T : Y \to CB(X)$ is a multivalued cyclic $(\lambda, \alpha + \beta < 1)$-interpolative Kannan type contraction. Then $T$ has a fixed point in $\cap_{i=1}^{p} A_i$.*

*Proof.* Take $x_1 \in A_1$. Following arguments similar to those given in Theorem 1.8, we construct a sequence $\{x_n\}$ in $X$ satisfying:

$$d(x_n, x_{n+1}) \leq qH(Tx_n, Tx_{n-1}), \qquad \forall n \in N.$$

From Definition 1.9, we have

$$d(x_n, x_{n+1}) \leq rH(Tx_n, Tx_{n-1}) \leq \sqrt{\lambda} d(x_{n-1}, Tx_{n-1})^{\alpha} d(x_x, Tx_n)^{\beta},$$

that is,

$$d(x_n, x_{n+1})^{1-\beta} \leq \sqrt{\lambda} d(x_{n-1}, Tx_{n-1})^{\alpha},$$

because $x_{n+1} \in Tx_n$. Moreover

$$d(x_n, x_{n+1}) \leq r^{\frac{1}{1-\beta}} d(x_{n-1}, Tx_{n-1})^{\frac{\alpha}{1-\beta}}.$$

Indeed, $\alpha + \beta < 1$, $d(x, Tx) \geq 1, \forall x \in X$. Thus

$$d(x_n, x_{n+1}) \leq rd(x_{n-1}, Tx_{n-1}) \leq rd(x_{n-1}, x_n).$$

Adopting the similar procedure as in the proof of Theorem 1.8, we can prove that $(x_n)$ is a Cauchy sequence. Furthermore, using the completeness of $X$, $(x_n)$ converges to fixed point $x \in \cap_{i=1}^{p} A_i$ of $T$. □

**Definition 1.11.** A mapping $T : Y \to CB(X)$ is called a multivalued cyclic $(\lambda, \alpha + \beta > 1)$-interpolative Kannan type contraction if
(i) $TA_i \subseteq A_{i+1}$, for all $1 \leq i \leq p$ where $A_{p+1} = A_1$ and $TA_p \subseteq A_1$.
(ii) For any $(x, y) \in A_i \times A_{i+1}, i = 1, 2, \ldots p$ and $A_{p+1} = A_1$ with $H(Tx, Ty) > 0$, there exist $\lambda \in (0, 1)$ such that

$$H(Tx, Ty) \leq \lambda d(x, Tx)^{\alpha} d(y, Ty)^{\beta},$$

is satisfied for all $x, y \notin \cap_{i=1}^{p} A_i$, $x, y \notin Fix(T)$ and $d(x, Tx) \neq 0, d(y, Ty) \neq 0$.

**Theorem 1.12.** *Let $X$ be a complete metric space, and $T : Y \to CB(X)$ a multivalued cyclic $(\lambda, \alpha+\beta > 1)$-interpolative Kannan type contraction. If there exists $x \in X$ such that $d(x, Tx) \leq 1$, then $T$ has a fixed point in $\cap_{i=1}^{p} A_i$.*

*Proof.* Let $x_1 \in X$ such that $d(x_1, Tx_1) \leq 1$. Without any loss of generality we assume that $x_1 \in A_1 \subseteq X$. Consider $x_2 \in Tx_1 \subseteq A_2$ and for $q = \frac{1}{\sqrt{\lambda}} > 1$. By Lemma 1.1, there exist $x_3 \in Tx_2 \subseteq A_3$ such that

$$d(x_2, x_3) \leq qH(Tx_1, Tx_2).$$

Also by Definition 1.11, we have

$$d(x_2, x_3) \leq q\lambda d(x_1, Tx_1)^{\alpha} d(x_2, Tx_2)^{\beta}.$$

As $x_3 \in Tx_2$ , $d(x_2, Tx_2) \leq d(x_2, x_3)$. Thus

$$d(x_2, x_3) \leq \sqrt{\lambda} d(x_1, Tx_1)^{\alpha} d(x_2, x_3)^{\beta},$$

$$d(x_2, x_3)^{1-\beta} \leq r d(x_1, Tx_1)^{\alpha},$$

that is,

$$d(x_2, x_3) \leq r^{\frac{1}{1-\beta}} d(x_1, Tx_1)^{\frac{\alpha}{1-\beta}}.$$

Since $\frac{\alpha}{1-\beta} > 1$ and $d(x_1, Tx_1) \leq 1$, we have $d(x_1, Tx_1)^{\frac{\alpha}{1-\beta}} \leq 1$ and $r^{\frac{1}{1-\beta}} \leq r$. Thus

$$d(x_2, x_3) \leq r.$$

Continuing the same way, we can construct a sequence by using Lemma 1.1 and Definition 1.11 such that

$$d(x_n, x_{n+1}) \leq r^n$$

for all $n \in N$. Note that,

$$d(x_n, x_m) \leq r^n + r^{n+1} + r^{n+2} + \ldots, + r^{m-1} = \sum_{i=n}^{m-1} r^i.$$

Hence $d(x_n, x_m) \to 0$ as $n, m \to \infty$. Thus $(x_n)$ is a Cauchy sequence in $X$, and completeness of $X$ yields that $(x_n)$ converges to some $x \in X$. Further, due to the cyclic nature of the mapping, the sequence $(x_n)$ is composed of sub-sequences $x_i \in A_i$, $mp \leq i \leq (m+1)p$ $\forall m \in N$. Thus the sub-sequences $\{x_{mp+i}\} \in A_i$ $1 \leq i \leq p$, $m \in N$. Moreover, the convergence of $(x_n)$ assures that each of the sub-sequence also converges to the same limit $x$, such that $x_{mp+i} \to x$, $\forall$ $1 \leq i \leq p$. Hence $x \in A_i$ for all $1 \leq i \leq p$ because each $A_i$ is closed. Thus $x \in \cap_{i=1}^{p} A_i$. Now, we show that $x$ is the fixed point of $T$. Note that,

$$\begin{aligned} 0 \leq d(x_{n+1}, Tx) &\leq H(Tx_n, Tx) \\ &\leq \lambda d(x_n, Tx_n)^{\alpha} d(x, Tx)^{\beta} \\ &\leq \lambda d(x_n, x_{n+1})^{\alpha} d(x, Tx)^{\beta} \\ &\leq \lambda r^{n\alpha} d(x, Tx)^{\beta}. \end{aligned}$$

On taking limit as $n \to \infty$, we obtain that

$$0 \leq \lim_{n\to\infty} d(x_{n+1}, Tx) \leq 0,$$

and

$$\lim_{n\to\infty} d(x_{n+1}, Tx) = d(x, Tx) = 0.$$

Thus $x \in Tx$, that is, $x \in \cap_{i=1}^{p} A_i$ is the fixed point of $T$. □

*Remark* 1.13. In Theorem 1.10, we have $\alpha + \beta < 1$ which implies that

$$H(Tx, Ty) \leq \lambda d(x, Tx)^{\alpha} d(y, Ty)^{\beta} \leq \lambda d(x, Tx)^{\alpha} d(y, Ty)^{1-\alpha},$$

since $d(y, Ty) \geq 1$. Thus, Theorem 1.10 is satisfied for $\alpha+\beta \leq 1$. However, it does not cover Theorem 1.8 which is more generally applicable without the restriction $d(x, Tx) \geq 1$. In Theorem 1.12 with $\alpha + \beta > 1$ implying $\alpha > 1 - \beta$, we have

$$H(Tx, Ty) \leq \lambda d(x, Tx)^{\alpha} d(y, Ty)^{\beta} \leq \lambda d(x, Tx)^{1-\beta} d(y, Ty)^{\beta},$$

because $d(x, Tx) \leq 1$. Thus, Theorem 1.12 is satisfied for $\alpha + \beta \geq 1$ but does not cover Theorem 1.8 which does not restrict with $d(x, Tx) \leq 1$.

### 1.2. **Perov intertopaltive Kannan type contraction.**

**Definition 1.14.** Let $(X, D)$ be a complete vector valued metric space with $D : X \times X \to \mathbb{R}^n$. A self mapping $T : X \to X$ is said to be Perov interpolative Kannan type contraction if there exist a matrix $A \in M_n(\mathbb{R}^+)$ which converges to zero or having spectral radius $\rho(A) < 1$ and $\alpha \in (0, 1)$ such that

$$D(Tx, Ty) \leq AD(x, Tx)^{\alpha} D(y, Ty)^{1-\alpha},$$

is satisfied for all $x, y \in X$ with $x, y \notin Fix(T)$.

**Theorem 1.15.** *Let $T : X \to X$ be Perov interpolative Kannan type contraction. Then $T$ has a fixed point.*

*Proof.* Let $x_0 \in X$, we construct a sequence $(x_n)$ as follows:
$x_{n+1} = T(x_n) = T^n(x_0)$ for all positive integers $n$. Without any loss of generality, we assume that $x_n \neq x_{n+1}$ for each nonnegative integer $n$. Thus, we have

$$D(x_{n+1}, x_{n+2}) = D(Tx_n, Tx_{n+1}) \leq AD(x_n, Tx_n)^{\alpha} D(x_{n+1}, Tx_{n+1})^{1-\alpha}.$$

This yields

$$D(x_{n+1}, x_{n+2})^{\alpha} \leq AD(x_n, Tx_n)^{\alpha}$$

$$D(x_{n+1}, x_{n+2}) \leq A^{1/\alpha} D(x_n, x_{n+1}).$$

Take $A^{1/\alpha} = B$. Hence

$$D(x_{n+1}, x_{n+2}) \leq BD(x_n, x_{n+1}). \tag{1.1}$$

In a similar fashion, we can write

$$D(x_n, x_{n+1}) = D(Tx_{n-1}, Tx_n) \leq AD(x_{n-1}, Tx_{n-1})^{\alpha} D(x_n, Tx_n)^{1-\alpha}$$

$$D(x_n, x_{n+1}) \leq A^{1/\alpha} D(x_{n-1}, x_n)$$

$$D(x_n, x_{n+1}) \leq BD(x_{n-1}, x_n). \tag{1.2}$$

By combining (1.1) and (1.2), we obtain

$$D(x_n, x_{n+1}) \leq B^2 D(x_{n-1}, x_n). \tag{1.3}$$

Following the arguments similar to those given above, we get the following relation for all $n \in N$

$$D(x_n, x_{n+1}) \leq B^n D(x_0, x_1). \tag{1.4}$$

Note that,

$$\begin{aligned} D(x_m, x_{m+p}) &\leq D(x_m, x_{m+1}) + D(x_{m+1}, x_{m+2}) + \cdots , D(x_{m+p-1}, x_{m+p}) \\ &\leq (B^m + B^{m+1} + B^{m+2} +, \cdots , B^{m+p-1}) D(x_0, x_1) \\ &\leq (B^m + B^{m+1} + B^{m+2} + \cdots) D(x_0, x_1) \\ &= B^m (I + B + B^2 + B^3 + \cdots) D(x_0, x_1). \end{aligned}$$

Since $B$ is convergent to zero, matrix $(I - B)$ is non-singular and

$$(I - B)^{-1} = I + B + B^2 + \cdots .$$

Therefore,

$$D(x_m, x_{m+p}) \leq B^m (I - B)^{-1} D(x_0, x_1),$$

As $B^m \to \theta_n$ on taking limit as $m \to \infty$. So $D(x_m, x_{m+p}) \to \mathbf{0}$ as $m \to \infty$. Hence, the sequence $(x_m)$ is a fundamental (Cauchy), and using the completeness of the space $(X, D)$, there exist $z \in X$ such that $D(x_n, z) \to \mathbf{0}$ as $n \to \infty$, that is,

$$\lim_{n\to\infty} D(x_n, z) = \mathbf{0}. \tag{1.5}$$

We now show that $z$ is a fixed point of $T$. Note that,

$$\begin{aligned} \mathbf{0} \leq D(z, Tz) &= \lim_{n\to\infty} D(x_n, Tz) \\ &= \lim_{n\to\infty} D(Tx_{n-1}, Tz) \\ &\leq A \lim_{n\to\infty} D(x_{m-1}, Tx_{m-1})^{\alpha} D(z, Tz)^{1-\alpha} = \mathbf{0}. \end{aligned}$$

Hence

$$D(z, Tz) = \mathbf{0} \implies z = Tz.$$

□

**Example 1.16.** Let $X = [0, \infty)$, and $D : X \times X \to R^2$ a metric defined as:

$$D(x, y) = \begin{pmatrix} d(x, y) \\ d(x, y) \end{pmatrix}, \qquad \text{where} \qquad d(x, y) = \begin{cases} x + y, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases}$$

Suppose that $T : X \to X$ is defined as follows:

$$Tx = \begin{cases} 1 & \text{if } x \in [0, 4], \\ e^{-2x} & \text{if } x \in (4, \infty). \end{cases}$$

Take, $A = \begin{pmatrix} 1/2 & 1/4 \\ 0 & 1/8 \end{pmatrix}$. Clearly, $\rho(A) = 1/2$. We now discuss the following cases.
Case I: If $x, y \in [0, 4]$, then

$$D(Tx, Ty) = \begin{pmatrix} d(Tx, Ty) \\ d(Tx, Ty) \end{pmatrix} = \begin{pmatrix} d(1, 1) \\ d(1, 1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Also

$$\begin{aligned} AD(x, Tx)^{1/2} D(y, Ty)^{1/2} &= \begin{pmatrix} 1/2 & 1/4 \\ 0 & 1/8 \end{pmatrix} \begin{pmatrix} d(x, Tx) \\ d(x, Tx) \end{pmatrix}^{1/2} \begin{pmatrix} d(y, Ty) \\ d(y, Ty) \end{pmatrix}^{1/2} \\ &= \begin{pmatrix} 1/2 & 1/4 \\ 0 & 1/8 \end{pmatrix} \begin{pmatrix} x + 1 \\ x + 1 \end{pmatrix}^{1/2} \begin{pmatrix} y + 1 \\ y + 1 \end{pmatrix}^{1/2} \\ &= \begin{pmatrix} 1/2 & 1/4 \\ 0 & 1/8 \end{pmatrix} \begin{pmatrix} (x + 1)^{1/2}(y + 1)^{1/2} \\ (x + 1)^{1/2}(y + 1)^{1/2} \end{pmatrix} \\ &= \begin{pmatrix} 3/4(x + 1)^{1/2}(y + 1)^{1/2} \\ 1/8(x + 1)^{1/2}(y + 1)^{1/2} \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \end{aligned}$$

Case II: If $x \in [0, 4]$, and $y \in (4, \infty)$, then

$$D(Tx, Ty) = \begin{pmatrix} d(Tx, Ty) \\ d(Tx, Ty) \end{pmatrix} = \begin{pmatrix} d(1, e^{-2y}) \\ d(1, e^{-2y}) \end{pmatrix} = \begin{pmatrix} 1 + e^{-2y} \\ 1 + e^{-2y} \end{pmatrix} \leq \begin{pmatrix} 1 + e^{-8} \\ 1 + e^{-8} \end{pmatrix}.$$

Also

$$\begin{aligned} AD(x,Tx)^{1/2}D(y,Ty)^{1/2} &= \begin{pmatrix} 1/2 & 1/4 \\ 0 & 1/8 \end{pmatrix} \begin{pmatrix} d(x,Tx) \\ d(x,Tx) \end{pmatrix}^{1/2} \begin{pmatrix} d(y,Ty) \\ d(y,Ty) \end{pmatrix}^{1/2} \\ &= \begin{pmatrix} 1/2 & 1/4 \\ 0 & 1/8 \end{pmatrix} \begin{pmatrix} (x+1)^{1/2}(y+e^{-2y})^{1/2} \\ (x+1)^{1/2}(y+e^{-2y})^{1/2} \end{pmatrix} \\ &= \begin{pmatrix} (3/4)(x+1)^{1/2}(y+e^{-2y})^{1/2} \\ (1/8)(x+1)^{1/2}(y+e^{-2y})^{1/2} \end{pmatrix} \\ &\geq \begin{pmatrix} (3/4)(4+e^{-8})^{1/2} \\ (1/8)(4+e^{-8})^{1/2} \end{pmatrix} \geq \begin{pmatrix} 1+e^{-8} \\ 1+e^{-8} \end{pmatrix}. \end{aligned}$$

Similarly, it holds for the case $x \in [0,4]$ and $y \in (4,\infty)$.

Case III: If $x, y \in (4,\infty)$, then

$$D(Tx,Ty) = \begin{pmatrix} d(Tx,Ty) \\ d(Tx,Ty) \end{pmatrix} = \begin{pmatrix} d(e^{-2x},e^{-2y}) \\ d(e^{-2x},e^{-2y}) \end{pmatrix} = \begin{pmatrix} e^{-2x}+e^{-2y} \\ e^{-2x}+e^{-2y} \end{pmatrix} \leq \begin{pmatrix} 2e^{-8} \\ 2e^{-8} \end{pmatrix}.$$

Also

$$\begin{aligned} AD(x,Tx)^{1/2}D(y,Ty)^{1/2} &= \begin{pmatrix} 1/2 & 1/4 \\ 0 & 1/8 \end{pmatrix} \begin{pmatrix} d(x,Tx) \\ d(x,Tx) \end{pmatrix}^{1/2} \begin{pmatrix} d(y,Ty) \\ d(y,Ty) \end{pmatrix}^{1/2} \\ &= \begin{pmatrix} 1/2 & 1/4 \\ 0 & 1/8 \end{pmatrix} \begin{pmatrix} x+1/e^{2x} \\ x+1/e^{2x} \end{pmatrix}^{1/2} \begin{pmatrix} y+1/e^{2y} \\ y+1/e^{2y} \end{pmatrix}^{1/2} \\ &= \begin{pmatrix} (3/4)(x+e^{-2x})^{1/2}(y+e^{-2y})^{1/2} \\ (1/8)(x+e^{-2x})^{1/2}(y+e^{-2y})^{1/2} \end{pmatrix} \\ &\geq \begin{pmatrix} (3/4)(4+e^{-8}) \\ (1/8)(4+e^{-8}) \end{pmatrix} \geq \begin{pmatrix} 2e^{-8} \\ 2e^{-8} \end{pmatrix}. \end{aligned}$$

Thus, in all the cases the required interpolative condition holds. Moreover, $x = 1$ is a fixed point of $T$.

**Conclusion.** If we set $A_i = X$ for all $1 \leq i \leq p$ and consider the collection of compact subsets, then the Theorem 1 of [8] becomes a special case of Theorem 1.8. Furthermore, if we take $n = 1$ in Theorem 1.15, then the result of [5] becomes a special case of our Theorem .

cc by nc sa

**Acknowledgements:** We thank the referees for their valuable feedback, which greatly improved the manuscript, and the section editor for their efficient handling of the article.

**Funding:** This research has not received external funding.

**Author Contributions:** Conceptualization, methodology, and software: A.R., M.A., and N.S.; validation: A.R. and N.S.; formal analysis: M.A. and N.S.; investigation: A.R., M.A., and N.S.; resources: A.R., M.A., and N.S.; data curation: N.S.; writing—original draft preparation: A.R. and N.S.; writing—review and editing: M.A.; visualization: A.R., M.A., and N.S.; supervision: A.R. and M.A.; project administration: A.R. and M.A. All authors have read and agreed to the published version of the manuscript.

## References

[1] G. Allaire and S. M. Kaber, Numerical linear algebra, Texts in Applied Mathematics, vol. 55, Springer, New York (2008).

[2] S. Banach, Sur les operations dans les ensembles abstraits et leur application aux equations integrales, Fundamenta Mathematicae 3, no. 1 (1922), 133–181.

[3] Y. Errai, E. M. Marhrani, and M. Aamri, Some remarks on fixed point theorems for interpolative Kannan contraction, Journal of Function Spaces 2020 (2020), 1–7.

[4] Y. U. Gaba, A. Hassen, and M. Nabil, $(\rho, \eta, \mu)$-Interpolative Kannan contractions I, Axioms 10, no. 3 (2021), 212.

[5] E. Karapınar, Revisiting the Kannan type contractions via interpolation, Advances in the Theory of Nonlinear Analysis and its Application 2, no. 2 (2018), 85–87.

[6] S. Khan and A. Raza, Interpolative contractive results for $m$-metric spaces, Advances in the Theory of Nonlinear Analysis and its Application 7, no. 2 (2023), 336–347.

[7] W. A. Kirk, P. S. Srinivasan, and P. Veeramani, Fixed points for mappings satisfying cyclical contractive conditions, Fixed point theory 4, no. 1 (2003), 79–89.

[8] N. Konwar, R. Srivastava, P. Debnath, and H. M. Srivastava, Some new results for a class of multivalued interpolative Kannan-type contractions, Axioms 11, no. 2 (2022), 76.

[9] S. B. Nadler Jr, Multi-valued contraction mappings, Pacific J. Math. 30, no. 2 (1969), 475–488.

[10] A. I. Perov, On the Cauchy problem for a system of ordinary differential equations, Pviblizhen. Met. Reshen. Differ. Uvavn 2 (1964), 115–134.

[11] A. I. Perov and A. V. Kibenko, On a certain general method for investigation of boundary value problems, Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya 30, no. 2 (1966), 249–264.

[12] R. S. Varga, Matrix iterative analysis, Springer Series in Computational Mathematics, vol. 27, Springer Science & Business Media, Berlin, (2000).