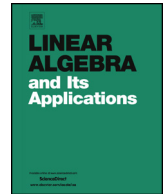




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Row completion of polynomial and rational matrices[☆]



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ABSTRACT

We characterize the existence of a polynomial (rational) matrix when its eigenstructure (complete structural data) and some of its rows are prescribed. For polynomial matrices, this problem was solved in [1] when the polynomial matrix has the same degree as the prescribed submatrix. In that paper, the following row completion problems were also solved arising when the eigenstructure was partially prescribed, keeping the restriction on the degree: the eigenstructure but the row (column) minimal indices, and the finite and/or infinite structures. Here we remove the restriction on the degree, allowing it to be greater than or equal to that of the submatrix. We also generalize the results to rational matrices.

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Completion

Obviously, the results obtained hold for the corresponding column completion problems.

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1. Introduction

An important problem in Matrix Theory is the matrix completion problem. It consists in characterizing the existence of a matrix with certain properties when a submatrix is prescribed. In fact, this problem includes many other ones depending on the type of matrices involved and the properties analyzed. In the last decades the research in the area has been very fruitful. See [1,10,14,15,17] and the references therein.

This work is devoted to the matrix completion problem for polynomial and rational matrices when the complete structural data (or some of them) of the polynomial or the rational matrix are prescribed and the submatrix is formed by some of its rows (columns). This study generalizes the results obtained in [1], where the row completion problem of a polynomial matrix is solved when the eigenstructure (or part of it) is prescribed and the degree of the completed matrix is the same as that of the prescribed submatrix.

The generalization addressed in this paper is two-folded. On the one hand, we allow that the degree of the completed polynomial matrix is greater than or equal to that of the prescribed submatrix. On the other hand, the results of [1] are generalized to rational matrices, solving the row completion problem when the complete structural data (or some of them) of the completed rational matrix are prescribed.

The eigenstructure of a polynomial matrix is formed by four types of invariants: the invariant factors, the partial multiplicities of ∞ , and the column and row minimal indices ([11]). The invariant factors form the finite structure of the matrix, the partial multiplicities of ∞ are the infinite structure, and the minimal indices, the singular structure. The prescription of all or part of these invariants leads to pose 15 different completion problems, some of them solved in [1], when the degree of the completed matrix is prescribed to coincide with that of the given submatrix. The remaining ones are solved in [2].

The complete structural data of a rational matrix is formed by the invariant rational functions (finite structure), the invariant orders at ∞ (infinite structure), and the column and row minimal indices (singular structure) ([4]). We show later that for polynomial matrices, knowing the eigenstructure is equivalent to knowing the complete structural data.

Once more, due to the high number of problems that the complete analysis of the study includes, we solve in this work some of the cases, generalizing the results in [1], and leaving the remaining ones for a future paper, hence generalizing [2]. One of the problems, the prescription of only the finite structure, was solved independently by E. Marques de Sá and R.C. Thompson in 1979 in the seminal papers [15,17], and the result was generalized to the rational case in [5].

The paper is organized as follows. Section 2 contains the notation, definitions, preliminary results, and the statement of the problems we deal with. In Section 3 we solve the problem when the complete structural data are prescribed. Finally, in Section 4 we solve the problem of the prescription of the complete structural data but the row (column) minimal indices (Subsection 4.1), and that of the prescription of the finite and/or infinite structures (Subsection 4.2).

2. Preliminaries

Let $\mathbb{F}$ be a field. The ring of polynomials in the indeterminate s with coefficients in $\mathbb{F}$ is denoted by $\mathbb{F}[s]$, $\mathbb{F}(s)$ is the field of fractions of $\mathbb{F}[s]$, i.e., the field of rational functions over $\mathbb{F}$, and $\mathbb{F}_{pr}(s)$ is the ring of proper rational functions, i.e., the rational functions with degree of the denominator at least the degree of the numerator. The ring of polynomials in two variables s, t with coefficients in $\mathbb{F}$ is denoted by $\mathbb{F}[s, t]$. A polynomial in $\mathbb{F}[s]$ is *monic* if its leading coefficient is 1. We say that a polynomial in $\mathbb{F}[s, t]$ is *monic* if it is monic with respect to the variable s . Given two polynomials α, β , by $\alpha \mid \beta$ we mean that α is a divisor of β , by $\text{lcm}(\alpha, \beta)$, the monic least common multiple of α and β , and by $\text{gcd}(\alpha, \beta)$, the monic greatest common divisor of α and β .

In this work we deal with finite sequences of integers $\mathbf{a} = (a_1, \dots, a_r)$ where $a_1 \geq \dots \geq a_r$. If $a_r \geq 0$, the sequence is called a *partition*. When necessary, we take $a_i = +\infty$ for $i < 1$ and $a_i = -\infty$ for $i > r$. If $b_1 \leq \dots \leq b_r$ is an increasing sequence of integers, we take $b_i = -\infty$ for $i < 1$ and $b_i = +\infty$ for $i > r$.

We also deal with polynomial chains $\alpha_1 \mid \dots \mid \alpha_r$, where $\alpha_i \in \mathbb{F}[s]$ or $\alpha_i \in \mathbb{F}[s, t]$, and take $\alpha_1 = 1$ for $i < 1$ and $\alpha_i = 0$ for $i > r$. If $\varphi_r \mid \dots \mid \varphi_1$, we take $\varphi_i = 1$ for $i > r$ and $\varphi_i = 0$ for $i < 1$.

We denote by $\mathbb{F}^{m \times n}$, $\mathbb{F}[s]^{m \times n}$, $\mathbb{F}(s)^{m \times n}$, and $\mathbb{F}_{pr}(s)^{m \times n}$ the vector spaces over $\mathbb{F}$ of $m \times n$ matrices with elements in $\mathbb{F}$, $\mathbb{F}[s]$, $\mathbb{F}(s)$, and $\mathbb{F}_{pr}(s)$, respectively. A matrix $U(s) \in \mathbb{F}[s]^{n \times n}$ is said *unimodular* if it has inverse in $\mathbb{F}[s]^{n \times n}$, while a matrix $B(s) \in \mathbb{F}_{pr}(s)^{n \times n}$ is said *biproper* if it has inverse in $\mathbb{F}_{pr}(s)^{n \times n}$.

Let $R(s) \in \mathbb{F}(s)^{m \times n}$ of $\text{rank}(R(s)) = r$. A canonical form for the unimodular equivalence of $R(s)$ is the *Smith–McMillan form*

$$\begin{bmatrix} \text{diag} \left(\frac{\eta_1(s)}{\varphi_1(s)}, \dots, \frac{\eta_r(s)}{\varphi_r(s)} \right) & 0 \\ 0 & 0 \end{bmatrix},$$

where $\eta_1(s) \mid \dots \mid \eta_r(s)$ and $\varphi_r(s) \mid \dots \mid \varphi_1(s)$ are monic polynomials and $\frac{\eta_1(s)}{\varphi_1(s)}, \dots, \frac{\eta_r(s)}{\varphi_r(s)}$ are irreducible rational functions called the *invariant rational functions* of $R(s)$. We also refer to them as the *finite structure* of $R(s)$. The polynomial $\varphi_1(s)$ is the monic least common denominator of the entries of $R(s)$ (see, for instance, [16, Chapter 3, Section 4]).

A canonical form for the equivalence at infinity of $R(s)$ is the *Smith–McMillan form at infinity*

$$\begin{bmatrix} \text{diag}(s^{-\tilde{p}_1}, \dots, s^{-\tilde{p}_r}) & 0 \\ 0 & 0 \end{bmatrix},$$

where $\tilde{p}_1 \leq \dots \leq \tilde{p}_r$ are integers called the *invariant orders at infinity* of $R(s)$ (see, for instance, [18]). In [4], the sequence of invariant orders at ∞ is called the *structural index sequence of $R(s)$ at ∞* .

We recall now the *singular structure* of a rational matrix. Denote by $\mathcal{N}_\ell(R(s))$ and $\mathcal{N}_r(R(s))$ the *left* and *right null-spaces* over $\mathbb{F}(s)$ of $R(s)$, respectively, i.e., if $R(s) \in \mathbb{F}(s)^{m \times n}$,

$$\begin{aligned} \mathcal{N}_\ell(R(s)) &= \{x(s) \in \mathbb{F}(s)^{m \times 1} : x(s)^T R(s) = 0\}, \\ \mathcal{N}_r(R(s)) &= \{x(s) \in \mathbb{F}(s)^{n \times 1} : R(s)x(s) = 0\}, \end{aligned}$$

which are vector subspaces of $\mathbb{F}(s)^{m \times 1}$ and $\mathbb{F}(s)^{n \times 1}$, respectively. For a subspace $\mathcal{V}$ of $\mathbb{F}(s)^{m \times 1}$ it is possible to find a basis consisting of vector polynomials; it is enough to take an arbitrary basis and multiply each vector by a least common multiple of the denominators of its entries. The *order* of a polynomial basis is defined as the sum of the degrees of its vectors (see [12]). A *minimal basis* of $\mathcal{V}$ is a polynomial basis with least order among the polynomial bases of $\mathcal{V}$. The degrees of the vector polynomials of a minimal basis, increasingly ordered, are always the same (see [12]), and are called the *minimal indices* of $\mathcal{V}$.

A *right (left) minimal basis* of a rational matrix $R(s)$ is a minimal basis of $\mathcal{N}_r(R(s))$ ($\mathcal{N}_\ell(R(s))$). The *right (left) minimal indices* of $R(s)$ are the minimal indices of $\mathcal{N}_r(R(s))$ ($\mathcal{N}_\ell(R(s))$). From now on in this paper, we work with the right (left) minimal indices decreasingly ordered, and we refer to them as the *column (row) minimal indices* of $R(s)$. Notice that a rational matrix $R(s) \in \mathbb{F}(s)^{m \times n}$ of $\text{rank}(R(s)) = r$ has $m - r$ row and $n - r$ column minimal indices.

For a rational matrix $R(s) \in \mathbb{F}(s)^{m \times n}$ of $\text{rank}(R(s)) = r$, the *complete structural data* consist of four components (see [4, Definition 2.15]): the invariant rational functions $\frac{\eta_1(s)}{\varphi_1(s)}, \dots, \frac{\eta_r(s)}{\varphi_r(s)}$, the invariant orders at infinity $\tilde{p}_1 \leq \dots \leq \tilde{p}_r$, the row minimal indices $(u_1, \dots, u_{m-r})$ and the column minimal indices $(c_1, \dots, c_{n-r})$. Observe that the complete structural data of a rational matrix determine its rank.

If the rational matrix is a polynomial matrix $P(s)$ of $\text{rank } P(s) = r$, then $\varphi_1(s) = \dots = \varphi_r(s) = 1$, the polynomials $\eta_1(s) \mid \dots \mid \eta_r(s)$ are the *invariant factors* of $R(s)$, and the Smith-McMillan form is its *Smith normal form* ([16, Chapter 1, Section 1]). Hence, the complete structural data of a polynomial matrix is formed by the invariant factors, the invariant orders at infinity, and the column and row minimal indices.

For polynomial matrices we introduce some other definitions. Let $\deg(P(s)) = d$, where $\deg(\cdot)$ stands for degree. The *reversal* of $P(s)$ is the polynomial matrix

$$\text{rev}(P)(t) = t^d P\left(\frac{1}{t}\right).$$

The *partial multiplicities of ∞* in $P(s)$ are defined as the partial multiplicities of 0 in $\text{rev}(P)(t)$ (see, for instance, [7]).

The invariant factors, the partial multiplicities of ∞ , the row minimal indices and the column minimal indices are known as the *eigenstructure* of the polynomial matrix $P(s) \in \mathbb{F}[s]^{m \times n}$ (see [11]).

Observe that the eigenstructure of a polynomial matrix determines its rank (it is the number of invariant factors, or the number of partial multiplicities of ∞ , and it is also equal to the number of columns (rows) minus the number of column (row) minimal indices).

In the literature, the invariant factors and the partial multiplicities of ∞ of a polynomial matrix $P(s)$ are often treated together as follows: Let $\alpha_1(s) \mid \cdots \mid \alpha_r(s)$ be the invariant factors and $e_1 \leq \cdots \leq e_r$ the partial multiplicities of ∞ of $P(s)$. The *homogeneous invariant factors of $P(s)$* are homogeneous polynomials in $\mathbb{F}[s, t]$, $\phi_1(s, t) \mid \cdots \mid \phi_r(s, t)$, defined as

$$\phi_i(s, t) = t^{e_i} t^{\deg(\alpha_i)} \alpha_i\left(\frac{s}{t}\right), \quad 1 \leq i \leq r.$$

Given $P(s), \bar{P}(s) \in \mathbb{F}[s]^{m \times n}$, we write $P(s) \approx \bar{P}(s)$ when they have the same eigenstructure. If $P(s), \bar{P}(s)$ are matrix pencils, $P(s) \approx \bar{P}(s)$ if and only if they are strictly equivalent ($P(s) \stackrel{s.e.}{\sim} \bar{P}(s)$), i.e., $\bar{P}(s) = SP(s)T$ for some non singular matrices S and T .

Prior to state our problems, we present some results related to the existence of polynomial or rational matrices with prescribed eigenstructure or complete structural data.

Theorem 2.1 ([1, Theorem 3.1], [7, Theorem 3.3] for infinite fields). *Let m, n, r be positive integers, $r \leq \min\{m, n\}$, and d a non negative integer. Let $\alpha_1(s) \mid \cdots \mid \alpha_r(s)$ be monic polynomials. Let $(e_r, \dots, e_1)$, $(c_1, \dots, c_{n-r})$, $(u_1, \dots, u_{m-r})$ be partitions. Then, there exists a polynomial matrix $P(s) \in \mathbb{F}[s]^{m \times n}$ of $\text{rank}(P(s)) = r$, $\deg(P(s)) = d$, with $\alpha_1(s), \dots, \alpha_r(s)$ as invariant factors, $e_1, \dots, e_r$ as partial multiplicities of ∞ , and $c_1, \dots, c_{n-r}$ and $u_1, \dots, u_{m-r}$ as column and row minimal indices, respectively, if and only if*

$$\begin{aligned} e_1 &= 0, \\ \sum_{i=1}^{n-r} c_i + \sum_{i=1}^{m-r} u_i + \sum_{i=1}^r e_i + \sum_{i=1}^r \deg(\alpha_i) &= rd. \end{aligned} \quad (1)$$

As a consequence of (1), the eigenstructure of a polynomial matrix determines its degree.

Given a polynomial matrix $P(s) \in \mathbb{F}[s]^{m \times n}$ of $\deg(P(s)) = d$, let $e_1 \leq \cdots \leq e_r$ be the partial multiplicities of ∞ in $P(s)$ and $p_1 \leq \cdots \leq p_r$ the invariant orders at ∞ of $P(s)$. Then (see [3, Proposition 6.14]),

$$e_i = p_i + d, \quad 1 \leq i \leq r.$$

As a consequence, $\deg(P(s)) = -p_1$. Hence, knowing the degree and the partial multiplicities of ∞ in $P(s)$ is the same as knowing its invariant orders at ∞ ; i.e., the information provided by the complete structural data is equivalent to that provided by the eigenstructure. Thus, we can restate Theorem 2.1 as follows.

Theorem 2.2. *Let $m, n, r \leq \min\{m, n\}$ be positive integers. Let $\alpha_1(s) \mid \cdots \mid \alpha_r(s)$ be monic polynomials. Let $p_1 \leq \cdots \leq p_r$ be integers and $(c_1, \dots, c_{n-r}), (u_1, \dots, u_{m-r})$ be partitions. Then, there exists a polynomial matrix $P(s) \in \mathbb{F}[s]^{m \times n}$ of $\text{rank}(P(s)) = r$, with $\alpha_1(s), \dots, \alpha_r(s)$ as invariant factors, $p_1, \dots, p_r$ as invariant orders at ∞ , and $c_1, \dots, c_{n-r}$ and $u_1, \dots, u_{m-r}$ as column and row minimal indices, respectively, if and only if*

$$\sum_{i=1}^{n-r} c_i + \sum_{i=1}^{m-r} u_i + \sum_{i=1}^r p_i + \sum_{i=1}^r \deg(\alpha_i) = 0.$$

Now we state our first problem, which is a generalization of the row completion problem of polynomial matrices solved in [1, Theorem 4.2].

Problem 1. Let $P(s) \in \mathbb{F}[s]^{m \times n}$ be a polynomial matrix. Find necessary and sufficient conditions for the existence of a polynomial matrix $W(s) \in \mathbb{F}[s]^{z \times n}$ such that $\begin{bmatrix} P(s) \\ W(s) \end{bmatrix}$ has prescribed complete structural data.

We would like to point out that the row completion problem studied in [1] requires that $\deg \begin{bmatrix} P(s) \\ W(s) \end{bmatrix} = \deg P(s)$. Here this restriction is removed, i.e., $\deg \begin{bmatrix} P(s) \\ W(s) \end{bmatrix} \geq \deg P(s)$.

It is our aim to also study the row completion problem for rational matrices. First of all we extend Theorem 2.2. The following lemma is essential to generalize to rational matrices some results obtained for polynomial matrices.

Lemma 2.3. *Let $R(s)$ be a rational matrix and let $\psi(s)$ be a monic polynomial multiple of the least common denominator of the entries in $R(s)$. Then, $\psi(s)R(s)$ is a polynomial matrix of the same rank as $R(s)$ and*

- (i) *the quotients $\frac{\eta_1(s)}{\varphi_1(s)}, \dots, \frac{\eta_r(s)}{\varphi_r(s)}$ are the invariant rational functions of $R(s)$ if and only if the polynomials $\frac{\psi(s)\eta_1(s)}{\varphi_1(s)}, \dots, \frac{\psi(s)\eta_r(s)}{\varphi_r(s)}$ are the invariant factors of $\psi(s)R(s)$.*
- (ii) *the integers $\tilde{p}_1, \dots, \tilde{p}_r$ are the invariant orders at ∞ of $R(s)$ if and only if $\tilde{p}_1 - \deg(\psi(s)), \dots, \tilde{p}_r - \deg(\psi(s))$ are the invariant orders at ∞ of $\psi(s)R(s)$.*
- (iii) *$\mathcal{N}_r(R(s)) = \mathcal{N}_r(\psi(s)R(s))$, $\mathcal{N}_\ell(R(s)) = \mathcal{N}_\ell(\psi(s)R(s))$ and, therefore, the minimal indices of $\psi(s)R(s)$ and of $R(s)$ are the same.*

Proof. Items (i) and (ii) can be easily derived from the Smith–McMillan forms. The proof of item (iii) is straightforward. $\square$

When $\mathbb{F}$ is an infinite field, Theorem 4.1 of [4] provides necessary and sufficient conditions for the existence of a rational matrix $R(s) \in \mathbb{F}(s)^{m \times n}$ with prescribed complete structural data. The proof is based on Theorem 3.3 of [7], which establishes an analogous result for polynomial matrices over infinite fields. This theorem was generalized to arbitrary fields in [1, Theorem 3.1]. Using the latter result, we obtain a generalization of [4, Theorem 4.1] to arbitrary fields.

Theorem 2.4. *Let $m, n, r \leq \min\{m, n\}$ be positive integers. Let $\eta_1(s) \mid \cdots \mid \eta_r(s)$ and $\varphi_r(s) \mid \cdots \mid \varphi_1(s)$ be monic polynomials such that $\frac{\eta_1(s)}{\varphi_1(s)}, \dots, \frac{\eta_r(s)}{\varphi_r(s)}$ are irreducible rational functions. Let $\tilde{p}_1 \leq \cdots \leq \tilde{p}_r$ be integers and $(c_1, \dots, c_{n-r}), (u_1, \dots, u_{m-r})$ partitions. Then, there exists a rational matrix $R(s) \in \mathbb{F}(s)^{m \times n}$, $\text{rank}(R(s)) = r$, with $\frac{\eta_1(s)}{\varphi_1(s)}, \dots, \frac{\eta_r(s)}{\varphi_r(s)}$ as invariant rational functions, $\tilde{p}_1, \dots, \tilde{p}_r$ as invariant orders at ∞ , and $c_1, \dots, c_{n-r}$ and $u_1, \dots, u_{m-r}$ as column and row minimal indices, respectively, if and only if*

$$\sum_{i=1}^{n-r} c_i + \sum_{i=1}^{m-r} u_i + \sum_{i=1}^r \tilde{p}_i + \sum_{i=1}^r \deg(\eta_i) - \sum_{i=1}^r \deg(\varphi_i) = 0.$$

Proof. The proof is analogous to that of [4, Theorem 4.1] using Theorem 3.1 of [1] instead of Theorem 3.3 of [7], and Lemma 2.3. $\square$

Now, we generalize to rational matrices Problem 1.

Problem 2. Let $R(s) \in \mathbb{F}(s)^{m \times n}$ be a rational matrix. Find necessary and sufficient conditions for the existence of a rational matrix $\widetilde{W}(s) \in \mathbb{F}(s)^{z \times n}$ such that $\begin{bmatrix} R(s) \\ \widetilde{W}(s) \end{bmatrix}$ has prescribed complete structural data.

We are also interested in solving row (column) completion problems when we prescribe part of the complete structural data, i.e., when one or some of the four types of invariants which form the complete structural data are prescribed.

Problem 3. Let $R(s) \in \mathbb{F}(s)^{m \times n}$ ($P(s) \in \mathbb{F}[s]^{m \times n}$). Find necessary and sufficient conditions for the existence of a matrix $\widetilde{W}(s) \in \mathbb{F}(s)^{z \times n}$ ($W(s) \in \mathbb{F}[s]^{z \times n}$) such that $\begin{bmatrix} R(s) \\ \widetilde{W}(s) \end{bmatrix}$ ($\begin{bmatrix} P(s) \\ W(s) \end{bmatrix}$) has part of the structural data prescribed.

A solution to Problem 3 for polynomial matrices when only the finite structure is prescribed follows from a well-known result: the characterization of the invariant factors of a polynomial matrix with a prescribed submatrix (see the next Theorem 2.5).

Theorem 2.5 ([6, Chapter 7], [15], [17]). *Let $P(s) \in \mathbb{F}[s]^{m \times n}$ and $Q(s) \in \mathbb{F}[s]^{(m+z) \times (n+q)}$ be polynomial matrices of $\text{rank}(P(s)) = r$ and $\text{rank}(Q(s)) = \bar{r}$, and let $\alpha_1(s) \mid \cdots \mid \alpha_r(s)$*

and $\beta_1(s) \mid \cdots \mid \beta_{\bar{r}}(s)$ be the invariant factors of $P(s)$ and $Q(s)$, respectively. There exist matrices $X(s) \in \mathbb{F}[s]^{m \times q}$, $Y(s) \in \mathbb{F}[s]^{z \times q}$, $W(s) \in \mathbb{F}[s]^{z \times n}$ such that $\begin{bmatrix} P(s) & X(s) \\ W(s) & Y(s) \end{bmatrix}$ is unimodularly equivalent to $Q(s)$ if and only if

$$\beta_i(s) \mid \alpha_i(s) \mid \beta_{i+z+q}(s), \quad 1 \leq i \leq r.$$

For the rational case and when only the finite structure is prescribed, the following result gives a solution to Problem 3.

Theorem 2.6 ([5, Theorem 1]). Let $R(s) \in \mathbb{F}(s)^{m \times n}$, $G(s) \in \mathbb{F}(s)^{(m+z) \times (n+q)}$ be rational matrices, $\text{rank}(R(s)) = r$, $\text{rank}(G(s)) = \bar{r}$. Let $\frac{\eta_1(s)}{\varphi_1(s)}, \dots, \frac{\eta_r(s)}{\varphi_r(s)}$ and $\frac{\epsilon_1(s)}{\psi_1(s)}, \dots, \frac{\epsilon_{\bar{r}}(s)}{\psi_{\bar{r}}(s)}$ be the invariant rational functions of $R(s)$ and $G(s)$, respectively. There exist matrices $X(s) \in \mathbb{F}(s)^{m \times q}$, $Y(s) \in \mathbb{F}(s)^{z \times q}$, $W(s) \in \mathbb{F}(s)^{z \times n}$ such that $\begin{bmatrix} R(s) & X(s) \\ W(s) & Y(s) \end{bmatrix}$ is unimodularly equivalent to $G(s)$ if and only if

$$\epsilon_i(s) \mid \eta_i(s) \mid \epsilon_{i+z+q}(s), \quad \psi_{i+z+q}(s) \mid \varphi_i(s) \mid \psi_i(s), \quad 1 \leq i \leq r.$$

3. Row (column) completion with prescribed complete structural data

The aim of this section is to present a solution to Problems 1 and 2.

Given a polynomial matrix $P(s)$, the *grade* of $P(s)$ is an integer which is at least as large as $\deg(P(s))$ (see [8]). We denote it by $\text{grade}(P(s))$.

Definition 3.1. Let $P(s) = P_g s^g + P_{g-1} s^{g-1} + \cdots + P_1 s + P_0 \in \mathbb{F}[s]^{m \times n}$ be a polynomial matrix of grade $g \geq 1$. The *first Frobenius companion form of $P(s)$ with respect to g* is the $(m + (g-1)n) \times gn$ pencil $C_{g,P}(s) = sX_1 + Y_1$ with

$$X_1 = \begin{bmatrix} P_g & & & \\ & I_n & & \\ & & \ddots & \\ & & & I_n \end{bmatrix} \quad \text{and} \quad Y_1 = \begin{bmatrix} P_{g-1} & P_{g-2} & \cdots & P_0 \\ -I_n & 0 & \cdots & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & -I_n & 0 \end{bmatrix}.$$

When $g = \deg(P(s))$ we omit “with respect to g ” and $C_{g,P}(s)$ is denoted by $C_P(s)$. Notice that when $g = 1$, $C_{g,P}(s) = P(s)$. The following lemma is a consequence of [8, Theorems 5.3 and 4.1].

Lemma 3.2. Let $P(s) \in \mathbb{F}[s]^{m \times n}$ be a polynomial matrix of $\text{grade}(P(s)) = g \geq 1$, and let $C_{g,P}(s)$ be its first Frobenius companion form with respect to g . Then,

1. If $\alpha_1(s), \dots, \alpha_r(s)$ are the invariant factors of $P(s)$, then the invariant factors of $C_{g,P}(s)$ are $1, \overset{(g-1)n}{\dots}, 1, \alpha_1(s), \dots, \alpha_r(s)$.

2. If $p_1, \dots, p_r$ are the invariant orders at ∞ of $P(s)$, then $-1, (g-1)^n, -1, g-1 + p_1, \dots, g-1 + p_r$ are the invariant orders at ∞ of $C_{g,P}(s)$.
3. If $c_1 \geq \dots \geq c_{n-r}$ are the column minimal indices of $P(s)$, then $c_1 + g - 1 \geq \dots \geq c_{n-r} + g - 1$ are the column minimal indices of $C_{g,P}(s)$.
4. If $u_1 \geq \dots \geq u_{m-r}$ are the row minimal indices of $P(s)$, then $u_1 \geq \dots \geq u_{m-r}$ are also the row minimal indices of $C_{g,P}(s)$.

As a consequence of Lemma 3.2 we obtain the next corollary.

Corollary 3.3. Let $P(s), \bar{P}(s) \in \mathbb{F}[s]^{m \times n}$ be polynomial matrices such that $\text{grade}(P(s)) = \text{grade}(\bar{P}(s)) = g \geq 1$, and let $C_{g,P}(s), C_{g,\bar{P}}(s)$ be their respective first Frobenius companion forms with respect to g . Then, $P(s) \approx \bar{P}(s)$ if and only if $C_{g,P}(s) \stackrel{s.e.}{\sim} C_{g,\bar{P}}(s)$.

In Theorem 4.3 of [10] (see Theorem 3.5 below) a solution of the row completion problem for matrix pencils is given. We state the result for non constant pencils. It involves the definition of the generalized majorization (see Definition 3.4 below).

Let $\mathbf{c} = (c_1, \dots, c_x)$ and $\mathbf{a} = (a_1, \dots, a_x)$ be two sequences of integers. It is said that $\mathbf{c}$ is *majorized* by $\mathbf{a}$ (denoted by $\mathbf{c} \prec \mathbf{a}$) if $\sum_{i=1}^k c_i \leq \sum_{i=1}^k a_i$ for $1 \leq k \leq x-1$ and $\sum_{i=1}^x c_i = \sum_{i=1}^x a_i$ (this is an extension to sequences of integers of the definition of majorization given for partitions in [13]).

Definition 3.4. [9, Definition 2] Let $\mathbf{d} = (d_1, \dots, d_{q-x})$, $\mathbf{a} = (a_1, \dots, a_x)$ and $\mathbf{c} = (c_1, \dots, c_q)$ be sequences of integers. We say that $\mathbf{c}$ is majorized by $\mathbf{d}$ and $\mathbf{a}$ ($\mathbf{c} \prec' (\mathbf{d}, \mathbf{a})$) if

$$d_i \geq c_{i+x}, \quad 1 \leq i \leq q-x, \quad (2)$$

$$\sum_{i=1}^{h_j} c_i - \sum_{i=1}^{h_j-j} d_i \leq \sum_{i=1}^j a_i, \quad 1 \leq j \leq x, \quad (3)$$

where $h_j = \min\{i : d_{i-j+1} < c_i\}$, $1 \leq j \leq x$ ($d_{q-x+1} = -\infty$),

$$\sum_{i=1}^q c_i = \sum_{i=1}^{q-x} d_i + \sum_{i=1}^x a_i. \quad (4)$$

In the case that $x = 0$, condition (3) disappears, and conditions (2) and (4) are equivalent to $\mathbf{c} = \mathbf{d}$. On the other hand, if $q = x$ then $\mathbf{c} \prec' (\mathbf{d}, \mathbf{a})$ is equivalent to $\mathbf{c} \prec \mathbf{a}$.

Theorem 3.5 ([10, Theorem 4.3]). (Prescription of the complete structural data for non constant matrix pencils) Let $C(s) \in \mathbb{F}[s]^{(\bar{r}+p) \times (\bar{r}+q)}$ be a matrix pencil, $\deg(C(s)) = 1$, $\text{rank}(C(s)) = \bar{r}$. Let $\bar{\phi}_1(s, t) \mid \dots \mid \bar{\phi}_{\bar{r}}(s, t)$ be its homogeneous invariant factors, $\bar{\mathbf{c}} = (\bar{c}_1, \dots, \bar{c}_q)$ its column minimal indices, and $\bar{\mathbf{u}} = (\bar{u}_1, \dots, \bar{u}_p)$ its row minimal indices, where $\bar{u}_1 \geq \dots \geq \bar{u}_\theta > \bar{u}_{\theta+1} = \dots = \bar{u}_p = 0$. Let x and y be non negative integers. Let

$D(s) \in \mathbb{F}[s]^{(\bar{r}+p+x+y) \times (\bar{r}+q)}$ be a matrix pencil, $\text{rank}(D(s)) = \bar{r} + x$. Let $\bar{\gamma}_1(s, t) \mid \cdots \mid \bar{\gamma}_{\bar{r}+x}(s, t)$ be its homogeneous invariant factors, $\bar{\mathbf{d}} = (\bar{d}_1, \dots, \bar{d}_{q-x})$ its column minimal indices, and $\bar{\mathbf{v}} = (\bar{v}_1, \dots, \bar{v}_{p+y})$ its row minimal indices, where $\bar{v}_1 \geq \cdots \geq \bar{v}_{\bar{\theta}} > \bar{v}_{\bar{\theta}+1} = \cdots = \bar{v}_{p+y} = 0$. There exists a pencil $A(s)$ such that $\begin{bmatrix} C(s) \\ A(s) \end{bmatrix} \stackrel{s.e.}{\sim} D(s)$ if and only if

$$\bar{\gamma}_i(s, t) \mid \bar{\phi}_i(s, t) \mid \bar{\gamma}_{i+x+y}(s, t), \quad 1 \leq i \leq \bar{r}, \quad (5)$$

$$\bar{\theta} \geq \theta, \quad (6)$$

$$\bar{\mathbf{c}} \prec' (\bar{\mathbf{d}}, \bar{\mathbf{a}}), \quad (7)$$

$$\bar{\mathbf{v}} \prec' (\bar{\mathbf{u}}, \bar{\mathbf{b}}), \quad (8)$$

$$\sum_{i=1}^{\bar{r}+x} \deg(\text{lcm}(\bar{\phi}_{i-x}, \bar{\gamma}_i)) \leq \sum_{i=1}^{p+y} \bar{v}_i - \sum_{i=1}^p \bar{u}_i + \sum_{i=1}^{\bar{r}+x} \deg(\bar{\gamma}_i), \quad (9)$$

where $\bar{\mathbf{a}} = (\bar{a}_1, \dots, \bar{a}_x)$ and $\bar{\mathbf{b}} = (\bar{b}_1, \dots, \bar{b}_y)$ are defined as

$$\begin{aligned} \sum_{i=1}^j \bar{a}_i &= \sum_{i=1}^{p+y} \bar{v}_i - \sum_{i=1}^p \bar{u}_i + \sum_{i=1}^{\bar{r}+x} \deg(\bar{\gamma}_i) \\ &\quad - \sum_{i=1}^{\bar{r}+x-j} \deg(\text{lcm}(\bar{\phi}_{i-x+j}, \bar{\gamma}_i)) - j, \quad 1 \leq j \leq x, \\ \sum_{i=1}^j \bar{b}_i &= \sum_{i=1}^{p+y} \bar{v}_i - \sum_{i=1}^p \bar{u}_i + \sum_{i=1}^{\bar{r}+x} \deg(\bar{\gamma}_i) \\ &\quad - \sum_{i=1}^{\bar{r}+x} \deg(\text{lcm}(\bar{\phi}_{i-x-j}, \bar{\gamma}_i)), \quad 1 \leq j \leq y. \end{aligned}$$

Remark 3.6.

1. Let $\bar{r}, x$ and y be non negative integers. Given two polynomial chains $\bar{\phi}_1(s, t) \mid \cdots \mid \bar{\phi}_{\bar{r}}(s, t)$ and $\bar{\gamma}_1(s, t) \mid \cdots \mid \bar{\gamma}_{\bar{r}+x}(s, t)$, by [9, Lemmas 1 and 2] we can see that for $1 \leq j \leq x-1$,

$$\begin{aligned} &\sum_{i=1}^{\bar{r}+x-j+1} \deg(\text{lcm}(\bar{\phi}_{i-x+j-1}, \bar{\gamma}_i)) - \sum_{i=1}^{\bar{r}+x-j} \deg(\text{lcm}(\bar{\phi}_{i-x+j}, \bar{\gamma}_i)) \\ &\geq \sum_{i=1}^{\bar{r}+x-j} \deg(\text{lcm}(\bar{\phi}_{i-x+j}, \bar{\gamma}_i)) - \sum_{i=1}^{\bar{r}+x-j-1} \deg(\text{lcm}(\bar{\phi}_{i-x+j+1}, \bar{\gamma}_i)), \end{aligned}$$

and for $1 \leq j \leq y-1$,

$$\begin{aligned} &\sum_{i=1}^{\bar{r}+x} \deg(\text{lcm}(\bar{\phi}_{i-x-j+1}, \bar{\gamma}_i)) - \sum_{i=1}^{\bar{r}+x} \deg(\text{lcm}(\bar{\phi}_{i-x-j}, \bar{\gamma}_i)) \\ &\geq \sum_{i=1}^{\bar{r}+x} \deg(\text{lcm}(\bar{\phi}_{i-x-j}, \bar{\gamma}_i)) - \sum_{i=1}^{\bar{r}+x} \deg(\text{lcm}(\bar{\phi}_{i-x-j-1}, \bar{\gamma}_i)). \end{aligned}$$

As a consequence, from (9), in Theorem 3.5 we obtain that $\bar{a}_1 \geq \cdots \geq \bar{a}_x$ and $\bar{b}_1 \geq \cdots \geq \bar{b}_y \geq 0$.

Along the paper, finite sequences of integers similar to $\bar{a}_1, \dots, \bar{a}_x$ or $\bar{b}_1, \dots, \bar{b}_y$, will be introduced. They will analogously be decreasing. We will omit the explanation.

2. In Theorem 3.5, let $\bar{\alpha}_1(s), \dots, \bar{\alpha}_{\bar{r}}(s)$ and $\bar{p}_1, \dots, \bar{p}_{\bar{r}}$ be the invariant factors and the invariant orders at ∞ of $C(s)$, respectively, and let $\bar{\beta}_1(s), \dots, \bar{\beta}_{\bar{r}+x}(s)$ and $\bar{q}_1, \dots, \bar{q}_{\bar{r}+x}$ be the invariant factors and the invariant orders at ∞ of $D(s)$, respectively. Then,

$$\begin{aligned}\bar{\phi}_i(s, t) &= t^{\bar{p}_i+1} t^{\deg(\bar{\alpha}_i)} \bar{\alpha}_i\left(\frac{s}{t}\right), \quad 1 \leq i \leq \bar{r}, \\ \bar{\gamma}_i(s, t) &= t^{\bar{q}_i+1} t^{\deg(\bar{\beta}_i)} \bar{\beta}_i\left(\frac{s}{t}\right), \quad 1 \leq i \leq \bar{r} + x.\end{aligned}$$

Hence, (5) is equivalent to

$$\bar{\beta}_i(s) \mid \bar{\alpha}_i(s) \mid \bar{\beta}_{i+x+y}(s), \quad 1 \leq i \leq \bar{r}, \quad (10)$$

$$\bar{q}_i \leq \bar{p}_i \leq \bar{q}_{i+x+y}, \quad 1 \leq i \leq \bar{r}, \quad (11)$$

and (9) is equivalent to

$$\begin{aligned}& \sum_{i=1}^{\bar{r}} \deg(\text{lcm}(\bar{\alpha}_i, \bar{\beta}_{i+x})) + \sum_{i=1}^{\bar{r}} \max\{\bar{p}_i, \bar{q}_{i+x}\} \\ & \leq \sum_{i=1}^{p+y} \bar{v}_i - \sum_{i=1}^p \bar{u}_i + \sum_{i=1}^{\bar{r}} \deg(\bar{\beta}_{i+x}) + \sum_{i=1}^{\bar{r}} \bar{q}_{i+x}.\end{aligned} \quad (12)$$

Moreover,

$$\begin{aligned}\sum_{i=1}^j \bar{a}_i &= \sum_{i=1}^{p+y} \bar{v}_i - \sum_{i=1}^p \bar{u}_i + \sum_{i=1}^{\bar{r}+j} \deg(\bar{\beta}_{i+x-j}) + \sum_{i=1}^{\bar{r}+j} \bar{q}_{i+x-j} \\ &\quad - \sum_{i=1}^{\bar{r}} \deg(\text{lcm}(\bar{\alpha}_i, \bar{\beta}_{i+x-j})) - \sum_{i=1}^{\bar{r}} \max\{\bar{p}_i, \bar{q}_{i+x-j}\}, \\ &\quad 1 \leq j \leq x, \\ \sum_{i=1}^j \bar{b}_i &= \sum_{i=1}^{p+y} \bar{v}_i - \sum_{i=1}^p \bar{u}_i + \sum_{i=1}^{\bar{r}-j} \deg(\bar{\beta}_{i+x+j}) + \sum_{i=1}^{\bar{r}-j} \bar{q}_{i+x+j} \\ &\quad - \sum_{i=1}^{\bar{r}-j} \deg(\text{lcm}(\bar{\alpha}_i, \bar{\beta}_{i+x+j})) - \sum_{i=1}^{\bar{r}-j} \max\{\bar{p}_i, \bar{q}_{i+x+j}\}, \\ &\quad 1 \leq j \leq y.\end{aligned}$$

Proposition 3.7. Let $P(s) \in \mathbb{F}[s]^{m \times n}$ and $Q(s) \in \mathbb{F}[s]^{(m+z) \times n}$ be such that $\deg(Q(s)) = g \geq \max\{\deg(P(s)), 1\}$. Let $C_{g,P}(s)$ be the first Frobenius companion form of $P(s)$ with respect to g and $C_Q(s)$ be the first Frobenius companion form of $Q(s)$. Then, there exists $W(s) \in \mathbb{F}[s]^{z \times n}$ such that $\begin{bmatrix} P(s) \\ W(s) \end{bmatrix} \approx Q(s)$ if and only if there exists a matrix pencil $A(s) \in \mathbb{F}[s]^{z \times gn}$ such that $\begin{bmatrix} C_{g,P}(s) \\ A(s) \end{bmatrix} \stackrel{s.e.}{\sim} C_Q(s)$.

Proof. The proof is completely analogous to that of [1, Proposition 4.1] exchanging degree and grade and applying Corollary 3.3. $\square$

Now, we can give a solution to Problem 1.

Theorem 3.8. (Prescription of the complete structural data for polynomial matrices) Let $P(s) \in \mathbb{F}[s]^{m \times n}$ be a polynomial matrix of $\text{rank}(P(s)) = r$. Let $\alpha_1(s) \mid \dots \mid \alpha_r(s)$ be

its invariant factors, $p_1, \dots, p_r$ its invariant orders at ∞ , $\mathbf{c} = (c_1, \dots, c_{n-r})$ its column minimal indices, and $\mathbf{u} = (u_1, \dots, u_{m-r})$ its row minimal, where $u_1 \geq \dots \geq u_\eta > u_{\eta+1} = \dots = u_{m-r} = 0$.

Let z, x be integers such that $0 \leq x \leq \min\{z, n-r\}$. Let $\beta_1(s) \mid \dots \mid \beta_{r+x}(s)$ be monic polynomials, $q_1 \leq \dots \leq q_{r+x}$ integers, and $\mathbf{d} = (d_1, \dots, d_{n-r-x})$ and $\mathbf{v} = (v_1, \dots, v_{m+z-r-x})$ two partitions, where $v_1 \geq \dots \geq v_{\bar{\eta}} > v_{\bar{\eta}+1} = \dots = v_{m+z-r-x} = 0$. There exists a polynomial matrix $W(s) \in \mathbb{F}[s]^{z \times n}$ such that $\text{rank} \left(\begin{bmatrix} P(s) \\ W(s) \end{bmatrix} \right) = r+x$ and $\begin{bmatrix} P(s) \\ W(s) \end{bmatrix}$ has $\beta_1(s), \dots, \beta_{r+x}(s)$ as invariant factors, $q_1, \dots, q_{r+x}$ as invariant orders at ∞ , $d_1, \dots, d_{n-r-x}$ as column minimal indices and $v_1, \dots, v_{m+z-r-x}$ as row minimal indices if and only if

$$\beta_i(s) \mid \alpha_i(s) \mid \beta_{i+z}(s), \quad 1 \leq i \leq r, \quad (13)$$

$$q_i \leq p_i \leq q_{i+z}, \quad 1 \leq i \leq r, \quad (14)$$

$$\bar{\eta} \geq \eta, \quad (15)$$

$$\mathbf{c} \prec' (\mathbf{d}, \mathbf{a}), \quad (16)$$

$$\mathbf{v} \prec' (\mathbf{u}, \mathbf{b}), \quad (17)$$

$$\begin{aligned} & \sum_{i=1}^r \deg(\text{lcm}(\alpha_i, \beta_{i+x})) + \sum_{i=1}^r \max\{p_i, q_{i+x}\} \\ & \leq \sum_{i=1}^{m+z-r-x} v_i - \sum_{i=1}^{m-r} u_i + \sum_{i=1}^r \deg(\beta_{i+x}) + \sum_{i=1}^r q_{i+x}, \end{aligned} \quad (18)$$

with equality when $x = 0$,

where $\mathbf{a} = (a_1, \dots, a_x)$ and $\mathbf{b} = (b_1, \dots, b_{z-x})$ are defined as

$$\begin{aligned} \sum_{i=1}^j a_i &= \sum_{i=1}^{m+z-r-x} v_i - \sum_{i=1}^{m-r} u_i + \sum_{i=1}^{r+j} \deg(\beta_{i+x-j}) + \sum_{i=1}^{r+j} q_{i+x-j} \\ &\quad - \sum_{i=1}^r \deg(\text{lcm}(\alpha_i, \beta_{i+x-j})) - \sum_{i=1}^r \max\{p_i, q_{i+x-j}\}, \end{aligned} \quad (19)$$

$$1 \leq j \leq x,$$

$$\begin{aligned} \sum_{i=1}^j b_i &= \sum_{i=1}^{m+z-r-x} v_i - \sum_{i=1}^{m-r} u_i + \sum_{i=1}^{r-j} \deg(\beta_{i+x+j}) + \sum_{i=1}^{r-j} q_{i+x+j} \\ &\quad - \sum_{i=1}^{r-j} \deg(\text{lcm}(\alpha_i, \beta_{i+x+j})) - \sum_{i=1}^{r-j} \max\{p_i, q_{i+x+j}\}, \end{aligned} \quad (20)$$

$$1 \leq j \leq z-x.$$

Proof. The proof is analogous to that of [1, Theorem 4.2]. Define $d = -p_1$ and $g = -q_1$. Then $\deg(P(s)) = d$.

When $g \geq d$, we can build $C_{g,P}(s)$, the first Frobenius companion form of $P(s)$ with respect to g . If $g \geq 1$, we will take

$$\bar{r} = (g-1)n + r, \quad y = z - x, \quad p = m - r = m + (g-1)n - \bar{r}, \quad q = n - r = gn - \bar{r}.$$

Assume that there exists a polynomial matrix $W(s) \in \mathbb{F}[s]^{z \times n}$ such that $Q(s) = \begin{bmatrix} P(s) \\ W(s) \end{bmatrix}$ has the prescribed invariants. Then, $\deg(Q(s)) = g \geq d$. If $g = 0$, then (13)–(18) trivially hold. If $g \geq 1$, let $C_{g,Q}(s) = C_Q(s)$ be the first Frobenius companion form of $Q(s)$. By Proposition 3.7, there exists a matrix pencil $A(s) \in \mathbb{F}[s]^{z \times gn}$ such that $\begin{bmatrix} C_{g,P}(s) \\ A(s) \end{bmatrix} \stackrel{s.e.}{\sim} C_Q(s)$.

Let $\bar{\alpha}_1(s), \dots, \bar{\alpha}_{\bar{r}}(s)$, $\bar{p}_1, \dots, \bar{p}_{\bar{r}}$, $\bar{\mathbf{c}} = (\bar{c}_1, \dots, \bar{c}_q)$ and $\bar{\mathbf{u}} = (\bar{u}_1, \dots, \bar{u}_p)$ be the invariant factors, invariant orders at ∞ , column minimal indices and row minimal indices of $C_{g,P}(s)$, where $\bar{u}_1 \geq \dots \geq \bar{u}_{\bar{\theta}} > \bar{u}_{\bar{\theta}+1} = \dots = \bar{u}_p = 0$ and let $\bar{\beta}_1(s), \dots, \bar{\beta}_{\bar{r}+x}(s)$, $\bar{q}_1, \dots, \bar{q}_{\bar{r}+x}$, $\bar{\mathbf{d}} = (\bar{d}_1, \dots, \bar{d}_{q-x})$ and $\bar{\mathbf{v}} = (\bar{v}_1, \dots, \bar{v}_{p+y})$ be the invariant factors, invariant orders at ∞ , column minimal indices and row minimal indices of $C_Q(s)$, where $\bar{v}_1 \geq \dots \geq \bar{v}_{\bar{\eta}} > \bar{v}_{\bar{\eta}+1} = \dots = \bar{v}_{p+y} = 0$. By Theorem 3.5 and Remark 3.6, conditions (6)–(8) and (10)–(12) hold. Applying Lemma 3.2, it is easy to see that (6)–(8) and (10)–(12) are equivalent to (13)–(18).

Assume now that (13)–(18) hold. Then, from (16), (18) if $x = 0$, (13) and (14), we get $\sum_{i=1}^{n-r} c_i - \sum_{i=1}^{n-r-x} d_i = \sum_{i=1}^x a_i = \sum_{i=1}^{m+z-r-x} v_i - \sum_{i=1}^{m-r} u_i + \sum_{i=1}^{r+x} \deg(\beta_i) + \sum_{i=1}^{r+x} q_i - \sum_{i=1}^r \deg(\alpha_i) - \sum_{i=1}^r p_i$. By Theorem 2.2 applied to $P(s)$, we obtain

$$0 = \sum_{i=1}^{n-r-x} d_i + \sum_{i=1}^{m+z-r-x} v_i + \sum_{i=1}^{r+x} \deg(\beta_i) + \sum_{i=1}^{r+x} q_i.$$

Applying again Theorem 2.2, we derive that there exists a polynomial matrix $Q(s) \in \mathbb{F}[s]^{(m+z) \times n}$, $\text{rank}(Q(s)) = r+x$, with $\beta_1(s), \dots, \beta_{r+x}(s)$ as invariant factors, $q_1, \dots, q_{r+x}$ as invariant orders at ∞ , and $d_1, \dots, d_{n-r-x}$ and $v_1, \dots, v_{m+z-r-x}$ as column and row minimal indices. Then, $\deg(Q(s)) = -q_1 = g$. From (14) we obtain $g = -q_1 \geq -p_1 = d$.

If $g = 0$, then choosing $W \in \mathbb{F}^{z \times n}$ such that $\text{rank} \begin{bmatrix} P \\ W \end{bmatrix} = r+x$, the matrix $\begin{bmatrix} P \\ W \end{bmatrix}$ has the prescribed invariants. If $g \geq 1$, let $C_Q(s)$ be the first Frobenius companion form of $Q(s)$ and let $\bar{\beta}_1(s), \dots, \bar{\beta}_{\bar{r}+x}(s)$, $\bar{q}_1, \dots, \bar{q}_{\bar{r}+x}$, $\bar{\mathbf{d}} = (\bar{d}_1, \dots, \bar{d}_{q-x})$ and $\bar{\mathbf{v}} = (\bar{v}_1, \dots, \bar{v}_{p+y})$ be the invariant factors, invariant orders at ∞ , column minimal indices and row minimal indices of $C_Q(s)$, respectively, where $\bar{v}_1 \geq \dots \geq \bar{v}_{\bar{\eta}} > \bar{v}_{\bar{\eta}+1} = \dots = \bar{v}_{p+y} = 0$. As in the proof of the necessity, (13)–(18) are equivalent to (6)–(8) and (10)–(12). The result follows from Theorem 3.5, Remark 3.6 and Proposition 3.7. $\square$

Remark 3.9. Under the conditions of Theorem 3.8, let $e_i = p_i - p_1$, $1 \leq i \leq r$, and $f_i = q_i - q_1$, $1 \leq i \leq r+x$. Then, conditions (14) and (18) become

$$f_i \leq e_i + p_1 - q_1 \leq f_{i+z}, \quad 1 \leq i \leq r,$$

and

$$\begin{aligned}
& \sum_{i=1}^r \deg(\text{lcm}(\alpha_i, \beta_{i+x})) + \sum_{i=1}^r \max\{e_i + p_1 - q_1, f_{i+x}\} \\
& \leq \sum_{i=1}^{m+z-r-x} v_i - \sum_{i=1}^{m-r} u_i + \sum_{i=1}^r \deg(\beta_{i+x}) + \sum_{i=1}^r f_{i+x}, \\
& \text{with equality when } x = 0,
\end{aligned}$$

respectively, and $\mathbf{a} = (a_1, \dots, a_x)$ and $\mathbf{b} = (b_1, \dots, b_{z-x})$ can be rewritten as

$$\begin{aligned}
\sum_{i=1}^j a_i &= \sum_{i=1}^{m+z-r-x} v_i - \sum_{i=1}^{m-r} u_i + \sum_{i=1}^{r+j} \deg(\beta_{i+x-j}) + \sum_{i=1}^{r+j} f_{i+x-j} \\
&\quad - \sum_{i=1}^r \deg(\text{lcm}(\alpha_i, \beta_{i+x-j})) - \sum_{i=1}^r \max\{e_i + p_1 - q_1, f_{i+x-j}\} \\
&\quad + jq_1, \quad 1 \leq j \leq x, \\
\sum_{i=1}^j b_i &= \sum_{i=1}^{m+z-r-x} v_i - \sum_{i=1}^{m-r} u_i + \sum_{i=1}^{r-j} \deg(\beta_{i+x+j}) + \sum_{i=1}^{r-j} f_{i+x+j} \\
&\quad - \sum_{i=1}^{r-j} \deg(\text{lcm}(\alpha_i, \beta_{i+x+j})) - \sum_{i=1}^{r-j} \max\{e_i + p_1 - q_1, f_{i+x+j}\}, \\
&\quad 1 \leq j \leq z-x.
\end{aligned}$$

Therefore, when $q_1 = p_1 = -\deg(P(s))$, from Theorem 3.8 we recover [1, Theorem 4.2].

In order to solve Problem 2 we will use the following technical lemma.

Lemma 3.10. *Let $\varphi(s), \eta(s), \psi(s), \epsilon(s), \pi(s) \in \mathbb{F}[s]$ such that $\varphi(s) \mid \pi(s)$, $\psi(s) \mid \pi(s)$, $\gcd(\varphi, \eta) = 1$ and $\gcd(\psi, \epsilon) = 1$. Then,*

$$\text{lcm}\left(\frac{\pi}{\varphi}\eta, \frac{\pi}{\psi}\epsilon\right) = \frac{\pi(s)}{\gcd(\varphi, \psi)} \text{lcm}(\eta, \epsilon).$$

Proof. Note that $h_1(s) = \frac{\varphi(s)}{\gcd(\varphi, \psi)} \frac{\text{lcm}(\eta, \epsilon)}{\eta(s)}$ and $h_2(s) = \frac{\psi(s)}{\gcd(\varphi, \psi)} \frac{\text{lcm}(\eta, \epsilon)}{\epsilon(s)}$ are polynomials. Then,

$$\frac{\pi(s)}{\varphi(s)} \eta(s) h_1(s) = \frac{\pi(s)}{\gcd(\varphi, \psi)} \text{lcm}(\eta, \epsilon), \quad \frac{\pi(s)}{\psi(s)} \epsilon(s) h_2(s) = \frac{\pi(s)}{\gcd(\varphi, \psi)} \text{lcm}(\eta, \epsilon),$$

and both $\frac{\pi(s)}{\varphi(s)} \eta(s)$ and $\frac{\pi(s)}{\psi(s)} \epsilon(s)$ are divisors of $\frac{\pi(s)}{\gcd(\varphi, \psi)} \text{lcm}(\eta, \epsilon)$. Therefore, $\text{lcm}\left(\frac{\pi}{\varphi}\eta, \frac{\pi}{\psi}\epsilon\right)$ divides $\frac{\pi(s)}{\gcd(\varphi, \psi)} \text{lcm}(\eta, \epsilon)$, i.e., there exists $q(s) \in \mathbb{F}[s]$ such that

$$\text{lcm}\left(\frac{\pi}{\varphi}\eta, \frac{\pi}{\psi}\epsilon\right) = \frac{\pi(s)}{\gcd(\varphi, \psi)} \text{lcm}(\eta, \epsilon) \frac{1}{q(s)}.$$

Let $\ell_1(s), \ell_2(s) \in \mathbb{F}[s]$ such that

$$\begin{aligned}
\frac{\pi(s)}{\varphi(s)} \eta(s) \ell_1(s) &= \frac{\pi(s)}{\gcd(\varphi, \psi)} \text{lcm}(\eta, \epsilon) \frac{1}{q(s)}, \\
\frac{\pi(s)}{\psi(s)} \epsilon(s) \ell_2(s) &= \frac{\pi(s)}{\gcd(\varphi, \psi)} \text{lcm}(\eta, \epsilon) \frac{1}{q(s)}.
\end{aligned}$$

Then, $q(s)$ is a divisor of $h_1(s)$ and of $h_2(s)$.

Let $x_1(s) = \gcd(\eta, \epsilon)$. Then $\eta(s) = x_1(s)x_2(s)$ and $\epsilon(s) = x_1(s)x_3(s)$ with $\gcd(x_2, x_3) = 1$. Thus, $\text{lcm}(\eta, \epsilon) = x_1(s)x_2(s)x_3(s)$, $\frac{\text{lcm}(\eta, \epsilon)}{\eta(s)} = x_3(s)$, and $\frac{\text{lcm}(\eta, \epsilon)}{\epsilon(s)} = x_2(s)$.

Note that $h_1(s) = \frac{\varphi(s)}{\gcd(\varphi, \psi)}x_3(s)$ and $h_2(s) = \frac{\psi(s)}{\gcd(\varphi, \psi)}x_2(s)$.

Let $q(s) = q_1(s)q_2(s)$ with $q_1(s), q_2(s) \in \mathbb{F}[s]$ such that $q_1(s)$ divides $\frac{\varphi(s)}{\gcd(\varphi, \psi)}$ and $q_2(s)$ divides $x_3(s)$. As $\gcd(\varphi, \eta) = 1$, $\gcd(q_1, \eta) = 1$ and $\gcd(q_1, x_2) = 1$. Thus, $q_1(s)$ divides $\frac{\psi(s)}{\gcd(\varphi, \psi)}$. Analogously, since $\gcd(x_2, x_3) = 1$, $\gcd(q_2, x_2) = 1$ and $q_2(s)$ divides $\frac{\psi(s)}{\gcd(\varphi, \psi)}$. It follows from $\gcd(\psi, \epsilon) = 1$ that $\gcd(q_2, \epsilon) = 1$ and $\gcd(q_2, x_3) = 1$. Thus, $q_2(s) = 1$, and $q(s) = q_1(s)$ is a divisor of both $\frac{\varphi(s)}{\gcd(\varphi, \psi)}$ and $\frac{\psi(s)}{\gcd(\varphi, \psi)}$. Hence, $q(s) = 1$ and the result follows. $\square$

In the sequel we use the following notation: given $\varphi(s), \eta(s), \psi(s), \epsilon(s) \in \mathbb{F}[s]$ such that $\gcd(\varphi, \eta) = 1$ and $\gcd(\psi, \epsilon) = 1$, and p, q integers, we denote

$$\Delta\left(\frac{\eta}{\varphi}, \frac{\epsilon}{\psi}, p, q\right) = \deg(\text{lcm}(\eta, \epsilon)) - \deg(\gcd(\varphi, \psi)) + \max\{p, q\},$$

$$\Delta\left(\frac{\eta}{\varphi}, \frac{\epsilon}{\psi}\right) = \deg(\text{lcm}(\eta, \epsilon)) - \deg(\gcd(\varphi, \psi)),$$

$$\Delta\left(\frac{\eta}{\varphi}, p\right) = \deg(\eta) - \deg(\varphi) + p,$$

$$\Delta\left(\frac{\eta}{\varphi}\right) = \deg(\eta) - \deg(\varphi).$$

Theorem 3.11. (Prescription of the complete structural data for rational matrices) *Let $R(s) \in \mathbb{F}(s)^{m \times n}$ be a rational matrix, $\text{rank}(R(s)) = r$. Let $\frac{\eta_1(s)}{\varphi_1(s)}, \dots, \frac{\eta_r(s)}{\varphi_r(s)}$ be its invariant rational functions, $\tilde{p}_1, \dots, \tilde{p}_r$ its invariant orders at ∞ , $\mathbf{c} = (c_1, \dots, c_{n-r})$ its column minimal indices, and $\mathbf{u} = (u_1, \dots, u_{m-r})$ its row minimal indices, where $u_1 \geq \dots \geq u_\eta > u_{\eta+1} = \dots = u_{m-r} = 0$.*

Let z, x be integers such that $0 \leq x \leq \min\{z, n-r\}$ and let $\epsilon_1(s) \mid \dots \mid \epsilon_{r+x}(s)$ and $\psi_{r+x}(s) \mid \dots \mid \psi_1(s)$ be monic polynomials such that $\frac{\epsilon_i(s)}{\psi_i(s)}$ are irreducible rational functions, $1 \leq i \leq r+x$. Let $\tilde{q}_1 \leq \dots \leq \tilde{q}_{r+x}$ be integers and $\mathbf{d} = (d_1, \dots, d_{n-r-x})$ and $\mathbf{v} = (v_1, \dots, v_{m+z-r-x})$ be two partitions, where $v_1 \geq \dots \geq v_{\tilde{\eta}} > v_{\tilde{\eta}+1} = \dots = v_{m+z-r-x} = 0$. There exists a rational matrix $\tilde{W}(s) \in \mathbb{F}(s)^{z \times n}$ such that $\text{rank}\left(\begin{bmatrix} R(s) \\ \tilde{W}(s) \end{bmatrix}\right) = r+x$ and $\begin{bmatrix} R(s) \\ \tilde{W}(s) \end{bmatrix}$ has $\frac{\epsilon_1(s)}{\psi_1(s)}, \dots, \frac{\epsilon_{r+x}(s)}{\psi_{r+x}(s)}$ as invariant rational functions, $\tilde{q}_1, \dots, \tilde{q}_{r+x}$ as invariant orders at ∞ , $d_1, \dots, d_{n-r-x}$ as column minimal indices and $v_1, \dots, v_{m+z-r-x}$ as row minimal indices if and only if (15),

$$\epsilon_i(s) \mid \eta_i(s) \mid \epsilon_{i+z}(s), \quad 1 \leq i \leq r, \quad (21)$$

$$\psi_{i+z}(s) \mid \varphi_i(s) \mid \psi_i(s), \quad 1 \leq i \leq r, \quad (22)$$

$$\tilde{q}_i \leq \tilde{p}_i \leq \tilde{q}_{i+z}, \quad 1 \leq i \leq r, \quad (23)$$

$$\mathbf{c} \prec' (\mathbf{d}, \tilde{\mathbf{a}}), \quad (24)$$

$$\mathbf{v} \prec' (\mathbf{u}, \tilde{\mathbf{b}}), \quad (25)$$

$$\sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i}, \frac{\epsilon_{i+x}}{\psi_{i+x}}, \tilde{p}_i, \tilde{q}_{i+x} \right) - \sum_{i=1}^r \Delta \left(\frac{\epsilon_{i+x}}{\psi_{i+x}}, \tilde{q}_{i+x} \right) \leq \sum_{i=1}^{m+z-r-x} v_i - \sum_{i=1}^{m-r} u_i, \quad (26)$$

with equality when $x = 0$,

where $\tilde{\mathbf{a}} = (\tilde{a}_1, \dots, \tilde{a}_x)$ and $\tilde{\mathbf{b}} = (\tilde{b}_1, \dots, \tilde{b}_{z-x})$ are defined as

$$\begin{aligned} \sum_{i=1}^j \tilde{a}_i &= \sum_{i=1}^{m+z-r-x} v_i - \sum_{i=1}^{m-r} u_i + \sum_{i=1}^{r+j} \Delta \left(\frac{\epsilon_{i+x-j}}{\psi_{i+x-j}}, \tilde{q}_{i+x-j} \right) \\ &\quad - \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i}, \frac{\epsilon_{i+x-j}}{\psi_{i+x-j}}, \tilde{p}_i, \tilde{q}_{i+x-j} \right), \quad 1 \leq j \leq x, \end{aligned} \quad (27)$$

$$\begin{aligned} \sum_{i=1}^j \tilde{b}_i &= \sum_{i=1}^{m+z-r-x} v_i - \sum_{i=1}^{m-r} u_i + \sum_{i=1}^{r-j} \Delta \left(\frac{\epsilon_{i+x+j}}{\psi_{i+x+j}}, \tilde{q}_{i+x+j} \right) \\ &\quad - \sum_{i=1}^{r-j} \Delta \left(\frac{\eta_i}{\varphi_i}, \frac{\epsilon_{i+x+j}}{\psi_{i+x+j}}, \tilde{p}_i, \tilde{q}_{i+x+j} \right), \quad 1 \leq j \leq z-x, \end{aligned} \quad (28)$$

Proof. We start with a remark assuming that $\varphi_1(s) \mid \psi_1(s)$. Define $d = \deg(\psi_1) - \tilde{p}_1$, $g = \deg(\psi_1) - \tilde{q}_1$,

$$\begin{aligned} \alpha_i(s) &= \psi_1(s) \frac{\eta_i(s)}{\varphi_i(s)}, \quad p_i = \tilde{p}_i - \deg(\psi_1), \quad 1 \leq i \leq r, \\ \beta_i(s) &= \psi_1(s) \frac{\epsilon_i(s)}{\psi_i(s)}, \quad q_i = \tilde{q}_i - \deg(\psi_1), \quad 1 \leq i \leq r+x. \end{aligned}$$

Then (13) is equivalent to

$$\epsilon_i(s)\varphi_i(s) \mid \eta_i(s)\psi_i(s), \quad \eta_i(s)\psi_{i+z}(s) \mid \epsilon_{i+z}(s)\varphi_i(s), \quad 1 \leq i \leq r.$$

As $\gcd(\epsilon_i, \psi_i) = 1$, $1 \leq i \leq r+x$ and $\gcd(\eta_i, \varphi_i) = 1$, $1 \leq i \leq r$, we derive that (13) is equivalent to (21) and (22). It is clear that (14) is equivalent to (23). Define also $\mathbf{a} = (a_1, \dots, a_x)$ and $\mathbf{b} = (b_1, \dots, b_{z-x})$ as in (19) and (20). Then, by Lemma 3.10, $\mathbf{a} = \tilde{\mathbf{a}}$ and $\mathbf{b} = \tilde{\mathbf{b}}$, hence (16) and (17) are equivalent to (24) and (25), respectively. Analogously, condition (18) is equivalent to (26).

Assume that there exists a rational matrix $\widetilde{W}(s) \in \mathbb{F}(s)^{z \times n}$ such that, if $G(s) = \begin{bmatrix} R(s) \\ \widetilde{W}(s) \end{bmatrix}$, then $G(s)$ has the prescribed structural data. Recall that $\psi_1(s)$ and $\varphi_1(s)$ are the monic least common denominator of the entries of $G(s)$ and $R(s)$, respectively. Thus, the matrix $\psi_1(s)G(s) = \begin{bmatrix} \psi_1(s)R(s) \\ \psi_1(s)\widetilde{W}(s) \end{bmatrix}$ is polynomial and $\varphi_1(s) \mid \psi_1(s)$. Let

$$P(s) = \psi_1(s)R(s), \quad Q(s) = \psi_1(s)G(s).$$

By Lemma 2.3 we know that $\text{rank}(P(s)) = r$, $\alpha_1(s), \dots, \alpha_r(s)$ are the invariant factors, $p_1, \dots, p_r$ the invariant orders at ∞ , $c_1, \dots, c_{n-r}$ the column minimal indices and

$u_1, \dots, u_{m-r}$ the row minimal indices of $P(s)$, and $\text{rank}(Q(s)) = r+x$, $\beta_1(s), \dots, \beta_{r+x}(s)$ are the invariant factors, $q_1, \dots, q_{r+x}$ the invariant orders at ∞ , $d_1, \dots, d_{n-r-x}$ the column minimal indices and $v_1, \dots, v_{m+z-r-x}$ the row minimal indices of $Q(s)$. By Theorem 3.8, (13)–(18) hold, where $\mathbf{a}$ and $\mathbf{b}$ are defined in (19) and (20), respectively. Equivalently, (15) and (21)–(26) hold, where $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ are defined as in (27) and (28), respectively.

Conversely, assume that (15) and (21)–(26) are satisfied. Then $\varphi_1(s) \mid \psi_1(s)$ and (13)–(18) hold. Let $P(s) = \psi_1(s)R(s)$. By Lemma 2.3, $\text{rank}(P(s)) = r$, $\alpha_1(s), \dots, \alpha_r(s)$ are the invariant factors, $p_1, \dots, p_r$ the invariant orders at ∞ , $c_1, \dots, c_{n-r}$ the column minimal indices and $u_1, \dots, u_{m-r}$ the row minimal indices of $P(s)$.

From (13)–(18) by Theorem 3.8 there exists a polynomial matrix $W(s) \in \mathbb{F}[s]^{z \times n}$ such that $\text{rank} \left(\begin{bmatrix} P(s) \\ W(s) \end{bmatrix} \right) = r+x$ and $\begin{bmatrix} P(s) \\ W(s) \end{bmatrix}$ has $\beta_1(s), \dots, \beta_{r+x}(s)$ as invariant factors, $q_1, \dots, q_{r+x}$ as invariant orders at ∞ , $d_1, \dots, d_{n-r-x}$ as column minimal indices and $v_1, \dots, v_{m+z-r-x}$ as row minimal indices. Let $\tilde{W}(s) = \frac{1}{\psi_1(s)}W(s)$. Then $\tilde{W}(s) \in \mathbb{F}(s)^{z \times n}$ and $\begin{bmatrix} P(s) \\ W(s) \end{bmatrix} = \psi_1(s) \begin{bmatrix} R(s) \\ \tilde{W}(s) \end{bmatrix}$. By Lemma 2.3, $\text{rank} \left(\begin{bmatrix} R(s) \\ \tilde{W}(s) \end{bmatrix} \right) = r+x$ and $\begin{bmatrix} R(s) \\ \tilde{W}(s) \end{bmatrix}$ has $\frac{\epsilon_1(s)}{\psi_1(s)}, \dots, \frac{\epsilon_{r+x}(s)}{\psi_{r+x}(s)}$ as invariant rational functions, $\tilde{q}_1, \dots, \tilde{q}_{r+x}$ as invariant orders at ∞ , $d_1, \dots, d_{n-r-x}$ as column minimal indices and $v_1, \dots, v_{m+z-r-x}$ as row minimal indices. $\square$

Remark 3.12. By Theorem 2.4 (see also [1, Remark 4.3]), if (15) and (21)–(26) hold, then

$$\sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i}, \frac{\epsilon_{i+x}}{\psi_{i+x}}, \tilde{p}_i, \tilde{q}_{i+x} \right) \leq \sum_{i=1}^{n-r} c_i - \sum_{i=1}^{n-r-x} d_i + \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i}, \tilde{p}_i \right) - \sum_{i=1}^x \Delta \left(\frac{\epsilon_i}{\psi_i}, \tilde{q}_i \right),$$

with equality when $x = z$,

(29)

and (24) and (25) hold for $\tilde{\mathbf{a}} = (\tilde{a}_1, \dots, \tilde{a}_x)$ and $\tilde{\mathbf{b}} = (\tilde{b}_1, \dots, \tilde{b}_{z-x})$ defined as

$$\begin{aligned} \sum_{i=1}^j \tilde{a}_i &= \sum_{i=1}^{n-r} c_i - \sum_{i=1}^{n-r-x} d_i + \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i}, \tilde{p}_i \right) - \sum_{i=1}^{x-j} \Delta \left(\frac{\epsilon_i}{\psi_i}, \tilde{q}_i \right) \\ &\quad - \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i}, \frac{\epsilon_{i+x-j}}{\psi_{i+x-j}}, \tilde{p}_i, \tilde{q}_{i+x-j} \right), \quad 1 \leq j \leq x, \end{aligned}$$

(30)

$$\begin{aligned} \sum_{i=1}^j \tilde{b}_i &= \sum_{i=1}^{n-r} c_i - \sum_{i=1}^{n-r-x} d_i + \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i}, \tilde{p}_i \right) - \sum_{i=1}^{x+j} \Delta \left(\frac{\epsilon_i}{\psi_i}, \tilde{q}_i \right) \\ &\quad - \sum_{i=1}^{r-j} \Delta \left(\frac{\eta_i}{\varphi_i}, \frac{\epsilon_{i+x+j}}{\psi_{i+x+j}}, \tilde{p}_i, \tilde{q}_{i+x+j} \right), \quad 1 \leq j \leq z-x. \end{aligned}$$

(31)

Conversely, (15), (21)–(25) and (29) with $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ defined as in (30) and (31), respectively, imply (15) and (21)–(26) with $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{b}}$ defined as in (27) and (28), respectively.

4. Row (column) completion with part of the structural data prescribed

In this section we first solve Problem 3 when the complete structural data but the row (column) minimal indices are prescribed (see Subsection 4.1). Afterwards, in Subsection 4.2, we solve Problem 3 when the finite and/or infinite structures are prescribed.

Given a rational matrix $R(s) \in \mathbb{F}(s)^{m \times n}$ with $\frac{\eta_1(s)}{\varphi_1(s)}, \dots, \frac{\eta_r(s)}{\varphi_r(s)}$ as invariant rational functions, when $\varphi_1(s) = 1$ the matrix $R(s) \in \mathbb{F}[s]^{m \times n}$ is polynomial with invariant factors $\eta_1(s), \dots, \eta_r(s)$. When we prescribe the invariant rational functions (Theorems 4.1, 4.2, 4.3 and 4.9) we present the results for rational matrices, and the polynomial cases are derived from them.

4.1. Prescription of the finite and infinite structures and column or row minimal indices

We present two results related to Problem 3. In Theorem 4.1 we prescribe the finite and infinite structures and column minimal indices, and in Theorem 4.2 we replace the column minimal indices by the row minimal indices. The proofs are analogous to those of [1, Sections 4.2, 4.3].

Theorem 4.1. (Prescription of the finite and infinite structures, and the column minimal indices) *Let $R(s) \in \mathbb{F}(s)^{m \times n}$ be a rational matrix, $\text{rank}(R(s)) = r$. Let $\frac{\eta_1(s)}{\varphi_1(s)}, \dots, \frac{\eta_r(s)}{\varphi_r(s)}$ be its invariant rational functions, $\tilde{p}_1, \dots, \tilde{p}_r$ its invariant orders at ∞ and $\mathbf{c} = (c_1, \dots, c_{n-r})$ its column minimal indices.*

Let z, x be integers such that $0 \leq x \leq \min\{z, n - r\}$ and let $\epsilon_1(s) \mid \dots \mid \epsilon_{r+x}(s)$ and $\psi_{r+x}(s) \mid \dots \mid \psi_1(s)$ be monic polynomials such that $\frac{\epsilon_i(s)}{\psi_i(s)}$ are irreducible rational functions, $1 \leq i \leq r + x$. Let $\tilde{q}_1 \leq \dots \leq \tilde{q}_{r+x}$ be integers and $\mathbf{d} = (d_1, \dots, d_{n-r-x})$ a partition. There exists a rational matrix $\widetilde{W}(s) \in \mathbb{F}(s)^{z \times n}$ such that $\text{rank} \left(\begin{bmatrix} R(s) \\ \widetilde{W}(s) \end{bmatrix} \right) = r + x$ and $\begin{bmatrix} R(s) \\ \widetilde{W}(s) \end{bmatrix}$ has $\frac{\epsilon_1(s)}{\psi_1(s)}, \dots, \frac{\epsilon_{r+x}(s)}{\psi_{r+x}(s)}$ as invariant rational functions, $\tilde{q}_1, \dots, \tilde{q}_{r+x}$ as invariant orders at ∞ and $d_1, \dots, d_{n-r-x}$ as column minimal indices if and only if (21)–(24) and (29), where $\tilde{\mathbf{a}} = (\tilde{a}_1, \dots, \tilde{a}_x)$ is defined as in (30).

Proof. It is analogous to the proof of Theorem 4.5 of [1]. $\square$

Theorem 4.2. (Prescription of the finite and infinite structures, and the row minimal indices) *Let $R(s) \in \mathbb{F}(s)^{m \times n}$ be a rational matrix, $\text{rank}(R(s)) = r$. Let $\frac{\eta_1(s)}{\varphi_1(s)}, \dots, \frac{\eta_r(s)}{\varphi_r(s)}$ be its invariant rational functions, $\tilde{p}_1, \dots, \tilde{p}_r$ its invariant orders at ∞ , $\mathbf{c} = (c_1, \dots, c_{n-r})$ its column minimal indices, and $\mathbf{u} = (u_1, \dots, u_{m-r})$ its row minimal indices, where $u_1 \geq \dots \geq u_\eta > u_{\eta+1} = \dots = u_{m-r} = 0$.*

Let z, x be integers such that $0 \leq x \leq \min\{z, n - r\}$ and let $\epsilon_1(s) \mid \dots \mid \epsilon_{r+x}(s)$ and $\psi_{r+x}(s) \mid \dots \mid \psi_1(s)$ be monic polynomials such that $\frac{\epsilon_i(s)}{\psi_i(s)}$ are irreducible rational functions, $1 \leq i \leq r + x$. Let $\tilde{q}_1 \leq \dots \leq \tilde{q}_{r+x}$ be integers and $\mathbf{v} = (v_1, \dots, v_{m+z-r-x})$ a

partition such that $v_1 \geq \dots \geq v_{\bar{\eta}} > v_{\bar{\eta}+1} = \dots = v_{m+z-r-x} = 0$. Let $\tilde{\mathbf{a}} = (\tilde{a}_1, \dots, \tilde{a}_x)$ and $\tilde{\mathbf{b}} = (\tilde{b}_1, \dots, \tilde{b}_{z-x})$ be defined as in (27) and (28), respectively.

1. If $x = n-r$, there exists a rational matrix $\widetilde{W}(s) \in \mathbb{F}(s)^{z \times n}$ such that $\text{rank} \left(\begin{bmatrix} R(s) \\ \widetilde{W}(s) \end{bmatrix} \right) = r+x$ and $\begin{bmatrix} R(s) \\ \widetilde{W}(s) \end{bmatrix}$ has $\frac{\epsilon_1(s)}{\psi_1(s)}, \dots, \frac{\epsilon_{r+x}(s)}{\psi_{r+x}(s)}$ as invariant rational functions, $\tilde{q}_1, \dots, \tilde{q}_{r+x}$ as invariant orders at ∞ and $v_1, \dots, v_{m+z-r-x}$ as row minimal indices if and only if (15), (21)–(23), (25), (26) and

$$\mathbf{c} \prec \tilde{\mathbf{a}}.$$

2. If $x < n-r$, there exists a rational matrix $\widetilde{W}(s) \in \mathbb{F}(s)^{z \times n}$ such that $\text{rank} \left(\begin{bmatrix} R(s) \\ \widetilde{W}(s) \end{bmatrix} \right) = r+x$ and $\begin{bmatrix} R(s) \\ \widetilde{W}(s) \end{bmatrix}$ has $\frac{\epsilon_1(s)}{\psi_1(s)}, \dots, \frac{\epsilon_{r+x}(s)}{\psi_{r+x}(s)}$ as invariant rational functions, $\tilde{q}_1, \dots, \tilde{q}_{r+x}$ as invariant orders at ∞ and $v_1, \dots, v_{m+z-r-x}$ as row minimal indices if and only if (15), (21)–(23), (25), (26),

$$\sum_{i=1}^{x+1} c_i - c_\ell \geq \sum_{i=1}^x \tilde{a}_i,$$

and

$$\sum_{i=j+2}^{x+1} c_i \geq \sum_{i=j+1}^x \tilde{a}_i, \quad \ell \leq j \leq x-1,$$

where $\ell = \min\{j \geq 1 : \sum_{i=1}^j c_i > \sum_{i=1}^j \tilde{a}_i\}$.

Proof. It is analogous to the proof of Theorem 4.8 of [1]. $\square$

4.2. Prescription of the finite and/or infinite structures

In this subsection we deal with Problem 3 when only the finite or the infinite structures are prescribed. First, we present a solution when both the finite and infinite structures are prescribed. Secondly, we give a solution when only the infinite structure is prescribed (in both polynomial and rational cases). As mentioned, the solution for the case where only the finite structure is prescribed is known (for the polynomial case when the degree is not prescribed see Theorem 2.5, and for the rational case see Theorem 2.6). In Theorem 4.9 we prescribe the finite structure and the first invariant order at infinity. Note that in the polynomial case prescribing the first invariant order at infinity is the same as prescribing the degree.

Although some of the proofs in this subsection are analogous to those presented in [1, Section 4.4], we write them for the convenience of the reader.

Theorem 4.3. (Prescription of the finite and infinite structures) Let $R(s) \in \mathbb{F}(s)^{m \times n}$ be a rational matrix, $\text{rank}(R(s)) = r$. Let $\frac{\eta_1(s)}{\varphi_1(s)}, \dots, \frac{\eta_r(s)}{\varphi_r(s)}$ be its invariant rational functions, $\tilde{p}_1, \dots, \tilde{p}_r$ its invariant orders at ∞ , $\mathbf{c} = (c_1, \dots, c_{n-r})$ its column minimal indices, and $\mathbf{u} = (u_1, \dots, u_{m-r})$ its row minimal indices.

Let z, x be integers such that $0 \leq x \leq \min\{z, n-r\}$, let $\epsilon_1(s) \mid \dots \mid \epsilon_{r+x}(s)$ and $\psi_{r+x}(s) \mid \dots \mid \psi_1(s)$ be monic polynomials such that $\frac{\epsilon_i(s)}{\psi_i(s)}$ are irreducible rational functions, $1 \leq i \leq r+x$, and let $\tilde{q}_1 \leq \dots \leq \tilde{q}_{r+x}$ be integers.

1. If $x < z$ or $x = z = n-r$, then there exists a rational matrix $\widetilde{W}(s) \in \mathbb{F}(s)^{z \times n}$ such that $\text{rank} \left(\begin{bmatrix} R(s) \\ \widetilde{W}(s) \end{bmatrix} \right) = r+x$ and $\begin{bmatrix} R(s) \\ \widetilde{W}(s) \end{bmatrix}$ has $\frac{\epsilon_1(s)}{\psi_1(s)}, \dots, \frac{\epsilon_{r+x}(s)}{\psi_{r+x}(s)}$ as invariant rational functions, $\tilde{q}_1, \dots, \tilde{q}_{r+x}$ as invariant orders at ∞ if and only if (21)–(23) and

$$\begin{aligned} & \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i}, \frac{\epsilon_{i+x-j}}{\psi_{i+x-j}}, \tilde{p}_i, \tilde{q}_{i+x-j} \right) + \sum_{i=1}^{x-j} \Delta \left(\frac{\epsilon_i}{\psi_i}, \tilde{q}_i \right) \\ & + \sum_{i=1}^{m-r} u_i + \sum_{i=1}^j c_i + \sum_{i=x+1}^{n-r} c_i \leq 0, \quad 0 \leq j \leq x-1, \end{aligned} \quad (32)$$

with equality for $j = 0$ when $x = z = n-r$.

2. If $x = z < n-r$, then there exists a rational matrix $\widetilde{W}(s) \in \mathbb{F}(s)^{z \times n}$ such that $\text{rank} \left(\begin{bmatrix} R(s) \\ \widetilde{W}(s) \end{bmatrix} \right) = r+x$ and $\begin{bmatrix} R(s) \\ \widetilde{W}(s) \end{bmatrix}$ has $\frac{\epsilon_1(s)}{\psi_1(s)}, \dots, \frac{\epsilon_{r+x}(s)}{\psi_{r+x}(s)}$ as invariant rational functions, $\tilde{q}_1, \dots, \tilde{q}_{r+x}$ as invariant orders at ∞ if and only if (21)–(23),

$$\sum_{i=1}^{x+1} c_i - c_\ell \geq \sum_{i=1}^x \tilde{a}'_i, \quad (33)$$

and

$$\sum_{i=j+2}^{x+1} c_i \geq \sum_{i=j+1}^x \tilde{a}'_i, \quad \ell \leq j \leq x-1, \quad (34)$$

where $\tilde{\mathbf{a}}' = (\tilde{a}'_1, \dots, \tilde{a}'_x)$ is defined as

$$\sum_{i=1}^j \tilde{a}'_i = \sum_{i=1}^{r+j} \Delta \left(\frac{\epsilon_{i+x-j}}{\psi_{i+x-j}}, \tilde{q}_{i+x-j} \right) - \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i}, \frac{\epsilon_{i+x-j}}{\psi_{i+x-j}}, \tilde{p}_i, \tilde{q}_{i+x-j} \right), \quad (35)$$

$1 \leq j \leq x,$

and $\ell = \min\{j \geq 1 : \sum_{i=1}^j c_i > \sum_{i=1}^j \tilde{a}'_i\}$.

Proof. The proof follows the scheme of that of Theorem 4.10 of [1], but it deserves some hints.

1. Case $x < z$ or $x = z = n - r$. Assume that there exists a rational matrix $\widetilde{W}(s) \in \mathbb{F}(s)^{z \times n}$ such that $\begin{bmatrix} R(s) \\ \widetilde{W}(s) \end{bmatrix}$ has the prescribed invariants. Let $\mathbf{d} = (d_1, \dots, d_{n-r-x})$ be the column minimal indices of $\begin{bmatrix} R(s) \\ \widetilde{W}(s) \end{bmatrix}$. By Theorem 4.1 and Remark 3.12, (21)–(24) and (29) hold, where $\tilde{\mathbf{a}} = (\tilde{a}_1, \dots, \tilde{a}_x)$ is defined as in (30). From (29) and Theorem 2.4 we obtain

$$\sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i}, \frac{\epsilon_{i+x}}{\psi_{i+x}}, \tilde{p}_i, \tilde{q}_{i+x} \right) + \sum_{i=1}^x \Delta \left(\frac{\epsilon_i}{\psi_i}, \tilde{q}_i \right) + \sum_{i=1}^{m-r} u_i + \sum_{i=1}^{n-r-x} d_i \leq 0,$$

with equality if $x = z$. From (24) we get $\sum_{i=1}^{n-r-x} d_i \geq \sum_{i=1}^{n-r-x} c_{i+x} = \sum_{i=x+1}^{n-r} c_i$. Therefore, (32) holds for $j = 0$.

For $1 \leq j \leq x - 1$, from (24), [1, Lemma 4.9], (30) and Theorem 2.4, we obtain

$$\begin{aligned} \sum_{i=1}^j c_i &\leq \sum_{i=1}^j \tilde{a}_i + \sum_{i=1}^{n-r-x} d_i - \sum_{i=x+1}^{n-r} c_i \\ &= \sum_{i=1}^{n-r-x} d_i - \sum_{i=x+1}^{n-r} c_i - \sum_{i=1}^{n-r-x} d_i - \sum_{i=1}^{m-r} u_i \\ &\quad - \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i}, \frac{\epsilon_{i+x-j}}{\psi_{i+x-j}}, \tilde{p}_i, \tilde{q}_{i+x-j} \right) - \sum_{i=1}^{x-j} \Delta \left(\frac{\epsilon_i}{\psi_i}, \tilde{q}_i \right). \end{aligned}$$

Thus, (32) holds.

Conversely, assume that (21)–(23) and (32) hold. Define $\hat{a}_1, \dots, \hat{a}_x$ as

$$\begin{aligned} \sum_{i=1}^j \hat{a}_i &= \sum_{i=1}^x c_i + \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i}, \tilde{p}_i \right) - \sum_{i=1}^{x-j} \Delta \left(\frac{\epsilon_i}{\psi_i}, \tilde{q}_i \right) \\ &\quad - \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i}, \frac{\epsilon_{i+x-j}}{\psi_{i+x-j}}, \tilde{p}_i, \tilde{q}_{i+x-j} \right), \quad 1 \leq j \leq x. \end{aligned}$$

By condition (32) for $j = 0$ and Theorem 2.4,

$$\sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i}, \frac{\epsilon_{i+x}}{\psi_{i+x}}, \tilde{p}_i, \tilde{q}_{i+x} \right) + \sum_{i=1}^x \Delta \left(\frac{\epsilon_i}{\psi_i}, \tilde{q}_i \right) \leq \sum_{i=1}^x c_i + \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i}, \tilde{p}_i \right);$$

hence

$$\hat{a}_1 \geq \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i}, \frac{\epsilon_{i+x}}{\psi_{i+x}}, \tilde{p}_i, \tilde{q}_{i+x} \right) - \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i}, \frac{\epsilon_{i+x-1}}{\psi_{i+x-1}}, \tilde{p}_i, \tilde{q}_{i+x-1} \right) + \Delta \left(\frac{\epsilon_x}{\psi_x}, \tilde{q}_x \right).$$

Taking into account Remark 3.6. 1, we have $\hat{a}_1 \geq \hat{a}_2 \geq \dots \geq \hat{a}_x$. Let $\hat{\mathbf{a}} = (\hat{a}_1, \dots, \hat{a}_x)$. By Theorem 2.4, for $1 \leq j \leq x$,

$$\begin{aligned} \sum_{i=1}^j \hat{a}_i &= - \sum_{i=x+1}^{n-r} c_i - \sum_{i=1}^{m-r} u_i - \sum_{i=1}^{x-j} \Delta \left(\frac{\epsilon_i}{\psi_i}, \tilde{q}_i \right) \\ &\quad - \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i}, \frac{\epsilon_{i+x-j}}{\psi_{i+x-j}}, \tilde{p}_i, \tilde{q}_{i+x-j} \right), \quad 1 \leq j \leq x. \end{aligned}$$

From (32) we obtain

$$\sum_{i=1}^j \hat{a}_i \geq \sum_{i=1}^j c_i, \quad 1 \leq j \leq x-1.$$

Moreover, from (21)–(23), we obtain $\sum_{i=1}^x \hat{a}_i = \sum_{i=1}^x c_i$.

If $n = r + x$, then $\mathbf{c} \prec \hat{\mathbf{a}}$, and let $\mathbf{d} = \emptyset$ so that $\mathbf{c} \prec' (\mathbf{d}, \hat{\mathbf{a}})$ holds. Otherwise, if $n > r + x$, by [1, Lemma 4.6] there exists a sequence of integers $\mathbf{d} = (d_1, \dots, d_{n-r-x})$ such that $\mathbf{c} \prec' (\mathbf{d}, \hat{\mathbf{a}})$, $d_i = c_{i+x}$ for $2 \leq i \leq n-r-x$, and $d_1 = \sum_{i=1}^{x+1} c_i - \sum_{i=1}^x \hat{a}_i = c_{x+1}$.

Let $\tilde{\mathbf{a}} = (\tilde{a}_1, \dots, \tilde{a}_x)$ be defined as in (30). Then $\tilde{\mathbf{a}} = \hat{\mathbf{a}}$, therefore (24) holds. Furthermore, from (32) for $j = 0$ and Theorem 2.4 we obtain

$$\begin{aligned} & \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i}, \frac{\epsilon_{i+x}}{\psi_{i+x}}, \tilde{p}_i, \tilde{q}_{i+x} \right) \\ & \leq \sum_{i=1}^{n-r} (c_i - c_{i+x}) + \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i}, \tilde{p}_i \right) - \sum_{i=1}^x \Delta \left(\frac{\epsilon_i}{\psi_i}, \tilde{q}_i \right) \\ & = \sum_{i=1}^{n-r} c_i - \sum_{i=1}^{n-r-x} d_i + \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i}, \tilde{p}_i \right) - \sum_{i=1}^x \Delta \left(\frac{\epsilon_i}{\psi_i}, \tilde{q}_i \right), \end{aligned}$$

with equality if $x = z = n - r$, i.e., (29) is satisfied. By Theorem 4.1, the result follows.

2. Case $x = z < n - r$. As $x = z$, observe that if there exists $\widetilde{W}(s) \in \mathbb{F}(s)^{z \times n}$ such that $\text{rank} \left(\begin{bmatrix} R(s) \\ \widetilde{W}(s) \end{bmatrix} \right) = r + x$, then the row minimal indices of $\begin{bmatrix} R(s) \\ \widetilde{W}(s) \end{bmatrix}$ are the row minimal indices of $R(s)$, i.e., $\mathbf{v} = \mathbf{u}$. For the sufficiency we prescribe $\mathbf{v} = \mathbf{u}$. The result follows from Theorem 4.2. $\square$

Remark 4.4. If $x = z < n - r$, conditions (21)–(23), (33) and (34) imply (32) (see [1, Remark 4.11]).

When we only prescribe the infinite structure we present two results, one for polynomial matrices and another one for rational matrices.

Theorem 4.5. (Prescription of the infinite structure for polynomial matrices) *Let $P(s) \in \mathbb{F}[s]^{m \times n}$ be a polynomial matrix, $\text{rank}(P(s)) = r$. Let $p_1, \dots, p_r$ be its invariant orders at ∞ and $\mathbf{c} = (c_1, \dots, c_{n-r})$ its column minimal indices.*

Let z, x be integers such that $0 \leq x \leq \min\{z, n-r\}$ and let $q_1 \leq \dots \leq q_{r+x}$ be integers. There exists a polynomial matrix $W(s) \in \mathbb{F}[s]^{z \times n}$ such that $\text{rank} \left(\begin{bmatrix} P(s) \\ W(s) \end{bmatrix} \right) = r + x$ and $\begin{bmatrix} P(s) \\ W(s) \end{bmatrix}$ has $q_1, \dots, q_{r+x}$ as invariant orders at ∞ if and only if (14) and

$$\sum_{i=1}^r \max\{p_i, q_{i+x-j}\} + \sum_{i=1}^{x-j} q_i - \sum_{i=1}^r p_i \leq \sum_{i=j+1}^x c_i, \quad 0 \leq j \leq x-1. \quad (36)$$

Proof. The proof is similar to that of Theorem 4.12 of [1]; we precise some calculations.

Let $\alpha_1(s), \dots, \alpha_r(s)$ be the invariant factors, and $\mathbf{u} = (u_1, \dots, u_{m-r})$ the row minimal indices of $P(s)$.

Assume that there is a polynomial matrix $W(s) \in \mathbb{F}[s]^{z \times n}$, $\text{rank} \left(\begin{bmatrix} P(s) \\ W(s) \end{bmatrix} \right) = r + x$, such that $\begin{bmatrix} P(s) \\ W(s) \end{bmatrix}$ has $q_1, \dots, q_{r+x}$ as invariant orders at ∞ . Let $\beta_1(s), \dots, \beta_{r+x}(s)$ be its invariant factors. By Theorem 4.3 and Remark 4.4 we obtain (13), (14) and

$$\begin{aligned} & \sum_{i=1}^r \deg(\text{lcm}(\alpha_i, \beta_{i+x-j})) + \sum_{i=1}^{x-j} \deg(\beta_i) + \sum_{i=1}^r \max\{p_i, q_{i+x-j}\} \\ & + \sum_{i=1}^{x-j} q_i + \sum_{i=1}^{m-r} u_i + \sum_{i=1}^j c_i + \sum_{i=x+1}^{n-r} c_i \leq 0, \quad 0 \leq j \leq x-1. \end{aligned} \quad (37)$$

We have

$$\sum_{i=1}^r \deg(\text{lcm}(\alpha_i, \beta_{i+x-j})) \geq \sum_{i=1}^r \deg(\alpha_i) \geq \sum_{i=1}^r \deg(\alpha_i) - \sum_{i=1}^{x-j} \deg(\beta_i).$$

Thus, from (37) we obtain

$$\begin{aligned} & \sum_{i=1}^r \deg(\alpha_i) + \sum_{i=1}^r \max\{p_i, q_{i+x-j}\} + \sum_{i=1}^{x-j} q_i \\ & + \sum_{i=1}^{m-r} u_i + \sum_{i=1}^j c_i + \sum_{i=x+1}^{n-r} c_i \leq 0, \quad 0 \leq j \leq x-1, \end{aligned}$$

which by Theorem 2.2 is equivalent to (36).

Conversely, assume that (14) and (36) hold. Let

$$t = \sum_{i=1}^r p_i + \sum_{i=1}^x c_i - \sum_{i=1}^r \max\{p_i, q_{i+x}\} - \sum_{i=1}^x q_i.$$

If $x = 0$, from (14) we obtain $t = 0$, and if $x > 0$, from (36) we have $t \geq 0$.

Define

$$\begin{aligned} \beta_i(s) &= 1, & 1 \leq i \leq x, \\ \beta_{i+x}(s) &= \alpha_i(s), & 1 \leq i \leq r-1, \\ \beta_{r+x}(s) &= \alpha_r(s)\tau(s), \end{aligned}$$

where $\tau(s)$ is a monic polynomial of $\deg(\tau) = t$. We have $\beta_1(s) \mid \dots \mid \beta_{r+x}(s)$, and (13) holds.

If $x > 1$, for $1 \leq j \leq x$, we have $\beta_{i+x-j}(s) \mid \beta_{i+x-1}(s) \mid \alpha_i(s)$, $1 \leq i \leq r$, therefore

$$\sum_{i=1}^r \deg(\text{lcm}(\alpha_i, \beta_{i+x-j})) + \sum_{i=1}^{x-j} \deg(\beta_i) = \begin{cases} \sum_{i=1}^r \deg(\alpha_i) + t, & j = 0, \\ \sum_{i=1}^r \deg(\alpha_i), & 1 \leq j \leq x. \end{cases}$$

Thus, from Theorem 2.2 and (36) we obtain

$$\begin{aligned} & \sum_{i=1}^r \deg(\text{lcm}(\alpha_i, \beta_{i+x-j})) + \sum_{i=1}^{x-j} \deg(\beta_i) + \sum_{i=1}^r \max\{p_i, q_{i+x-j}\} + \sum_{i=1}^{x-j} q_i \\ & + \sum_{i=1}^{m-r} u_i + \sum_{i=1}^j c_i + \sum_{i=x+1}^{n-r} c_i \\ & = \begin{cases} t - \sum_{i=1}^r p_i - \sum_{i=1}^x c_i + \sum_{i=1}^r \max\{p_i, q_{i+x}\} + \sum_{i=1}^x q_i = 0, & j = 0, \\ \sum_{i=1}^r \max\{p_i, q_{i+x-j}\} + \sum_{i=1}^{x-j} q_i - \sum_{i=1}^r p_i - \sum_{i=j+1}^x c_i \leq 0, & 1 \leq j \leq x-1. \end{cases} \end{aligned} \quad (38)$$

If $x < z$ or $x = z = n - r$ the result follows from Theorem 4.3 (item 1).

If $x = z < n - r$, let $\tilde{\mathbf{a}}' = (\tilde{a}'_1, \dots, \tilde{a}'_x)$ be defined as

$$\begin{aligned} \sum_{i=1}^j \tilde{a}'_i &= \sum_{i=1}^{r+j} \deg(\beta_{i+x-j}) - \sum_{i=1}^r \deg(\text{lcm}(\alpha_i, \beta_{i+x-j})) \\ &+ \sum_{i=1}^{r+j} q_{i+x-j} - \sum_{i=1}^r \max\{p_i, q_{i+x-j}\}, \quad 1 \leq j \leq x. \end{aligned}$$

From (13), (14) and (38) we have

$$\begin{aligned} \sum_{i=1}^x \tilde{a}'_i &= \sum_{i=1}^{r+x} \deg(\beta_i) - \sum_{i=1}^r \deg(\alpha_i) + \sum_{i=1}^{r+x} q_i - \sum_{i=1}^r p_i \\ &= \sum_{i=1}^x c_i - \sum_{i=1}^r \max\{p_i, q_{i+x}\} + \sum_{i=1}^r q_{i+x} = \sum_{i=1}^x c_i. \end{aligned}$$

Let $j \in \{1, \dots, x-1\}$. Then

$$\begin{aligned} \sum_{i=1}^j \tilde{a}'_i &= \sum_{i=1}^{r+x} \deg(\beta_i) - \sum_{i=1}^r \deg(\text{lcm}(\alpha_i, \beta_{i+x-j})) - \sum_{i=1}^{x-j} \deg(\beta_i) \\ &+ \sum_{i=1}^{r+x} q_i - \sum_{i=1}^r \max\{p_i, q_{i+x-j}\} - \sum_{i=1}^{x-j} q_i \\ &= \sum_{i=1}^r \deg(\alpha_i) + \sum_{i=1}^r p_i + \sum_{i=1}^x c_i - \sum_{i=1}^r \deg(\text{lcm}(\alpha_i, \beta_{i+x-j})) \\ &- \sum_{i=1}^{x-j} \deg(\beta_i) - \sum_{i=1}^r \max\{p_i, q_{i+x-j}\} - \sum_{i=1}^{x-j} q_i \\ &= \sum_{i=1}^r p_i + \sum_{i=1}^x c_i - \sum_{i=1}^r \max\{p_i, q_{i+x-j}\} - \sum_{i=1}^{x-j} q_i. \end{aligned}$$

From (36) we obtain

$$\sum_{i=1}^j \tilde{a}'_i \geq \sum_{i=1}^x c_i - \sum_{i=j+1}^x c_i = \sum_{i=1}^j c_i, \quad 1 \leq j \leq x-1.$$

Therefore,

$$\min\{j \geq 1 : \sum_{i=1}^j c_i > \sum_{i=1}^j \tilde{a}'_i\} = x+1.$$

The result follows from Theorem 4.3 (item 2). $\square$

The technique used in the following theorem is different from the one used in the previous results. We first present a remark.

Remark 4.6. Let $R(s)$ be a rational matrix with $\tilde{p}_1, \dots, \tilde{p}_r$ as invariant orders at ∞ . If $\frac{\hat{\eta}_1(s)}{\hat{\varphi}_1(s)}, \dots, \frac{\hat{\eta}_r(s)}{\hat{\varphi}_r(s)}$ are the invariant rational functions of $R(\frac{1}{s})$, then

$$\frac{\hat{\eta}_i(s)}{\hat{\varphi}_i(s)} = s^{\tilde{p}_i} \frac{\hat{\eta}'_i(s)}{\hat{\varphi}'_i(s)}, \quad 1 \leq i \leq r,$$

where the triples of polynomials $(\hat{\eta}'_i(s), \hat{\varphi}'_i(s), s)$ are pairwise coprime for $1 \leq i \leq r$ (see [4, p. 724] or [3, Proposition 6.11]).

Theorem 4.7. (Prescription of the infinite structure for rational matrices) *Let $R(s) \in \mathbb{F}(s)^{m \times n}$ be a rational matrix, $\text{rank}(R(s)) = r$, and let $\tilde{p}_1, \dots, \tilde{p}_r$ be its invariant orders at ∞ .*

Let z, x be integers such that $0 \leq x \leq \min\{z, n-r\}$ and let $\tilde{q}_1 \leq \dots \leq \tilde{q}_{r+x}$ be integers. There exists a rational matrix $\widetilde{W}(s) \in \mathbb{F}(s)^{z \times n}$ such that $\text{rank} \left(\begin{bmatrix} R(s) \\ \widetilde{W}(s) \end{bmatrix} \right) = r+x$ and $\left[\begin{smallmatrix} R(s) \\ \widetilde{W}(s) \end{smallmatrix} \right]$ has $\tilde{q}_1, \dots, \tilde{q}_{r+x}$ as invariant orders at ∞ if and only if (23) holds.

Proof. The necessity follows from Theorem 3.11.

For the sufficiency, assume that (23) holds. Let $\frac{\hat{\eta}_1(s)}{\hat{\varphi}_1(s)}, \dots, \frac{\hat{\eta}_r(s)}{\hat{\varphi}_r(s)}$ be the invariant rational functions of $R(\frac{1}{s})$. We can write

$$\frac{\hat{\eta}_i(s)}{\hat{\varphi}_i(s)} = s^{\tilde{p}_i} \frac{\hat{\eta}'_i(s)}{\hat{\varphi}'_i(s)}, \quad 1 \leq i \leq r,$$

where the triples of polynomials $(\hat{\eta}'_i(s), \hat{\varphi}'_i(s), s)$ are pairwise coprime (see Remark 4.6). Notice that

$$\begin{aligned} \hat{\eta}_i(s) &= s^{\tilde{p}_i} \hat{\eta}'_i(s), & \hat{\varphi}_i(s) &= \hat{\varphi}'_i(s), & \text{if } \tilde{p}_i \geq 0, \\ \hat{\eta}_i(s) &= \hat{\eta}'_i(s), & \hat{\varphi}_i(s) &= s^{-\tilde{p}_i} \hat{\varphi}'_i(s), & \text{if } \tilde{p}_i < 0, \end{aligned}$$

and

$$\hat{\eta}'_i(s) \mid \hat{\eta}'_{i+1}(s), \quad \hat{\varphi}'_{i+1}(s) \mid \hat{\varphi}'_i(s), \quad 1 \leq i \leq r-1. \quad (39)$$

For $1 \leq i \leq x$, define

$$\begin{aligned} \hat{\epsilon}_i(s) &= s^{\tilde{q}_i}, & \hat{\psi}_i(s) &= \hat{\varphi}'_1(s), & \text{if } \tilde{q}_i \geq 0, \\ \hat{\epsilon}_i(s) &= 1, & \hat{\psi}_i(s) &= s^{-\tilde{q}_i} \hat{\varphi}'_1(s), & \text{if } \tilde{q}_i < 0, \end{aligned}$$

and for $1 \leq i \leq r$,

$$\begin{aligned} \hat{\epsilon}_{i+x}(s) &= s^{\tilde{q}_{i+x}} \hat{\eta}'_i(s), & \hat{\psi}_{i+x}(s) &= \hat{\varphi}'_i(s), & \text{if } \tilde{q}_{i+x} \geq 0, \\ \hat{\epsilon}_{i+x}(s) &= \hat{\eta}'_i(s), & \hat{\psi}_{i+x}(s) &= s^{-\tilde{q}_{i+x}} \hat{\varphi}'_i(s), & \text{if } \tilde{q}_{i+x} < 0. \end{aligned}$$

Then,

$$\begin{aligned}\frac{\hat{\epsilon}_i(s)}{\hat{\psi}_i(s)} &= s^{\tilde{q}_i} \frac{1}{\hat{\phi}_1'(s)}, & 1 \leq i \leq x, \\ \frac{\hat{\epsilon}_{i+x}(s)}{\hat{\psi}_{i+x}(s)} &= s^{\tilde{q}_{i+x}} \frac{\hat{\eta}_i'(s)}{\hat{\phi}_i'(s)}, & 1 \leq i \leq r.\end{aligned}$$

Since $\tilde{q}_i \leq \tilde{q}_{i+1}$, $1 \leq i \leq r+x-1$, from (39) we obtain

$$\hat{\epsilon}_i(s) \mid \hat{\epsilon}_{i+1}(s), \quad \hat{\psi}_{i+1}(s) \mid \hat{\psi}_i(s), \quad 1 \leq i \leq r+x-1,$$

and from (23) we obtain (recall that for $i > r+x$ we take $\epsilon_i(s) = 0$ and $\psi_i(s) = 1$)

$$\hat{\epsilon}_i(s) \mid \hat{\eta}_i(s) \mid \hat{\epsilon}_{i+z}(s), \quad \hat{\psi}_{i+z}(s) \mid \hat{\phi}_i(s) \mid \hat{\psi}_i(s), \quad 1 \leq i \leq r.$$

By Theorem 2.6, there is a rational matrix $\widehat{W}(s) \in \mathbb{F}(s)^{z \times n}$, $\text{rank} \left(\left[\begin{smallmatrix} R(\frac{1}{s}) \\ \widehat{W}(s) \end{smallmatrix} \right] \right) = r+x$,

such that $\left[\begin{smallmatrix} R(\frac{1}{s}) \\ \widehat{W}(s) \end{smallmatrix} \right]$ has $\frac{\hat{\epsilon}_1(s)}{\hat{\psi}_1(s)}, \dots, \frac{\hat{\epsilon}_{r+x}(s)}{\hat{\psi}_{r+x}(s)}$ as invariant rational functions.

Let $\widetilde{W}(s) = \widehat{W}(\frac{1}{s})$ and $\widetilde{Q}(s) = \left[\begin{smallmatrix} R(s) \\ \widetilde{W}(s) \end{smallmatrix} \right]$. Then, $\widetilde{Q}(\frac{1}{s}) = \left[\begin{smallmatrix} R(\frac{1}{s}) \\ \widehat{W}(s) \end{smallmatrix} \right]$ and $\tilde{q}_1, \dots, \tilde{q}_{r+x}$ are the invariant orders at ∞ of $\widetilde{Q}(s)$ (Remark 4.6). $\square$

The next example shows the difference between the rational and polynomial cases when prescribing the infinite structure.

Example 4.8. Let $P(s) = [s \ 0] \in \mathbb{F}[s]^{1 \times 2}$. The matrix $P(s)$ has $\alpha_1(s) = s$ as invariant factor, $p_1 = -1$ as invariant order at ∞ and $c_1 = 0$ as column minimal index.

Let $z = x = 1$, and $q_1 = -1, q_2 = +1$. Then (14) holds, but (36) is not satisfied. Therefore, there is no polynomial matrix $Q(s) = \left[\begin{smallmatrix} P(s) \\ W(s) \end{smallmatrix} \right] \in \mathbb{F}[s]^{2 \times 2}$ of $\text{rank } Q(s) = 2$ with $q_1 = -1, q_2 = +1$ as invariant orders at ∞ . If there were such a polynomial matrix, then by Theorem 3.8, the invariant factors $\beta_1(s), \beta_2(s)$ of $Q(s)$ would satisfy $\alpha_1(s) = s \mid \beta_2(s)$ and $\deg(\beta_1) + \deg(\beta_2) = 0$, which leads to a contradiction.

However, if we allow the completion to be rational, it is possible to obtain the desired invariants. For example, the rational matrix $\widetilde{Q}(s) = \left[\begin{smallmatrix} s & 0 \\ 0 & \frac{1}{s} \end{smallmatrix} \right] \in \mathbb{F}(s)^{2 \times 2}$ has $\tilde{q}_1 = -1, \tilde{q}_2 = +1$ as invariant orders at ∞ .

As mentioned, a solution to the row completion problem for polynomial matrices when the finite structure is prescribed follows from Theorem 2.5. In this theorem no condition is imposed on the invariant orders at ∞ , and therefore on the degree of the completed matrix $Q(s)$. To prescribe the degree of $Q(s)$, we must prescribe the first order at ∞ of $Q(s)$, $q_1 = -\deg(Q(s))$, which shall satisfy $q_1 = -\deg(Q(s)) \leq -\deg(P(s)) = p_1$.

Theorem 2.5 was later generalized to rational matrices in Theorem 2.6. In the next theorem, we give a solution to the row completion problem for rational matrices when the finite structure and the first invariant order at ∞ of the completed matrix is prescribed.

Theorem 4.9. (Prescription of the finite structure and the first invariant order at ∞)
 Let $R(s) \in \mathbb{F}(s)^{m \times n}$ be a rational matrix, $\text{rank}(R(s)) = r$. Let $\frac{\eta_1(s)}{\varphi_1(s)}, \dots, \frac{\eta_r(s)}{\varphi_r(s)}$ be its invariant rational functions, $\tilde{p}_1, \dots, \tilde{p}_r$ its invariant orders at ∞ , $\mathbf{c} = (c_1, \dots, c_{n-r})$ its column minimal indices.

Let z, x be integers such that $0 \leq x \leq \min\{z, n - r\}$, let $\epsilon_1(s) \mid \dots \mid \epsilon_{r+x}(s)$ and $\psi_{r+x}(s) \mid \dots \mid \psi_1(s)$ be monic polynomials such that $\frac{\epsilon_i(s)}{\psi_i(s)}$ are irreducible rational functions, $1 \leq i \leq r + x$, and let $\tilde{q}_1$ be an integer $\tilde{q}_1 \leq \tilde{p}_1$. There exists a rational matrix $\widetilde{W}(s) \in \mathbb{F}(s)^{z \times n}$ such that $\text{rank} \left(\begin{bmatrix} R(s) \\ \widetilde{W}(s) \end{bmatrix} \right) = r + x$ and $\begin{bmatrix} R(s) \\ \widetilde{W}(s) \end{bmatrix}$ has $\frac{\epsilon_1(s)}{\psi_1(s)}, \dots, \frac{\epsilon_{r+x}(s)}{\psi_{r+x}(s)}$ as invariant rational functions and $\tilde{q}_1$ as first invariant order at ∞ if and only if (21), (22) and

$$\begin{aligned} & \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i}, \frac{\epsilon_{i+x-j}}{\psi_{i+x-j}} \right) + \sum_{i=1}^{x-j} \Delta \left(\frac{\epsilon_i}{\psi_i} \right) - \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i} \right) \\ & \leq \sum_{i=j+1}^x c_i + (j - x)\tilde{q}_1, \quad 0 \leq j \leq x - 1. \end{aligned} \quad (40)$$

Proof. The proof is similar to that of Theorem 4.5 and [1, Theorem 4.12]. We also precise the steps.

Let $\mathbf{u} = (u_1, \dots, u_{m-r})$ be the row minimal indices of $R(s)$. Assume that there exists $\widetilde{W}(s) \in \mathbb{F}(s)^{z \times n}$ such that $\text{rank} \left(\begin{bmatrix} R(s) \\ \widetilde{W}(s) \end{bmatrix} \right) = r + x$ and $\begin{bmatrix} R(s) \\ \widetilde{W}(s) \end{bmatrix}$ has $\frac{\epsilon_1(s)}{\psi_1(s)}, \dots, \frac{\epsilon_{r+x}(s)}{\psi_{r+x}(s)}$ as invariant rational functions and $\tilde{q}_1 \leq \dots \leq \tilde{q}_{r+x}$ as invariant orders at ∞ .

By Theorem 4.3 and Remark 4.4 we obtain (21)–(23) and

$$\begin{aligned} & \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i}, \frac{\epsilon_{i+x-j}}{\psi_{i+x-j}} \right) + \sum_{i=1}^{x-j} \Delta \left(\frac{\epsilon_i}{\psi_i} \right) + \sum_{i=1}^r \max\{\tilde{p}_i, \tilde{q}_{i+x-j}\} \\ & + \sum_{i=1}^{x-j} \tilde{q}_i + \sum_{i=1}^{m-r} u_i + \sum_{i=1}^j c_i + \sum_{i=x+1}^{n-r} c_i \leq 0, \quad 0 \leq j \leq x - 1. \end{aligned} \quad (41)$$

We have

$$\sum_{i=1}^r \max\{\tilde{p}_i, \tilde{q}_{i+x-j}\} + \sum_{i=1}^{x-j} \tilde{q}_i \geq \sum_{i=1}^r \tilde{p}_i + (x - j)\tilde{q}_1, \quad 0 \leq j \leq x - 1.$$

Thus, from (41) we obtain

$$\begin{aligned} & \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i}, \frac{\epsilon_{i+x-j}}{\psi_{i+x-j}} \right) + \sum_{i=1}^{x-j} \Delta \left(\frac{\epsilon_i}{\psi_i} \right) + \sum_{i=1}^r \tilde{p}_i \\ & + \sum_{i=1}^{m-r} u_i + \sum_{i=1}^j c_i + \sum_{i=x+1}^{n-r} c_i \leq (j - x)\tilde{q}_1, \quad 0 \leq j \leq x - 1, \end{aligned}$$

which by Theorem 2.4 is equivalent to (40).

Conversely, assume that (21), (22) and (40) hold. Let

$$\tilde{t} = \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i} \right) - \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i}, \frac{\epsilon_{i+x}}{\psi_{i+x}} \right) - \sum_{i=1}^x \Delta \left(\frac{\epsilon_i}{\psi_i} \right) + \sum_{i=1}^x c_i - x\tilde{q}_1.$$

If $x = 0$, from (21) and (22) we obtain $\tilde{t} = 0$. If $x > 0$, from (40) we have $\tilde{t} \geq 0$.

Define

$$\begin{aligned} \tilde{q}_i &= \tilde{q}_1, & 1 \leq i \leq x, \\ \tilde{q}_{i+x} &= \tilde{p}_i, & 1 \leq i \leq r-1, \\ \tilde{q}_{r+x} &= \tilde{p}_r + \tilde{t}. \end{aligned}$$

As $\tilde{q}_1 \leq \tilde{p}_1$ and $\tilde{t} \geq 0$, we have $\tilde{q}_1 \leq \dots \leq \tilde{q}_{r+x}$, and (23) holds.

If $x > 1$, for $1 \leq j \leq x$, we have $\tilde{q}_{i+x-j} \leq \tilde{q}_{i+x-1} \leq \tilde{p}_i$, $1 \leq i \leq r$, therefore

$$\sum_{i=1}^r \max\{\tilde{p}_i, \tilde{q}_{i+x-j}\} + \sum_{i=1}^{x-j} \tilde{q}_i = \begin{cases} \sum_{i=1}^r \tilde{p}_i + \tilde{t} + x\tilde{q}_1, & j = 0, \\ \sum_{i=1}^r \tilde{p}_i + (x-j)\tilde{q}_1, & 1 \leq j \leq x. \end{cases}$$

Thus, from Theorem 2.4 we obtain

$$\begin{aligned} & \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i}, \frac{\epsilon_{i+x}}{\psi_{i+x}}, \tilde{p}_i, \tilde{q}_{i+x} \right) + \sum_{i=1}^x \Delta \left(\frac{\epsilon_i}{\psi_i}, \tilde{q}_i \right) + \sum_{i=1}^{m-r} u_i + \sum_{i=x+1}^{n-r} c_i \\ &= \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i}, \frac{\epsilon_{i+x}}{\psi_{i+x}} \right) + \sum_{i=1}^x \Delta \left(\frac{\epsilon_i}{\psi_i} \right) + \tilde{t} + x\tilde{q}_1 - \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i} \right) - \sum_{i=1}^x c_i \\ &= 0, \end{aligned} \quad (42)$$

and from Theorem 2.4 and (40)

$$\begin{aligned} & \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i}, \frac{\epsilon_{i+x-j}}{\psi_{i+x-j}}, \tilde{p}_i, \tilde{q}_{i+x-j} \right) + \sum_{i=1}^{x-j} \Delta \left(\frac{\epsilon_i}{\psi_i}, \tilde{q}_i \right) + \sum_{i=1}^{m-r} u_i + \sum_{i=1}^j c_i \\ &+ \sum_{i=x+1}^{n-r} c_i = \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i}, \frac{\epsilon_{i+x-j}}{\psi_{i+x-j}} \right) + \sum_{i=1}^{x-j} \Delta \left(\frac{\epsilon_i}{\psi_i} \right) + \sum_{i=1}^r \tilde{p}_i + (x-j)\tilde{q}_1 \\ &+ \sum_{i=1}^{m-r} u_i + \sum_{i=1}^j c_i + \sum_{i=x+1}^{n-r} c_i = \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i}, \frac{\epsilon_{i+x-j}}{\psi_{i+x-j}} \right) + \sum_{i=1}^{x-j} \Delta \left(\frac{\epsilon_i}{\psi_i} \right) \\ &- \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i} \right) - \sum_{i=j+1}^x c_i + (x-j)\tilde{q}_1 \leq 0, \quad 1 \leq j \leq x-1. \end{aligned}$$

If $x < z$ or $x = z = n - r$ the result follows from Theorem 4.3 (item 1).

If $x = z < n - r$, let $\tilde{\mathbf{a}}' = (\tilde{a}'_1, \dots, \tilde{a}'_x)$ be defined as in (35). From (21), (22), (23) and (42) we have

$$\begin{aligned} \sum_{i=1}^x \tilde{a}'_i &= \sum_{i=1}^{r+x} \Delta \left(\frac{\epsilon_i}{\psi_i}, \tilde{q}_i \right) - \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i}, \tilde{p}_i \right) \\ &= \tilde{t} + x\tilde{q}_1 + \sum_{i=1}^{r+x} \Delta \left(\frac{\epsilon_i}{\psi_i} \right) - \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i} \right) \\ &= \sum_{i=1}^x c_i - \sum_{i=1}^r \Delta \left(\frac{\eta_i}{\varphi_i}, \frac{\epsilon_{i+x}}{\psi_{i+x}} \right) + \sum_{i=1}^r \Delta \left(\frac{\epsilon_{i+x}}{\psi_{i+x}} \right) = \sum_{i=1}^x c_i. \end{aligned}$$

Let $j \in \{1, \dots, x-1\}$. Then

$$\begin{aligned} \sum_{i=1}^j \tilde{a}'_i &= \sum_{i=1}^{r+x} \Delta\left(\frac{\epsilon_i}{\psi_i}, \tilde{q}_i\right) - \sum_{i=1}^r \Delta\left(\frac{\eta_i}{\varphi_i}, \frac{\epsilon_{i+x-j}}{\psi_{i+x-j}}, \tilde{p}_i, \tilde{q}_{i+x-j}\right) \\ &\quad - \sum_{i=1}^{x-j} \Delta\left(\frac{\epsilon_i}{\psi_i}, \tilde{q}_i\right) = \sum_{i=1}^r \Delta\left(\frac{\eta_i}{\varphi_i}, \tilde{p}_i\right) + \sum_{i=1}^x c_i \\ &\quad - \sum_{i=1}^r \Delta\left(\frac{\eta_i}{\varphi_i}, \frac{\epsilon_{i+x-j}}{\psi_{i+x-j}}, \tilde{p}_i, \tilde{q}_{i+x-j}\right) - \sum_{i=1}^{x-j} \Delta\left(\frac{\epsilon_i}{\psi_i}, \tilde{q}_i\right) \\ &= \sum_{i=1}^r \Delta\left(\frac{\eta_i}{\varphi_i}\right) + \sum_{i=1}^x c_i - \sum_{i=1}^r \Delta\left(\frac{\eta_i}{\varphi_i}, \frac{\epsilon_{i+x-j}}{\psi_{i+x-j}}\right) \\ &\quad - \sum_{i=1}^{x-j} \Delta\left(\frac{\epsilon_i}{\psi_i}\right) + (j-x)\tilde{q}_1. \end{aligned}$$

From (40) we obtain

$$\sum_{i=1}^j \tilde{a}'_i \geq \sum_{i=1}^x c_i - \sum_{i=j+1}^x c_i = \sum_{i=1}^j c_i, \quad 1 \leq j \leq x-1.$$

Therefore,

$$\min\{j \geq 1 : \sum_{i=1}^j c_i > \sum_{i=i}^j a'_i\} = x+1.$$

The result follows from Theorem 4.3 (item 2). $\square$

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

References

- [1] A. Amparan, I. Baragaña, S. Marcaida, A. Roca, Row or column completion of polynomial matrices of given degree, *SIAM J. Matrix Anal. Appl.* 45 (1) (2024) 478–503.
- [2] A. Amparan, I. Baragaña, S. Marcaida, A. Roca, Row or column completion of polynomial matrices of given degree II, *Linear Algebra Appl.* 708 (2025) 252–279.
- [3] A. Amparan, S. Marcaida, I. Zaballa, Finite and infinite structures of rational matrices: a local approach, *Electron. J. Linear Algebra* 30 (2015) 196–226.
- [4] L.M. Anguas, F.M. Dopico, R. Hollister, D.S. Mackey, Van Dooren's index sum theorem and rational matrices with prescribed structural data, *SIAM J. Matrix Anal. Appl.* 40 (2) (2019) 720–738.
- [5] I. Baragaña, Interlacing inequalities for regular pencils, *Linear Algebra Appl.* 121 (1989) 521–531.
- [6] N. Bourbaki, *Eléments de Mathématique. Livre II*, Herman, Paris, 1964.
- [7] F. De Terán, F.M. Dopico, P.V. Dooren, Matrix polynomials with completely prescribed eigenstructure, *SIAM J. Matrix Anal. Appl.* 36 (1) (2015) 302–328.

- [8] F. De Terán, F.M. Dopico, D.S. Mackey, Spectral equivalence of matrix polynomials and the index sum theorem, *Linear Algebra Appl.* 459 (2014) 264–333.
- [9] M. Dodig, M. Stošić, On convexity of polynomial paths and generalized majorizations, *Electron. J. Comb.* 17 (2010) 1.
- [10] M. Dodig, M. Stošić, The general matrix completion problem: a minimal case, *SIAM J. Matrix Anal. Appl.* 40 (1) (2019) 347–369.
- [11] P.V. Dooren, P. Dewilde, The eigenstructure of an arbitrary polynomial matrix: Computational aspects, *Linear Algebra Appl.* 50 (1983) 545–579.
- [12] G.D. Forney, Minimal bases of rational vector spaces with applications to multivariable linear systems, *SIAM J. Control* 13 (3) (1975) 143–520.
- [13] G.H. Hardy, J.E. Littlewood, G. Polya, *Inequalities*, Cambridge University Press, London, 1988.
- [14] J. Loiseau, S. Mondié, I. Zaballa, P. Zagalak, Assigning the Kronecker invariants of a matrix pencil by row or column completion, *Linear Algebra Appl.* 278 (1998) 327–336.
- [15] E. Marques de Sá, Imbedding conditions for λ -matrices, *Linear Algebra Appl.* 24 (1979) 33–50.
- [16] H.H. Rosenbrock, *State-Space and Multivariable Theory*, Thomas Nelson and Sons, London, 1970.
- [17] R. Thompson, Interlacing inequalities for invariant factors, *Linear Algebra Appl.* 24 (1979) 1–31.
- [18] A.I.G. Vardulakis, *Linear Multivariable Control*, John Wiley and Sons, New York, 1991.