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Cellular-$\mathcal{P}$ spaces for some Lindelöf-type properties

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Abstract

In this paper, we study two classes of topological spaces: cellular-almost Lindelöf spaces and cellular-weakly Lindelöf spaces. We prove that the classes of cellular-Lindelöf, cellular-weakly Lindelöf and cellular-almost Lindelöf are distinct. In addition, we present a comparative study of these classes.

We also establish some cardinality results. In particular, we prove that, under the assumption of $2^{<\mathfrak{c}} = \mathfrak{c}$, every normal sequential cellular-weakly Lindelöf space of character at most $\mathfrak{c}$ has cardinality at most the continuum. This result generalizes a result by Bella and Spadaro. Furthermore, we prove that if a space X is normal, satisfies the DCCC property, possesses a symmetric g-function g with the property that $\bigcap\{g((n,x)) : n \in \omega\} = \{x\}$ for each $x \in X$ and $H\psi(X) = \omega$, then its cardinality is bounded by $\mathfrak{c}$.

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1. Introduction and preliminaries

The concept of a cellular-Lindelöf space, introduced by A. Bella and S. Spadaro in [2], is a generalization of the notions of Lindelöf, almost discrete Lindelöf space and CCC-spaces. Recall that a topological space X is cellular-Lindelöf if, for each cellular family $\mathcal{U} \in X$, there exists a Lindelöf subspace Y of X such that $U \cap Y \neq \emptyset$ for every $U \in \mathcal{U}$. This notion has attracted significant attention, leading to new results and related questions. For instance, Xuan and Song obtained various results for this class of spaces and studied their connections with other topological classes (see [16] and [17]).

In a more general way, if $\mathcal{P}$ is a topological property, then a topological space X is said to be cellular-$\mathcal{P}$ if for each cellular family $\mathcal{U} \in X$, there exists a subspace Y of X satisfying property $\mathcal{P}$ such that $U \cap Y \neq \emptyset$ for every $U \in \mathcal{U}$. In [1], the authors formally introduced the notion of cellular-$\mathcal{P}$ space and also established some general properties concerning these classes of spaces. Recently, some authors have focused on cellular-compact and cellular-countably compact spaces (see [14], [1] and [9]). In this paper, we extend the study of these “cellular-Lindelöf-like" properties by considering the classes of cellular-almost Lindelöf spaces and cellular-weakly Lindelöf spaces. We make an analysis between the cellular-$\mathcal{P}$ classes, for $\mathcal{P}$ = Lindelöf, $\mathcal{P}$ = almost lindelöf and $\mathcal{P}$ = weakly Lindelöf.

Throughout this paper, all spaces are assumed to be Hausdorff unless otherwise is stated. Given a topological space X and a set $A \subseteq X$, the closure of A is denoted by $\overline{A}$ or $cl_X(A)$. The cardinality of a set X is denoted by $|X|$. If $\mathcal{P}$ is a topological property then $X \in [\mathcal{P}]$ means that X is a topological space with property $\mathcal{P}$. A property $\mathcal{P}$ is said to be preserved under continuity if for any topological spaces X and Y, and for any continuous function $f : X \to Y$, $X \in [\mathcal{P}]$ implies $f[X] \in [\mathcal{P}]$. Moreover, if $\mathcal{P}$ and $\mathcal{Q}$ are properties, we will say that $\mathcal{P}$ implies $\mathcal{Q}$ if whenever X has property $\mathcal{P}$, it also has property $\mathcal{Q}$. This is denoted by $[\mathcal{P}] \subseteq [\mathcal{Q}]$. Also, if $\mathcal{P}$ is a topological property, we define c-$\mathcal{P}$ as the cellular-$\mathcal{P}$ property.

The extent of a topological space X, denoted by $e(X)$, is defined as the supremum of the cardinalities of closed discrete subsets of X.

A cellular family $\mathcal{U}$ in a topological space X is a collection of nonempty, pairwise disjoint open subsets of X.

A space X is a Lindelöf space or X has the Lindelöf property (denoted by L) if for every open cover of X there exists a countable subcover. X is an almost Lindelöf space or X has the almost Lindelöf property (denoted by aL) if for every open cover $\mathcal{U}$ of X there is a countable subfamily $\mathcal{V} \subseteq \mathcal{U}$ such that $X = \bigcup\{\overline{V} : V \in \mathcal{V}\}$. Finally, X is a weakly Lindelöf space or X has the weakly Lindelöf property (denoted by wL) if for every open cover $\mathcal{U}$ of X, there exists a countable subfamily $\mathcal{V} \subseteq \mathcal{U}$ such that $X = \overline{\bigcup \mathcal{V}}$.

Recall that X has the countable chain condition (denoted by CCC) if every cellular family in X is countable and X has the discrete countable chain condition (denoted by DCCC) if any discrete family of nonempty open subsets of X is countable.

In the proof of Theorem 3.5 we will use the elementary submodels technique. For a proper study of this technique, we refer the reader to [6]. We will use the usual notations; $H(\theta)$ represents the collection of sets whose transitive closure has cardinality $< \theta$. If M is an elementary submodel of $H(\theta)$, we will denote this fact by $M \prec H(\theta)$.

Definition 1.1. Let X be a space. The Hausdorff pseudo-character of X, denoted by $H\psi(X)$, is the smallest infinite cardinal κ such that for each $x \in X$, there exists a collection $\mathcal{B}_x = \{V_{\alpha,x} : \alpha < \kappa\}$ of open neighborhoods of x satisfying the following condition: if $x, y \in X$ with $x \neq y$, then there exists $\alpha, \beta < \kappa$ such that $V_{\alpha,x} \cap V_{\beta,y} = \emptyset$.

Definition 1.2. Let (X, τ) be a topological space.

(1) A g-*function* for X is a map $g : \omega \times X \longrightarrow \tau$ such that for every $x \in X$ and each $n \in \omega$ we have $x \in g((n, x))$ and $g((n + 1, x)) \subseteq g((n, x))$.
(2) A g-function g is said to be *symmetric* if for every $n \in \omega$ and $x, y \in X$, it holds that $y \in g((n, x))$ if and only if $x \in g((n, y))$

It is straightforward to verify the following well-known properties for a given property $\mathcal{P}$:

(1) $[\mathcal{P}] \subseteq [c\text{-}\mathcal{P}]$.
(2) $X \in [c\text{-}\mathcal{P}]$ if and only if, for every cellular family of basic open sets in X, there exists a subspace Y of X such that $Y \in [\mathcal{P}]$ and Y intersects all elements in the family.
(3) If X is a discrete space, then $X \in [\mathcal{P}]$ if and only if $X \in [c\text{-}\mathcal{P}]$.
(4) If Y is a dense subspace of X such that $Y \in [\mathcal{P}]$, then $X \in [c\text{-}\mathcal{P}]$.
(5) If $f : X \longrightarrow Y$ is a continuous function and $\mathcal{P}$ is a property that is preserved under continuity, then $X \in [c\text{-}\mathcal{P}]$ implies $f[X] \in [c\text{-}\mathcal{P}]$.
(6) If $\mathcal{P}$ and $\mathcal{Q}$ are topological properties such that $[\mathcal{P}] \subseteq [\mathcal{Q}]$, then $[c\text{-}\mathcal{P}] \subseteq [c\text{-}\mathcal{Q}]$.

The following result establishes that if a property $\mathcal{P}$ is preserved under clopen subspaces, then the cellular-$\mathcal{P}$ property is also preserved under clopen subspaces.

Proposition 1.3. *Let $\mathcal{P}$ be a property that is preserved under clopen subspaces. Then the following are equivalent:*

(1) $X \in [c\text{-}\mathcal{P}]$*;*
(2) $Y \in [c\text{-}\mathcal{P}]$ *for any Y clopen subspace of X.*

Proof. Suppose that $X \in [c\text{-}\mathcal{P}]$. Let Y be a clopen subspace of X and let $\mathcal{A}$ be a cellular family in Y. Clearly, $\mathcal{A}$ is a cellular family in X. Thus, there exists a subspace Z of X such that $Z \in [\mathcal{P}]$ and $A \cap Z \neq \emptyset$ for $A \in \mathcal{A}$. Since $Z \cap Y$ is a clopen subset of Z, it follows that $Z \cap Y \in [\mathcal{P}]$. Now, if $A \in \mathcal{A}$ then

$$(Z \cap Y) \cap A = Z \cap A \neq \emptyset.$$

The other implication follows from the fact that X is a clopen subspace of itself. □

Since every topological property preserved under closed subspaces is also preserved under clopen subspaces, the following is proved.

Corollary 1.4. *Let $\mathcal{P}$ be a property that is preserved under closed subspaces. Then the following are equivalent:*

(1) $X \in [c\text{-}\mathcal{P}]$*;*
(2) $Y \in [c\text{-}\mathcal{P}]$ *for any Y clopen subspace of X.*

We will say that a topological property $\mathcal{P}$ is *preserved under disjoint countable sums* if, for every disjoint countable collection of topological spaces, say $\{X_n : n \in \omega\}$, the following holds: if $X_n \in [\mathcal{P}]$ for each $n \in \omega$, then $\bigoplus_{n\in\omega} X_n \in [\mathcal{P}]$.

Proposition 1.5. *Let $\mathcal{P}$ be a topological property and let $\{X_n : n \in \omega\}$ be a disjoint countable collection of topological spaces.*

(1) *If $\mathcal{P}$ is preserved under disjoint countable sums and $X_n \in [c\text{-}\mathcal{P}]$, for each $n \in \omega$, then $\bigoplus_{n\in\omega} X_n \in [c\text{-}\mathcal{P}]$.*
(2) *If $\mathcal{P}$ is preserved under clopen subspaces, then $\bigoplus_{n\in\omega} X_n \in [c\text{-}\mathcal{P}]$ implies that $X_n \in [c\text{-}\mathcal{P}]$ for each $n \in \omega$.*

Proof. (1) Let $X_n \in [c\text{-}\mathcal{P}]$ for each $n \in \omega$ and let $\mathcal{P}$ be a property preserved under disjoint countable sums. Let $\mathcal{U}$ be a cellular family in $X = \bigoplus_{n\in\omega} X_n$. For each $n \in \omega$ we define $\mathcal{U}'_n = \{U \cap X_n : U \in \mathcal{U} \text{ and } U \cap X_n \neq \emptyset\}$. If $\mathcal{U}'_n = \emptyset$, we set $\mathcal{U}_n = \{X_n\}$; otherwise, let $\mathcal{U}_n = \mathcal{U}'_n$. Then, for every $n \in \mathbb{N}$ we have that $\mathcal{U}_n$ is a cellular family in X_n. Thus, there exists a subspace of X_n, denoted by L_n, which satisfies property $\mathcal{P}$ and intersects all elements of the family U_n. Therefore, $L = \bigoplus_{n\in\omega} L_n$ is a subspace of X such that $L \in [\mathcal{P}]$. If $U \in \mathcal{U}$, then there exists $n \in \omega$ such that $U \cap X_n \neq \emptyset$. Thus,

$$\emptyset \neq L_n \cap (U \cap X_n) \subseteq L_n \cap U \subseteq L \cap U.$$

It follows that X is a cellular-$\mathcal{P}$ space.

(2) For each $n \in \omega$, X_n is a clopen subset of $X = \bigoplus_{n\in\omega} X_n$. Therefore, by Proposition 1.3, it can be deduced that $X_n \in [c\text{-}\mathcal{P}]$. □

A natural question that arises is: For which properties $\mathcal{P}$ is it true that if X and Y are cellular-$\mathcal{P}$ spaces, then their product $X \times Y$ is cellular-$\mathcal{P}$?

The Example 3.8 in [17] shows that the Lindelöf property does not satisfy this condition. Despite this, they show that the product of a cellular-Lindelöf space with a separable space is also a cellular-Lindelöf space. The following result generalizes this result to cellular-$\mathcal{P}$ spaces, provided that the property $\mathcal{P}$ satisfies a couple of additional conditions.

Proposition 1.6. *Let $\mathcal{P}$ be a property preserved under continuity. Additionally, let's assume that for any space Z, if $\{L_n : n \in \omega\}$ is a family of subspaces of Z and $L_n \in [\mathcal{P}]$ for each $n \in \omega$, then $\bigcup_{n\in\omega} L_n \in [\mathcal{P}]$. If $X \in [c\text{-}\mathcal{P}]$ and Y is a separable space, then $X \times Y \in [c\text{-}\mathcal{P}]$.*

Proof. Let $\mathcal{U} = \{U_\alpha \times V_\alpha : \alpha < \kappa\}$ be a cellular family of basic open sets in $X \times Y$. Since Y is separable, there exists a dense subset $D = \{d_n : n \in \omega\}$ of Y. We define a cellular family $\mathcal{U}_n$ in X for each $n \in \omega$ as follows: if there exists $\beta < \kappa$ such that $d_n \in V_\beta$, then $\mathcal{U}_n = \{U_\alpha : d_n \in V_\alpha \text{ and } \alpha < \kappa\}$; otherwise $\mathcal{U}_n = \{X\}$. It is clear that $\mathcal{U}_n$ is a cellular family in X. Hence, there exists $L_n \subseteq X$ such that $L_n \in [\mathcal{P}]$ and L_n intersects all elements in $\mathcal{U}_n$.

It follows that $L = \bigcup_{n\in\omega} L_n \in [\mathcal{P}]$. Clearly, for each $n \in \omega$, $L \times \{d_n\} \in [\mathcal{P}]$. Consequently, we obtain that $L \times D = \bigcup_{n\in\omega}(L \times \{d_n\}) \in [\mathcal{P}]$.

Finally, we prove that $L \times D$ intersects every element of $\mathcal{U}$. Let $\alpha < \kappa$, then there exists $n_0 \in \omega$ such that $d_{n_0} \in V_\alpha$. Since $U_\alpha \in \mathcal{U}_{n_0}$ and L_{n_0} intersects all elements of $\mathcal{U}_{n_0}$, it follwos that $L_{n_0} \cap U_\alpha \neq \emptyset$. Therefore, $(L \times D) \cap (U_\alpha \times V_\alpha) \neq \emptyset$. □

Recall that the underlying set of $AD(X)$, the Alexandroff duplicate of X, is $X \times \{0, 1\}$, where each point of $X \times \{1\}$ is isolated and a basic neighborhood of a point $(x, 0) \in X \times \{0\}$ has the form $[(U \times \{0\}) \cup (U \times \{1\})] \backslash \{(x, 1)\}$, where U is a neighborhood of x in X.

We say that a property $\mathcal{P}$ is countably discrete if the fact that a discrete space X satisfies $\mathcal{P}$, implies that X is countable.

Proposition 1.7. *Let $\mathcal{P}$ be a countably discrete topological property that is preserved under clopen subspaces. If $AD(X) \in [\mathcal{P}]$, then $e(X) = \omega$.*

Proof. Suppose that $e(X) > \omega$ and let L be a closed and discrete subset of X such that $|L| > \omega$. Then, $L \times \{1\}$ is a discrete open subset of $AD(X)$ and has cardinality greater than ω.

Moreover, $L \times \{1\}$ is a closed subset of $AD(X)$. Indeed, let $p \in AD(X) \setminus (L \times \{1\})$. If $p = (x, 1)$, for some $x \in X \setminus L$, then $V_x = \{p\}$ it is an open neighborhood of p contained in $AD(X) \setminus (L \times \{1\})$. Now, let us consider the case when $p = (x, 0)$ for some $x \in X$. Since L is a closed subset of X, if $x \notin L$ then we can choose a neighborhood V_x of x such that $V_x \subseteq X \setminus L$. Clearly, $(V_x \times \{0\}) \cup ((V_x \setminus \{x\}) \times \{1\})$ is a neighborhood of $(x, 0)$ in $AD(X)$ that does not intersect $L \times \{1\}$. If $x \in L$, we can choose a neighborhood V_x of x such that $V_x \cap L = \{x\}$ because L is a discrete space. In this case, $V_x \times \{0\}$ serves as the required neighborhood. Therefore, $L \times \{1\}$ is a clopen subset of $AD(X)$.

Finally, if we assume that $AD(X) \in [\mathcal{P}]$, then it follows that $L \times \{1\} \in [\mathcal{P}]$. Since $L \times \{1\}$ is discrete and $\mathcal{P}$ is a countable discrete property, we obtain that $L \times \{1\}$ is countable, which is a contradiction. □

For every topological property $\mathcal{P}$, if $\mathcal{P}$ is preserved under clopen subspaces and countably discrete, then the property c-$\mathcal{P}$ is preserved under clopen subspaces and is countably discrete. Hence, we have the next corollary.

Corollary 1.8. *Let $\mathcal{P}$ be a topological property that is countably discrete and preserved under clopen subspaces. If $AD(X) \in [c\text{-}\mathcal{P}]$, then $e(X) = \omega$.*

2. Cellular-$\mathcal{P}$, for Lindelöf type properties

In this section we study the behavior of cellular-$\mathcal{P}$ spaces, for $\mathcal{P} \in \{L, aL, wL\}$, and the relationships among the classes $[c\text{-}L]$, $[c\text{-}aL]$, and $[c\text{-}wL]$.

The Lindelöf, almost Lindelöf, and weakly Lindelöf properties have been studied in the literature. It is known that $[L] \subsetneq [aL] \subsetneq [wL]$ (for further details on these properties, we refer the reader to [13]). This implies that $[c\text{-}L] \subseteq [c\text{-}aL]$ and $[c\text{-}aL] \subseteq [c\text{-}wL]$. Since $[wL] \subseteq [DCCC]$, then $[c\text{-}wL] \subseteq [c\text{-}DCCC]$.

If $X \in [c\text{-}DCCC]$ and $\mathcal{U}$ is a discrete family in X, then $\mathcal{U}$ is a cellular family. Therefore, there exists a subspace Y of X such that $Y \in [DCCC]$ and Y intersects all elements of $\mathcal{U}$. The family $\mathcal{U}$ determines a cellular family in Y of the same cardinality as $\mathcal{U}$, so $\mathcal{U}$ is countable. Since $[DCCC] \subseteq [c\text{-}DCCC]$, we conclude that $[DCCC] = [c\text{-}DCCC]$. A similar argument shows that $[CCC] = [c\text{-}CCC]$.

The following diagram summarizes the relationships among Lindelöf, almost Lindelöf, weakly Lindelöf, and $DCCC$ spaces, together with their cellular versions:

$$\begin{tikzcd}
L \arrow[r] \arrow[d] & aL \arrow[r] \arrow[d] & wL \arrow[r] \arrow[d] & DCCC \arrow[d, leftrightarrow] \\
c\text{-}L \arrow[r] & c\text{-}aL \arrow[r] & c\text{-}wL \arrow[r] & c\text{-}DCCC
\end{tikzcd}$$

Now, we will show that the classes of spaces $[c\text{-}L]$, $[c\text{-}aL]$ and $[c\text{-}wL]$ are different classes of topological spaces.

Example 2.1. There exists a Hausdorff cellular-Lindelöf space that is not a Lindelöf space.

Let $X = [(\omega_1 + 1) \times (\omega + 1)] \setminus \{(\omega_1, n) : n \in \omega\}$. For every $x \in X$ let $\mathcal{B}(x)$ be a neighborhood system, defined as follows:

(1) if $x = (\alpha, n) \in \omega_1 \times \omega$, let

$$\mathcal{B}(x) = \{\{x\}\};$$

(2) if $x = (\alpha, \omega) \in \omega_1 \times \{\omega\}$, for $n < \omega$ we define $U_n(x)$ a neighborhood of x as follows

$$U_n(x) = \{(\alpha, m) : n \leq m \leq \omega\},$$

and let

$$\mathcal{B}(x) = \{U_n(x) : n < \omega\}.$$

(3) if $x = (\omega_1, \omega)$, for $\alpha < \omega_1$ we define $U_\alpha(x)$ a neighborhood of x as follows

$$U_\alpha(x) = \{(\beta, n) : \alpha \leq \beta < \omega_1 \text{ y } n < \omega\}.$$

We now consider the set

$$\mathcal{B}(x) = \{U_\alpha(x) : \alpha < \omega_1\}.$$

Recall that a topological space X is nearly-Lindelöf if for every open cover $\{U_\lambda : \lambda \in \Lambda\}$, there exists $\{\lambda_n : n \in \omega\} \subseteq \Lambda$ such that $X = \bigcup\{\text{int}(\overline{U_{\lambda_n}}) : n \in \omega\}$, where $\text{int}(U)$ denotes the interior of the set U. In [4], Cammaroto and Santoro prove that X is a Hausdorff nearly-Lindelöf space which is not Lindelöf.

Furthermore, the space X is cellular-Lindelöf since $(\omega_1 \times \omega) \cup \{(\omega_1, \omega)\}$ is a dense Lindelöf subspace of X. In consequence, this subspace intersects all elements of any cellular family in X.

Example 2.2. There exists a Tychonoff cellular-almost Lindelöf space that is not an almost Lindelöf space.

Let $\mathbb{S}$ be the Sorgenfrey line. It is a well-known fact that $\mathbb{S}$ is a Lindelöf space such that $\mathbb{S} \times \mathbb{S}$ is not Lindelöf. Since $\mathbb{S}$ is a Tychonoff space, then the Sorgenfrey plane $\mathbb{S} \times \mathbb{S}$ is also Tychonoff. Thus, $\mathbb{S} \times \mathbb{S}$ cannot be an almost Lindelöf space. Now, $\mathbb{Q} \times \mathbb{Q}$ is a countable dense subspace of $\mathbb{S} \times \mathbb{S}$. Therefore, $\mathbb{S} \times \mathbb{S}$ is a cellular-almost Lindelöf space.

Example 2.3. There exists a Hausdorff cellular-weakly Lindelöf P-space that is not weakly Lindelöf.

Let $\mathbb{D} = \{0, 1\}$ and define $T = \{x \in \mathbb{D}^{\mathfrak{c}^+} : \varnothing \neq x^{-1}[\{1\}] \in [\mathfrak{c}^+]^{<\omega}\}$. Let X be the G_δ-modification of T. In [15], Tkachuk showed that X is a Hausdorff cellular-Lindelöf P-space but not weakly Lindelöf. In consequence, X is a cellular-weakly Lindelöf space that is not weakly Lindelöf.

Example 2.4. There exists a Hausdorff cellular-almost Lindelöf space that is not a cellular-Lindelöf space.

Let us consider the set ω_1 with the discrete topology and let $\kappa(\omega_1)$ be the Katětov extension of ω_1. Since $\kappa(\omega_1)$ is an H-closed space, $\kappa(\omega_1)$ is almost Lindelöf. Therefore $\kappa(\omega_1)$ is cellular-almost Lindelöf.

Let $\mathcal{A} = \{A_\alpha : \alpha < \omega_1\}$ be a pairwise disjoint family of subsets of ω_1. For $\alpha < \omega_1$, let P_α be a free ultrafilter in ω_1 such that $A_\alpha \in P_\alpha$ and let us consider $\mathcal{U} = \{A_\alpha \cup \{P_\alpha\} : \alpha < \omega_1\}$ and $\mathcal{F} = \{P_\alpha : \alpha < \omega_1\}$. It is clear that $\mathcal{U}$ is a cellular family in $\kappa(\omega_1)$. We claim that there is no Lindelöf subspace of $\kappa(\omega_1)$ that intersects all elements of $\mathcal{U}$. Suppose that L is a Lindelöf subspace of $\kappa(\omega_1)$ intersecting all elements of $\mathcal{U}$. Since the set of free ultrafilters in $\kappa(\omega_1)$ is closed and discrete in $\kappa(\omega_1)$, then $\mathcal{F} \cap L$ is closed and discrete in L. Therefore, $\mathcal{F} \cap L$ is countable. Let $\alpha_0 = \sup\{\alpha < \omega_1 : P_\alpha \in \mathcal{F} \cap L\} + 1$. For each $\alpha > \alpha_0$ let $x_\alpha \in A_\alpha \cap L$ and define $A = \{x_\alpha : \alpha_0 < \alpha < \omega_1\}$. If x is an accumulation point of A in L, then $x \notin \omega_1$. Thus, x is an ultrafilter. Let $x = P_\beta$, for some $\beta \leq \alpha_0$. However, $(A_\beta \cup \{P_\beta\}) \cap L$ intersects at most one element of A. It follows that A is closed and discrete in L. However, this is in contradiction with the assumption that L is Lindelöf.

Example 2.5. There exists a Tychonoff cellular-weakly Lindelöf space that is not a cellular-almost Lindelöf space.

Let $A \subseteq \mathbb{R} \setminus \mathbb{Q}$ be a countable dense subset in $\mathbb{R}$, let κ be a non-countable infinite cardinal, $Q = \mathbb{Q} \times \kappa$ and define

$$X = Q \cup \{(a, \kappa) : a \in A\}.$$

We define a neighborhood system on X as follows. Let $x \in X$:

(1) if $x = (q, \alpha) \in Q$, for each $n \in \mathbb{N}$ let

$$U_n(x) = (B_{\frac{1}{n}}(q) \cap \mathbb{Q}) \times \{\alpha\}$$

be a neighborhood of x, where $B_{\frac{1}{n}}(q)$ is the usual ball of $\mathbb{R}$ (with the Euclidean metric) of radius $\frac{1}{n}$ and center q. Let:

$$\mathcal{B}(x) = \{U_n(x) : n \in \mathbb{N}\}.$$

(2) if $x = (a, \kappa)$, with $a \in A$, for each $F \in [\kappa]^{<\omega}$ and $n \in \mathbb{N}$ let

$$U_{n,F}(x) = [(B_{\frac{1}{n}}(a) \cap A) \times \{\kappa\}] \cup [(B_{\frac{1}{n}}(a) \cap \mathbb{Q}) \times (\kappa \setminus F)]$$

be a neighborhood for x. Finally, define

$$\mathcal{B}(x) = \{U_{n,F}(x) : n \in \mathbb{N} \text{ and } F \in [\kappa]^{<\omega}\}.$$

In [16], it was shown that X is a Hausdorff zero-dimensional and weakly Lindelöf space that does not satisfy the cellular-Lindelöf property. Since X is a weakly Lindelöf space, then X is a cellular-weakly Lindelöf space.

Every regular almost Lindelöf space is a Lindelöf space. Therefore, every regular cellular-almost Lindelöf space is cellular-Lindelöf. Thus, X cannot be cellular-almost Lindelöf.

Example 2.6. There exists a regular *DCCC* space that is not a cellular-weakly Lindelöf space.

Let

$$X = (\omega_1 + 1) \times \omega_1.$$

Given that $\omega_1 + 1$ is compact and ω_1 is countably compact with the usual topology, it follows that X is also countably compact. Since X is regular then X is *DCCC*, by the Theorem 2.1.8 of [5].

We claim that

$$C = \{\{(\alpha, \alpha)\} : \alpha \text{ is a successor ordinal and } \alpha < \omega_1\}$$

is a cellular family such that there is no a weakly Lindelöf subspace that intersects all elements of C. Indeed, let L be a weakly Lindelöf subspace of X. Since the projection on the second coordinate, $\pi_2 : (\omega_1+1)\times\omega_1 \longrightarrow \omega_1$ is continuous and the property of weakly Lindelöf is preserved under continuity, then $\pi_2(L)$ is a weakly Lindelöf space. In ω_1 every weakly Lindelöf space is bounded. Let $\alpha_0 = \sup \pi_2(L)$, then $(\alpha_0 + 1, \alpha_0 + 1) \notin L$. Therefore X is not cellular-weakly Lindelöf space.

The previous examples allow us to construct the following diagram:

$$\begin{tikzcd}
L \arrow[r, shift left] \arrow[d, shift left] & aL \arrow[l, shift left, "/" marking, red] \arrow[r, shift left] \arrow[d, shift left] & wL \arrow[l, shift left, "/" marking, red] \arrow[r, shift left] \arrow[d, shift left] & DCCC \arrow[l, shift left, "/" marking, red] \arrow[d, leftrightarrow] \\
c\text{-}L \arrow[u, shift left, "/" marking, red] \arrow[r, shift left] & c\text{-}aL \arrow[u, shift left, "/" marking, red] \arrow[l, shift left, "/" marking, red] \arrow[r, shift left] & c\text{-}wL \arrow[u, shift left, "/" marking, red] \arrow[l, shift left, "/" marking, red] \arrow[r, shift left] & c\text{-}DCCC \arrow[l, shift left, "/" marking, red]
\end{tikzcd}$$

The diagram above naturally raises questions about whether a cellular-weakly Lindelöf (cellular-almost Lindelöf) space is weakly Lindelöf (almost Lindelöf).

Since the properties of being almost Lindelöf and weakly Lindelöf are properties which are preserved under clopen subspaces (see [13]), we can conclude from Proposition 1.3 that:

Corollary 2.7 ([16], for $\mathcal{P} = L$)**.** *Every clopen subset of a cellular-$\mathcal{P}$ spaces is cellular-$\mathcal{P}$, for* $\mathcal{P} \in \{L,\ aL,\ wL\}$.

Proposition 1.5 states that if $\mathcal{P}$ is preserved under disjoint countable sums, then c-$\mathcal{P}$ is also preserved under the same condition. Given that the almost Lindelöf (aL) and weakly Lindelöf (wL) properties are preserved under clopen subspaces and disjoint countable sums, we obtain the following result.

Corollary 2.8. *Let $\mathcal{P} \in \{$almost Lindelöf, weakly Lindelöf$\}$ and let $\{X_n : n \in \omega\}$ be a disjoint family of topological spaces. Then, $\bigoplus_{n\in\omega} X_n \in [c\text{-}\mathcal{P}]$ if and only if $X_n \in [c\text{-}\mathcal{P}]$ for each $n \in \omega$.*

It is well known that the product of cellular-Lindelöf spaces is not necessarily cellular-Lindelöf. Furthermore, in [7] A. Dow and R.M. Stephenson Jr. provided an example of a cellular-Lindelöf space and a compact space whose product is not cellular-Lindelöf. Next, we provide topological spaces X and Y that are cellular-$\mathcal{P}$ for $\mathcal{P} \in \{aL, wL\}$ but whose product $X \times Y$ is not cellular-$\mathcal{P}$.

Recall that a subset A of $\mathbb{R}$ is totally imperfect [10] if A does not contain a copy of the Cantor set. Bernstein's Theorem [10] states that there exists a subset A of $\mathbb{R}$ such that $|A| = |\mathbb{R} \setminus A| = \mathfrak{c}$ and both A and $\mathbb{R} \setminus A$ are totally imperfect sets.

Example 2.9. There exist two Tychonoff cellular-almost Lindelöf spaces X and Y such that $X \times Y$ is not cellular-almost Lindelöf.

Proof. Let A be the subset of $\mathbb{R}$ as in Bernstein Theorem. Let X and Y be subsets of the Alexandroff duplicate $AD(\mathbb{R})$ defined as follows:

$$X = (A \times \{0\}) \cup (\mathbb{R} \times \{1\}) \text{ and } Y = ((\mathbb{R} \setminus A) \times \{0\}) \cup (\mathbb{R} \times \{1\}).$$

In [11], A.D. Rojas-Sánchez, and Á. Tamariz-Mascarúa proved that X and Y are Tychonoff Lindelöf spaces. Thus, X and Y are cellular-almost Lindelöf.

It is not difficult to prove that $D = \{(p, p) : p \in \mathbb{R} \times \{1\}\}$ is an uncountable clopen discrete subspace of $X \times Y$. Hence, by Proposition 2.4 we conclude that $X \times Y$ is not cellular-almost Lindelöf. □

The next result is a consequence of Proposition 1.6.

Corollary 2.10 ([16], for $\mathcal{P} = L$)**.** *If $\mathcal{P} \in$ {Lindelöf, almot Lindelöf, weakly Lindelöf}, $X \in [c\text{-}\mathcal{P}]$ and Y is a separable space, then $X \times Y \in [c\text{-}\mathcal{P}]$.*

Question 2.11. *Is the product of a compact space and a cellular-almost Lindelöf (cellular-weakly Lindelöf) cellular-almost Lindelöf (cellular-weakly Lindelöf)?*

The following example illustrates that in general, the Alexandroff duplicate of a cellular-$\mathcal{P}$ space is not necessarily a cellular-$\mathcal{P}$ space, for $P \in \{L, aL, wL\}$.

Example 2.12. There is a Tychonoff space X such that $X \in [c\text{-}\mathcal{P}]$ and its Alexandroff duplicate $AD(X) \notin [c\text{-}\mathcal{P}]$, for $\mathcal{P} \in \{\text{L,aL,wL}\}$.

Let D be a discrete space of cardinality ω_1. Suppose that $D = \{d_\alpha : \alpha < \omega_1\}$ and let βD be the Stone-Čech compactification of D. Define

$$X = (\beta D \times (\omega + 1)) \setminus ((\beta D \setminus D) \times \{\omega\}),$$

with the subspace topology. Since $\beta D \times \omega$ is σ-compact, then $\beta D \times \omega \in [\mathcal{P}]$. Furthermore, $\beta D \times \omega$ is dense in X, which implies that $X \in [c\text{-}\mathcal{P}]$. Now, since $A = \{(d_\alpha, \omega) : \alpha < \omega_1\}$ is closed and discrete in X, then $A \times \{1\}$ is a clopen uncountable subset in $AD(X)$; hence, by Proposition 1.7, we conclude that X is a Tychonoff space such that $X \in [c\text{-}\mathcal{P}]$ and $AD(X) \notin [c\text{-}\mathcal{P}]$.

The Sorgenfrey plane, $\mathbb{S} \times \mathbb{S}$, is a cellular-almost Lindelöf space with uncountable extent. Hence, by Corollary 1.8, its Alexandroff duplicate is not cellular-almost Lindelöf. Also, the topological spaces considered in the examples 2.1 and 2.3 are a cellular-$\mathcal{P}$ spaces such that their Alexandroff duplicate is not cellular-$\mathcal{P}$, for $\mathcal{P} = L$ and $\mathcal{P} = wL$, respectively. In [12], Song and Zhang prove that $X \in [L]$ is equivalent to $AD(X) \in [L]$. Consequently, if $X \in [L]$, then $AD(X) \in [c\text{-}L]$.

Question 2.13. *Let $\mathcal{P} \in \{aL, wL\}$ and X be a $\mathcal{P}$ space. Is $AD(X)$ a cellular-$\mathcal{P}$ space?*

Remark 2.14. Let X be the space from the previous example and let $AD(X)$ his Alexandroff duplicate. Define $f : AD(X) \to X$ by $f((x, 0)) = f((x, 1)) = x$ for each $x \in X$. Then f is a closed 2-to-1 continuous map. Thus, the preimage of cellular-$\mathcal{P}$ spaces under a closed 2-to-1 continuous map is not necessarily cellular-$\mathcal{P}$, for $\mathcal{P} \in \{aL, wL\}$.

3. Cardinality of the cellular-$\mathcal{P}$ spaces

The following result is a generalization of Theorem 5.2 given by Xuan and Song in [16]. So, for the sake of completeness we include the proof with the appropriated modifications.

Theorem 3.1. *Let X be a normal and DCCC space with $H\psi(X) = \omega$ and a symmetric g-function g such that for each $x \in X$, $\bigcap\{g((n, x)) : n \in \omega\} = \{x\}$. Then $|X| \leq \mathfrak{c}$.*

Proof. For each $x \in X$ let $\mathcal{B}_x = \{V_{n,x} : n \in \omega\}$ be the family of neighborhoods of x given by the Hausdorff pseudo-character. Suppose that $|X| > \mathfrak{c}$ and let

$$P_{n,m,k} = \{\{x, y\} \in [X]^2 : V_{n,x} \cap V_{m,y} = \emptyset \text{ and } y \notin g((k, x))\}.$$

It is straightforward to prove that $[X]^2 = \bigcup_{n,m,k\in\omega} P_{n,m,k}$.

By the Erdős-Rado lemma (Theorem 2.3, [8]), there exist $n_0, m_0, k_0 \in \omega$ and $S \subseteq X$ infinite and non-countable such that $[S]^2 \subseteq P_{n_0,m_0,k_0}$. Let us assume that S is not closed and discrete in X. Then, there exists an accumulation point $x \in X$ of S. Consequently, there is $y \in (S \setminus \{x\}) \cap g((k_0, x))$. Given that g is symmetric, it follows that $x \in g((k_0, y))$. Since X is T_1, it follows that there exists $z \in S \setminus \{y\} g((k_0, y))$, which is a contradiction.

For each $x \in X$ let $U_x = V_{n_0,x} \cap V_{m_0,x}$ and $\mathcal{U} = \{U_x : x \in S\}$, then $\mathcal{U}$ is a cellular family in X. By the normality of X, there exists an open set W such that $S \subseteq W \subseteq \overline{W} \subseteq \bigcup \mathcal{U}$. It is not difficult to prove that the family of sets, $\{U_x \cap W : x \in S\}$, is discrete in the space X.

Since X has the $DCCC$ property, it follows that $\{U_x \cap W : x \in S\}$ is countable, which contradicts that S is uncountable. $\square$

If $\mathcal{P}$ is a topological property, then $[\mathcal{P}] \subseteq [DCCC]$ if and only if $[c\text{-}\mathcal{P}] \subseteq [DCCC]$.

Corollary 3.2 ([16], Theorem 5.2)**.** *Let X be a normal cellular-Lindelöf space with $H\psi(X) = \omega$ and a symmetric g-function g such that for each $x \in X$, $\bigcap\{g((n,x)) : n \in \omega\}=\{x\}$. Then $|X| \leq \mathfrak{c}$.*

Corollary 3.3. *Let X be a normal space with $H\psi(X) = \omega$ and a symmetric g-function g such that for each $x \in X$, $\bigcap\{g((n,x)) : n \in \omega\}=\{x\}$. If $X \in [c\text{-}\mathcal{P}]$ and $[\mathcal{P}] \subseteq [DCCC]$, then $|X| \leq \mathfrak{c}$, (in particular, if $\mathcal{P} \in \{c\text{-}aL, c\text{-}wL\}$).*

In [3] A. Bella and S. Spadaro showed that every monotonically normal cellular-Lindelöf first-countable space has cardinality at most $\mathfrak{c}$. Furthermore, they showed that under CH every normal first-countable cellular-Lindelöf space is weakly Lindelöf. The following results generalize these results to cellular-weakly Lindelöf spaces. For the sake of completeness, we present its proof.

Lemma 3.4. *Let X be a cellular-weakly Lindelöf space and let $\mathcal{U}$ be an open cover of X such that $|\mathcal{U}| = \mathfrak{c}$. There exists $\mathcal{V} \in [\mathcal{U}]^{<\mathfrak{c}}$ such that $X \subseteq \overline{\bigcup \mathcal{V}}$.*

Proof. Suppose that $\mathcal{U} = \{U_\alpha : \alpha < \mathfrak{c}\}$ is an open cover of X such that if $\mathcal{V} \in [\mathcal{U}]^{<\mathfrak{c}}$, then $X \setminus \overline{\bigcup \mathcal{V}} \neq \emptyset$. Now, let $\{\alpha_\beta : \beta < \mathfrak{c}\}$ be a strictly increasing sequence of ordinals such that $V_\beta = U_{\alpha_{\beta+1}} \setminus \overline{\bigcup\{U_\gamma : \gamma \leq \alpha_\beta\}}$ is a nonempty open set, for every $\beta < \mathfrak{c}$. Since $\mathcal{V} = \{V_\beta : \beta < \mathfrak{c}\}$ is a cellular family and X is a cellular weakly Lindelöf space, then there exists $L \subseteq X$ such that L is a weakly Lindelöf space and $L \cap V_\beta \neq \emptyset$. Therefore, there exists $\{\gamma_n : n < \omega\} \subseteq \mathfrak{c}$ such that $L \subseteq \overline{\bigcup\{U_{\gamma_n} : n \in \omega\}}$. Given that $cf(\mathfrak{c}) > \omega$, there exists $\beta < \mathfrak{c}$ such that $\sup\{\gamma_n : n \in \omega\} < \beta$. Consequently $L \subseteq \overline{\bigcup\{U_\gamma : \gamma \leq \beta\}}$, so $L \cap V_\beta = \emptyset$. However, this contradicts the election of L. $\square$

Theorem 3.5 ($2^{<\mathfrak{c}} = \mathfrak{c}$)**.** *Let X be a normal sequential cellular-weakly Lindelöf space such that $\chi(X) \leq \mathfrak{c}$. Then $|X| \leq \mathfrak{c}$.*

Proof. Let θ be a regular large enough cardinal, and let M be an elementary submodel of $H(\theta)$ such that $X, \tau \in M$, $\mathfrak{c} + 1 \subseteq M$, M is closed under sequences of cardinality less than $\mathfrak{c}$ and $|M| = \mathfrak{c}$.

Let $E = M \cap X$, $x \in X$ and (x_n) be a sequence in E such that $x_n \to x$. Since M is closed under ω-sequences, then $(x_n) \in M$. By elementary, $x \in M$, so $x \in E$. The sequenciality of the space implies that E is closed.

We claim that $X \subseteq E$. Suppose by contradiction that $z \in X \setminus M$. It follows from the regularity of X that there exists an open set U such that $E \subseteq U \subseteq \overline{U} \subseteq X \setminus \{z\}$. By elementary, for every $x \in X$ let $\mathcal{B}(x) \in M$

be a neighbourhood system for x such that $|\mathcal{B}(x)| \leq \mathfrak{c}$. Since $\mathfrak{c}+1 \subseteq M$, then $\mathcal{B}(x) \subseteq M$. Hence, there exists $U_x \in \tau \cap M$ such that $x \in U_x$ and $U_x \subseteq U$. Let $V = \bigcup_{x\in E} U_x$. Now E and $X \setminus V$ are disjoint closed sets, so there are $F, G \in \tau$ such that $E \subseteq F$, $X \setminus V \subseteq G$ and $F \cap G = \emptyset$. Note that $\overline{V} \subseteq X \setminus \{z\}$, so $z \notin \overline{\bigcup_{x\in E} U_x}$.

Since $\{U_x : x \in E\} \cup \{G\}$ is an open cover having cardinality $\mathfrak{c}$, so by the Lemma 3.4, there exists $C \in [E]^{<\mathfrak{c}}$ such that $X \subseteq \overline{\bigcup_{x\in C} U_x} \cup \overline{G}$. But $E \cap \overline{G} = \emptyset$, so $E \subseteq \overline{\bigcup\{U_x : x \in C\}}$. Given that M is closed under κ-sequences, for $\kappa < \mathfrak{c}$, and $|C| < \mathfrak{c}$, then

$$M \vDash X \subseteq \overline{\bigcup\{U_x : x \in C\}}.$$

By elementary,

$$H(\theta) \vDash X \subseteq \overline{\bigcup\{U_x : x \in C\}}.$$

But $z \notin \overline{\bigcup_{x\in E} U_x}$. We conclude that $X \subseteq E \subseteq M$. □

Corollary 3.6. *(CH) Every normal first-countable cellular-weakly Lindelöf space is a weakly Lindelöf space.*

Proof. Since $2^\omega = \omega_1$ implies $2^{<\mathfrak{c}} = \mathfrak{c}$, and by the Theorem 3.5 X has cardinality at most $\mathfrak{c}$. By Lemma 3.4, every open cover of X is either countable or has a countable subcollection with a dense union. □

Question 3.7. *Under what additional conditions is a Hausdorff (regular or Tychonoff) cellular-weakly Lindelöf space weakly Lindelöf?*

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