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Fast orbital convergence reveals more hypercyclic vectors

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Abstract

Let X be an infinite dimensional separable Banach space, $T : X \to X$ be a hypercyclic operator, and $x \in X$ be a (frequently) hypercyclic vector of T. We show that if the terms from the T-orbit of x converge to a vector y sufficiently fast, then y is also a hypercyclic vector of T. As a corollary, we deduce that if T is a frequently hypercyclic operator with spectral radius $r(T) = 1$, then $\lim_{n\to\infty} \|T^n x\|^{1/n} = 1$ for every frequently hypercyclic vector x of T. Some related observations are also made.

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1. Introduction

Consider an infinite dimensional separable Banach space X (over $\mathbb{R}$ or $\mathbb{C}$) and a bounded linear operator $T : X \to X$. We will be addressing the following problem: if x is a (frequently) hypercyclic vector of T and if the terms from the T-orbit of x converge to a vector $y \in X$ very fast, can we conclude that y is a hypercyclic vector of T?

Let $HC(T)$, $FHC(T)$, and $UFHC(T)$ denote respectively the collections of hypercyclic vectors, frequently hypercyclic vectors, and upper frequently hypercyclic vectors of T (precise definitions are given below). Let $r(T)$ denote the spectral radius of T. A subset Y of X is said to be *residual* in X if Y contains

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a dense G_δ subset of X (equivalently if $X \setminus Y$ is of first category in X). The norm on X will be denoted as $\|\cdot\|$. Our aim in this note is to prove the following Theorem.

Theorem 1.1. *Let X be an infinite dimensional separable Banach space and $T : X \to X$ be a bounded linear operator.*
(1) Assume T is frequently hypercyclic, and let $x \in FHC(T)$. Then the set
$Y_1 := \{y \in X : \liminf_{n\to\infty} \|T^n x - y\|^{1/n} = 0\}$ is a dense G_δ subset of X, and $Y_1 \subset UFHC(T) \subset HC(T)$.
(2) Assume T is frequently hypercyclic, $r(T) = 1$, and let $x \in FHC(T)$. Then
$Y_2 := \{y \in X : \liminf_{n\to\infty} \|T^n x - y\|^{1/n} < 1\}$ is a residual subset of X, and $Y_2 \subset UFHC(T) \subset HC(T)$.
(3) Assume T is hypercyclic, and let $x \in HC(T)$. Then there is a strictly increasing function $\alpha : \mathbb{N} \to \mathbb{N}$ such that $Y_3 := \{y \in X : \liminf_{n\to\infty} \|T^n x - y\|^{1/\alpha(n^2)} < 1/r(T)\}$ is a residual subset of X, and $Y_3 \subset HC(T)$.

We remark here that by Theorem 6.45 of [3], there are frequently hypercyclic operators T on infinite dimensional separable Hilbert spaces with $r(T) = 1$. Indeed, the frequently hypercyclic operator T given by Theorem 6.45 of [3] has the property that $\sup\{\|T^n\| : n \in M\} < \infty$ for some infinite set $M \subset \mathbb{N}$, and this implies $r(T) = 1$ (we have $r(T) \leq 1$ by the spectral radius formula, and $r(T) \geq 1$ by hypercyclicity - see Theorem B.3 and Proposition 5.3 in [5]). The reader may refer to the books [3, 5] to get an overview of various features of hypercyclicity. The notion of a frequently hypercyclic operator was introduced by Bayart and Grivaux [2], and this notion relates the theory of Linear Dynamics to Ergodic Theory and Combinatorial Number Theory. Different types of quantitative studies pertaining to hypercyclicity and frequent hypercyclicity have been carried out by researchers in the recent past; see [4, 6], Chapter V of the book [8], the article [9] and the references therein. For some quantitative studies in Topological Dynamics somewhat related to Theorem 1.1, see [7].

To facilitate our investigations, a few definitions are needed, which we write down now. For a subset $A \subset \mathbb{N}$, its *lower density* $Ld(A)$ and *upper density* $Ud(A)$ are defined as follows:

$$Ld(A) = \liminf_{n\to\infty} \frac{|\{1, 2, \ldots, n\} \cap A|}{n}, \tag{1.1}$$

$$Ud(A) = \limsup_{n\to\infty} \frac{|\{1, 2, \ldots, n\} \cap A|}{n}. \tag{1.2}$$

Just for the notational convenience of writing the technical parts of our proofs, it is useful to introduce a generalization of the notion of $Ld(A)$ and $Ud(A)$. For a strictly increasing function $\alpha : \mathbb{N} \to \mathbb{N}$, we define the *lower density* $Ld(\alpha, A)$ and *upper density* $Ud(\alpha, A)$ of $A \subset \mathbb{N}$ with respect to α as follows:

$$Ld(\alpha, A) = \liminf_{n\to\infty} \frac{|\{1, 2, \ldots, \alpha(n)\} \cap A|}{n}, \tag{1.3}$$

$$Ud(\alpha, A) = \limsup_{n\to\infty} \frac{|\{1, 2, \ldots, \alpha(n)\} \cap A|}{n}. \tag{1.4}$$

When α is the identity map I of $\mathbb{N}$, the quantities $Ld(\alpha, A)$ and $Ud(\alpha, A)$ reduce respectively to $Ld(A)$ and $Ud(A)$. Observe that while $Ld(A)$ and $Ud(A)$ are bounded above by 1, there is no upper bound to $Ld(\alpha, A)$ and $Ud(\alpha, A)$: if $\alpha(n) = n^2$ for every $n \in \mathbb{N}$, then $Ld(\alpha, \mathbb{N}) = Ud(\alpha, \mathbb{N}) = \infty$.

We say (X, T) is a *linear dynamical system* if X is an infinite dimensional separable Banach space and $T : X \to X$ is a bounded linear operator. Let (X, T) be a linear dynamical system. The *T-orbit* of a vector $x \in X$ is the set $\{T^n x : n = 0, 1, 2, \ldots\}$. For $x \in X$ and $U \subset X$, let

$$N_T(x, U) = \{n \in \mathbb{N} : T^n x \in U\}. \tag{1.5}$$

A vector $x \in X$ is said to be a *hypercyclic vector* of T if $N_T(x, U) \neq \varnothing$ for every nonempty open set $U \subset X$. Let $HC(T)$ denote the collection of hypercyclic vectors of T. If $HC(T) \neq \varnothing$, then T is called a *hypercyclic operator*. When $\alpha : \mathbb{N} \to \mathbb{N}$ is a strictly increasing function, we say a vector $x \in X$ is an *α-frequently hypercyclic vector* of T if $Ld(\alpha, N_T(x, U)) > 0$ for every nonempty open set $U \subset X$, and an *upper α-frequently hypercyclic vector* of T if $Ud(\alpha, N_T(x, U)) > 0$ for every nonempty open set $U \subset X$. Let $FHC(\alpha, T)$ and $UFHC(\alpha, T)$ denote respectively the collections of α-frequently hypercyclic vectors of T and upper α-frequently hypercyclic vectors of T.

The main use of the set $UFHC(\alpha, T)$ in our proofs will be the fact that the obvious inclusion $UFHC(\alpha, T) \subset HC(T)$ holds.

Let $FHC(T) = FHC(I, T)$ and $UFHC(T) = UFHC(I, T)$ (where I is the identity map of $\mathbb{N}$), which are respectively the collections of *frequently hypercyclic vectors* of T and *upper frequently hypercyclic vectors* of T. If $FHC(T) \neq \varnothing$, then T is called a *frequently hypercyclic operator*. Clearly, $FHC(T) \subset UFHC(T) \subset HC(T)$, and thus every frequently hypercyclic operator is hypercyclic.

To analyze the speed of convergence of terms from the T-orbit of a (frequently) hypercyclic vector x, we will use certain functions $s : \mathbb{N} \times [0, \infty) \to [0, \infty)$, where the requirement on s will be that $s(n, \cdot) : [0, \infty) \to [0, \infty)$ is an increasing homeomorphism for every $n \in \mathbb{N}$. Observe that this condition implies $s(n, 0) = 0$ for each $n \in \mathbb{N}$. A typical example of such a function is $s : \mathbb{N} \times [0, \infty) \to [0, \infty)$ given by $s(n, t) = t^{1/n}$.

2. Proof of the main Theorem

For a nonempty set $A \subset X$ and $z \in X$, let $dist(z, A) = \inf\{\|z - a\| : a \in A\}$.

Lemma 2.1. *Let X be an infinite dimensional separable Banach space, and $s : \mathbb{N} \times [0, \infty) \to [0, \infty)$ be a function such that $s(n, \cdot) : [0, \infty) \to [0, \infty)$ is an increasing homeomorphism for every $n \in \mathbb{N}$. For each*

$n \in \mathbb{N}$, let $A_n \subset X$ be a nonempty set and $T_n : X \to X$ be a bounded linear operator. Then we have the following:
(1) The set $Y_0 := \{y \in X : \liminf_{n\to\infty} s(n, dist(y, T_n(A_n))) = 0\}$ is a dense G_δ subset of X if $\bigcup_{n=k}^{\infty} T_n(A_n)$ is dense in X for each $k \in \mathbb{N}$.
(2) The set $X_0 := \{x \in X : \liminf_{n\to\infty} s(n, dist(T_n x, A_n)) = 0\}$ is a dense G_δ subset of X if $\bigcup_{n=k}^{\infty} T_n^{-1}(A_n)$ is dense in X for each $k \in \mathbb{N}$.

Proof. (1) Note that $Y_0 = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} V_{k,n}$, where $V_{k,n} := \{y \in X : s(n, dist(y, T_n(A_n))) < 1/k\}$. Since the distance function and $s(n, \cdot)$ are continuous, each $V_{k,n}$ is open in X. Moreover, $T_n(A_n) \subset V_{k,n}$ because $s(n, 0) = 0$, and therefore by the denseness of $\bigcup_{n=k}^{\infty} T_n(A_n)$ from the hypothesis, we get that $\bigcup_{n=k}^{\infty} V_{k,n}$ is dense in X for each $k \in \mathbb{N}$. We conclude by Baire category theorem that Y_0 is a dense G_δ subset of X.
(2) Note that $X_0 = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} U_{k,n}$, where $U_{k,n} := \{x \in X : s(n, dist(T_n x, A_n)) < 1/k\}$. Since T_n, the distance function, and $s(n, \cdot)$ are continuous, each $U_{k,n}$ is open in X. Moreover, $T_n^{-1}(A_n) \subset U_{k,n}$ because $s(n, 0) = 0$, and therefore by the denseness of $\bigcup_{n=k}^{\infty} T_n^{-1}(A_n)$ from the hypothesis, we get that $\bigcup_{n=k}^{\infty} U_{k,n}$ is dense in X for each $k \in \mathbb{N}$. We conclude by Baire category theorem that X_0 is a dense G_δ subset of X. □

For the proof of Theorem 1.1, what we need is part (1) of Lemma 2.1. Some applications of part (2) of Lemma 2.1 will be presented at the end of the article.

The next two results are the key tools required in the proof of our main Theorem.

Lemma 2.2. *Let (X, T) be a hypercyclic linear dynamical system, and $\alpha, \beta : \mathbb{N} \to \mathbb{N}$ be two strictly increasing functions such that $\limsup_{n\to\infty} \dfrac{\alpha(kn)}{\beta(n)} < \infty$ for every $k \in \mathbb{N}$. Let $x \in FHC(\alpha, T)$ and $y \in X$.*
(1) If $\liminf_{n\to\infty} \|T^n x - y\|^{1/\beta(n)} = 0$, then $y \in UFHC(\alpha, T) \subset HC(T)$.
(2) If $r(T) = 1$ and $\liminf_{n\to\infty} \|T^n x - y\|^{1/\beta(n)} < 1$, then $y \in UFHC(\alpha, T) \subset HC(T)$.

Proof. We will prove both (1) and (2) together. By the hypercyclicity of T, we have that $\|T\| > 1$ and $r(T) \geq 1$ (see Proposition 5.8 and Proposition 5.3 in [5]). For $z \in X$ and $\varepsilon > 0$, let $B(z, \varepsilon)$ denote the open ball in X centered at z and having radius ε. Suppose for a contradiction that $y \notin UFHC(\alpha, T)$. Then there are $z \in X$ and $\varepsilon \in (0, 1)$ such that

$$Ud(\alpha, N_T(y, B(z, 2\varepsilon))) = 0. \tag{2.1}$$

By the assumptions on α, β, and the assumption that $x \in FHC(\alpha, T)$, we may choose $C > 0$ and integers $k \geq 4$ and $p \geq 1$ such that

$$\alpha(kn) \leq C\beta(n) \text{ for every } n \geq p, \tag{2.2}$$

and

$$Ld(\alpha, N_T(x, B(z,\varepsilon))) > \frac{4}{k}. \tag{2.3}$$

By taking p large enough, we may also assume in view of (2.1) that

$$|\{1 \le j \le \alpha(kn) : \|T^j y - z\| < 2\varepsilon\}| \le n \text{ for every } n \ge p. \tag{2.4}$$

The assumptions of both (1) and (2) in this Lemma imply $\liminf_{n\to\infty} \|T^n x - y\|^{1/\beta(n)} < r(T)^{-C}$. Hence we may choose $\delta \in (0,1)$ close to 1 and an infinite set $M_0 \subset \mathbb{N} \cap [p, \infty)$ such that

$$\|T^m x - y\|^{1/\beta(m)} < r(T)^{-C} \delta^{2C} \text{ for every } m \in M_0. \tag{2.5}$$

Since $\delta^{C\beta(m)} < \varepsilon < 1$ for all sufficiently large m, we may suppose after discarding finitely many members of M_0 that

$$\|T^m x - y\| < r(T)^{-C\beta(m)} \delta^{C\beta(m)} \varepsilon \text{ for every } m \in M_0. \tag{2.6}$$

Since $\lim_{n\to\infty} \|T^n\|^{1/n} = r(T) < r(T)\delta^{-1}$ by the spectral radius furmula (see Theorem B.3 in [5]), there is $q \in \mathbb{N}$ such that

$$\|T^n\| < r(T)^n \delta^{-n} \text{ for every } n \ge q. \tag{2.7}$$

Let $M = M_0 \cap [q, \infty)$ and note that $m \ge \max\{p, q\}$ for every $m \in M$.

For any $m \in M$ and $1 \le j \le \alpha(km) - m$, we see by (2.7) and (2.6) that

$$\|T^{2m+j} x - T^{m+j} y\| \le \|T^{m+j}\| \|T^m x - y\| < r(T)^{m+j} \delta^{-(m+j)} r(T)^{-C\beta(m)} \delta^{C\beta(m)} \varepsilon. \tag{2.8}$$

For $m \in M$ and $1 \le j \le \alpha(km) - m$, we have $m + j \le \alpha(km) \le C\beta(m)$ by (2.2). As $\delta < 1 \le r(T)$, we deduce from (2.8) that

$$\|T^{2m+j} x - T^{m+j} y\| < \varepsilon \text{ for every } m \in M \text{ and } 1 \le j \le \alpha(km) - m. \tag{2.9}$$

Combining (2.4) and (2.9), we obtain by the triangle inequality that

$$|\{1 \le j \le \alpha(km) - m : \|T^{2m+j} x - z\| < \varepsilon\}| \le m \text{ for every } m \in M. \tag{2.10}$$

We explain this step a little more. Fix $m \in M$, and note that $m \ge p$ by the choice of M. Let $J_m = \{1 \le j \le \alpha(km) - m : \|T^{2m+j} x - z\| < \varepsilon\}$ and $I_m = \{1 \le i \le \alpha(km) : \|T^i y - z\| < 2\varepsilon\}$, the sets appearing in (2.10) and (2.4) respectively. If $j \in J_m$, then by (2.9) and the definition of J_m, we have that

$\|T^{m+j}y - z\| \leq \|T^{m+j}y - T^{2m+j}x\| + \|T^{2m+j}x - z\| < \varepsilon + \varepsilon = 2\varepsilon$; and moreover $m + j \leq \alpha(km)$. In other words, $m + j \in I_m$ for every $j \in J_m$. Since $|I_m| \leq m$ by (2.4), we must have $|J_m| \leq m$, which is precisely the statement in (2.10).

Hence

$$|\{1 \leq i \leq \alpha(km) : \|T^i x - z\| < \varepsilon\}| \leq 2m + m = 3m \text{ for every } m \in M. \tag{2.11}$$

This would imply that $Ld(\alpha, N_T(x, B(z, \varepsilon))) \leq \dfrac{3}{k}$, which contradicts the choice of k in (2.3). □

Lemma 2.3. *Let (X, T) be a hypercyclic linear dynamical system, and $\alpha, \beta : \mathbb{N} \to \mathbb{N}$ be two strictly increasing functions with the property that for each $k \in \mathbb{N}$, we have that $\alpha(kn) \leq \beta(n)$ for all but finitely many $n \in \mathbb{N}$. If $x \in FHC(\alpha, T)$, $y \in X$, and $\liminf_{n\to\infty} \|T^n x - y\|^{1/\beta(n)} < 1/r(T)$, then $y \in UFHC(\alpha, T) \subset HC(T)$.*

Proof. This result follows from the proof of Lemma 2.2 by taking $C = 1$ in (2.2). □

Now we are in a position to establish Theorem 1.1.

Proof of Theorem 1.1. (1) Consider $x \in FHC(T)$. Since $FHC(T) \subset HC(T)$ and $T(HC(T)) \subset HC(T)$, we have that $\{T^n x : n \geq k\}$ is dense in X for each $k \in \mathbb{N}$. Applying part (1) of Lemma 2.1 with $s(n, t) := t^{1/n}$, $T_n := T^n$, and $A_n := \{x\}$ for every $n \in \mathbb{N}$, we deduce that the given set Y_1 is a dense G_δ subset of X. Taking α and β to be the identity map of $\mathbb{N}$, we see that the hypothesis of Lemma 2.2 is satisfied. Hence $Y_1 \subset UFHC(T) \subset HC(T)$ by part (1) of Lemma 2.2.

(2) Consider $x \in FHC(T)$. The dense G_δ subset Y_1 of X from part (1) is included in Y_2 and hence Y_2 is residual in X. Taking α and β to be the identity map of $\mathbb{N}$, we see that the hypothesis of Lemma 2.2 is satisfied. Hence $Y_2 \subset UFHC(T) \subset HC(T)$ by part (2) of Lemma 2.2.

(3) Consider $x \in HC(T)$. Let $\{B_p : p \in \mathbb{N}\}$ be a countable base of open balls for X. For each $p \in \mathbb{N}$, write $N_T(x, B_p) = \{k(p, 1) < k(p, 2) < k(p, 3) < \cdots\}$. Define $\alpha : \mathbb{N} \to \mathbb{N}$ as $\alpha(n) = \max\{k(p, n) : 1 \leq p \leq n\}$. Clearly, α is strictly increasing and

$$|\{1, 2, \ldots, \alpha(n)\} \cap N_T(x, B_p)| \geq n \text{ for every } p \in \mathbb{N} \text{ and every } n \geq p. \tag{2.12}$$

Therefore, $Ld(\alpha, N_T(x, B_p)) \geq 1$ for every $p \in \mathbb{N}$. As $\{B_p : p \in \mathbb{N}\}$ is a base for X, we conclude that $x \in FHC(\alpha, T)$. Let $Y_0 = \{y \in X : \liminf_{n\to\infty} \|T^n x - y\|^{1/\alpha(n^2)} = 0\}$. Applying part (1) of Lemma 2.1 with $s(n, t) := t^{1/\alpha(n^2)}$, $T_n := T^n$, and $A_n := \{x\}$ for every $n \in \mathbb{N}$, we deduce that Y_0 is a dense G_δ subset of X. The inclusion $Y_0 \subset Y_3$ implies that Y_3 is residual in X. Define $\beta : \mathbb{N} \to \mathbb{N}$ as

$\beta(n) = \alpha(n^2)$. Then β is strictly increasing, and $\alpha(kn) \le \alpha(n^2) = \beta(n)$ for every $k \in \mathbb{N}$ and $n \ge k$. Hence $Y_3 \subset UFHC(\alpha, T) \subset HC(T)$ by Lemma 2.3. □

Corollary 2.4. *Let (X, T) be a frequently hypercyclic linear dynamical system with $r(T) = 1$. Then $\lim_{n\to\infty} \|T^n x\|^{1/n} = 1$ for every $x \in FHC(T)$.*

Proof. Let $x \in FHC(T)$. Since $x \neq 0$, we have that $\lim_{n\to\infty} \|x\|^{1/n} = 1$. Combining this with the inequality $\|T^n x\| \le \|T^n\|\|x\|$ and using the spectral radius formula (Theorem B.3 in [5]), we get that $\limsup_{n\to\infty} \|T^n x\|^{1/n} \le r(T) = 1$. If $\liminf_{n\to\infty} \|T^n x\|^{1/n} < 1$, then part (2) of Theorem 1.1 would imply that $0 \in UFHC(T) \subset HC(T)$, which is a contradiction. Thus we must have $\lim_{n\to\infty} \|T^n x\|^{1/n} = 1$. □

3. Additional observations

We now indicate some applications of part (2) of Lemma 2.1. The Proposition below is stated only for the special case where $s(n, t) = t^{1/n}$, but is more generally true for any function $s : \mathbb{N} \times [0, \infty) \to [0, \infty)$ such that $s(n, \cdot) : [0, \infty) \to [0, \infty)$ is an increasing homeomorphism for every $n \in \mathbb{N}$. The notation $\ker(T)$ stands for the kernel of the operator T.

Proposition 3.1. *Let (X, T) be a linear dynamical system.*
(1) If the set of periodic points of T is dense in X, then $\{x \in X : \liminf_{n\to\infty} \|T^n x - x\|^{1/n} = 0\}$ is a dense G_δ subset of X.
(2) If $\bigcup_{n=1}^{\infty} \ker(T^n)$ is dense in X, then $\{x \in X : \liminf_{n\to\infty} \|T^n x\|^{1/n} = 0\}$ is a dense G_δ subset of X.
(3) Assume T^n is not identically zero for every $n \in \mathbb{N}$. If $y \in X$ is such that $\bigcup_{n=1}^{\infty} T^{-n}(y)$ is dense in X, then $\{x \in X : \liminf_{n\to\infty} \|T^n x - y\|^{1/n} = 0\}$ is a dense G_δ subset of X.
(4) Assume that T is hypercyclic and $y \in HC(T)$. Then for every $p < q$ in $\mathbb{N}$, the set $\{x \in X : \liminf_{n\to\infty} \|T^{pn} x - T^{qn} y\|^{1/n} = 0\}$ is a dense G_δ subset of X.

Proof. In all the applications of Lemma 2.1(2) below, the function $s : \mathbb{N} \times [0, \infty) \to [0, \infty)$ is to be taken as $s(n, t) = t^{1/n}$.
(1) Apply Lemma 2.1(2) with $T_n := T^n - I$ and $A_n := \{0\}$.
(2) Apply Lemma 2.1(2) with $T_n := T^n$ and $A_n := \{0\}$.
(3) First we claim that the closed set $T^{-n}(y)$ is nowhere dense in X for each $n \in \mathbb{N}$. If $T^{-n}(y)$ is not nowhere dense, then there exist $u \in X$ and $\varepsilon > 0$ with $B(u, \varepsilon) \subset T^{-n}(y)$; and then $T^n(B(0, \varepsilon)) = T^n(B(u, \varepsilon) - u) = \{0\}$, which would imply $T^n \equiv 0$, a contradiction. This proves the claim. From the hypothesis and the claim, it follows that $\bigcup_{n=k}^{\infty} T^{-n}(y)$ is dense in X for every $k \in \mathbb{N}$. Now apply Lemma 2.1(2) with $T_n := T^n$ and $A_n := \{y\}$.
(4) $y \in HC(T^{q-p})$ by [1]. Apply Lemma 2.1(2) with $T_n := T^{pn}$ and $A_n := \{T^{qn} y\}$. □

In this article, we have presented a few results in the setting of Linear Chaos demonstrating that various interesting conclusions can be drawn by studying the speed of convergence of terms from the orbit of a (frequently) hypercyclic vector. Possibly there is much more to discover in this line of research.

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