Some remarks on Phelps property U of a Banach space into $C(K)$ spaces

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Abstract

A subspace X of a Banach space Y has *Property U* whenever every continuous linear functional on X has a unique norm-preserving (i.e., Hahn–Banach) extension to Y (Phelps, 1960). Throughout this document, we introduce and develop a systematic study of the existence of *U-embeddings* between Banach spaces X and Y, that is, isometric embeddings of X into Y whose ranges have property U. In particular, we focus on the case $Y = C(K)$, where K is a compact Hausdorff topological space. We provide results for X a reflexive or another $C(K)$ space.

1 Introduction

The uniqueness of norm-preserving extensions of continuous linear forms on a closed subspace is an old and broadly studied topic within the Geometry of Banach spaces field. Recall that a subspace X of a Banach space Y has the Phelps property U if every continuous linear form on X admits a unique norm-preserving extension to Y. Prior to the systematic study of that property, initiated by Phelps in [21], the Banach spaces satisfying that all their subspaces have property U was characterized in [12].

Over the years, this topic has been developed in many directions. We mention a few of them without the intention of being exhaustive. Phelps himself [21] found a characterization of subspaces with U property in terms of its annihilator, Lima [16] —and Oja and Põldvere [17] afterward— found intrinsic, in terms of the geometry of Y, characterizations of them. The natural

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question of whether the map that sends a functional to its Hahn–Banach extension is linear has also been well studied, and it can be traced back to [22]. One of the main directions is the study of a Banach space having property U into its bidual, which started with Smith and Sullivan [23, 24] and was continued in recent works by Oja, Viil, and Werner [18] and Cobollo, Guirao, and Montesinos [5]. Through the last thirty years, many other mathematicians have contributed to the development of the understanding of this topic. Still today, it is an active line of research as evidenced by the recent works [26], [15], [8], and [7].

This paper focuses on those Banach spaces that are isometric to a subspace of $Y = C(K)$ —for certain compact space K— with property U. It is organized into six sections. Section 1 introduces the topic and fixes the common ground and well-known and well-established results needed throughout this note. Section 2 introduces the notion of U-embedding and presents some direct consequences. It is worth pointing out that this is a device that, being essentially a change of notation, allows us to focus on the nature of the Banach space itself and not on the precise way they are fixed subspaces. In Section 3, we focus on the case $Y = C(K)$, establishing facial structure consequences regarding U-embeddings. In Section 4, some necessary conditions for the existence of U-embeddings are found. Section 5 gives some negative results on U-embeddings into $C(K)$ spaces. In Section 6, we present our main results. Namely, in Subsection 6.1, a necessary condition for the existence of a wU-embedding into $C(K)$ is presented —Theorem 6.6. In Subsection 6.2, we give a characterization of finite-dimensional spaces (the proof of which works for the reflexive case) that can be U-embedded into a C(K) space —Theorem 6.7. In Subsection 6.3, some characterizations and some straightforward applications are proved for those $C(K)$ spaces that can be U-embedded into another $C(S)$ space —Theorems 6.14 and 6.21. Finally, in Subsection 6.4, some applications of the previous results are given.

Notation and preliminaries

All topological spaces here are supposed to be Hausdorff.

Let X be a (real) Banach space (henceforth referred to as a *space*). We denote by B_X its closed unit ball and by S_X its unit sphere. The word *subspace* will refer to a *closed linear* subspace of a given space. Given a space Y and a subspace X, let us denote by $i\colon X \to Y$ the canonical inclusion from X into Y and by $q\colon Y^* \to X^*$ its adjoint (a quotient map).

Recall that the norm of a Banach space, being a convex function, satisfies that it is Gâteaux differentiable at $x \neq 0$ if and only if the subdifferential of

the norm at x is a singleton —see [11, Definition 7.11 and Theorem 7.17].
Given a compact topological space K, $C(K)$ is the Banach space of all real-valued continuous functions on K endowed with the supremum norm. If K is, moreover, a subset of a locally convex space, $A(K)$ ($\subset C(K)$) is the closed subspace of all affine functions. The symbol δ will indistinctly denote the mapping from K to $M(K)$ or to $A(K)^*$ that to $k \in K$ associates $\delta_k := \delta(k)$ such that $\langle f, \delta_k \rangle = f(k)$ for $k \in K$ and $f \in C(K)$ or $f \in A(K)$. The set $\delta(K)$ is, in both cases, homeomorphic to K when endowed with the restriction of the w^*-topology.
Given two subsets of a set S, $A \dot{\cup} B$ will denote $A \cup B$ whenever $A \cap B = \emptyset$.
For additional notation and all non-defined concepts, we refer to [11].

Given $x^* \in X^*$, the set of all continuous linear extensions of x^* to Y is $q^{-1}(x^*)$, and the set of all Hahn–Banach (i.e., norm-preserving) extensions of x^* to Y is

$$\mathcal{HB}(x^*) := q^{-1}(x^*) \cap \|x^*\| S_{Y^*} = (y^* + X^{\perp}) \cap \|x^*\| S_{Y^*}. \tag{1}$$

The set of all Hahn–Banach extensions of the elements in a subset $A \subset X^*$ will be denoted by $\mathcal{HB}(A)$, i.e.,

$$\mathcal{HB}(A) := \{y^* \in Y^* : \ y^* \in \mathcal{HB}(x^*) \text{ for some } x^* \in A\}$$

The following is the central concept treated here:

Definition 1.1. *Let X be a subspace of a Banach space Y. It is said that X has* **property U in** Y *(or X* **is U-embedded in** Y*) whenever $\mathcal{HB}(x^*)$ is a singleton for every $x^* \in S_{X^*}$.*

This property was introduced by Phelps in [21]. He started a systematic study of property U in certain classical Banach spaces. Later, Smith and Sullivan considered the property for the particular case $Y := X^{**}$ under the name of *Hahn–Banach smoothness* —see [23, 24].

As we mentioned previously, it was in 1939 when Taylor —see [25]— studied the question of identifying which Banach spaces Y satisfy that any of its subspaces has property U in Y. Foguel provided the complete characterization. We include it here for later reference and omit its proof.

Proposition 1.2 (Taylor–Foguel, [12])**.** *Let $(Y, \|\cdot\|)$ be a Banach space. The dual norm $\|\cdot\|^*$ on Y^* is strictly convex if, and only if, every subspace of Y has property U in Y.*

Remark 1.3. The set of extreme points of a subset A of a space X will be denoted by Ext A. Assume that X is a subspace of a Banach space Y and that X has property U in Y. As well known and easy to see, every element $e_1^* \in \text{Ext } B_{X^*}$ is the image through q of an element $e_0^* \in \text{Ext } B_{Y^*}$. In this case, this element e_0^* is unique. This shows that $\mathcal{HB}(\text{Ext } B_{X^*}) \subset \text{Ext } B_{Y^*}$. ®

A rewording of Definition 1.1 above is that a subspace X of a space Y has property U in Y if, and only if, $q|_{\mathcal{HB}(S_{X^*})} \colon \mathcal{HB}(S_{X^*}) \to S_{X^*}$ is one-to-one. It is an easy observation that, in this case, $q|_{\mathcal{HB}(S_{X^*})}$ is a w^*-w^*-homeomorphism from $\mathcal{HB}(S_{X^*})$ onto S_{X^*}. This is clearly equivalent to having property U.

Faces of the dual unit ball

Let X be a Banach space. Following the usual convention, a **face** of B_{X^*} is a nonempty convex set $\mathfrak{F} \subset B_{X^*}$ with the following property: If $y^*, z^* \in B_{X^*}$ satisfy $(y^*, z^*) \cap \mathfrak{F} \neq \emptyset$, then $[y^*, z^*] \subset \mathfrak{F}$. A face of B_{X^*} is called w**-exposed** (respect. w^***-exposed**) when it is of the form

$$\mathfrak{F}(x^{**}) = \{x^* \in S_{X^*} : \langle x^{**}, x^* \rangle = 1\} \tag{2}$$

for some $x^{**} \in S_{X^{**}}$ (respect. for some $x^{**} \in S_X$). Remarks 1.4 and 1.5 below collect some easy and well-known facta —together with some definitions— about faces of B_{X^*} and, in particular, of $B_{C(K)^*}$, for later reference.

Remark 1.4. On faces of B_{X^*}

(i) There are faces of B_{X^*} that are not $\|\cdot\|$-closed (and then not w^*-closed), as $\mathfrak{F} := \text{conv}\,\{e_n^* : n \in \mathbb{N}\} \subset B_{\ell_1}$ (here, e_n^* is the n-th vector of the canonical basis of ℓ_1).

(ii) Every proper face of B_{X^*} is a subset of a w-exposed face, in particular, of S_{X^*}.

(iii) In the non-reflexive case, $\mathfrak{F}(x^{**})$ can be empty: Just take an element $x^{**} \in S_{X^{**}}$ that does not attain its norm.

(iv) If $x \in S_X$, the set $\mathfrak{F}(x)$ (always nonempty) is the subdifferential of the norm on X at the point x —see, e.g., [11, Definition 7.11 and Lemma 7.19].

(v) A face can be w^*-compact and yet not a subset of a w^*-exposed face of B_{X^*}. Indeed, every separable Banach space has an equivalent norm $\|\cdot\|$ such that $B_{(X^*,\|\cdot\|^*)}$ is strictly convex —see [9]. Thus, if X is not reflexive, James' theorem provides an element $x^* \in S_{(X^*,\|\cdot\|^*)}$ that does not attain its norm; since it is an extreme point of B_{X^*}, the set $\{x^*\}$ is a (w^*-closed) face of $B_{(X^*,\|\cdot\|^*)}$ that is not a subset of a w^*-exposed face of $B_{(X^*,\|\cdot\|^*)}$. ®

Remark 1.5. On faces of $B_{C(K)^*}$

(i) In case $X := C(K)$ for some compact topological space K, *every proper w^*-compact face $\mathfrak{F}$ of $B_{M(K)}$ is a (Choquet) simplex* (for the definition, see, e.g., [20]). This is probably well known. Yet, let us sketch the argument:

Observe first that $\mathfrak{F}$ is always contained in $\mathfrak{F}(g)$ for some $g \in B_{C(K)}$ (compare with (v) in Remark 1.4 above). To show this, notice that

$$\text{Ext } \mathfrak{F} = \text{Ext } B_{M(K)} \cap \mathfrak{F} \subset \text{Ext } B_{M(K)} = \{\pm\delta_k : \ k \in K\}. \tag{3}$$

In particular, Ext $\mathfrak{F}$ is w^*-compact. Thus, we may consider the two (disjoint and w^*-compact) subsets of K

$$K^+_{\mathfrak{F}} := \{k \in K : \ \delta_k \in \text{Ext } \mathfrak{F}\} \ \text{ and } \ K^-_{\mathfrak{F}} := \{k \in K : \ -\delta_k \in \text{Ext } \mathfrak{F}\}.$$

By the Krein–Milman theorem, we get

$$\mathfrak{F} = \overline{\text{conv}}\,(\text{Ext } \mathfrak{F}) = \overline{\text{conv}}\,(\delta(K^+_{\mathfrak{F}}) \cup -\delta(K^-_{\mathfrak{F}})). \tag{4}$$

Now, if $g \in S_{C(K)}$ satisfies $g|_{K^+_{\mathfrak{F}}} \equiv 1$ and $g|_{K^-_{\mathfrak{F}}} \equiv -1$, we get from (4) that $\mathfrak{F} \subset \mathfrak{F}(g)$.

Every proper w^*-exposed face $\mathfrak{F}(f)$ of $B_{C(K)^*}$ is affinely homeomorphic to the set of probability measures —i.e., $\mathfrak{F}(1)$, where 1 is here the constant-one function— on a compact subset of K. Thus, each $\mathfrak{F}(f)$ is a w^*-compact simplex (see, e.g., [1, Theorem II.3.6].

Notice, finally, that the family $\mathcal{C} := \{\mathfrak{F}(f) : \ f \in S_{C(K)}, \ \mathfrak{F} \subset \mathfrak{F}(f)\}$ is downward directed, and $\mathfrak{F} = \bigcap \mathcal{C}$.

It follows from the preceding observations and [20, Theorem 16.3] that $\mathfrak{F}$ is a simplex, too.

(ii) The proper w^*-compact faces are precisely the w^*-exposed faces if, and only if, every closed subset of K is a zero-set.

(iii) The subject of faces of $C(K)^*$ will be revisited in Subsection 3.3 below. ®

Remark 1.6. On faces of $A(K)^*$

Assume now that K is a compact convex subset of a locally convex space. Let 1 denote the constant-1 function on K. It is simple to show that $\mathfrak{F}(1)$ in its w^*-topology is affine homeomorphic to K, that Ext $B_{A(K)^*} = q\delta(\text{Ext }(K)) \cup (-q\delta(\text{Ext }(K)))$, and that Ext $\mathfrak{F}(1) = q\delta(\text{Ext }(K))$. This description allows for showing that, if $\sharp K \geq 2$, then the canonical injection $j : A(K) \to C(K)$ is not a U-embedding —even not a wU-embedding, see the next entry—. Indeed, if $k_1 \neq k_2$ are two points in K, then $\delta_{\frac{k_1+k_2}{2}}$ and $\frac{\delta_{k_1}+\delta_{k_2}}{2}$ in $S_{M(K)}$ are two different Hahn–Banach extensions of the same element in $S_{A(K)^*}$. ®

Norm-attaining functionals: Property wU

Let Y be a Banach space and X a subspace of Y. Denote by $\text{NA}(X)$ the subset of X^* of all norm-attaining functionals. If $x_0^* \in \text{NA}(X) \cap S_{X^*}$ and $x_0 \in S_X$ is such that $\langle x_0, x_0^* \rangle = 1$, then

$$q^{-1}(x_0^*) \subset \{y^* \in Y^* : \ \langle x_0, y^* \rangle = 1\}. \tag{5}$$

The following is a weakening of property U. Certainly if X is a reflexive subspace of Y, the two properties for X coincide:

Definition 1.7. *A subspace X of a space Y has* **property wU** (**weak U**) *in Y (or X* **is wU-embedded in** *Y) whenever $\mathcal{HB}(x^*)$ is a singleton for every $x^* \in \text{NA}(X)$.*

The origin of this property can be traced back to Smith and Sullivan [23] and Lima [16], under the name *weak Hahn–Banach smoothness*. To our knowledge, the term "wU" appeared for the first time in [4] as the analogous version of property U for norm-attaining elements and has also been studied in the recent [7, 8].
When X has property wU in Y, then $q|_{\mathfrak{F}(i(x_0))}$ is a w^*-w^*-homeomorphism from $\mathfrak{F}(i(x_0))$ onto $\mathfrak{F}(x_0)$ for every $x_0 \in S_X$. Again, this is equivalent to having property wU.
The w^*-w^*-homeomorphisms between w^*-exposed faces are given by the restrictions of the linear operator q. This means that not only the topological but somehow also the affine structure of the (convex) w^*-exposed faces $\mathfrak{F}(i(x_0))$ and $\mathfrak{F}(x_0)$ is preserved. This applies, for example, to the following situation: If $n \in \mathbb{N} \cup \{0\}$, a point $x_0 \in S_X$ is said to be **of n-almost Gâteaux smoothness** whenever $\mathfrak{F}(x_0)$ has affine dimension n (thus, 0-almost Gâteaux smoothness is just Gâteaux smoothness —see [11, Theorem 7.17]). We have then that a wU-embedding sends points of n-almost Gâteaux

smoothness to points of n-almost Gâteaux smoothness. Even more, $\mathfrak{F}(i(x_0))$ and $\mathfrak{F}(x_0)$ share the same extremal structure.

Example 1.8. For $C(K)$, where K is a compact Hausdorff space, the w^*-exposed face determined by an n-almost Gâteaux smooth point in $S_{C(K)}$ would be an n-simplex —since the set of extreme points of $B_{C(K)^*}$ is affinely independent (see also Remark 1.5 above). Therefore, we can only hope to U-embed (wU-embed) Banach spaces X into $C(K)$ spaces in case all n-almost Gâteaux smooth points in S_X define w^*-exposed faces in B_{X^*} affinely isomorphic to an n-simplex. For $n = 1$, this leads to a necessary condition for U-embeddings in $C(K)$-spaces —see Proposition 4.2 below.

The following is an almost trivial consequence of previous observations:

Corollary 1.9. *A Banach space Y contains a U-embedded one-dimensional subspace if, and only if, Y has at least a Gâteaux-smooth point.*

Gâteaux smoothness appears as the wU-counterpart of strict convexity on the dual (this assertion is justified by looking at Proposition 1.10 below, that should be compared with Proposition 1.2 above).

Proposition 1.10. *Let Y be a Banach space. Its norm is Gâteaux differentiable if, and only if, every subspace X of Y has property wU in Y.*

Proof. Assume first that $\|\cdot\|$ on Y is Gâteaux differentiable. Let X be a subspace of Y and $i\colon X \to Y$ be the canonical inclusion mapping. Given $x_0^* \in NA(X) \cap S_{X^*}$ that attains its norm at $x_0 \in S_X$, we have —see (1) and (5) above— that

$$\mathcal{HB}(x_0^*) = q^{-1}(x_0^*) \cap S_{Y^*} \subset \mathfrak{F}(i(x_0)), \tag{6}$$

The last set in (6) is a singleton due to the Gâteaux differentiability of the norm at $i(x_0)$. Indeed, being the norm a convex function, Gâteaux differentiability at $i(x_0)$ is equivalent to the subdifferential of the norm at $i(x_0)$ being a singleton —see [11, Theorem 7.17]. This shows the necessity. For the sufficiency, assume that $\|\cdot\|$ on Y is not Gâteaux differentiable at some point $x \in Y$. Then we can find two distinct points y_1^* and y_2^* in S_{Y^*} such that $\langle x, y_1^*\rangle = \langle x, y_2^*\rangle = 1$. Thus, if $X := \ker(y_1^* - y_2^*)$, we have $y_1^*|_X = y_2^*|_X$, and, since $x \in X$, both y_1^* and y_2^* belong to $\mathrm{NA}(X)$. $\square$

In Example 1.8 above, we could already appreciate one of the particularities when embedding into $C(K)$ spaces, which is the main topic of the paper. The structural behavior of this kind of space will lead us to develop some specific tools, but it is worth noting that without further development, we can recover one of Phelps' results for the $C(K)$ case.

Corollary 1.11 (Phelps, [21, Theorem 3.2])**.** *Let K be a compact Hausdorff space, and let X be a finite-dimensional subspace of $(C(K), \|\cdot\|_\infty)$. Let $\dim(X) = n$. Assume that X has property U in $C(K)$. Then every x^* in S_{X^*} is a convex combination of at most n elements $\pm q(\delta_k)$, $k \in K$. In particular, for every $x \in X$ it holds that $\|x\| = |x(k)|$ for at most n points $k \in K$.*

Proof. It is enough to apply the classical Minkowski–Carathéodory result on the convex hull of a finite-dimensional set, together with the fact that the set of extreme points of $S_{C(K)^*}$ is $\{\pm\delta_k : \ k \in K\}$. □

The other main result for $C(K)$ spaces provided by Phelps in [21] is Proposition 1.12 below. Let K be a compact space and $Z \subset K$ a closed subset. The **closed ideal of** $C(K)$ **defined by** Z is the linear subspace $I_Z(K) := \{f \in C(K) : f|_Z = 0\}$.

Proposition 1.12 (Phelps, [21, Lemma 3.1])**.** *Let K be a compact space and $Z \subset K$ a closed subset. Then, the closed ideal $I_Z(K)$ defined by Z has property U in $C(K)$.*

Phelps provided a direct proof of Proposition 1.12. Nowadays, it can be obtained as a consequence of the study of M-ideals since those have property U, and the M-ideals in $C(K)$ spaces are precisely the closed ideals of $C(K)$ —see [14].

2 U- and wU-embeddings

The following definition is just a formal extension of the notion of U- and wU- embeddability; it allows for some notational flexibility:

Definition 2.1. *Let X and Y be Banach spaces. Let $T\colon X \to Y$ be a linear isometry into. Given $y^* \in Y^*$ and $x^* \in X^*$, we say that y^* is an **extension of x^* through T** whenever $T^*(y^*) = x^*$. If additionally $\|y^*\| = \|x^*\|$ we will say that y^** **is a Hahn-Banach extension of** x^* **(through** T**)***. We say that T is a* **U-embedding** *(a* **wU-embedding***) whenever every $x^* \in S_{X^*}$ (respectively, every $x^* \in S_{X^*} \cap \mathrm{NA}(X)$) has a unique Hahn–Banach extension through T.*

In other words, T is a **U-embedding** (a **wU-embedding**) if, and only if, $T(X)$ is U-embedded (respectively, wU-embedded) in Y.

Definition 2.2. *Given a linear isometry into $T\colon X \to Y$ and a subset $A \subset X^*$, we define its* **set of Hahn–Banach extensions through** T *as:*

$$\mathcal{HB}_T(A) := \{y^* \in Y^* : \; y^* \in \mathcal{HB}_T(x^*) \text{ for some } x^* \in A\}.$$

Remark 2.3. It is worth stating that if the set $A \subset X^*$ is symmetric, so it is $\mathcal{HB}_T(A)$. ®

The corresponding characterization of the two properties of T in Definition 2.1 in terms of homeomorphisms is the following:

Proposition 2.4. *Let $T\colon X \to Y$ be an isometry into. Then,*

(i) *T is a U-embedding if, and only if, $T^*|_{\mathcal{HB}_T(S_{X^*})}\colon \mathcal{HB}_T(S_{X^*}) \to S_{X^*}$ is a w^*-w^*-homeomorphism.*

(ii) *T is a wU-embedding if, and only if, ${T^*}_{|\mathfrak{F}(T(x))}$ is a w^*-w^*-homeomorphism from $\mathfrak{F}(T(x))$ onto $\mathfrak{F}(x)$, for every $x \in S_X$.*

We will end this section by establishing an easy result about composition of U-embeddings:

Lemma 2.5. *Let $i\colon X \to Y$ and $j\colon Y \to Z$ be isometries into. Assume that j is a U-embedding. Then, the composition $j \circ i$ is a U-embedding if, and only if, i is a U-embedding.*

Notice that if $j \circ i$ is a U-embedding, so it is i; however, it is not possible to conclude that j should be a U-embedding, too (arguing in terms of subspaces, an element y^* can have two distinct Hahn–Banach extensions to Z that coincide on X). In the other direction, the previous lemma concludes that if i and j are both U-embeddings, so it is $j \circ i$.

3 Isometries into $C(K)$ and first consequences

3.1 A specific set-up for the $C(K)$ case

As usual, if K is a compact topological space, the space $C(K)$ of all the real-valued continuous functions defined on K is endowed with the supremum norm. Its dual $M(K)$ is the space of all regular Borel measures on K, and K is identified (topologically) with the subset $\delta K := \{\delta_k : \; k \in K\}$ of $S_{M(K)}$, endowed with the restriction of the w^*-topology on $M(K)$. We shall often consider δ as a function that sends $k \in K$ to $\delta_k \in M(K)$; thus, δ_k and $\delta(k)$ will denote the same element.

Remark 3.1. If X is a Banach space, the mapping $j\colon X \to C((B_{X^*}, w^*))$ given by $j(x) := x|_{(B_{X^*}, w^*)}$ is a linear isometric embedding. It is not a wU-embedding, even in a strong way: no $x^* \in S_{X^*}$ has a unique Hahn–Banach extension to $C((B_{X^*}, w^*))$. Indeed, if $x^* \in S_{X^*}$, both δ_{x^*} and $-\delta_{-x^*}$ are clearly two distinct Hahn–Banach extensions of x^* to $C((B_{X^*}, w^*))$. ®

Due to Remark 3.1, it is natural to try to find other —natural— embeddings of a Banach space into a $C(K)$ space.

For further reference, let us quote here that, *given a Banach space X and a compact space K, the space $B(X, C(K))$ of all bounded linear mappings from X into $C(K)$ and the space $C(K, (X^*, w^*))$ of all continuous functions from K into (X^*, w^*) are linearly isometric* —see, e.g., [10].
Indeed, the mapping that sends $T \in B(X, C(K))$ to $F_T \in C(K, (X^*, w^*))$ given by

$$F_T(k) = T^*(\delta_k), \text{ for } k \in K \tag{7}$$

is the sought linear isometry onto, as it is easy to show. If we identify, as usual, K and δK, then F_T is just $T^*|_{\delta K}$. Observe that the inverse image of $F \in C(K, (X^*, w^*))$ under this isometry is its **associated operator** $T_F \in B(X, C(K))$ given by

$$T_F(x)(k) = \langle x, F(k)\rangle, \text{ for } x \in X \text{ and } k \in K. \tag{8}$$

We are interested in the subset of $B(X, C(K))$ consisting of isometries into. Lemma 3.2 characterizes these operators T in terms of properties of the associated mapping F in $C(K, (X^*, w^*))$. First, we need two definitions.

Let X be a Banach space. A subset $\mathrm{JB} \subset B_{X^*}$ is said to be a **James boundary for** X if every $x \in S_X$ attains its norm at some point of JB. For example, the set $\operatorname{Ext} B_{X^*}$ of all extreme points of B_{X^*} is a James boundary for X. The Separation Theorem shows that $\overline{\operatorname{conv}}\,(\mathrm{JB}) = B_{X^*}$ for every James boundary JB. A subset N of B_{X^*} is said to be **1-norming for** X if $\sup_{x^* \in N} |\langle x, x^*\rangle| = 1$ for every $x \in S_X$. Clearly, every James boundary is 1-norming, and every 1-norming set N satisfies $\overline{\operatorname{conv}}\,(N \cup -N) = B_{X^*}$, so, in particular, N is w^*-linearly dense in X^*.
A simple observation is that, in case that $T\colon X \to Y$ is a linear isometry into and JB is a James boundary for Y (N is a 1-norming set for X) then $T^*(\mathrm{JB})$ is a James boundary for X (respectively, $T^*(N)$ is a 1-norming set for X).

Lemma 3.2. *Let K be a compact Hausdorff space, and let X be a Banach space. Let $F \in C(K, (X^*, w^*))$, and let $T_F \in B(X, C(K))$ be its associated operator defined in* (8)*. Then, the following are equivalent:*

(i) T_F *is a linear isometry into.*
(ii) $F(K)$ *is a James boundary for* X*.*
(iii) $F(K)$ *is* 1*-norming.*
(iv) $\text{Ext } B_{X^*} \subset F(K) \cup -F(K) \subset B_{X^*}$.

Proof. (i)⇒(ii) follows form the previous observation, since δK is obviously a James boundary for $C(K)$.
(ii)⇒(iii) is always true.
(iii)⇒(iv) is a consequence of the fact that $F(K)$ is a w^*-closed subset of B_{X^*} such that $\overline{\text{conv}}\,(F(K) \cup -F(K)) = B_{X^*}$, and Milman's converse to the classical Krein–Milman Theorem —see, e.g., [11, Theorem 3.66].
(iii)⇒(i) is clear, and (iv)⇒(iii) follows from the fact that $\text{Ext } B_{X^*}$ is 1-norming for X. □

Remark 3.3. Let $T\colon X \to C(K)$ be a linear isometric embedding. Put $K_0 := F_T(K)$ $(= T^*(\delta K) \subset B_{X^*})$, and endow it with the restriction of the w^*-topology. According to Lemma 3.2, K_0 is a James boundary for X. In particular $\overline{\text{conv}}\,(\pm K_0) = B_{X^*}$. Put $C(f) := f \circ F_T$ for $f \in C(K_0)$. This defines a linear mapping $C\colon C(K_0) \to C(K)$. Observe that

$$\begin{aligned}\|C(f)\|_\infty &= \sup_{k\in K} |C(f)(k)| \\ &= \sup_{k\in K} |(f \circ T^*)(\delta_k)| = \sup_{k_0\in K_0} |f(k_0)| = \|f\|_\infty,\end{aligned}$$

This shows that $C\colon C(K_0) \to C(K)$ is a linear isometry. Let $R\colon X \to C(K_0)$ be defined as $R(x) := x|_{K_0}$. Thus, R is a linear isometry from X into $C(K_0)$. Clearly, $T = C \circ R$. It is obvious that if T is a U-embedding, then so it is R. ®

3.2 The U-core

Definition 3.4. *Let* $T\colon X \to C(K)$ *be a linear isometry. We define its* **U-core** *as the set*

$$\mathfrak{C}_T := F_T^{-1}(\text{Ext } B_{X^*}) \subset K,$$

where F_T *is defined in* (7)*.*

For later reference, let us list some properties of the U-core. We omit the easy proofs.

(i) $\delta(\mathfrak{C}_T) = \mathcal{HB}_T(\text{Ext } B_{X^*}) \cap \delta(K) = (T^*)^{-1}(\text{Ext } B_{X^*}) \cap \delta(K)$.

(ii) As T is an isometry, item (iv) of Lemma 3.2 implies that $\mathfrak{C}_T$ is non-empty. Indeed, every extreme point of B_{X^*} is the image under T^* of an extreme point of $M(K)$, i.e., a point of the form δ_k or $-\delta_k$, for some $k \in K$. The result follows as Ext B_{X^*} is symmetric.

(iii) Since $\delta(\mathfrak{C}_T) \subset \delta(K)$, it follows that $\overline{\delta(\mathfrak{C}_T)}^{w^*} \cap \overline{-\delta(\mathfrak{C}_T)}^{w^*} = \emptyset$.

(iv) Put

$$\mathrm{Ext}^+ := T^*(\delta(\mathfrak{C}_T)) \ (= F_T(\mathfrak{C}_T)),$$

and

$$\mathrm{Ext}^- := T^*(-\delta(\mathfrak{C}_T)) \ (= -F_T(\mathfrak{C}_T)).$$

Notice that $\mathrm{Ext}^- = -\mathrm{Ext}^+$, and that Ext $B_{X^*} = \mathrm{Ext}^+ \cup \mathrm{Ext}^-$.

(v) $F_T(K) \cap \mathrm{Ext}\ B_{X^*} = \mathrm{Ext}^+$.

(vi) Since T^* is w^*-w^*-continuous, $T^*\left(\overline{\mathcal{HB}_T(\mathrm{Ext}\ B_{X^*})}^{w^*}\right) = \overline{\mathrm{Ext}\ B_{X^*}}^{w^*}$, and $F_T\left(\overline{\mathfrak{C}_T}\right) = \overline{\mathrm{Ext}^+}^{w^*}$.

(vii) From (iv) and (vi) it follows that

$$\overline{\mathrm{Ext}\ B_{X^*}}^{w^*} = \overline{\mathrm{Ext}^+}^{w^*} \cup -\overline{\mathrm{Ext}^+}^{w^*} \ \ (= F_T(\overline{\mathfrak{C}_T}) \cup -F_T(\overline{\mathfrak{C}_T})).$$

(viii) If T is a U-embedding, then $\delta(\mathfrak{C}_T) \subset \mathcal{HB}_T(S_{X^*})$. Since the restriction $T^*|_{\mathcal{HB}_T(S_{X^*})} \colon \mathcal{HB}_T(S_{X^*}) \to S_{X^*}$ is a w^*-w^*-homeomorphism —see Proposition 2.4 above— we have $\mathrm{Ext}^+ \cap \mathrm{Ext}^- = \emptyset$. Thus, Ext $B_{X^*} = \mathrm{Ext}^+ \dot\cup \mathrm{Ext}^-$.

3.3 Face structure of the dual of $C(K)$-spaces

Fix a compact Hausdorff space K. The Riesz representation theorem gives a handful description of the w^*-exposed face $\mathfrak{F}(f)$ determined by a function $f \in S_{C(K)}$ —see (2) for the definition of $\mathfrak{F}(f)$. This is the content of Lemma 3.5 below. A measure μ on K is said to be **supported on a Borel set** $B \subset K$ whenever $\mu(C) = 0$ for every closed $C \subset K$ disjoint with B. Given $f \in C(K)$, put

$$\sigma^+(f) := \{t \in K : \ f(t) = 1\},$$

and

$$\sigma^-(f) := \{t \in K : \ f(t) = -1\}.$$

Lemma 3.5 ([3])**.** *Let K be a compact Hausdorff space. Fix $f \in S_{C(K)}$. A measure $\mu \in S_{M(K)}$ belongs to the w^*-exposed face $\mathfrak{F}(f)$ if, and only, if its positive part μ^+ is supported on $\sigma^+(f)$ and its negative part μ^- is supported on $\sigma^-(f)$.*

Given a space X and a point $x \in S_X$, we observed in Remark 1.4 that the extreme points of the w^*-exposed face $\mathfrak{F}(x)$ are precisely the extreme points of B_{X^*} sitting in $\mathfrak{F}(x)$. In particular, if $X = C(K)$ for a compact topological space K, and $f \in S_{C(K)}$, then

$$\text{Ext } \mathfrak{F}(f) = \delta\left(\sigma^+(f)\right) \dot\cup \left(-\delta\left(\sigma^-(f)\right)\right). \tag{9}$$

Notice that each of the sets in the disjoint union in (9) is a closed set. If there exists a linear isometric embedding T from a space X into $C(K)$, we get in particular that, for $x \in S_X$,

$$\begin{aligned} \text{Ext } \mathfrak{F}(Tx) &= \{\delta_t : \langle Tx, \delta_t\rangle = 1\} \dot\cup \{-\delta_t : \langle Tx, -\delta_t\rangle = 1\} \\ &= \{\delta_t : |\langle Tx, \delta_t\rangle| = 1\} \\ &= \delta(\sigma^+(Tx)) \dot\cup - \delta(\sigma^-(Tx)). \end{aligned} \tag{10}$$

It is easy to see that

$$\sigma^+(Tx) = F_T^{-1}(\mathfrak{F}(x)), \text{ and } \sigma^-(Tx) = F_T^{-1}(\mathfrak{F}(-x)).$$

Remark 3.6. Assume that X is a Banach space wU-embedded in $C(K)$. If $x \in X$, it follows from (10) above that Ext $\mathfrak{F}(Tx)$ is w^*-closed. A consequence of Proposition 2.4 is that Ext $\mathfrak{F}(x)$ is w^*-closed, too. This simple remark plays a relevant role in some of the arguments below. ®

Due to Lemma 3.5, we have the following result, whose proof will be omitted.

Corollary 3.7. *Let $T: X \to C(K)$ be an isometric embedding and let $x \in S_X$. A necessary and sufficient condition for $\mu \in S_{C(K)^*}$ to belong to $\mathfrak{F}(T(x))$ is that its positive part μ^+ is supported on $F_T^{-1}(\mathfrak{F}(x))$ and its negative part μ^- is supported on $F_T^{-1}(\mathfrak{F}(-x))$.*

A particular kind of isometric embedding into $C(K)$-spaces is of interest. Namely, given a w^*-closed $K \subset B_{X^*}$, we can define the map $u_K: X \to C(K)$ by $u_K(x)(k) := \langle x, k\rangle$, for every $x \in X$ and every $k \in K$. The map u_K is an isometric embedding if and only if $\overline{\text{conv}}\,(\pm K) = B_{X^*}$ (Milman's converse to the Krein–Milman theorem shows that this happens if, and only if, Ext $B_{X^*} \subset \pm K$). If this is the case, given $x \in S_X$, it holds that

$$\begin{aligned} \sigma^+(u_K(x)) &= F_{u_K}^{-1}(\mathfrak{F}(x)) = K \cap \mathfrak{F}(x), \\ \sigma^-(u_K(x)) &= F_{u_K}^{-1}(\mathfrak{F}(-x)) = K \cap \mathfrak{F}(-x). \end{aligned}$$

Let us also state, for later reference, the translation of Corollary 3.7 for this particular isometric embedding.

Corollary 3.8. *Let X be a Banach space and $K \subset B_{X^*}$ be w^*-closed with $\overline{\operatorname{conv}}\,(\pm K) = B_{X^*}$. Given $x \in S_X$, a necessary and sufficient condition for $\mu \in S_{C(K)^*}$ to belong to $\mathfrak{F}(u_K(x))$ is that its positive part μ^+ is supported on $K \cap \mathfrak{F}(x)$ and its negative part μ^- is supported on $K \cap \mathfrak{F}(-x)$.*

4 Some necessary conditions for U- and wU-embeddability into $C(K)$

Let $T\colon X \to C(K)$ be a wU-embedding. A consequence of Proposition 2.4 above is that the duals of the spaces X and $C(K)$ must share a similar face structure. Some of the facts listed in Remarks 1.4 and 1.5 above make it natural to focus on the following concepts:

Definition 4.1 ([1])**.** *Simplices having a closed set of extreme points will be called* **Bauer simplices***. A* **(Bauer) simplexoid** —see Definition 16.8 in [20]— *is a compact convex set in a locally convex space whose proper closed faces are (Bauer) simplices. For a given Banach space X, we will say that its dual unit ball B_{X^*} is a* **quasi-Bauer simplexoid** *provided that every w^*-exposed face $\mathfrak{F}(x)$ with $x \in X$ is a Bauer simplex.*

As it was mentioned in Remark 1.5 above, the w^*-compact set $B_{C(K)^*}$ is a Bauer simplexoid. Thus, we have the following necessary condition for wU-embeddability into a $C(K)$-space:

Proposition 4.2. *Let X be a Banach space. If X is wU-embeddable in a space $C(K)$, then B_{X^*} is a quasi-Bauer simplexoid.*

Remark 4.3. As stated after Remark 1.3, a U-embedding of X into $C(K)$ would provide an affine w^*-w^*-homeomorphism between $\mathcal{HB}(S_{X^*}) \subset B_{C(K)^*}$ and S_{X^*}, so B_{X^*} inherits the property of being a Bauer simplexoid. The weaker notion of a quasi-Bauer simplexoid appears as a natural necessary condition of wU-embeddability into a $C(K)$ space due to the fact that, in this case, nothing can be said about w^*-closed faces of B_{X^*} that are not in any w^*-exposed face —see item (v) in Remark 1.4. ®

Next, we present a necessary condition for T_F being a U-embedding.

Lemma 4.4. *Given $F \in C(K, (X^*, w^*))$, a necessary condition for the operator $T_F\colon X \to C(K)$ to be a U-embedding is that $F(K)$ is a James boundary and that, given $t, s \in K$, $t \neq s$, such that $F(t) = F(s)$ (or, alternatively, $F(t) = -F(s)$), then $\|F(t)\| < 1$.*

Proof. The first condition is a consequence of Lemma 3.2. The second follows from the fact that, if $\|F(t)\| = 1$, then δ_t and δ_s (δ_t and $-\delta_s$, respectively), are two different Hahn–Banach extensions of $F(t)$. $\square$

The following result adds some extra information to Lemma 4.4 above:

Proposition 4.5. *Let $T\colon X \to C(K)$ be a U-embedding and $\mathfrak{C}_T \subset K$ be its U-core. Then $\|F_T(t)\| < 1$ for every $t \in K \setminus \overline{\mathfrak{C}_T}$.*

Proof. Let $k \in K \setminus \overline{\mathfrak{C}_T}$. Assume that $\|F_T(k)\| = 1$. Then $\delta_k \in \mathcal{HB}(S_{X^*})$. Find $f \in C(K)$ with range in $[0,1]$ such that $f(k) = 1$ and $f(c) = 0$ for each $c \in \overline{\mathfrak{C}_T}$. To simplify the notation, put $\mathrm{Ext} := \mathrm{Ext}\, B_{X^*}$ and write Ext^+ and Ext^- as in (iv), Subsection 3.2. Recall that $\mathrm{Ext} = \mathrm{Ext}^+ \dot\cup \mathrm{Ext}^-$, and that $\mathrm{Ext}^- = -\mathrm{Ext}^+$. Find a net $\{x_i^* :\ i \in I,\ \leq\}$ in $\mathrm{conv}\,(\mathrm{Ext})$ that w^*-converges to $F_T(k)$. For $i \in I$ put

$$x_i^* = \sum_{e^*\in \mathrm{Ext}^+} \alpha_{e^*}^i e^* + \sum_{e^*\in \mathrm{Ext}^-} \alpha_{e^*}^i e^*, \text{ where}$$
$$\sum_{e^*\in \mathrm{Ext}} \alpha_{e^*}^i = 1,\ \alpha_{e^*}^i \geq 0,\ \text{for } e^* \in \mathrm{Ext},$$

and the sum above has only a finite number of non-zero summands. Let us define

$$\mu_i := \sum_{e^*\in \mathrm{Ext}^+} \alpha_{e^*}^i \delta_{k_{e^*}} + \sum_{e^*\in \mathrm{Ext}^-} \alpha_{e^*}^i (-\delta_{k_{e^*}}),$$

where k_{e^*} is the (unique) element in K such that $F_T(k_{e^*}) = e^*$ if $e^* \in \mathrm{Ext}^+$ and $T^*(-\delta_{k_{e^*}}) = e^*$ if $e^* \in \mathrm{Ext}^-$. Observe that $\mu_i \in \mathrm{conv}\,(\pm\delta K) \subset B_{M(K)}$, that

$$T^*(\mu_i) = \sum_{e^*\in \mathrm{Ext}^+} \alpha_{e^*}^i e^* + \sum_{e^*\in \mathrm{Ext}^-} \alpha_{e^*}^i e^* = x_i^*,$$

and also

$$\begin{aligned}\langle f, \mu_i\rangle &= \sum_{e^*\in \mathrm{Ext}^+} \alpha_{e^*}^i \langle f, \delta_{k_{e^*}}\rangle + \sum_{e^*\in \mathrm{Ext}^-} \alpha_{e^*}^i \langle f, -\delta_{k_{e^*}}\rangle \\ &= \sum_{e^*\in \mathrm{Ext}^+} \alpha_{e^*}^i f(k_{e^*}) - \sum_{e^*\in \mathrm{Ext}^-} \alpha_{e^*}^i f(k_{e^*}) = 0\end{aligned}$$

for each $i \in I$. If we let μ ($\in B_{M(K)}$) be a w^*-cluster point of $\{\mu_i : i \in I,\ \leq\}$, then we have $\langle f, \mu\rangle = 0$ and $T^*\mu = F_T(k)$. Since $\|F_T(k)\| = 1$, we get that $\mu \in S_{M(K)}$. Thus, μ is a Hahn–Banach extension of $F_T(k)$, which is different from δ_k, a contradiction. $\square$

Example 4.6. Given a compact topological space K and a closed set $Z \subset K$, Proposition 1.12 above says that the closed ideal $I_Z(K)$ has property U in $C(K)$. For $T\colon I_Z(K) \to C(K)$ the canonical inclusion mapping, the U-core $\mathfrak{C}_T$ is just $K \setminus Z$. Clearly, $F_T(z) = 0$ for every $z \in Z$ (compare with Proposition 4.5 above). Thus, if Z does not reduces to a point, the w^*-w^*-homeomorphism $F_T\colon \mathfrak{C}_T \to F_T(\mathfrak{C}_T)$ cannot be extended to $\overline{\mathfrak{C}_T}$.

We will end this section by proving the existence of a specific type of subset in B_{X^*} that plays a role in the assumption of the U-embeddability of a space X into a space $C(K)$.

Definition 4.7. *Given a Banach space X, a w^*-*closed *set $E \subset B_{X^*}$ will be called* **U-suitable** *if it satisfies the following properties:*
(i) $E \cap (-E) \cap S_{X^*} = \emptyset$;
(ii) $E \cup (-E) = \overline{\mathrm{Ext}\ B_{X^*}}^{w^*}$.
If, moreover, it satisfies the following condition:
(iii) *for every $x \in S_X$, $E \cap \mathfrak{F}(x) = E \cap \overline{\mathrm{Ext}\ \mathfrak{F}(x)}^{w^*}$,*
*we will say that E is a **proper** U-suitable set.*

Observe that if E is a U-suitable set of X, then $E \cup (-E)$ is a James boundary for X.

The most natural example of a proper U-suitable set E is δK for the Banach space $X := C(K)$, where K is a compact topological space. In this case, $E \subset S_{X^*}$. The same applies to the Banach space $A(K)$.

The next result is almost clear. Its proof is routine and shall be omitted. It hints at why the concept of U-suitable set has been defined that way.

Proposition 4.8. *Let X and Y be two Banach spaces. Let $T : X \to Y$ be a linear isometry into. If T is a U-embedding and $E_Y \subset B_{Y^*}$ is a (proper) U-suitable set for Y, then $E_X := \overline{T^*(E_Y) \cap \mathrm{Ext}\ B_{X^*}}^{w^*}$ is a (proper) U-suitable set for X.*

Recall that $\mathrm{Ext}^+ = \overline{F_T(K) \cap \mathrm{Ext}}^{w^*}$, where F_T was defined in (7).

Corollary 4.9. *Let $T\colon X \to C(K)$ be a U-embedding. Then, the set $(F_T(\overline{\mathfrak{C}_T}) =)\ E_X := \overline{\mathrm{Ext}^+}^{w^*}$ $(\subset B_{X^*})$ is a proper U-suitable set for X.*

Remark 4.10. Observe that for (iii) in Definition 4.7 for the set E_X in Proposition 4.8 and Corollary 4.9, wU-embeddability of X into $C(K)$ suffices.

Remark 4.11. The same argument that proves Corollary 4.9 shows that if K is a convex compact subset of a locally convex space and $T : X \to A(K)$ is a U-embedding, then $E := \overline{\text{Ext}^{\ +}}^{w^*}$ is again a proper U-suitable set. It is worth to mention that every $C(K)$ space is linearly isometric to $A(B_{M(K)})$. Thus, Corollary 4.9 above could be deduced from the corresponding $A(K)$-version. ®

5 Spaces that cannot be U-embedded in $C(K)$

We obtained in the previous section a necessary condition for a Banach space X to be U-embedded into a $C(K)$ space, namely, the existence of a U-suitable set. This allows us to provide some classes of spaces that are not U-embeddable into a space $C(K)$.
In that direction, the feeling that whenever X is U-embedded into a space $C(K)$, then B_{X^*} cannot have too many extreme points, is supported by the following result, a consequence of Corollary 4.9 above

Corollary 5.1. *Let X be a Banach space such that $(\overline{\text{Ext } B_{X^*}}^{w^*} \cap S_{X^*}, w^*)$ is connected. Then X cannot be U-embedded into any $C(K)$-space.*

Proof. Assume that X is U-embedded in a space $C(K)$. Corollary 4.9 and (i) and (ii) in Definition 4.7 show that $(\overline{\text{Ext } B_{X^*}}^{w^*} \cap S_{X^*}, w^*)$ is disconnected. □

Corollary 5.2. *No Gâteaux smooth Banach space (with dimension greater than* 1*) can be U-embedded into a $C(K)$ space.*

Proof. If X is a Gâteaux smooth Banach space then $\text{NA}(X) \cap S_{X^*} \subset \text{Ext } B_{X^*}$ —since all w^*-exposed faces are singletons by [11, Theorem 7.17]— . By the Bishop–Phelps theorem, $\text{NA}(X) \cap S_{X^*}$ is norm-dense in S_{X^*} and, therefore,

$$S_{X^*} = \overline{\text{NA}(X) \cap S_{X^*}}^{\|\cdot\|} \subset \overline{\text{NA}(X) \cap S_{X^*}}^{w^*} \subset \overline{\text{Ext } B_{X^*}}^{w^*} \ (\subset B_{X^*}).$$

Thus, $\overline{\text{Ext } B_{X^*}}^{w^*} \cap S_{X^*} = S_{X^*}$, which is obviously connected in the w^*-topology. The result follows from Corollary 5.1. □

We will see later —see Corollary 6.9— that for two-dimensional subspaces, being non-Gâteaux smooth is equivalent to U-embeddability into $C(K)$-spaces.

6 Construction of U-embeddings

6.1 A sufficient condition in terms of Bauer simplices

Let us first show —as a consequence of Bauer's characterization of extreme points of compact convex subsets of locally convex spaces— that the existence of a U-suitable set $E \subset B_{X^*}$ gives uniqueness of the extension for extreme points of the unit ball of X^* that are simultaneously norm-attaining, through its associated isometric embedding u_E.

Proposition 6.1. *Let X be a Banach space for which a U-suitable set E exists. Then the canonical restriction-to-E map $u_E\colon X \to C(E)$ is an isometric embedding such that for every $x^* \in \operatorname{Ext} B_{X^*} \cap \mathrm{NA}(X)$ there exists a unique norm-one measure μ on E such that $u_E^*(\mu) = x^*$. Precisely, if $x^* \in E$ then $\mu = \delta_{x^*}$, and if $x^* \in -E$ then $\mu = -\delta_{-x^*}$.*

Proof. That $u_E\colon X \to C(E)$ is an isometric embedding was already noticed above —see the paragraph after Corollary 3.7. Fix $x^* \in \operatorname{Ext} B_{X^*} \cap \mathrm{NA}(X)$ that attains its norm at, say, $x \in S_X$. Assume that ν in $S_{M(E)}$ is such that $u_E^*(\nu) = x^*$. By Corollary 3.8, it holds that ν^+ is supported on $E \cap \mathfrak{F}(x)$ and ν^- is supported on $E \cap \mathfrak{F}(-x)$. Define the measure μ on B_{X^*} by

$$\mu(A) := \nu^+(A \cap E \cap \mathfrak{F}(x)) + \nu^-(-A \cap E \cap \mathfrak{F}(-x)),$$

for every Borel subset A of B_{X^*}. It is clear that μ is a probability measure on B_{X^*} —it is well defined due to the fact that $\mathfrak{F}(x)$ and $\mathfrak{F}(-x)$ are w^*-closed. It is also easy to see that for every $z \in X$, it holds that

$$\begin{aligned}\int z\, d\mu &= \int_{E\cap\mathfrak{F}(x)} z\, d\mu + \int_{-E\cap\mathfrak{F}(x)} z\, d\mu = \\ &= \int_{E\cap\mathfrak{F}(x)} z\, d\nu^+ - \int_{E\cap\mathfrak{F}(-x)} z\, d\nu^- = \\ &= \int z\, d\nu = \langle u_E(z), \nu\rangle = \langle z, x^*\rangle.\end{aligned}$$

By Bauer theorem [20, Proposition 1.4] applied to the convex and w^*-compact set B_{X^*}, we deduce that $\mu = \delta_{x^*}$, which implies that $\nu = \delta_{x^*}$ in case x^* belongs to E, and that $\nu = -\delta_{-x^*}$ in case x^* belongs to $-E$. $\square$

Definition 6.2 (See Propositions 1.1 and 1.2 in [20])**.** *Let $K \subset S$ be a compact convex subset of the locally convex space $(S, \mathcal{T})$. There exists an affine, onto, and w^*-$\mathcal{T}$-continuous map $r_K\colon P(\overline{\operatorname{Ext}\,(K)}) \to K$ (where $P(\overline{\operatorname{Ext}\,(K)})$*

is the set of all probability measures on $\overline{\text{Ext }(K)}$*) such that, for every* $f \in S^*$ *and every* $\mu \in P(\overline{\text{Ext }(K)})$,

$$\int f\, d\mu = \langle r_K(\mu), f \rangle.$$

The map r_K *is called the* **resultant map**. *The point* $r_K(\mu)$ $(\in K)$ *is usually called the* **resultant** *—or* **barycenter***— of the measure* μ.

In the concrete case where X is a Banach space and $S = (X^*, w^*)$, since $S^* = X$ it follows that for every $x^* \in K$ the preimage $(r_K)^{-1}(x^*)$ is precisely the set of representing probability measures of x^* on K; that is, a probability measure μ belongs to $(r_K)^{-1}(x^*)$ if, and only if, it satisfies $\langle y, x^* \rangle = \int_K y\, d\mu$ for every $y \in X$ —in a similar way, though not completely analogous, to the Riesz representation theorem in the case $X = C(K)$.

Let X be a Banach space. Given $x \in S_X$, put

$$\mathfrak{D}(x) := \overline{\text{Ext } \mathfrak{F}(x)}^{w^*} \ \ (\subset \mathfrak{F}(x) \subset S_{X^*})\,.$$

Certainly, $\mathfrak{F}(x) = \overline{\text{conv}}^{\,w^*}(\mathfrak{D}(x))$.

Put

$$\mathfrak{D}^+(x) := \mathfrak{D}(x) \cap E, \text{ and } \mathfrak{D}^-(x) := \mathfrak{D}(x) \cap -E.$$

Thus,

$$\mathfrak{D}(x) = \mathfrak{D}^+(x) \,\dot{\cup}\, \mathfrak{D}^-(x), \tag{11}$$

and both sets, $\mathfrak{D}^+(x)$ and $\mathfrak{D}^-(x)$, are w^*-closed.
The following lemma collects some easy facta for later reference. Its proof shall be omitted.

Lemma 6.3. *Let* E *be a U-suitable set for a Banach space* X*, and let* $u_E \colon X \to C(E)$ *be the natural isometric embedding defined above. Let* $x \in S_X$*. Then*

(i) $\text{Ext } \mathfrak{F}\,(u_E(x)) = \delta\,(E \cap \mathfrak{F}(x)) \,\dot{\cup} - \delta\,(E \cap \mathfrak{F}(-x))$.

(ii) $\text{Ext } \mathfrak{F}(x) \subset \mathfrak{D}(x) \subset (E \cap \mathfrak{F}(x)) \,\dot{\cup}\, (-E \cap \mathfrak{F}(x))$.

If E *is, moreover, proper, then*

(iii) $\mathfrak{D}(x) = (E \cap \mathfrak{F}(x)) \,\dot{\cup}\, (-E \cap \mathfrak{F}(x))$*. More precisely,* $\mathfrak{D}^+(x) = E \cap \mathfrak{F}(x)$ *and* $\mathfrak{D}^-(x) = -E \cap \mathfrak{F}(x)$.

We continue now the work done in Subsection 3.3.

Let X be a Banach space. Fix $x \in S_X$. To simplify the notation in the following two results, we shall write

$$\mathfrak{F} := \mathfrak{F}(x) \ (\subset B_{X^*})$$

and

$$\mathfrak{F}_u := \mathfrak{F}\left(u_E(x)\right) \ (\subset B_{M(E)}).$$

Lemma 6.4. *Let X be a Banach space. Let us assume that $E \subset B_{X^*}$ is a U-suitable set for X. Let $x \in S_X$. Then, the resultant mapping $r_{\mathfrak{F}_u} \colon P(\mathrm{Ext}\ \mathfrak{F}_u) \to \mathfrak{F}_u$ is one-to-one and onto.*

Proof. Let us prove first injectivity. It is enough to show that

$$r_{\mathfrak{F}_u}(\mu)(B) = \mu\left(\delta(B \cap \mathfrak{F})\right) - \mu\left(-\delta\left(B \cap \mathfrak{F}(-x)\right)\right) \tag{12}$$

for every compact subset B of E and every $\mu \in P(\mathrm{Ext}\ \mathfrak{F}_u)$. Let us sketch the proof. The reader may easily fill in the ε's missing. The set $\mathfrak{F}_u$ is a w^*-compact convex subset of the locally convex space $(M(E), w^*)$. Thus, given $f \in C(E)$, the definition of resultant shows that

$$\int f\, d\mu = \int_{\mathrm{Ext}\ \mathfrak{F}_u} f\, d\mu = \langle f, r_{\mathfrak{F}_u}(\mu)\rangle. \tag{13}$$

Find $g \in C(E)$ such that $g(e^*) = 1$ for $e^* \in B$ and that is almost 0 on $E \setminus B$. Notice that the way g acts on $M(E)$ shows that $\langle g, -\delta_{e^*}\rangle = -g(e^*)$ for all $e^* \in E$. Then

$$\begin{aligned} r_{\mathfrak{F}_u}(\mu)(B) &\approx \langle g, r_{\mathfrak{F}_u}(\mu)\rangle \\ &= \int_{\mathrm{Ext}\ \mathfrak{F}_u} g\, d\mu = \int_{\delta(E\cap\mathfrak{F})} g\, d\mu + \int_{-\delta(E\cap\mathfrak{F}(-x))} g\, d\mu \\ &\approx \mu\left(\delta(B \cap \mathfrak{F})\right) - \mu\left(-\delta\left(B \cap \mathfrak{F}(-x)\right)\right), \end{aligned}$$

where the first equality comes from (13). This (almost) shows (12) and then the sought injectivity.

Surjectivity is a consequence of a general representation theorem in this case since Ext $\mathfrak{F}_u$ is w^*-closed. □

Lemma 6.5. *Let X be a Banach space. Let us assume that $E \subset B_{X^*}$ is a U-suitable set for X. Then, for every $x \in S_X$ there is a one-to-one and w^*-w^*-continuous mapping $P \colon P\left(\mathfrak{D}(x)\right) \to P(\mathrm{Ext}\ \mathfrak{F}_u)$. Moreover, if E is proper, then P is onto.*

Proof. Let $\mu \in P\left(\mathfrak{D}(x)\right)$. We shall define a regular Borel measure $\widehat{\mu} := P(\mu)$ on Ext $\mathfrak{F}_u$. It is enough to define $\widehat{\mu}$ on the family of all closed (i.e., compact) subsets $\widehat{A}$ of Ext $\mathfrak{F}_u$. For such a set $\widehat{A}$, put

$$\widehat{\mu}(\widehat{A}) := \mu\left(u_E^*(\widehat{A}) \cap \mathfrak{D}(x)\right) \tag{14}$$

It is well defined since $u_E^*(\widehat{A})$ is a w^*-compact set. We claim that $\widehat{\mu}$ is a probability measure on Ext $\mathfrak{F}_u$. Indeed, $\widehat{\mu}$ is non-negative, and

$$\widehat{\mu}(\text{Ext } \mathfrak{F}_u) = \mu\left(\mathfrak{D}(x)\right) = 1. \tag{15}$$

Let us prove that P is w^*-w^*-continuous. To this end, observe that the mapping $u_E^* \colon M(E) \to X^*$ is w^*-w^*-continuous, and $u_E^*|_{\text{Ext } \mathfrak{F}_u}$ maps the w^*-compact set Ext $\mathfrak{F}_u$ onto the w^*-compact set $(\pm E) \cap \mathfrak{F}$ in a one-to-one way. Precisely, $u_E^*(\delta_{e^*}) = e^*$ for $e^* \in E \cap \mathfrak{F}$, and $u_E^*(-\delta_{e^*}) = -e^*$ for $e^* \in E \cap \mathfrak{F}(-x)$. Its inverse, denoted by $\Delta \colon (\pm E) \cap \mathfrak{F} \to \text{Ext } \mathfrak{F}_u$, is hence also w^*-w^*-continuous.

Let $\{\mu_i\}$ be a net in $P(\mathfrak{D}(x))$ that w^*-converges to $\mu \in P(\mathfrak{D}(x))$. Let $f \in C\left(\text{Ext } \mathfrak{F}\left(u_E(x)\right)\right)$. Put $g := f \circ \Delta$. It follows that $g \in C\left(\mathfrak{D}(x)\right)$. Thus, $\langle g, \mu_i \rangle \to \langle g, \mu \rangle$, hence $\langle f, P(\mu_i) \rangle \to \langle f, P(\mu) \rangle$, as we wanted to show. From (ii) in Lemma 6.3, it follows easily that P is one-to-one. If E is, moreover, proper, the surjectivity of P follows from the fact —see (i) and (iii) in Lemma 6.3— that $\mathfrak{D}(x)$ and Ext $\mathfrak{F}_u$ are homeomorphic in their w^*-topologies. □

The following result provides a sufficient condition for the existence of a wU-embedding. Observe that the quasi-Bauer simplexoid condition on B_{X^*} is unavoidable by Proposition 4.2

Theorem 6.6. *Let X be a Banach space admitting a proper U-suitable set E and whose dual ball is a quasi-Bauer simplexoid. Then the map u_E is a wU-embedding from X into $C(E)$.*

Proof. Let us assume that B_{X^*} is a quasi-Bauer simplexoid and that E is a U-suitable set for X. It is clear that u_E is an isometric embedding. We claim that u_E is a wU-embedding. Indeed, fix $x \in S_X$. By Lemma 6.5 there exits a bijective function $\gamma \colon \mathfrak{F}(u_E(x)) \to P(\mathfrak{D}(x))$ such that $u_E^* = r_x \circ \gamma$. If a face is a Bauer simplex, then the probabilities in $P(\mathfrak{D}(x))$ are maximal (incidentally, this is an equivalence) since they are supported on the extreme points of the face. Then, Choquet–Meyer uniqueness theorem implies that the resultant is bijective from the maximal probability measures onto $\mathfrak{F}(x)$. □

6.2 Finite-dimensional spaces

Along this section, X will be a finite-dimensional Banach space. We are interested in characterizing when it can be U-embedded into a $C(K)$-space. Observe that in this case —as finite-dimensional spaces are reflexive— U-embeddings and wU-embeddings coincide. Also, notice that any simplexoid in a finite-dimensional space is a Bauer simplexoid.

By Proposition 4.2, a necessary condition on X to be U-embedded into some $C(K)$-space is that its w^*-exposed faces $\mathfrak{F}(x)$ are Bauer simplices. In this case, we can rely on [21, Theorem 3.2]: Given $x \in S_X$, there exist a set $\{t_i\}_{i=1}^k \subset K$ and signs $\{\varepsilon_i\}_{i=1}^k$, with $k \leq n$ such that $\mathfrak{F}(x)$ is affine homeomorphic to the convex hull of the affinely independent set $\{\varepsilon_i \delta_{t_i}^K\}_{i=1}^k$ ($\subset S_{M(K)}$).

Theorem 6.7. *A finite-dimensional Banach space X can be U-embedded into a $C(K)$-space if, and only if, it admits a proper U-suitable set, and its dual unit ball is a simplexoid.*

Proof. One direction follows from Theorem 6.6. The other follows from Proposition 4.2 and Corollary 4.9. □

Since any ball in $\mathbb{R}^2$ is a simplexoid and every U-suitable set is proper —the set of extreme points of B_{X^*} is closed— we deduce the following corollary.

Corollary 6.8. *A two-dimensional Banach space X can be U-embedded into a $C(K)$-space if, and only if, it admits a U-suitable set.*

Again a concrete geometric property of two-dimensional spaces gives the following characterization.

Corollary 6.9. *A two-dimensional Banach space X can be U-embedded into a $C(K)$-space if, and only if, it is not Gâteaux smooth.*

Proof. On one hand, we know that a Banach space (of dimension greater than one) that can be U-embedded into a $C(K)$-space can not be Gâteaux by Corollary 5.2. On the other hand, assume that X is not Gâteaux and pick $x \in S_X$ where the norm is not Gâteaux smooth. By [11, Theorem 7.17], we can find x^* and y^* two extreme points of $\mathfrak{F}(x)$ and set $z^* = (x^* + y^*)/2$. Take now $z \in S_X$ such that $z^*(z) = 0$. Then, it is easy to see that $E = \{e^* \in \text{Ext } B_{X^*} : e^*(z) > 0\}$ is a U-suitable set for X. The conclusion follows from Corollary 6.8 □

Thus, a consequence of Proposition 6.23 below and the argument in the proof of Corollary 6.9, the space $C[0,1]$ is "U-embedding"-universal for the 2-dimensional non-Gâteaux spaces. Indeed, every such space has a U-suitable set simultaneously a retract and a zero-set of the interval $[0,1]$. We can then use Proposition 6.23 below.

Remark 6.10. The corollaries above are obtained as a consequence of some previous theorems. It is worth noticing that, for finite-dimensional spaces, it is not difficult to get them through explicit computations. For example, if we assume X is a finite-dimensional polyhedral space with simplexoid dual ball, then we also know that B_{X^*} has an even number of finite extreme points, say $2n$. Thus, one can just take $E \subset \text{Ext } B_{X^*}$ a selection of n "proper" extreme points (namely, for each $x^* \in \text{Ext } B_{X^*}$, we can just take either x^* or $-x^*$). Trivially, such set E satisfies the conditions of a U-suitable set, and thus X can be embedded in $C(E) \equiv C(\{1, 2, ..., n\})$ through the operator u_E. Also, even though it is obtained as a particular consequence of Theorem 6.7, it is easy to explicitly prove that every $x^* \in X^*$ has a unique Hahn–Banach extension through u_E. ®

6.3 Pairs of $C(K)$-spaces

Through this subsection, we characterize the pairs of compact Hausdorff spaces (K, S) for which $C(K)$ can be U-embedded into $C(S)$. The following result shows that the natural isometric embeddings between $C(K)$-spaces are never U-embeddings.

Lemma 6.11. *Let $h\colon S \to K$ be a continuous onto map between the compact Hausdorff spaces S and K. The isometric embedding $C_h\colon C(K) \to C(S)$ given by $C_h(f) = f \circ h$ for $h \in C(K)$ is a U-embedding if, and only if, h is one-to-one —so S and K are homeomorphic,— and if, and only if, C_h is onto.*

Proof. It is easy to see that the isometry C_h coincides with T_F for $F(s) := \delta_{h(s)}$, $s \in S$ —see equations (7) and (8). If h is not one-to-one then there exist s_1 and s_2 in S such that $s_1 \neq s_2$ and $h(s_1) = h(s_2)$. Thus, $F(s_1) = F(s_2) \in S_{M(K)}$. An appeal to Lemma 4.4 gives that T_F $(= C_h)$ is not a U-embedding.
Notice that C_h is always one-to-one. Thus, C_h is onto if, and only if, it is an isometry from $C(K)$ onto $C(S)$. This happens if, and only if, S and K are homeomorphic. □

Let us recall the following result from [3]. We need first a definition:

Definition 6.12. *Let S be a Hausdorff compact topological space. Let $f \in C(S)$ and $0 < \sigma < \varepsilon$. Let $u^f_{\sigma,\varepsilon} \in B_{C(S)}$ such that*

$$u^f_{\sigma,\varepsilon}|_{\{f \geq 1-\sigma\}} \equiv 1 \quad \text{and} \quad u^f_{\sigma,\varepsilon}|_{\{f \leq 1-\varepsilon\}} \equiv 0,$$

and let $v^f_{\sigma,\varepsilon} \in B_{C(S)}$ such that

$$v^f_{\sigma,\varepsilon}|_{\{f \leq -1+\sigma\}} \equiv 1 \quad \text{and} \quad v^f_{\sigma,\varepsilon}|_{\{f \geq -1+\varepsilon\}} \equiv 0.$$

Given $\mu \in M(S)$, let us define $\mu^{f,1}_{\sigma,\varepsilon}$ and $\mu^{f,2}_{\sigma,\varepsilon}$ in $M(S)$ by

$$\mu^{f,1}_{\sigma,\varepsilon}(g) := \int_S g.u^{f,1}_{\sigma,\varepsilon}\, d\mu \quad \text{and} \quad \mu^{f,2}_{\sigma,\varepsilon}(g) := \int_S g.v^{f,1}_{\sigma,\varepsilon}\, d\mu \quad \text{for all } g \in C(S).$$

Lemma 6.13 ([3, Lemma 2.8])**.** *Let S be a Hausdorff compact topological space. Let $f \in B_{C(S)}$ and $\mu \in B_{M(S)}$. Then, for every $0 < \sigma < \varepsilon < 1$, it holds that*

(i) $\|\mu^{f,1}_{\sigma,\varepsilon}\| \le 1$ *and* $\|\mu^{f,1}_{\sigma,\varepsilon}\| \le 1$.
(ii) $\|(\mu^{f,1}_{\sigma,\varepsilon})^+\| + \|(\mu^{f,2}_{\sigma,\varepsilon})^-\| \ge 1 - (1 - \langle f, \mu\rangle)/\sigma$.
(iii) $\|(\mu^{f,1}_{\sigma,\varepsilon})^-\| + \|(\mu^{f,2}_{\sigma,\varepsilon})^+\| \le (1 - \langle f, \mu\rangle)/\sigma$.
(iv) $\|\mu - \mu^{f,1}_{\sigma,\varepsilon} - \mu^{f,2}_{\sigma,\varepsilon}\| \le (1 - \langle f, \mu\rangle)/\sigma$.

The following will be a useful characterization.

Theorem 6.14. *Let K and S be compact Hausdorff spaces. An isometric embedding $T\colon C(K) \to C(S)$ is a U-embedding if, and only if, there exist a closed subset S_0 of S, a homeomorphism $h\colon K \to S_0$, and a continuous function $\varepsilon\colon K \to \{\pm 1\}$ such that*

(i) $F_T(h(k)) = \varepsilon(k)\delta^K_k$ *for every $k \in K$, and*
(ii) $\|F_T(s)\| < 1$ *for every $s \in S \setminus S_0$.*

Proof. Let us assume first that T is a U-embedding. Put

$$S_0 := F_T^{-1}(\delta(K)) \ (\subset S). \tag{16}$$

Obviously, S_0 is closed. Proposition 2.4 above shows that $F_T|_{S_0}\colon S_0 \to \delta(K)$ is a w^*-w^*-homeomorphism from S_0 onto $\delta(K)$. Put $h := (F_T|_{S_0})^{-1} \circ \delta^K\colon K \to S_0$. It is a w^*-w^*-homeomorphism from K onto S_0. The equation $T^*(\delta_{h(k)}) = \varepsilon(k)\delta^K_k$ defines a mapping $\varepsilon = K \to \{\pm 1\}$. It is easy to show that ε is continuous. Finally, the fact that $\|F_T(s)\| < 1$ for every $s \in S \setminus S_0$ follows from Proposition 4.5.

Let us now show a sketch of a proof for the reciprocal. Assume $T\colon C(K) \to C(S)$ is an isometric embedding with the properties in the statement.

Pick $\mu \in S_{M(K)}$ and take ν_1 and ν_2 in $S_{M(S)}$ such that $T^*(\nu_1) = T^*(\nu_2) = \mu$. Choose a sequence of functions $\{f_n\}_n \subset S_{C(K)}$ such that $\{\langle f_n, \mu\rangle\}_n$ is increasingly convergent to 1. This, in particular, means that the sequence $\{\langle T(f_n), \nu_i\rangle\}_n$ also increasingly converges to 1 for $i = 1, 2$. Let Z be a compact subset of $S\setminus h(K)$ such that $\nu_i\left((S \setminus h(K)) \setminus Z\right)$ is small for $i = 1, 2$. Since $\|F_T(s)\| < 1$ for every $s \in S\setminus h(K)$, we have $\sup\{\|F_T(s)\| : s \in Z\} < 1$. Then we can find $n_0 \in \mathbb{N}$ such that $\sigma^+(T(f_n)) \cup \sigma^-(T(f_n)) \subset S\setminus Z$ for every $n \ge n_0$. An easy consequence of Lemma 6.13 shows then that the measures

ν_i, $i = 1, 2$, are supported on $h(K)$. Given any function $f \in S_{C(S)}$, consider the function $g = \varepsilon \cdot (f \circ h)$, that belongs to $B_{C(K)}$. Then, for $s \in h(K)$ it holds that there exists $t \in K$ with $s = h(t)$ and

$$T(g)(s) = \langle g, F_T(h(t)) \rangle = \langle g, \varepsilon(t)\delta_t^K \rangle = \varepsilon(t)g(t) = f(s).$$

Since the support of ν_i is contained in $h(K)$, it follows that

$$\langle f, \nu_i \rangle = \langle T(g), \nu_i \rangle = \langle g, T^*(\nu_i) \rangle = \langle g, \mu \rangle,$$

which, in particular, implies that $\nu_1 = \nu_2$ as we wanted to show. □

Remark 6.15. Observe that in case that a linear isometry $T\colon C(K) \to C(S)$ is a U-embedding, then the set S_0 defined in (16) is the U-core $\mathfrak{C}_T$ of T —see Definition 3.4 above. Indeed, given $k \in K$ there is a single $s_0 \in S_0$ such that either $T^*(\delta_{s_0}) = \delta_k$ or $T^*(-\delta_{s_0}) = \delta_k$. This element is precisely $h(k)$. No element in $\delta(S \setminus S_0)$ can be applied to some δ_k for $k \in K$, since $\|T^*(\delta_s)\| < 1$. ®

The following example is a consequence of Theorem 6.14. It is interesting due to the fact that the space of convergent sequences c has property β but it is not polyhedral.

Corollary 6.16. *The Banach space c can be U-embedded into $C[0,1]$.*

Proof. Define K as the one-point compactification of $\mathbb{N}$. Then c is isometrically isomorphic to $C(K)$. Set $F\colon [0,1] \to B_{M(K)}$ by $F(0) = \delta_\infty$, $F(1/n) = \delta_n$, and for $t \in [1/(n+1), 1/n]$ the point $F(t)$ runs along the Bezier curve with control points δ_{n+1}, 0 and δ_n starting at δ_{n+1} and ending at δ_n, for $n \in \mathbb{N}$. The function F is w^*-continuous and defines an isometric embedding T_F as in equation (8) above. Since $\|F(t)\| < 1$ whenever $t \in [0,1]$ with $t \neq 1/n$ or $t \neq 0$, the function $h\colon K \to [0,1]$ defined as $h(n) = 1/n$ for $n \in \mathbb{N}$ and $h(\infty) = 0$ is a homeomorphism from K into $S_0 := \{0, 1, 1/2, 1/3, \ldots\}$ $(\subset [0,1])$ that satisfies the set of sufficient conditions in Theorem 6.14. This shows the statement. □

As mentioned in Proposition 1.12, Phelps [21] showed that given a closed subset A of a compact Hausdorff space K, the ideal $I_K(A) \subset C(K)$, consisting of those continuous function vanishing at every point of A, has property U in $C(K)$. As a consequence of this and the transitivity of the U-embeddings —see Lemma 2.5— we have the following corollary.

Corollary 6.17. *The Banach space c_0 can be U-embedded into $C[0,1]$.*

Remark 6.18. Corollary 6.17 shows something that should be clear from the beginning: The property of being U-embedded into a Banach space is not stable by renormings. Indeed, c_0 has an equivalent Gâteaux smooth norm $\|\cdot\|_1$, as it does every separable Banach space —see [13]. However, $(c_0, \|\cdot\|_1)$ cannot be embedded in any $C(K)$ space —see Corollary 5.2 above. ®

A topological subspace Z of a compact space S is called a **zero-set** whenever there exists $f \in C(S)$ such that $Z = \{t \in S : \ f(t) = 0\}$. The next lemma is well known; we omit the proof.

Lemma 6.19. *Let Z be a compact subspace of a normal space S. Then the following conditions are equivalent:*
(i) *Z is a zero-set in S.*
(ii) *Z is a G_δ-set in S.*

Corollary 6.20. *Let S and K be two compact Hausdorff spaces. Assume that there exists a U-embedding T from $C(K)$ into $C(S)$. Then the homeomorphic copy of K inside S given by Theorem* 6.14 *(i.e., the U-core $\mathfrak{C}_T$, see Remark* 6.15*) is a zero-set in S and so a G_δ-set in S.*

Proof. Let $h\colon K \to h(K) =: S_0$ $(\subset S)$ and $\varepsilon\colon K \to \{\pm 1\}$ be the functions given by Theorem 6.14. Observe that

$$\langle T\varepsilon, \delta_{s_0}\rangle = \langle \varepsilon, T^*\delta_{h(k)}\rangle = \langle \varepsilon, \varepsilon(k)\delta_k\rangle = (\varepsilon(k))^2 = 1, \quad \text{for } s_0 = h(k) \in S_0,$$

while

$$|\langle T\varepsilon, \delta_s\rangle| = |\langle \varepsilon, T^*\delta_s\rangle| \leq \|T^*\delta_s\| < 1, \quad \text{for } s \in S \setminus S_0.$$

This shows the statement. □

Following [19] and given a continuous embedding $h\colon K \to S$ between compact spaces, a linear operator $T\colon C(K) \to C(S)$ is said to be a **linear extension operator** for h whenever $F_T \circ h = \delta^K$. This terminology allows us to restate Theorem 6.14 as follows.

Theorem 6.21. *Let K and S be compact spaces. The following conditions are equivalent:*
(i) *There exists a U-embedding $T\colon C(K) \to C(S)$.*
(ii) *There exists a continuous embedding $h\colon K \to S$ admitting a norm-one extension operator with $h(K)$ a G_δ-set of S.*

Proof. (i)⇒(ii). Assume that $T\colon C(K) \to C(S)$ is a U-embedding. By Theorem 6.14, there exists an homeomorphic embedding $h\colon K \to S$ and a continuous map $\varepsilon\colon K \to \{\pm 1\}$ such that $F_T \circ h = \varepsilon \cdot \delta^K$ and $\|F_T(s)\| < 1$ for $s \in S \setminus h(K)$. Define $v\colon C(K) \to C(S)$ by $v(f) = T(\varepsilon \cdot f)$ for every $f \in C(K)$. Then, it is easy to check that $F_v \circ h = \delta^K$; thus v is a norm-one linear extension operator for h. The fact that $h(K)$ is a G_δ-set in S follows from Corollary 6.20 above.
(ii)⇒(i). Assume now that $h\colon K \to S$ is a continuous embedding admitting a norm-one extension operator $T\colon C(K) \to C(S)$ and such that $h(K)$ is a G_δ set of S. By using Theorem 6.14 with ε the constant function 1, we only need to check that $\|F_T(s)\| < 1$ for every $s \in S \setminus h(K)$. However, this last condition may be not satisfied, for instance, if T is regular —see [19, Definition 2.1]. Now, since $h(K)$ is a G_δ set of S, it is a zero-set of S by Lemma 6.19. Pick $g \in C(S)$ whose range is within $[0,1]$ and such that $h(K) = \{s \in S : \ g(s) = 1\}$. Define $v\colon C(K) \to C(S)$ by $v(f) := g \cdot T(f)$ for every $f \in C(K)$. Now, given $s \in S$ and $f \in C(K)$, it holds

$$\langle f, v^*(\delta_s^S)\rangle = \langle v(f), \delta_s^S\rangle = \langle g \cdot T(f), \delta_s^S\rangle = g(s)\langle T(f), \delta_s^S\rangle = g(s)\langle f, T^*(\delta_s^S)\rangle,$$

so $v^*(\delta_s^S) = g(s)T^*(\delta_s^S)$ for every $s \in S$. In case $s = h(t)$ for some $t \in K$, then

$$F_v(h(t)) = g(h(t))\delta_t^K = \delta_t^K,$$

since $g(h(t)) = 1$. In other case, for $s \in S \setminus h(K)$ it holds that $0 \le g(s) < 1$, so

$$\|F_v(s)\| = \|g(s)F_T(s)\| = |g(s)|\|F_T(s)\| < 1,$$

since T is norm-one. Now, an appeal to Theorem 6.14 shows that v is a U-embedding and the proof is over. □

Definition 6.22. *Let S be a compact Hausdorff space. We say that a subset $K \subset S$ is a* **retract** *of S if there exists a continuous onto mapping $r\colon S \to K$ such that $r(k) = k$ for every $k \in K$.*

The following result should be compared with Lemma 6.11.

Proposition 6.23. *Let S be a compact Hausdorff space and $K \subset S$ a retract which is a zero set. Then, there exists a U-embedding from $C(K)$ into $C(S)$.*

Proof. Let $r\colon S \to K$ be a retraction and $f \in C(S)$ with $K = f^{-1}(\{0\})$. We can assume without loss of generality that $0 \le f(s) \le 1$ for every $s \in S$. Set $F\colon S \to (M(K), w^*)$ defined for every $s \in S$ by

$$F(s) = \frac{1 - f(s)}{1 + f(s)}\delta_{r(s)}.$$

Since r is a retraction, we have $F(k) = \delta_k$ for every $k \in K$. If $s \in S \setminus K$ then $0 \le \|F(s)\| < 1$. Theorem 6.14 concludes that the associated isomorphism $T_F \colon C(K) \to C(S)$ is a U-embedding. $\square$

Remark 6.24. Proposition 6.23 above could also be obtained as a combination of [19, Proposition 3.3] and Theorem 6.21.

An easy corollary of the former proposition is the following:

Corollary 6.25. *Let $\alpha < \kappa$ be two ordinals. Then $C([0, \alpha])$ can be U-embedded into $C([0, \kappa])$.*

Proof. $[0, \alpha]$ is a retract of $[0, \kappa]$ and $[0, \alpha]$ is clearly a zero set in $[0, \kappa]$. $\square$

Corollary 6.26. *Let $h \colon K \to S$ be a neighborhood coretraction —i.e., for each $t \in K$, there is a closed neighborhood of t, say $V \subset K$, such that $h(V)$ is a retract of S. Then, $C(K)$ can be U-embedded into $C(S)$.*

Proof. This is an straightforward consequence of [19, Proposition 3.5] and Theorem 6.21. $\square$

Remark 6.27. Theorem 6.21 allows us to translate some of the results in [19] in terms of U-embeddings between C-spaces. For instance, we can show [19, Proposition 2.9] that $C(K, \mathbb{C})$ can be U-embedded into $C(S, \mathbb{C})$ if and only if $C(K)$ is U-embedded into $C(S)$. ®

6.4 Some applications

The following result is an easy but interesting corollary of Theorem 6.14.

Corollary 6.28. *A one-dimensional Banach space can be U-embedded into a $C(K)$ space if, and only if, K contains at least a G_δ-point.*

Proof. Theorem 6.14 states that $\mathbb{R} = C(\{1\})$ can be U-embedded into $C(K)$ if and only if there is a function $f \colon K \to [-1, 1]$ such that $\{t : |f(t)| = 1\}$ has cardinality exactly 1. This is equivalent to saying that there exists $t \in K$ such that $\{t\}$ is a zero-set of $C(K)$. A singleton in a completely regular space is a zero-set if and only if it is a G_δ-set. The proof is over. $\square$

Previous corollary can also be deduced from Corollary 1.9 and the fact that the points of Gâteaux smoothness in $C(K)$ are peaking functions (extremals of $B_{C(K)^*}$ are $\pm\delta(K)$ and Gâteaux equivales to w^*-exposed faces being singletons —see [11, Theorem 7.17]). As a simple application of it, we present the following necessary condition for U-embeddability (see also Proposition 6.31 below).

Corollary 6.29. *Let K and S be two compact spaces such that $C(K)$ is U-embedded into $C(S)$. If K has at least a G_δ-point, so does S.*

Proof. This is a consequence of Lemma 2.5 and the previous corollary. $\square$

It is known that a Valdivia compact K can be constructed without G_δ-points —see [6]— which leads to the fact that for any compact S with at least a G_δ-point (say, e.g., a metrizable or a convex space), the Banach space $C(S)$ can not be U-embedded into such $C(K)$ space.

Corollary 6.28 above can be generalized to a finite number of G_δ-points. Corollary 6.20 gives one direction. Namely, if ℓ_∞^N can be U-embedded into $C(K)$ then, K contains at least N G_δ-points. The reciprocal is also true:

Proposition 6.30. *Let N be a natural number and K be a compact Hausdorff space. Then the space ℓ_∞^N can be U-embedded into $C(K)$ if, and only if, K contains at least N G_δ-points.*

Proof. As we mentioned, one direction is given by Corollary 6.20 above. Let us now assume that K contains at least N G_δ-points $\{p_i\}_{i=1}^N \subset K$. We are going to construct a U-embedding from $C(\{p_i\}_{i=1}^N)$ —which is isometrically isomorphic to ℓ_∞^N— into $C(K)$. Choose for each of the points a family $\{G_n^i\}_{n=1}^\infty$ of decreasing open sets such that $\{p_i\} = \bigcap_n G_n^i$ for every $1 \leq i \leq N$. Due to the fact that K is Hausdorff, we can also assume that there exist a collection of pairwise disjoint open sets $\{U^i\}_{i=1}^N$ such that $G_n^i \subset U^i$ for every $n \geq 1$ and every $1 \leq i \leq N$. By the Urysohn lemma, there exists a continuous function $g_n \colon K \to [0,1]$ such that $g_n(p_i) = 1$ for each $1 \leq i \leq N$ and $g_n(t) = 0$ for every $t \notin \bigcup_i G_n^i$. Using this function, define $h_n \colon K \to B_{\ell_1^N}$ by $h_n(t) = g_n(t)\delta_{p_i}$ whenever t belongs to G_n^i and $0 \in \ell_1$ whenever t does not belong to the union $\bigcup_i G_n^i$. It is easy to see that this function is continuous (it can be seen to be continuous in each of $N+1$ closed sets covering K). Now consider the function $h = \sum_n \frac{1}{2^n} h_n$. It is a well-defined function from K to B_{ℓ_1} and is continuous —the partial sums are uniformly convergent over K, and we assume the topology of the norm on the unit ball. Now, observe that $\|h(t)\| = 1$ if and only if t belongs to $\{p_1, \ldots, p_N\}$ in which case $h(p_i) = \delta_{p_i}$ for every $1 \leq i \leq N$. By Theorem 6.14, the proof is over. $\square$

Related to the following result, see Corollary 6.29 above.

Proposition 6.31. *If K is a compact topological space such that no singleton is a G_δ, then $C(K)$ has no separable U-subspace.*

Proof. If $Y \subset C(K)$ is separable then, by Mazur's theorem, Y has a smooth point, which, since Y is a U-subspace, is now a smooth point of $C(K)$, inducing a singleton G_δ set in K. $\square$

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