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On topological groups with a weak q-point

This paper is dedicated to Professors Shou Lin and Peng-Fei Yan on the occasions of their 65th and 60th birthdays, respectively.

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ABSTRACT

In this article, firstly, we introduce the concepts of weak q-spaces and weak sq-spaces. Some properties of these spaces are discussed. Secondly, we study weak q-spaces and weak sq-spaces in topological groups in terms of preimages of submetrizable spaces. We show that a topological group G is a weak q-space (resp., weak sq-space) if and only if it is an open countable-compact (resp., sequential-compact) preimage of a submetrizable space. Finally, we give some characterizations of weakly feathered, weak q-spaces and weak sq-spaces in topological groups in coset spaces as follows:

(1) *Let H be a closed neutral subgroup of a topological group G. Then G/H is weakly feathered if and only if G/H is an open perfect preimage of a submetrizable space.*
(2) *Let H be a closed neutral subgroup of a topological group G. Then G/H is a weak q-space (resp., weak sq-space) if and only if G/H is an open countable-compact (resp., sequential-compact) preimage of a submetrizable space.*

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1. Introduction

All topological groups and spaces in this paper are Hausdorff unless stated otherwise.

A group G endowed with a topology τ is called a *paratopological group* if multiplication in G is continuous as a mapping of $G \times G$ to G, where $G \times G$ carries the usual product topology. A *topological group* is a paratopological group with continuous inversion. Given a topological group G and its subgroup H, the quotient space G/H means that the left coset with the quotient topology.

Feathered topological groups are very important in topological algebra. A topological group G is *feathered* [3, p.235] if G has a non-empty compact set K of countable character in G. A Tychnoff space X is *Čech-complete* if X is homeomorphic to a G_δ-set in a compact space, where a subset $A \subseteq X$ is called a G_δ-set if A is the intersections of countable open sets in X. All Čech-complete topological groups are feathered. It is also clear that all metrizable topological groups are feathered. B.A. Pasynkov established the following result in [11]:

Theorem 1.1 ([11])**.** *A topological group G is feathered if and only if G contains a compact subgroup H such that the quotient space G/H is metrizable.*

Recall that a continuous mapping f from a space X onto a space Y is *compact* [15] if for each $y \in Y$, $f^{-1}(y)$ is a compact set in X. f is called a *perfect* mapping [15] if f is a closed compact mapping.

In fact, based on the proof of Theorem 1.1, they have established the following result:

Theorem 1.2. *A topological group G is feathered if and only if G is an open perfect preimage of a metrizable space.*

Recall that a continuous mapping f from a space X onto a space Y is *countable-compact* [15], if for each $y \in Y$, $f^{-1}(y)$ is a countably compact set in X. f is called a *quasi-perfect* mapping [6, Definition 2.1.3], if f is a closed countable-compact mapping. A point $x \in X$ is called a *q-point* [9] of a space X if there exists a sequence $\{U_n : n \in \omega\}$ of open neighborhoods of x in X such that every sequence $\{x_n\}_{n\in\omega}$ of points in X such that $x_n \in U_n$ for each $n \in \omega$ has a point of accumulation in X. We also call $\{U_n : n \in \omega\}$ a *q-sequence* at x. A space X is said to be a *q-space* if every point of it is a q-point.

Recently, similar to feathered topological groups, Peng and Liu obtained the following result:

Theorem 1.3 ([12, Theorem 6])**.** *A topological group G is a q-space if and only if G is an open quasi-perfect preimage of a metrizable space.*

In the same paper, Peng and Liu also introduced the class of strong q-spaces.

A point $x \in X$ is called a *strong q-point* [12] of a space X if there exists a sequence $\{U_n : n \in \omega\}$ of open neighborhoods of x in X such that every sequence $\{x_n : n \in \omega\}$ of points in X such that $x_n \in U_n$ for each $n \in \omega$ has a convergent subsequence. We also call $\{U_n : n \in \omega\}$ a *strong q-sequence* at x. A space X is said to be a *strong q-space* if every point in X is a strong q-point.

Lin, Xie and Chen in [7] introduced the class of strict q-spaces as following: A point $x \in X$ is called a *strict q-point* [7] of a space X if there exists a q-sequence $\{U_n : n \in \omega\}$ at x such that $\bigcap_{n\in\omega} U_n$ is a sequentially compact set. We also call $\{U_n : n \in \omega\}$ a *strict q-sequence* at x. A space X is said to be a *strict q-space* if every point in X is a strict q-point.

To give a characterization of topological groups which are strict q-spaces and strong q-spaces. Lin, Xie and Chen introduced the following mappings in the same paper.

A continuous mapping $f : X \to Y$ is called *sequential-perfect* [7] if f is closed and $f^{-1}(y)$ is a sequentially compact set for each $y \in Y$. A continuous mapping $f : X \to Y$ is called *strongly sequential-perfect* [7] if f is closed and $f^{-1}(F)$ is a sequentially compact set for each sequentially compact set $F \subseteq Y$.

Lin, Xie and Chen established the following results:

Theorem 1.4 ([7, Theorem 2.4])**.** *A topological group G is a strict q-space if and only if G is an open sequential-perfect preimage of a metrizable space.*

Theorem 1.5 ([7, Theorem 2.6])**.** *A topological group G is a strong q-space if and only if G is an open strongly sequential-perfect preimage of a metrizable space.*

Recall that a topological space X has countable pseudocharacter if every singleton $\{x\}$ is a G_δ-set. A space X is submetrizable if it admits a coarser metrizable topology. It is clear that every submetrizable space is Hausdorff and has countable pseudocharacter. It is well known that a topological group G is submetrizable if and only if G has countable pseudocharacter [2, Proposition 1.7].

Recall that a topological group G is *weakly feathered* if G contains a non-empty compact G_δ-set F [3, p.248].

The following result is a characterization of weakly feathered topological groups.

Theorem 1.6 ([3, Exercise 4.3.f])**.** *A topological group G is weakly feathered if and only if it contains a compact subgroups H such that the left coset space G/H is submetrizable.*

In view of the definitions of feathered topological groups and q-spaces, we naturally introduce the concepts of weak q-spaces and weak sq-spaces (see Definitions 2.1 and 2.2) in Section 2.

Hence, in view of Theorem 1.2 and Theorem 1.6, it is natural to ask how to characterize weak q-spaces and weak sq-spaces in topological groups by certain preimages of submetrizable spaces. This is the first aim of our paper.

Recall that X is of *point-countable type* if for each $x \in X$, there exists a compact subset F_x of X such that $x \in F_x$ and $\chi(F_x, X) \leqslant \omega$ [1]. Recently, Ling in [8] gave some characterizations of spaces of point-countable type and q-spaces in coset spaces.

Theorem 1.7 ([8, Proposition 2.1.3])**.** *If H is a closed neutral subgroup of a topological group G, then G/H has point-countable type if and only if G/H is an open perfect preimage of a metrizable space.*

Theorem 1.8 ([8, Theorem 4.2.6])**.** *If H is a closed neutral subgroup of a topological group G, then G/H is a q-space if and only if G/H is an open quasi-perfect preimage of a metrizable space.*

Chen and Zhao in [4] gave some characterizations of strict q-spaces and strong q-spaces in coset spaces.

Theorem 1.9 ([4, Theorem 3.2])**.** *Let H be a closed neutral subgroup of a topological group G. Then G/H is a strict q-space if and only if G/H is an open sequential-perfect preimage of a metrizable space.*

Theorem 1.10 ([4, Theorem 3.3])**.** *Let H be a closed neutral subgroup of a topological group G. Then G/H is a strong q-space if and only if G/H is an open strongly sequential-perfect preimage of a metrizable space.*

In view of Theorems 1.9 and 1.10, it is natural to ask how to characterize weakly feathered, weak q-spaces and weak sq-spaces in coset spaces. This is the second aim of our paper.

This paper is organized as follows.

In Section 2, firstly, we give the definitions of weak q-spaces and weak sq-spaces and the relationship among spaces with certain weak q-sequences. Next, we study some properties of weak q-spaces and weak sq-spaces. It is proved that: (1) Every closed subspace of a weak q-space (resp., weak sq-space) is a weak q-space (resp., weak sq-space); (2) If X is a weak q-space and Y is a weak sq-space, then $X \times Y$ is a weak q-space; (3) The product of countably many regular weak sq-spaces is a weak sq-space; (4) Every quasi-perfect pre-image of a weak q-space is a weak q-space; (5) Every strong sequential-perfect pre-image of a weak sq-space is a weak sq-space.

In Section 3, we give some characterizations of weak q-spaces and weak sq-spaces in topological groups in terms of preimages of submetrizable spaces. Firstly, we introduce the concept of sequential-compact mapping and prove that G is a weak q-space (resp., weak sq-space) if and only if there is a closed countably compact (resp., sequentially compact) subgroup H of G such that the quotient space G/H is submetrizable if and only if G is an open countable-compact (resp., sequential-compact) preimage of a submetrizable space if and only if G contains a non-empty countably compact (resp., sequentially compact) set which is a G_δ-set. (see Theorem 3.2).

In Section 4, we give some characterizations of weak q-spaces and weak sq-spaces in topological groups in coset spaces in terms of preimages of submetrizable spaces. Firstly, we show that if H is a closed neutral subgroup of a topological group G such that G/H is weakly feathered, then there exists a closed subgroup K of G with $H \subseteq K$ such that K/H is compact if and only if G/H is an open perfect preimage of G/K if and only if G/K is submetrizable (see Proposition 4.2). Next, it is showed that: (1) Let H be a closed neutral subgroup of a topological group G. Then G/H is weakly feathered if and only if G/H is an open perfect preimage of a submetrizable space (see Theorem 4.3); (2) Let H be a closed neutral subgroup of a topological group G. Then G/H is a weak q-space (resp., weak sq-space) if and only if G/H is an open countable-compact (resp., sequential-compact) preimage of a submetrizable space (see Theorem 4.5).

2. Definitions and basic properties

Definition 2.1. A space X is a *weak q-space* if for each $x \in X$, there exists a non-empty countably compact G_δ-set F such that $x \in F$. We call $\{U_n : n \in \omega\}$ of open sets in X such that $F = \bigcap_{n\in\omega} U_n$ a *weak q-sequence* at x. Also, we call that x is a weak q-point.

Definition 2.2. A space X is a *weak sq-space* if for each $x \in X$, there exists a non-empty sequentially compact G_δ-set F such that $x \in F$. We call $\{U_n : n \in \omega\}$ of open sets in X such that $F = \bigcap_{n\in\omega} U_n$ a *weak sq-sequence* at x. Also, we call that x is a weak sq-point.

The following diagram demonstrates the relations between spaces defined above.

$$\begin{tikzcd}
\text{strong } q\text{-spaces} \arrow[r] \arrow[d] & q\text{-spaces} \arrow[d] & \text{point-countable type} \arrow[l] \\
\text{weak } sq\text{-spaces} \arrow[r] & \text{weak } q\text{-spaces} &
\end{tikzcd}$$

Fig.1.1 The relationship among spaces with certain weak q-sequences

Note that all inverses of the above relationships are not true.

Example 2.3 ([7, Example 1.3])**.** There exists a q-space which is not a strong q-space.

Example 2.4 ([12, Example 11])**.** There exists a strong q-space which is not a space of point-countable type.

Example 2.5. Every regular space with countable pseudocharacter, but not first-countable, is a weak sq-space which is not a q-space, since a regular space X is a q-space of countable pseudocharacter, X is a first-countable space.

The following Example 2.6 shows that there is a weak q-space but not a weak sq-space, since every regular compact space is a weak q-space.

Example 2.6. There is a regular compact space but not a weak sq-space.

Proof. Let $\beta\mathbb{N}$ be the Stone-Čech compactification of the discrete space $\mathbb{N}$. We know that there does not exist a non-trivial convergent sequence in $\beta\mathbb{N}$ [5, Corollary 3.6.15]. Thus every sequence $\{x_n\}_{n\in\omega}$ of points in $\beta\mathbb{N}$ with $x_i \neq x_j$ whenever $i \neq j$ has no convergent subsequence. It is obvious that every point $x \in \beta\mathbb{N} \setminus \mathbb{N}$ is not a G_δ-set. Hence, every point $x \in \beta\mathbb{N} \setminus \mathbb{N}$ dose not have a weak sq-sequence at x. □

Now we study the hereditary of weak q-spaces and weak sq-spaces.

Lemma 2.7. [5, Theorem 3.10.4 and Theorem 3.10.33] *Every closed subspace of a countably compact (resp., sequentially compact) space is countably compact (resp., sequentially compact).*

By Lemma 2.7, we can easily obtain the following result:

Proposition 2.8. *Every closed subspace of a weak q-space (resp., weak sq-space) is a weak q-space (resp., weak sq-space).*

Next we study the Cartesian products of weak q-spaces and weak sq-spaces.

Lemma 2.9 ([16, Proposition 3.4])**.**
(1) *The product of a sequentially compact space and a countably compact space is countably compact.*
(2) *The product of countably many sequentially compact spaces is sequentially compact.*

Proposition 2.10. *If X is a weak q-space and Y is a weak sq-space, then $X \times Y$ is a weak q-space.*

Proof. Let $\langle x, y\rangle$ be any point of $X \times Y$. Since $x \in X$ and X is a weak q-space, there excits a non-empty countably compact G_δ-set F_1 such that $x \in F_1$ and $F_1 = \bigcap_{n\in\omega} U_n$, where $\{U_n : n \in \omega\}$ is a set of countable open sets in X. Since $y \in Y$ and Y is a weak sq-space, there excits a non-empty countably compact G_δ-set F_2 such that $y \in F_2$ and $F_2 = \bigcap_{n\in\omega} V_n$, where $\{V_n : n \in \omega\}$ is a set of countable open sets in Y. It is clear that $\langle x, y\rangle \in F_1 \times F_2 = \bigcap_{n\in\omega} V_n \times \bigcap_{n\in\omega} U_n = \bigcap_{n\in\omega} V_n \times U_n$, where $V_n \times U_n$ is a set of countable open sets in $X \times Y$. Then $F_1 \times F_2$ is a G_δ-set. According to Lemma 2.9, $F_1 \times F_2$ is countably compact. We have thus proved that $X \times Y$ is a weak q-space. □

Proposition 2.11. *The product of countably many regular weak sq-spaces is a weak sq-space.*

Proof. For each $n \in \omega$, let X_n be a regular weak sq-space and $X = \prod_{n\in\omega} X_n$. Let $x = \langle x_n : n \in \omega\rangle$ be any point of X. For each point x_n in X_n, there exists a non-empty sequentially compact G_δ-set F_n such that $x_n \in F_n$. Let $F = \prod_{n\in\omega} F_n$. Then $x \in F$ and F is a sequentially compact subspace of X by (b) of Lemma 2.9. Next, we prove that F is a G_δ-set in X. Since F_n is a G_δ-set in X_n for each $n \in \omega$, take a family $\{U_{n,m} : m \in \omega\}$ of countable open sets in X_n such that $F_n = \bigcap_{m\in\omega} U_{n,m}$. Then one can easily show that $\bigcap_{n\in\omega}\bigcap_{m\in\omega}(U_{m,n} \times \prod_{n\neq n'} X_{n'}) = F$, which implies that F is a G_δ-set in X. Then X is a weak sq-space. □

To study the mapping preservation of weak q-spaces and weak sq-spaces, we give a useful lemma.

Lemma 2.12. *If a continuous mapping f from a space X onto a space Y is quasi-perfect, then $f^{-1}(K)$ is countably compact in X, for each countably compact set K in Y.*

Proposition 2.13. *Every quasi-perfect pre-image of a weak q-space is a weak q-space.*

Proof. Let Y be a weak q-space and $f : X \to Y$ be a quasi-perfect mapping. To show that X is a weak q-space it is only to show that for each point $x \in X$, there exists a non-empty countably compact G_δ-set H such that $x \in H$. Since Y is a weak q-space, there is a weak q-sequence $\{U_n : n \in \omega\}$ at $f(x)$ such that $F = \bigcap_{n\in\omega} U_n$ is a countably compact G_δ-set. Put $H = f^{-1}(F)$. By Lemma 2.12, H is a countably compact G_δ-set containing x. Thus we have proved that weak q-spaces preserve quasi-perfect pre-image. □

By the definition of strongly sequential-perfect mapping and the proof process of Proposition 2.13. We can obtain the following result:

Proposition 2.14. *Every strongly sequential-perfect pre-image of a weak sq-space is a weak sq-space.*

Morita in [10] introduced the important class of M-spaces and gave characterations of them as follows: A space X is an M-space (resp., paracompact M-space) if and only if there exists a quasi-perfect (resp., perfect) map from X onto a metrizable space. Thus, we pose the following question:

Question 2.15. *How to characterize a topological space X which is a countable-compact (resp., sequential-compact) pre-image of a metrizable space M?*

3. Topological groups with weak q-points

In this section, we give some "internal" characterizations of weak q-spaces and weak sq-spaces in topological groups.

We give a preliminary lemma which will be used later.

A subgroup H of a topological group G is called *admissible* [14, Definition 2.19] if there exists a sequence $\{U_n : n \in \omega\}$ of open symmetric neighborhoods of the identity in G such that $U_{n+1}^3 \subseteq U_n$

for each $n \in \omega$ and $H = \bigcap_{n\in\omega} U_n$. It is easy to see that every open neighborhood of e in G contains an admissible subgroup and that the family of admissible subgroups of G is closed under countable intersections. In particular, every G_δ-set P in G with $e \in P$ contains an admissible subgroup.

Lemma 3.1 ([3, Lemma 6.10.7])**.** *If H is an admissible subgroup of a topological group G, then the left coset space G/H is submetrizable.*

To characterize a topological group which is a weak sq-space, we introduce the following concept: A continuous mapping $f : X \to Y$ is called *sequential-compact* if $f^{-1}(y)$ is a sequentially compact set, for each $y \in Y$. Now we give some characterizations of topological groups which are weak q-spaces and weak sq-spaces.

Theorem 3.2. *Let G be a topological group with the identity e. Then the following statements are equivalent:*

(1) *G is a weak q-space (resp., weak sq-space);*
(2) *there is a closed countably compact (resp., sequentially compact) subgroup H of G such that the quotient space G/H is submetrizable;*
(3) *G is an open countable-compact (resp., sequential-compact) preimage of a submetrizable space;*
(4) *G contains a non-empty countably compact (resp., sequentially compact) set which is a G_δ-set.*

Proof. (1) ⇒ (2). If G is a weak q-space (resp., weak sq-space), then there is a weak q-sequence (resp., weak sq-sequence) $\{U_n : n \in \omega\}$ at e such that $F = \bigcap_{n\in\omega} U_n$ is a countably compact (resp., sequentially compact) G_δ-set. Let U be an open neighborhood of e. We define by induction a sequence $\{V_n : n \in \omega\}$ of symmetric open neighborhoods of e in G satisfying the following conditions:

(a) $V_0 \subseteq U \cap U_0$;
(b) $V_{n+1}^3 \subseteq V_n$ and $V_{n+1} \subseteq U_{n+1}$ for each $n \in \omega$.

Then $\overline{V_{n+1}} \subseteq V_n$ for each $n \in \omega$. Put $H = \bigcap_{n\in\omega} V_n$. Then H is a closed admissible subgroup in G. By Lemma 3.1, the quotient space G/H is submetrizable.

(2) ⇒ (3). Put $M = G/H$ and f is the natural quotient mapping from G to M. For each $f(x) \in M$, $f^{-1}(f(x)) = xH$ is a countably compact (resp., sequentially compact) set. Thus, f is a countable-compact (resp., sequential-compact) mapping.

(3) ⇒ (4). Let M be a submetrizable space and $f : G \to M$ be a countable-compact (resp., sequential-compact) mapping. Since M has countable pseudocharacter, there is a sequence $\{U_n : n \in \omega\}$ of open neighborhoods of $f(e)$ in M satisfying that $f(e) = \bigcap_{n\in\omega} U_n$. For each $n \in \omega$, put $V_n = f^{-1}(U_n)$ and $F = f^{-1}f(e)$. Then $F = \bigcap_{n\in\omega} V_n$ is a non-empty countably compact (resp., sequentially compact) G_δ-set of G.

(4) $\Rightarrow$ (1). Let H be a non-empty countably compact (resp., sequentially compact) set which is a countably compact (resp., sequentially compact) G_δ-set and $\{U_n : n \in \omega\}$ is an open neighborhood sequence of H such that $H = \bigcap_{n\in\omega} U_n$. By the homogeneity of G, we can assume that $e \in H$. It is obvious that $\{U_n : n \in \omega\}$ is a weak q-sequence (resp., weak sq-sequence) at e. Thus one can easily show that the sequence $\{gU_n : n \in \omega\}$ is a weak q-sequence (resp., weak sq-sequence) at g for each $g \in G$. We have proved that G is a weak q-space (resp., weak sq-space). $\square$

It is still an open problem that let G and H be topological groups and suppose that both groups are q-spaces. Is $G \times H$ a q-space [3, Open problem 6.9.4]? We pose the following question:

Question 3.3. *Let G and H be topological groups and suppose that both groups are weak q-spaces. Is $G \times H$ a weak q-space?*

4. The cosets spaces of topological groups with weak q-points

A subgroup H of a topological group G is called *neutral* [13] if for every open neighborhood U of the identity e in G, there exists an open neighborhood V of e such that $HV \subseteq UH$ (equivalently, $VH \subseteq HU$).

Lemma 4.1 ([4, Lemma 2.11])**.** *Let H and K be closed subgroups of a topological group G with $H \subset K$. Then the natural quotient mapping of G/H onto G/K is open.*

We start with the following proposition which plays an important role below.

Proposition 4.2. *Let H be a closed neutral subgroup of a topological group G. If G/H is weakly feathered, then there exists a closed subgroup K of G with $H \subseteq K$ such that the following conclusions are satisfied:*

(1) *K/H is compact;*
(2) *G/H is an open perfect preimage of G/K;*
(3) *G/K is submetrizable.*

Proof. (1) Let $p : G \to G/H$ be the natural quotient mapping. Since G/H is weakly feathered, we fix a sequence $\{U_n : n \in \omega\}$ of decreasing symmetric open neighborhoods of the identity e in G such that $\bigcap_{n\in\omega} p(U_n)$ is compact in G/H. We define by induction a sequence $\{W_n : n \in \omega\}$ of symmetric open neighborhoods of e in G satisfying (a): $(HW_{n+1}H)^2 \subseteq HW_nH \cap U_nH$, for each $n \in \omega$. The induction process of the sequence is as follows: Assume that $\{W_i : i \leqslant n\}$ satisfying (a) has been constructed. For W_n, take a symmetric open neighbourhoods V_1 of the identity in G such that $V_1^2 \subseteq W_n$. Since H is neutral, for V_1, there is a symmetric open neighbourhood W'_{n+1} of e such that $W'_{n+1}H \subseteq HV_1$ and that $W'_{n+1} \subseteq V_1$.

Thus, we can obtain that

$$(HW'_{n+1}H)^2 = HW'_{n+1}HW'_{n+1}H \subseteq HHV_1W'_{n+1}H \subseteq HV_1^2H \subseteq HW_nH.$$

Since H is neutral, for U_n, there is an open neighbourhood V_2 of e such that $HV_2 \subseteq U_nH$. For V_2, there is an open neighbourhood V_3 of e such that $V_3^2 \subseteq V_2$. Since H is neutral, for V_3, there is a symmetric open neighbourhood W''_{n+1} of e such that $HW''_{n+1} \subseteq V_3H$ and that $W''_{n+1} \subseteq V_3$. Thus, we can obtain that

$$(HW''_{n+1}H)^2 = HW''_{n+1}HW''_{n+1}H \subseteq HW''_{n+1}V_3HH \subseteq HV_3^2H \subseteq HV_2H \subseteq U_nHH = U_nH.$$

Put $W_{n+1} = W'_{n+1} \cap W''_{n+1}$, thus we have $(HW_{n+1}H)^2 \subseteq HW_nH \cap U_nH$ and W_{n+1} is symmetric. So $\{W_i : i \leqslant n+1\}$ satisfies (a). Continuing this construction process, we can obtain a sequence $\{W_n : n \in \omega\}$ of symmetric open neighborhoods of e in G satisfying (a): $(HW_{n+1}H)^2 \subseteq HW_nH \cap U_nH$, for each $n \in \omega$. Let $V_n = HW_nH$, for each $n \in \omega$. Put $K = \bigcap_{n\in\omega} V_n$. Since $V_{n+1}^2 \subset V_n$, for each $n \in \omega$ we have $\overline{V_{n+1}} \subset V_n$. By the definition of K it follows that $K = \bigcap_{n\in\omega} V_n = \bigcap_{n\in\omega} \overline{V_{n+1}}$ is a closed subgroup of G with $H \subseteq K$. Since $K = p^{-1}(p(K))$ is closed and p is a quotient mapping, $p(K)$ is closed in G/H. Because $K/H = p(K) \subset \bigcap_{n\in\omega} p(U_n)$ and the closed hereditary of compactness, we have that K/H is compact.

(2) Let $p : G \to G/H$ and $q : G \to G/K$ be the natural quotient mapping. Since $H \subset K$, there exists a mapping $\pi : G/H \to G/K$ satisfying the equality $q = \pi \circ p$.

$$\begin{tikzcd} G \arrow[r, "p"] \arrow[dr, "q"'] & G/H \arrow[d, "\pi"] \\ & G/K \end{tikzcd}$$

It is obvious that π is also a natural quotient mapping. Firstly, we show that π is closed. Let F be any closed subset of G/H, we need to show that $\pi(F)$ is closed in G/K. Put $E = p^{-1}(F)$. Since q is a quotient mapping, it is only to show that $q^{-1}(\pi(F))$ is closed in G. Since $q = \pi \circ p$, $q(E) = \pi(p(E)) = \pi(F)$ is holding. Thus, we obtain that $q^{-1}(\pi(F)) = q^{-1}(q(E)) = EK$. Now we shall prove that EK is closed in G, which implies that $q^{-1}(\pi(F))$ is closed in G. Take any $x \notin EK$, which is equivalent to $K \cap x^{-1}E = \varnothing$. We note that $H \subseteq K$. Thus $p^{-1}(p(K)) = KH = K$. This implies that $p(K) \cap p(x^{-1}E) = \varnothing$. Since $E = p^{-1}(F)$, we obtain that $EH = p^{-1}(F) = E$. This implies that $x^{-1}EH = x^{-1}E$, which implies that $p^{-1}(p(x^{-1}E)) = x^{-1}E$. Since p is a quotient mapping and $x^{-1}E$ is closed in G (by E being closed in G), $p(x^{-1}E)$ is closed in G/H.

For each $k \in K$, there exists an open neighbourhood V_k of e such that $p(V_k^2k) \cap p(x^{-1}E) = \varnothing$. Since $p(K)$ is compact and $\{p(V_k^2k) : k \in K)\}$ is an open cover of $p(K)$, there exists $\{k_1, k_2, ..., k_n\} \subseteq K$ such that

$p(K) \subseteq \bigcup_{i\leq n} p(V_{k_i}k_i)$. Put $U = \bigcap_{i\leq n} V_{k_i}$. Then $p(UK) \cap p(x^{-1}E) = \varnothing$. In fact, for each $k \in K$, there exists $i \leq n$ such that $p(k) \in p(V_{k_i}k_i)$. This implies that $k \in V_{k_i}k_iH$. It follows that $p(Uk) \subseteq p(UV_{k_i}k_iH) = p(UV_{k_i}k_i) \subseteq p(V_{k_i}^2 k_i)$. Since $p(V_{k_i}^2 k_i) \cap p(x^{-1}E) = \varnothing$, it implies that $p(Uk) \cap p(x^{-1}E) = \varnothing$ for each $k \in K$. Therefore, we have that $p(UK) \cap p(x^{-1}E) = \varnothing$. Note that $p^{-1}(p(x^{-1}E)) = x^{-1}EH = x^{-1}E$. Thus we obtain that $UK \cap x^{-1}E = \varnothing$ by $p(Uk) \cap p(x^{-1}E) = \varnothing$. i.e. $xU \cap EK = \varnothing$. This implies that $G\backslash EK$ is open in G. Thus EK is closed in G which implies that $\pi(F)$ is closed in G/K. Thus we have proved that the mapping π is a closed mapping.

Next, we show that $\pi^{-1}(y)$ is compact for each $y \in G/K$. Take an $x \in G$ such that $y = q(x)$. Since $q = \pi \circ p$, it is easy to see that

$$\pi^{-1}(y) = p(q^{-1}(y)) = p(xK).$$

Since $p^{-1}(p(xK)) = xKH = xK$ and $p^{-1}(p(K)) = KH = K$ by $H \subseteq K$, one can easily show that $p(xK)$ is homeomorphic to $p(K)$. It follows that $p(xK)$ is a compact subset of G/H. By Lemma 4.1, π is open. This proves that π is an open perfect mapping.

(3) By the definition of K, we have that K is an admissible subgroup of a topological group G. By Lemma 3.1, G/K is submetrizable. □

We give a characterization of weakly feathered topological groups in coset spaces.

Theorem 4.3. *Let H be a closed neutral subgroup of a topological group G. Then G/H is weakly feathered if and only if G/H is an open perfect preimage of a submetrizable space.*

Proof. Necessity. It can be obtained by Proposition 4.2.

Sufficiency. Let f be an open perfect mapping of G/H onto a submetrizable space M and $e^* = p(e)$, where $p : G \to G/H$ is the natural quotient mapping and e is an identity of G. Since M has countable pseudocharacter, there is a sequence $\{U_n : n \in \omega\}$ of open neighborhoods of $f(e^*)$ in M satisfies that $f(e^*) = \bigcap_{n\in\omega} U_n$. For each $n \in \omega$, put $V_n = f^{-1}(U_n)$ and $F = f^{-1}f(e^*)$, then $F = \bigcap_{n\in\omega} V_n$ is a non-empty compact G_δ-set of G/H. By the homogeneity of G/H, we can assume that $e^* \in F$. Thus one can easily show that $f^{-1}f(g) = \bigcap_{n\in\omega} gV_n$ is a non-empty compact G_δ-set for each $g \in G/H$. We have proved that G/H is weakly feathered. □

We establish the proposition similar to Proposition 4.2 in weak q-spaces and weak sq-spaces.

Proposition 4.4. *Let H be a closed neutral subgroup of a topological group G. If G/H is a weak q-space (resp., weak sq-space), then there exists a closed subgroup K of G with $H \subseteq K$ such that the following conclusions are satisfied:*

(1) *K/H is countably compact (resp., sequentially compact);*

(2) *G/H is an open countable-compact (resp., sequential-compact) preimage of G/K;*
(3) *G/K is submetrizable.*

Proof. The proof process is similar to Proposition 4.2. □

According to Proposition 4.4, we give characterizations of weak q-spaces and weak sq-spaces in coset spaces similar to Theorem 4.3.

Theorem 4.5. *Let H be a closed neutral subgroup of a topological group G. Then G/H is a weak q-space (resp., weak sq-space) if and only if G/H is an open countable-compact (resp., sequential-compact) preimage of a submetrizable space.*

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